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Whittaker-Shintani functions on the symplectic group of Fourier-Jacobi type

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0. Introduction

The Whittaker function is one of the fundamental tools in the theory of automorphic L-functions as is seen in the work of Jacquet and Langlands [6] (see also Bump's exposition [3]). In a recent paper [4], Bump showed an integral expression of the spin L-function $L_{\mathrm{spin}}(s, F)$ of a Siegel cusp form F of degree 3 in terms of its associated Whittaker function W_F and proved analytic continuation and functional equation of $L_{\mathrm{spin}}(s, F) \times$(a local factor at ∞) under the assumption $W_F \neq 0$. Unfortunately W_F vanishes if F is a *holomorphic* Siegel cusp form of degree not less than two and hence Bump's method does not work for holomorphic forms.

In [13], Shintani introduced Whittaker functions *of Fourier-Jacobi type* on Sp_{n+1}, which we call *Whittaker-Shintani functions* (briefly, *WS functions*) in this paper. One of the advantages of introducing such a modified Whittaker function is explained by the fact that the WS function associated with a holomorphic cusp form is not identically zero under a certain mild assumption on the form (see Corollary 5.3). Shintani investigated their basic properties and made several fundamental conjectures (see Conjectures 0.1 and 0.2 below). The purpose of this paper is to give an affirmative answer to his conjectures.

To compare the WS function with the original Whittaker function, we let F be a cusp form on $G^*(\mathbf{Q})\backslash G^*(\mathbf{A})$, where $\mathbf{A}$ is the adele ring of $\mathbf{Q}$ and $G^* = Sp_{n+1}$. Let $\psi_{\mathbf{A}}$ be the additive character of $\mathbf{A}$ trivial on $\mathbf{Q}$ with $\psi_{\mathbf{A}}(x_\infty) = \exp(2\pi i x_\infty)$ for $x_\infty \in \mathbf{R}$. Let $N^* = \left\{\begin{bmatrix} 1_{n+1} & B \\ 0 & 1_{n+1}\end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{bmatrix} \middle| A \text{ is an upper unipotent matrix of degree } (n+1) \text{ and } B = {}^tB \in M_{n+1}\right\}$ be a maximal unipotent subgroup of G^* and let ψ_{N^*} be the additive character of $N^*(\mathbf{A})$ trivial on $N^*(\mathbf{Q})$ defined by

$$\psi_{N^*}\left(\begin{bmatrix} 1_{n+1} & B \\ 0 & 1_{n+1}\end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{bmatrix}\right) = \psi_{\mathbf{A}}(b_{n+1,n+1}) \prod_{i=1}^{n} \psi_{\mathbf{A}}(a_{i,i+1}).$$

Then the *Whittaker function* associated with F is given by

$$W_F(g^*) = \int_{N^*(\mathbf{Q})\backslash N^*(\mathbf{A})} F(n^* g^*)\overline{\psi_{N^*}(n^*)}\,\mathrm{d}n^*. \tag{0.1}$$

Suppose that F is a common eigenform under the Hecke operators. Then, by the uniqueness property of local Whittaker functions (see [7, Proposition 2.1]), we obtain the following Euler decomposition for W_F:

$$W_F\left(\prod_v g_v^*\right) = \prod_v W_F^{(v)}(g_v^*), \tag{0.2}$$

where $W_F^{(v)}$ is a local Whittaker function on $G^*(\mathbf{Q}_v)$ for each prime v of $\mathbf{Q}$. It is well-known that $W_F \equiv 0$ if $n \geqslant 1$ and if F corresponds to a holomorphic cusp form.

In a similar manner, we can define Whittaker functions on reductive groups and the Euler decomposition (0.2) holds in general. Note that an explicit formula for local Whittaker functions is available by the work of Shintani [12] for GL_n, and by independent works of Kato ([7]; for split groups) and of Casselman-Shalika ([5]; for unramified groups).

To define WS functions, we let

$$\mathbb{G} = \left\{ \begin{bmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{bmatrix} \in G^* \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n \right\}$$

be the Jacobi group of degree n. Let f be a Jacobi cusp form on $\mathbb{G}$. Thus f is a function on $\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})$ satisfying

$$f\left(\begin{bmatrix} 1 & 0 & \kappa & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_n \end{bmatrix} \mathfrak{g}\right) = \psi_{\mathbf{A}}(\kappa) f(\mathfrak{g}) \quad (\kappa \in \mathbf{A},\ \mathfrak{g} \in \mathbb{G}(\mathbf{A}))$$

together with further nice conditions (for detail, see [8, §1]). For a pair (F,f) of a Siegel cusp form of degree $(n+1)$ and a Jacobi cusp form of degree n, the WS function $W_{F,f}$ is defined as follows:

$$W_{F,f}(g^*) = \int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} F(\mathbb{x} g^*)\overline{f(\mathbb{x})}\,\mathrm{d}\mathbb{x} \quad (g^* \in G^*(\mathbf{A})). \tag{0.3}$$

As an analogue of (0.2), Shintani conjectured the following:

CONJECTURE 0.1. If F and f are both common eigenforms under the Hecke operators, then

$$W_{F,f}\left(\prod_v g_v^*\right) = \prod_v W_{F,f}^{(v)}(g_v^*), \tag{0.4}$$

where $W_{F,f}^{(v)}$ *is a local WS function on* $G^*(\mathbf{Q}_v)$ for each prime v of $\mathbf{Q}$.

He also conjectured an integral expression of a quotient of the standard zeta functions of F and f in terms of the associated WS function:

CONJECTURE 0.2. Under the same assumption as in Conjecture 0.1, we have

$$\int_{\mathbf{A}_f^\times} W_{F,f}\left(\begin{bmatrix} t & & & \\ & 1_n & & \\ & & t^{-1} & \\ & & & 1_n \end{bmatrix}\right)|t|_{\mathbf{A}}^{s-n-1}\,\mathrm{d}^\times t$$

$$= \zeta(2s)^{-1}\overline{\mathbf{D}(\bar{s} + \tfrac{1}{2}, f)}^{-1} D(s, F) W_{F,f}(e)$$

where $\mathbf{A}_f^\times$ denotes the finite part of the idele group $\mathbf{A}^\times$, $|t|_{\mathbf{A}}$ the idele norm of $t \in \mathbf{A}^\times$ and $D(s, F)$ (resp. $\mathbf{D}(s,f)$) the standard zeta function attached to F (resp. to f).

In this paper, we establish these two conjectures (see Theorem 4.1 and Theorem 6.2).

We now explain a brief account of the paper. In §1, after recalling several basic properties of Hecke algebras of the symplectic group and of the Jacobi group, we introduce the space of local Whittaker-Shintani functions after Shintani and state one of the main results, the uniqueness of local WS functions (Theorem 1.2). To prove this, we first recall Shintani's results on the support of WS functions in §2. The proof is completed in §3 by showing the fact that the values of a WS function satisfy a system of difference equations with at most one solution. In §4, we give a proof of Conjecture 0.1 by applying the uniqueness theorem to each local component of the global WS function. In §5, we calculate explicitly the infinite component of the WS function associated with holomorphic cusp forms. Conjecture 0.2 is proved in the last section. The proof is based on some calculations of spherical functions on G^* and $\mathbb{G}$.

In the forthcoming paper, we will present an explicit formula for local WS functions on $\mathrm{Sp}_2(\mathbf{Q}_p)$. It is still an open problem to give an explicit formula for general n.

Notation

Let $r \geqslant 1$ be an integer. We let $\mathrm{Sp}_r = \{g \in \mathrm{GL}_{2r} | {}^t g J_r g = J_r\}$ be the symplectic group of degree r where $J_r = \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}$. Put

$$\mathbf{n}_r(x) = \begin{pmatrix} 1_r & x \\ 0 & 1_r \end{pmatrix}, \quad \mathbf{d}_r(y) = \begin{pmatrix} y & 0 \\ 0 & {}^t y^{-1} \end{pmatrix} \in \mathrm{Sp}_r (x \in \mathrm{Sym}_r,\ y \in \mathrm{GL}_r),$$

where Sym_r stands for the space of symmetric matrices of degree r. We denote by U_r the group of upper unipotent matrices of degree r and put

$$\mathrm{diag}(t_1, \ldots, t_r) = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_r \end{bmatrix} \in \mathrm{GL}_r.$$

We let $\Lambda_r = \{(a_1, \ldots, a_r) \in \mathbf{Z}^r | a_1 \geqslant \cdots \geqslant a_r \geqslant 0\}$ and set $\mathbf{e}[x] = \exp(2\pi i x)$ for $x \in \mathbf{C}$.

Throughout the paper, we fix an integer $n \geqslant 1$ and write G^* and G for Sp_{n+1} and Sp_n, respectively. We make a convention that each element of G^* (resp. G) is always denoted by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ $\left(\text{resp. } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$. The group G is embedded in G^* by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{bmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{bmatrix}.$$

Let $H(\mathbf{Q}) = \left\{(\lambda, \mu, \kappa) := \mathbf{n}_{n+1}\left(\begin{pmatrix} \kappa & \mu \\ {}^t\mu & 0 \end{pmatrix}\right) \cdot \mathbf{d}_{n+1}\left(\begin{pmatrix} 1 & \lambda \\ 0 & 1_n \end{pmatrix}\right) \in G^* \,\middle|\, \lambda, \mu \in \mathbf{Q}^n, \kappa \in \mathbf{Q}\right\}$ be the *Heisenberg group of degree n.* Then H is an algebraic subgroup of G^* normalized by G. Denote by $\mathbb{G} = \mathbb{G}_{n,1}$ the semi-direct product $H \cdot G$ of H and G in G^*. The non-reductive algebraic group $\mathbb{G}$ is called the *Jacobi group of degree n.* The center of $\mathbb{G}$ is $Z(\mathbb{G}) = \{(0, 0, \kappa)\}$. For simplicity, the typical elements $\mathbf{n}_{n+1}(X)$ and $\mathbf{d}_{n+1}(Y)$ of G^* are denoted by $\mathbf{n}^*(X)$ and $\mathbf{d}^*(Y)$, respectively ($X \in \mathrm{Sym}_{n+1}$, $Y \in \mathrm{GL}_{n+1}$). We also denote by $\mathbf{n}(x)$ and $\mathbf{d}(y)$ the elements $\mathbf{n}_n(x)$ and $\mathbf{d}_n(y)$ of G for $x \in \mathrm{Sym}_n$ and $y \in \mathrm{GL}_n$, respectively.

1. Local Whittaker-Shintani functions

Let E be a finite extension of $\mathbf{Q}_p$ and $\mathcal{o} = \mathcal{o}_E$ the ring of integers of E. Fix a prime element π of E and put

$$q = \#(\mathcal{o}/\pi\mathcal{o}). \tag{1.1}$$

Let ψ be an additive character of E with conductor $\mathcal{o}$. For an algebraic group X, we use the same notation X to denote the group $X(E)$ of E-rational points of X throughout §§1–3. Put $K^* = G^*(\mathcal{o})$ and $\mathbb{K} = \mathbb{G}(\mathcal{o})$.

Let $\mathscr{H}^* = \mathscr{H}(G^*, K^*)$ be the Hecke algebra of (G^*, K^*), the space of bi K^*-invariant functions on G^* with compact support. As usual, the multiplication is defined by

$$(\Phi_1 * \Phi_2)(g^*) = \int_{G^*} \Phi_1(g^* x^{*-1}) \Phi_2(x^*)\, dx^*, \tag{1.2}$$

where dx^* is the Haar measure on G^* normalized by $\int_{K^*} dx^* = 1$.

Let $\boldsymbol{\mathscr{H}} = \boldsymbol{\mathscr{H}}_\psi(\mathbb{G}, \mathbb{K})$ be the Hecke algebra of $(\mathbb{G}, \mathbb{K})$ with respect to the additive character ψ:

$$\begin{aligned}\boldsymbol{\mathscr{H}}_\psi(\mathbb{G}, \mathbb{K}) = \{\varphi\colon \mathbb{G} \to \mathbf{C} \mid\ & \\ &\text{(i) } \varphi((0, 0, \kappa)\Bbbk \mathbb{g} \Bbbk') = \psi(\kappa)\varphi(\mathbb{g})(\kappa \in E,\ \Bbbk,\ \Bbbk' \in \mathbb{K},\ \mathbb{g} \in \mathbb{G}). \\ &\text{(ii) } \varphi \text{ is compactly supported modulo } Z(\mathbb{G})\}.\end{aligned} \tag{1.3}$$

The multiplication is given by

$$(\varphi_1 * \varphi_2)(\mathbb{g}) = \int_{Z(\mathbb{G})\backslash\mathbb{G}} \varphi_1(\mathbb{g}\mathbb{x}^{-1})\varphi_2(\mathbb{x})\, d\mathbb{x}, \tag{1.4}$$

where $d\mathbb{x}$ is the Haar measure on $Z(\mathbb{G})\backslash\mathbb{G}$ normalized by $\int_{Z(\mathbb{G})\backslash Z(\mathbb{G})\mathbb{K}} d\mathbb{x} = 1$.

To describe Satake homomorphisms of the Hecke algebras $\mathscr{H}^*$ and $\boldsymbol{\mathscr{H}}$, let

$$N^* = \{\mathbf{n}^*(X)\mathbf{d}^*(Y) \mid X \in \mathrm{Sym}_{n+1},\ Y \in U_{n+1}\} \subset G^*,$$

$$\mathbb{N} = \{(0, \mu, 0)\mathbf{n}(x)\mathbf{d}(y) \mid \mu \in E^n,\ x \in \mathrm{Sym}_n,\ y \in U_n\} \subset \mathbb{G},$$

$$T^* = \left\{ \mathbf{d}^*\left(\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{n+1} \end{bmatrix}\right) \middle|\, t_i \in E^\times \right\} \subset G^*,$$

and

$$T = \left\{ \mathbf{d}\left(\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix}\right) \middle| t_i \in E^\times \right\} \subset G.$$

We normalize Haar measures dn^*, $d\mathbb{n}$, dt^* and dt on N^*, $\mathbb{N}$, T^* and T, respectively, so as to be

$$\int_{N^* \cap K^*} dn^* = \int_{\mathbb{N} \cap \mathbb{K}} d\mathbb{n} = \int_{T^*(\mathfrak{o})} dt^* = \int_{T(\mathfrak{o})} dt = 1. \tag{1.5}$$

Put $\delta_{N^*}(t^*) = d(t^* n^* t^{*-1})/dn^*$ and $\delta_{\mathbb{N}}(t) = d(t\mathbb{n}t^{-1})/d\mathbb{n}$ for $t^* \in T^*$ and $t \in T$.

Let $X_0(E^\times)$ be the group of unramified characters of $E^\times$. For $\chi = (\chi_1, \ldots, \chi_{n+1}) \in X_0(E^\times)^{n+1}$ (resp. $\xi = (\xi_1, \ldots, \xi_n) \in X_0(E^\times)^n$) and $\Phi \in \mathscr{H}^*$ (resp. $\varphi \in \mathbf{\mathscr{H}}$), we set

$$\chi^\wedge(\Phi) = \int_{T^*} \chi^{-1}(t^*)\Phi^\sim(t^*)\, dt \in \mathbf{C}, \tag{1.6}$$

$$\xi^\wedge(\varphi) = \int_{T} \xi^{-1}(t)\varphi^\sim(t)\, dt \in \mathbf{C}$$

where

$$\Phi^\sim(t^*) = \delta_{N^*}(t^*)^{-1/2} \int_{N^*} \Phi(n^* t^*)\, dn^* \qquad (t^* \in T^*), \tag{1.7}$$

$$\varphi^\sim(t) = \delta_{\mathbb{N}}(t)^{-1/2} \int_{\mathbb{N}} \Phi(\mathbb{n} t)\, d\mathbb{n} \qquad (t \in T).$$

Then $\Phi \to \chi^\wedge(\Phi)$ (resp. $\varphi \to \xi^\wedge(\varphi)$) gives rise to a $\mathbf{C}$-algebra homomorphism of $\mathscr{H}^*$ to $\mathbf{C}$ (resp. $\mathbf{\mathscr{H}}$ to $\mathbf{C}$). It is known that every homomorphism of $\mathscr{H}^*$ (resp. $\mathbf{\mathscr{H}}$) to $\mathbf{C}$ coincides with $\chi^\wedge$ (resp. $\xi^\wedge$) for some $\chi \in X_0(E^\times)^{n+1}$ (resp. $\xi \in X_0(E^\times)^n$) (these results are due to Satake and Shintani; for proofs, see [11] and [8]).

Let $C_\psi(\mathbb{K}\backslash G^*/K^*)$ be the space of functions F on G^* satisfying $F((0, 0, \kappa)\mathbb{k} g^* k^*) = \psi(\kappa)F(g^*)$ for $\kappa \in E$, $\mathbb{k} \in \mathbb{K}$, $k^* \in K^*$ and $g^* \in G^*$. Then the Hecke algebras $\mathscr{H}^*$ and $\mathbf{\mathscr{H}}$ act on $C_\psi(\mathbb{K}\backslash G^*/K^*)$ on the right and left respectively, as follows:

$$(\varphi * F * \Phi)(g^*) = \int_{Z(\mathbb{G})\backslash \mathbb{G}} d\mathbb{x} \int_{G^*} dx^*\, \overline{\varphi(\mathbb{x})}\, F(\mathbb{x} g^* x^{*-1})\Phi(x^*) \tag{1.8}$$

$(F \in C_\psi(\mathbb{K}\backslash G^*/K^*),\ \varphi \in \mathbf{\mathscr{H}},\ \Phi \in \mathscr{H}^*)$.

In [13], Shintani introduced a certain space of local Whittaker functions on G^* *of Fourier-Jacobi type*: For $(\chi, \xi) \in X_0(E^\times)^{n+1} \times X_0(E^\times)^n$, set

$$\mathrm{WS}_\psi(\chi, \xi) = \{W \in C_\psi(\mathbb{K}\backslash G^*/K^*) | \varphi * W * \Phi = \overline{\xi^\wedge(\varphi)}\chi^\wedge(\Phi) \cdot W \text{ for every } \varphi \in \mathscr{H} \text{ and } \Phi \in \mathscr{H}^*\}. \tag{1.9}$$

We call $\mathrm{WS}_\psi(\chi, \xi)$ the space of *Whittaker-Shintani functions associated with* (χ, ξ) (briefly, *WS-functions*). We can now state Shintani's fundamental conjecture on local WS functions [13]:

CONJECTURE 1.1. $\dim_{\mathbf{C}} \mathrm{WS}_\psi(\chi, \xi) = 1$.

In this paper we prove a half part of this conjecture, namely the uniqueness of WS functions:

THEOREM 1.2. $\dim_{\mathbf{C}} \mathrm{WS}_\psi(\chi, \xi) \leqslant 1$.

2. Support of Whittaker-Shintani functions

In this section, we recall Shintani's result on the support of WS functions. For $\mathbf{f} = (f_1, \ldots, f_{n+1}) \in \mathbf{Z}^{n+1}$ and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbf{Z}^n$, put

$$\Pi_{\mathbf{f}} = \mathbf{d}^*\left(\begin{bmatrix} \pi^{f_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{f_{n+1}} \end{bmatrix}\right) \in G^*, \qquad \pi_{\mathbf{m}} = \mathbf{d}\left(\begin{bmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_n} \end{bmatrix}\right) \in G. \tag{2.1}$$

LEMMA 2.1. *Let* $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$ *and* $\mathbf{f} = (f_1, \ldots, f_{n+1}) \in \mathbf{Z}^{n+1}$ *with* $f_2, \ldots, f_{n+1} \geqslant 0$. *Then* $F((\lambda, \mu, \kappa)\Pi_{\mathbf{f}}) \neq 0$ *implies* $f_1 \geqslant 0$ *and* $\mu \in \mathscr{o}^n$.

Proof. Put $g^* = (\lambda, \mu, \kappa)\Pi_{\mathbf{f}}$ and assume $F(g^*) \neq 0$. For every $\kappa_1 \in \mathscr{o}$, we have $F(g^*) = F(g^*(0, 0, \kappa_1)) = F((0, 0, \pi^{2f_1}\kappa_1)g^*) = \psi(\pi^{2f_1}\kappa_1)F(g^*)$. This proves $f_1 \geqslant 0$. Suppose $\mu = (\mu_1, \ldots, \mu_n) \notin \mathscr{o}^n$. We may suppose that $\mu_1 \in \pi^{-r}\mathscr{o}^\times$ with $r > 0$ without loss of generality. Then, for every $\varepsilon \in \mathscr{o}^\times$,

$$\begin{aligned}
F(g^*) &= F\left(g^{*t}\mathbf{n}\left(\begin{pmatrix} \pi^{2f_2+r}\varepsilon & 0 \\ 0 & 0 \end{pmatrix}\right)\right) = F\left((\lambda, \mu, \kappa)^t\mathbf{n}\left(\begin{pmatrix} \pi^r\varepsilon & 0 \\ 0 & 0 \end{pmatrix}\right)\Pi_{\mathbf{f}}\right) \\
&= F\left({}^t\mathbf{n}\begin{pmatrix} \pi^r\varepsilon & 0 \\ 0 & 0 \end{pmatrix}(\lambda + (\pi^r\mu_1\varepsilon, 0, \ldots, 0), \mu, \kappa + \pi^r\varepsilon\mu_1^2)\Pi_{\mathbf{f}}\right) \\
&= F(((\pi^r\mu_1\varepsilon, 0, \ldots, 0), 0, -\pi^r\varepsilon\mu_1^2)g^*) \\
&= \psi(-\pi^r\varepsilon\mu_1^2)F(g^*),
\end{aligned}$$

where ${}^t\mathbf{n}(x) = \begin{pmatrix} 1_n & 0 \\ x & 1_n \end{pmatrix} = \begin{bmatrix} 1 & & & \\ & 1_n & & \\ & & 1 & \\ & x & & 1_n \end{bmatrix}$. This is a contradiction, since we can choose $\varepsilon \in \mathcal{o}^\times$ so that $\psi(\pi^r \varepsilon \mu_1^2) \neq 1$. $\square$

Let

$$\Lambda^* = \{(f, \mathbf{m}; \mathbf{r}) \in \mathbf{Z} \times \mathbf{Z}^n \times \mathbf{Z}^n \mid f \geqslant 0 \text{ and } \mathbf{m}, \mathbf{r}, \mathbf{m} - \mathbf{r} \in \Lambda_n\} \tag{2.2}$$

denote the set of "dominant" vectors in $\mathbf{Z}^{2n+1}$ and "$<$" the usual dictionary order in Λ^*. For $\mathbf{r} \in \mathbf{Z}^n$, put

$$h(\mathbf{r}) = ((\pi^{-r_1}, \dots, \pi^{-r_n}), 0, 0) \in H. \tag{2.3}$$

PROPOSITION 2.2. *The support of every* $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$ *is contained in*

$$\bigcup_{(f, \mathbf{m}; \mathbf{r}) \in \Lambda^*} Z(\mathbb{G})\mathbb{K} \cdot h(\mathbf{r})\Pi_{f,\mathbf{m}} \cdot K^*.$$

Proof. Let $g^* \in G^*$ be in the support of $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$. By Iwasawa decomposition for G^* and Cartan decomposition for G, g^* can be written in the form $k \cdot h\Pi_{f,\mathbf{m}} \cdot k^*$ ($k \in K$, $h \in H$, $f \in \mathbf{Z}$, $\mathbf{m} \in \Lambda_n$, $k^* \in K^*$). In view of Lemma 2.1, we may assume $g^* = h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}$ with $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n) \in \mathbf{Z}^n$ ($\rho_i \geqslant 0$), $f \geqslant 0$ and $\mathbf{m} \in \Lambda_n$. We now show

$$\rho_i \leqslant m_i (1 \leqslant i \leqslant n). \tag{2.4}$$

Suppose, say, $\rho_1 > m_1$. Then, for $\varepsilon \in \mathcal{o}^\times$,

$$\begin{aligned} F(h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}) &= F\left(h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}\mathbf{n}\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}\right)\right) = F\left(h(\boldsymbol{\rho})\mathbf{n}\left(\begin{pmatrix} \pi^{2m_1}\varepsilon & 0 \\ 0 & 0 \end{pmatrix}\right)\Pi_{f,\mathbf{m}}\right) \\ &= F(((\pi^{-\rho_1}, \dots, \pi^{-\rho_n}), (\pi^{2m_1 - \rho_1}\varepsilon, 0, \dots, 0), \pi^{2m_1 - 2\rho_1}\varepsilon) \cdot \Pi_{f,\mathbf{m}}). \end{aligned}$$

By Lemma 2.1, we have $2m_1 - \rho_1 \geqslant 0$ and hence

$$F(h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}) = \psi(\pi^{2(m_1 - \rho_1)}\varepsilon)F(h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}).$$

This is a contradiction, since we can choose $\varepsilon \in \mathcal{o}^\times$ so that $\psi(\pi^{2(m_1 - \rho_1)}\varepsilon) \neq 1$. The remaining part of the claim is similarly proved. The proposition is now an immediate consequence of the following lemma, which will be also used in the next section.

LEMMA 2.3. *Let* $(f, \mathbf{m}; \mathbf{r}) \in \Lambda^*$. *If* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in \mathbf{Z}^n$ *satisfies* $0 \leqslant \rho_i \leqslant r_i$ $(1 \leqslant i \leqslant n)$, *then there exists* $\mathbf{r}' \in \Lambda_n$ *satisfying the following conditions*:

(a) $(f, \mathbf{m}; \mathbf{r}') \in \Lambda^*$
(b) $(f, \mathbf{m}; \mathbf{r}') \leqslant (f, \mathbf{m}; \mathbf{r})$
(c) $h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}} \in \mathbb{K} \cdot h(\mathbf{r}')\Pi_{f,\mathbf{m}} \cdot K^*$

(*hence* $F(h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}}) = F(h(\mathbf{r}')\Pi_{f,\mathbf{m}})$ *for every* $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$).

Proof. We first consider the case where $n = 2$. Suppose that $\rho_1 \geqslant \rho_2$ and $m_1 - \rho_1 \geqslant m_2 - \rho_2$. In this case, $\mathbf{r}' = \boldsymbol{\rho}$ satisfies the conditions of the lemma. Next suppose that $\rho_1 \geqslant \rho_2$ and $m_1 - \rho_1 < m_2 - \rho_2$. Since

$$h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}} \cdot \mathbf{d}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \mathbf{d}\left(\begin{pmatrix} 1 & \pi^{m_1 - m_2} \\ 0 & 1 \end{pmatrix}\right) \cdot ((\pi^{-\rho_1}, \pi^{-(\rho_1 - m_1 + m_2)}), 0, 0)\Pi_{f,\mathbf{m}},$$

we have $h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}} \in \mathbb{K} \cdot h(\rho_1, \ \rho_1 - m_1 + m_2)\Pi_{f,\mathbf{m}} \cdot K^*$. Note that $0 \leqslant \rho_1 - m_1 + m_2 \leqslant r_1 - m_1 + m_2 \leqslant r_2$ and that $m_1 - \rho_1 \geqslant m_2 - (\rho_1 - m_1 + m_2)$. Hence $\mathbf{r}' = (\rho_1, \rho_1 - m_1 + m_2)$ satisfies the conditions (a), (b) and (c).

Next suppose $\rho_1 < \rho_2$. Since

$$\begin{aligned} &\mathbf{d}\left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right) \cdot h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}} \\ &\quad = ((\pi^{-\rho_1} + \pi^{-\rho_2}, \pi^{-\rho_2}), 0, 0) \cdot \Pi_{f,\mathbf{m}} \cdot \mathbf{d}\left(\begin{pmatrix} 1 & 0 \\ -\pi^{m_1 - m_2} & 1 \end{pmatrix}\right), \end{aligned}$$

we have $h(\boldsymbol{\rho})\Pi_{f,\mathbf{m}} \in \mathbb{K} \cdot h(\rho_2, \ \rho_2)\Pi_{f,\mathbf{m}} \cdot K^*$. Thus we can take $\mathbf{r}' = (\rho_2, \rho_2)$. The lemma for $n = 2$ has been proved.

We now consider the general case. By repeating the second argument, we may assume that $\rho_1 \geqslant \cdots \geqslant \rho_n$. For such a $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$, we denote by $i = i(\boldsymbol{\rho})$ the smallest integer i such that $m_{i-1} - \rho_{i-1} < m_i - \rho_i$. By the first argument, we may replace ρ_i by $\rho_i' = \rho_{i-1} - m_{i-1} + m_i$. It is obvious that $\rho_i' \geqslant \rho_{i+1}$ and $i(\rho_1, \ldots, \rho_{i-1}, \ \rho_i', \ \rho_{i+1}, \ldots, \rho_n) > i(\boldsymbol{\rho})$. The lemma is proved by repeating this process. □

3. Uniqueness of local Whittaker-Shintani functions

For simplicity, we write $W(f, \mathbf{m}; \mathbf{r})$ for $W(h(\mathbf{r})\Pi_{f,\mathbf{m}})$ for $W \in \mathrm{WS}_\psi(\chi, \xi)$ and

$(f, \mathbf{m}; \mathbf{r}) \in \mathbf{Z} \times \mathbf{Z}^n \times \mathbf{Z}^n$. To prove Theorem 1.2, it is sufficient to show that, for $(f, \mathbf{m}; \mathbf{r}) \in \Lambda^*$, the value $W(f, \mathbf{m}; \mathbf{r})$ depends only on χ, ξ and $W(e) = W(0, \mathbf{0}; \mathbf{0})$ (e = the identity element of G^*, $\mathbf{0} = (0, \ldots, 0) \in \mathbf{Z}^n$). In fact, we prove the following result, from which the above assertion is derived.

THEOREM 3.1. *Let* $W \in \mathrm{WS}_\psi(\chi, \xi)$ *and* $(f, \mathbf{m}; \mathbf{r}) \in \Lambda^*$. *Then*

$$W(f, \mathbf{m}; \mathbf{r}) = \sum_{\substack{(f', \mathbf{m}'; \mathbf{r}') \in \Lambda^* \\ (f', \mathbf{m}'; \mathbf{r}') < (f, \mathbf{m}; \mathbf{r})}} c(f', \mathbf{m}'; \mathbf{r}') \cdot W(f', \mathbf{m}'; \mathbf{r}'), \tag{3.1}$$

where $c(f', \mathbf{m}'; \mathbf{r}')$ *is a constant depending only on* $(f', \mathbf{m}'; \mathbf{r}')$, χ *and* ξ.

COROLLARY 3.2. *If* $W \in \mathrm{WS}_\psi(\chi, \xi)$ *is not identically zero, then* $W(e) \neq 0$.

The proof of Theorem 3.1 is divided into three steps. We first consider the case of $f \geqslant m_1$ (it is equivalent to $(f, \mathbf{m}) \in \Lambda_{n+1}$) and $\mathbf{r} = \mathbf{0}$. To study the action of $\mathscr{H}^*$ on $C_\psi(\mathbb{K}\backslash G^*/K^*)$, we put $H(\alpha, \boldsymbol{\beta}; f, \mathbf{m}) = \{h \in H | h \cdot \Pi_{\alpha,\beta} \in K^* \Pi_{f,\mathbf{m}} K^*\}$ for $(\alpha, \boldsymbol{\beta})$, $(f, \mathbf{m}) \in \mathbf{Z} \times \Lambda_n$. Then the subset $H(\alpha, \boldsymbol{\beta}; f, \mathbf{m})$ of H is right $\Pi_{\alpha,\boldsymbol{\beta}} H(\mathcal{O}) \Pi_{\alpha,\boldsymbol{\beta}}^{-1}$-invariant. We fix a complete set $A(\alpha, \boldsymbol{\beta}; f, \mathbf{m})$ of representatives of $H(\alpha, \boldsymbol{\beta}; f, \mathbf{m})/(\Pi_{\alpha,\boldsymbol{\beta}} H(\mathcal{O}) \Pi_{\alpha,\boldsymbol{\beta}}^{-1})$. For $\boldsymbol{\beta} \in \Lambda_n$, put

$$\deg(K \pi_{\boldsymbol{\beta}} K) = \#(K \backslash K \pi_{\boldsymbol{\beta}} K) \tag{3.2}$$

and choose $k'_{\boldsymbol{\beta},i} \in K$ $(1 \leqslant i \leqslant \deg(K \pi_{\boldsymbol{\beta}} K))$ so that

$$K \pi_{\boldsymbol{\beta}} K = K \pi_{-\boldsymbol{\beta}} K = \bigcup_i K \pi_{-\boldsymbol{\beta}} k'_{\boldsymbol{\beta},i} \text{ (disjoint union).}$$

LEMMA 3.3. *Assume that* $(f, \mathbf{m}) \in \Lambda_{n+1}$. *Then*

$$K^* \Pi_{f,\mathbf{m}} K^* = \bigcup_\alpha \bigcup_{\boldsymbol{\beta}} \bigcup_i \bigcup_h K^* (h \Pi_{\alpha,\boldsymbol{\beta}})^{-1} k'_{\boldsymbol{\beta},i} \quad \textit{(disjoint union)}.$$

Here $-f \leqslant \alpha \leqslant f$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ *runs over* Λ_n *with* $0 \leqslant \beta_j \leqslant f$ $(1 \leqslant j \leqslant n)$, $1 \leqslant i \leqslant \deg(K \pi \boldsymbol{\beta} K)$ *and* h *runs over* $A(\alpha, \boldsymbol{\beta}; f, \mathbf{m})$.

Proof. Let $g^* \in K^* \Pi_{f,\mathbf{m}} K^*$. By Iwasawa decomposition for G^*, we have $g^* = k^* \Pi_{-\alpha,\mathbf{0}} g h'$ $(k^* \in K^*, \alpha \in \mathbf{Z}, g \in G, h' \in H)$. Let $g \in K \pi_{\boldsymbol{\beta}} K$ $(\boldsymbol{\beta} \in \Lambda_n)$. Then $g = k \pi_{-\boldsymbol{\beta}} k'_{\boldsymbol{\beta},i}$ with some $k \in K$ and i $(1 \leqslant i \leqslant \deg(K \pi_{\boldsymbol{\beta}} K))$. Thus we have $g^* = k^* k \Pi_{-\alpha,-\boldsymbol{\beta}} k'_{\boldsymbol{\beta},i} h' = k^* k (h \Pi_{\alpha,\boldsymbol{\beta}})^{-1} k'_{\boldsymbol{\beta},i}$ with $h = k'_{\boldsymbol{\beta},i} h'^{-1} k'^{-1}_{\boldsymbol{\beta},i} \in H$. Since $f \geqslant m_1$, we have $-f \leqslant \alpha \leqslant f$ and $0 \leqslant \beta_i \leqslant f$. The remaining part is easily verified. □

LEMMA 3.4. *Let* $(f, \mathbf{m}) \in \Lambda_{n+1}$ *and* $\boldsymbol{\beta} \in \Lambda_n$. *Then we have*

$$H(f, \boldsymbol{\beta}; f, \mathbf{m}) = \begin{cases} H(\mathfrak{o}) & \text{if } \boldsymbol{\beta} = \mathbf{m} \\ \phi & \text{otherwise} \end{cases}.$$

Proof. Let

$$h = (\lambda, \mu, \kappa) = \begin{bmatrix} 1 & \lambda & \kappa - \mu^t\lambda & \mu \\ 0 & 1_n & {}^t\mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t\lambda & 1_n \end{bmatrix} \in H(f, \boldsymbol{\beta}; f, \mathbf{m}).$$

By the assumption $f \geqslant m_1$, $\pi^f \cdot h\Pi_{f,\boldsymbol{\beta}}$ is an integral matrix. Since the $(n+2)$-th column of $\pi^f \cdot h\Pi_{f,\boldsymbol{\beta}}$ is ${}^t(\kappa - \mu^t\lambda, \mu, 1, -\lambda)$, we see that $h \in H(\mathfrak{o})$ and $\boldsymbol{\beta} = \mathbf{m}$. □

For $(f, \mathbf{m}) \in \Lambda_{n+1}$, we denote by $\Phi_{f,\mathbf{m}} \in \mathscr{H}^*$ the characteristic function of $K^*\Pi_{f,\mathbf{m}}K^*$. The following result follows from Lemma 3.3 and Lemma 3.4.

LEMMA 3.5. *For* $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$ *and* $(f, \mathbf{m}) \in \Lambda_{n+1}$, *we have*

$$(F * \Phi_{f,\mathbf{m}})(e) = q^{(2n+2)f} \cdot F(\Pi_{f,\mathbf{m}}) + \sum_{\alpha, \boldsymbol{\beta}, h} F(h\Pi_{\alpha,\boldsymbol{\beta}}),$$

where $0 \leqslant \alpha \leqslant f - 1$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ *runs over* Λ_n *with* $0 \leqslant \beta_i \leqslant f (1 \leqslant i \leqslant n)$ *and* h *over* $A(\alpha, \boldsymbol{\beta}; f, \mathbf{m})$.

PROPOSITION 3.6 (Step 1). *The assertion* (3.1) *in Theorem* 3.1 *holds for* $(f, \mathbf{m}; \mathbf{0})$ *with* $(f, \mathbf{m}) \in \Lambda_{n+1}$.

Proof. We first observe $(W * \Phi_{f,\mathbf{m}})(e) = \chi^\wedge(\Phi_{f,\mathbf{m}}) \cdot W(e)$ for $W \in \mathrm{WS}_\psi(\chi, \xi)$. If $W(h\Pi_{\alpha,\boldsymbol{\beta}}) \neq 0$, we can find $\mathbf{r} \in \Lambda_n$ such that $W(h\Pi_{\alpha,\boldsymbol{\beta}}) = c \cdot W(\alpha, \boldsymbol{\beta}; \mathbf{r}) (c \in \mathbf{C})$ and that $(\alpha, \boldsymbol{\beta}; \mathbf{r}) \in \Lambda^*$ by Lemma 2.3. The proposition is now a direct consequence of Lemma 3.5 (note that $(f, \mathbf{m}; \mathbf{0}) > (\alpha, \boldsymbol{\beta}; \mathbf{r})$ if $0 \leqslant \alpha \leqslant f - 1$). □

To consider the case $f < m_1$, we investigate the action of $\mathscr{H}$ on $C_\psi(\mathbb{K}\backslash G^*/K^*)$. For $\mathbf{r} \in \Lambda_n$, we denote by $\varphi_\mathbf{r}$ the element of $\mathscr{H}$ with support $Z(\mathbb{G})\mathbb{K}\pi_\mathbf{r}\mathbb{K}$ satisfying $\varphi_\mathbf{r}(\pi_\mathbf{r}) = 1$. Then $\{\varphi_\mathbf{r} | \mathbf{r} \in \Lambda_n\}$ forms a $\mathbf{C}$-basis of $\mathscr{H}$ (see [8, §4]). We let $B(\mathbf{r})$ be a complete set of representatives of $\mathfrak{o}^n \cdot \mathrm{diag}(\pi^{-r_1}, \ldots, \pi^{-r_n})/\mathfrak{o}^n$.

LEMMA 3.7. *For* $\mathbf{r} \in \Lambda_n$, $F \in C_\psi(\mathbb{K}\backslash G^*/K^*)$ *and* $g^* \in G^*$, *we have*

$$(\varphi_\mathbf{r} * F)(g^*) = \sum_{\lambda \in B(\mathbf{r})} \sum_{1 \leqslant i \leqslant \deg(K\pi_\mathbf{r}K)} F((\lambda, 0, 0)\pi_\mathbf{r} k_{\mathbf{r},i} g^*),$$

where $K\pi_\mathbf{r}K = \bigcup_{1 \leqslant i \leqslant \deg(K\pi_\mathbf{r}K)} K\pi_\mathbf{r} k_{\mathbf{r},i}$ *(disjoint union;* $k_{\mathbf{r},i} \in K$*).*

Proof. Since $\mathbb{K} = K \cdot H(\mathcal{o}) = H(\mathcal{o}) \cdot K$, every $\mathfrak{g} \in Z(\mathbb{G})\mathbb{K}\pi_{\mathbf{r}}\mathbb{K}$ can be written in the form $k \cdot (\lambda, \mu, \kappa)\pi_{\mathbf{r}} k_{\mathbf{r},i}$ ($k \in K$, $\lambda \in \mathcal{o}^n \cdot \mathrm{diag}(\pi^{-r_1}, \dots, \pi^{-r_n})$, $\mu \in \mathcal{o}^n$, $\kappa \in E$). Thus

$$Z(\mathbb{G})\mathbb{K}\pi_{\mathbf{r}}\mathbb{K} = \bigcup_{\lambda \in B(\mathbf{r})} \bigcup_{1 \leqslant i \leqslant \deg(K\pi_{\mathbf{r}}K)} Z(\mathbb{G})\mathbb{K} \cdot (\lambda, 0, 0)\pi_{\mathbf{r}} k_{\mathbf{r},i} \quad \text{(disjoint union).}$$

Since $\varphi_{\mathbf{r}}((\lambda, 0, 0)\pi_{\mathbf{r}}) = \varphi_{\mathbf{r}}(\pi_{\mathbf{r}}((\lambda_1 \pi^{r_1}, \dots, \lambda_n \pi^{r_n}), 0, 0)) = 1$, we have done. □

REMARK. Let $\mathbf{m} \in \Lambda_n$ and $\mathbf{r} \in \mathbf{Z}^n$ satisfy $0 \leqslant r_i \leqslant m_i$ ($i = 1, \dots, n$). By Lemma 3.7, we easily see

$$W(0, \mathbf{m}; \mathbf{r}) = q^{-m_1 - \cdots - m_n}(\deg(K\pi_{\mathbf{m}}K))^{-1}\overline{\xi^{\wedge}(\varphi_{\mathbf{m}})}W(e) \quad (W \in \mathrm{WS}_{\psi}(\mathbb{K}\backslash G^*/K^*)). \tag{3.3}$$

Let $\mathbf{r} \in \Lambda_n$. If

$$r_1 = \cdots = r_{n_1} = \rho_1, r_{n_1+1} = \cdots = r_{n_1+n_2} = \rho_2, \dots, r_{n_1+\cdots+n_{j-1}+1} = \cdots = r_n = \rho_j$$

with $\rho_1 > \rho_2 > \cdots > \rho_j \geqslant 0$, we write simply $\mathbf{r} = (\rho_1^{(n_1)}, \dots, \rho_j^{(n_j)})$ ($n_1 + \cdots + n_j = n$). The following lemma plays a crucial role in later discussion.

LEMMA 3.8. *Let* $\mathbf{r} = (\rho_1^{(n_1)}, \dots, \rho_j^{(n_j)}) \in \Lambda_n$ $(\rho_1 > \rho_2 > \cdots > \rho_j \geqslant 0)$, $\mathbf{m} \in \Lambda_n$ *and assume that* $\mathbf{r} \neq \mathbf{0}$. *Suppose that* $g \in G$ *satisfies* (i) $g \in K\pi_{\mathbf{m}}K$ *and* (ii) $\pi^{\mathbf{r}} g \in K\pi_{\mathbf{r}+\mathbf{m}}K$. *Then we have* $g = \mathbf{d}(y)\mathbf{n}(x)\pi_{\mathbf{m}}k$ *with*

$$x \in \mathrm{Sym}_n(\mathcal{o}),\ y = \begin{bmatrix} \gamma_1 & & * \\ & \ddots & \\ 0 & & \gamma_j \end{bmatrix} \in \mathrm{GL}_n(\mathcal{o}) \ (\gamma_i \in \mathrm{GL}_{n_i}(\mathcal{o}),\ 1 \leqslant i \leqslant j)$$

and $k \in K$.

Proof. By Iwasawa decomposition for G and Cartan decomposition for GL_{n_1}, $g \in G$ is written in the form

$$g = \mathbf{d}\left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & 1_{n'} \end{pmatrix}\right) \times \begin{bmatrix} 1_{n_1} & u & z & v \\ 0 & 1_{n'} & {}^t v & 0 \\ 0 & 0 & 1_{n_1} & 0 \\ 0 & 0 & -{}^t u & 1_{n'} \end{bmatrix} \times \mathbf{d}\left(\begin{pmatrix} p(\boldsymbol{\alpha}) & 0 \\ 0 & 1_{n'} \end{pmatrix}\right)$$
$$\times \begin{bmatrix} 1_{n_1} & & & \\ & a & & b \\ & & 1_{n_1} & \\ & & & {}^t a^{-1} \end{bmatrix} k$$

where $\gamma_1 \in GL_{n_1}(\mathcal{O})$, $n' = n - n_1$, u, $v \in M_{n_1,n'}$, $z + v{}^t u \in \mathrm{Sym}_{n_1}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n_1}) \in \Lambda_{n_1}$, $p(\boldsymbol{\alpha}) = \mathrm{diag}(\pi^{\alpha_1}, \ldots, \pi^{\alpha_{n_1}})$, $\begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix} \in \mathrm{Sp}_{n'}$ and $k \in K$.

We claim that $\alpha_1 = m_1$ and that the first row of (u, z, v) is integral. If this claim holds, the proof of the lemma is completed by induction on n. For a matrix A with coefficients in $\mathcal{O}$, we denote by π-rank(A) the rank of A modulo π in the finite field $\mathbb{F}_q = \mathcal{O}/\pi\mathcal{O}$. Put

$$g' = \mathbf{d}\left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & 1_{n'} \end{pmatrix}\right)^{-1} gk^{-1} = \begin{bmatrix} p(\boldsymbol{\alpha}) & ua & zp(\boldsymbol{\alpha})^{-1} & ub + v{}^t a^{-1} \\ 0 & a & {}^t vp(\boldsymbol{\alpha})^{-1} & b \\ 0 & 0 & p(\boldsymbol{\alpha})^{-1} & 0 \\ 0 & 0 & -{}^t up(\boldsymbol{\beta})^{-1} & {}^t a^{-1} \end{bmatrix}.$$

The assumption (i) implies that $\pi^{m_1} g'$ is integral. Moreover we have π-rank$(\pi^{m_1 + \rho_1} \cdot \pi_{\mathbf{r}} g') \geqslant 1$ by the assumption (ii). Since $\rho_1 > \cdots > \rho_j \geqslant 0$, we have

$$\pi^{\rho_1} \cdot \pi_{\mathbf{r}} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1_{n_1} & \\ & & & 0 \end{bmatrix} \text{ modulo } \pi$$

and hence

$$\pi\text{-rank}(\pi^{m_1 + \rho_1} \cdot \pi_{\mathbf{r}} g') = \pi\text{-rank} \begin{bmatrix} \pi^{m_1 - \alpha_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_1 - \alpha_{n_1}} \end{bmatrix},$$

which implies $\alpha_1 = m_1$. The second claim follows from the first one and the fact that $\pi^{m_1} g$ is integral. □

LEMMA 3.9. *Let* $\mathbf{r}$, $\mathbf{m} \in \Lambda_n$ *and* $f \in \mathbf{Z}$. *Assume* $\mathbf{r} \neq \mathbf{0}$ *and* $f \geqslant 0$. *Then, for* $F \in C_\psi(\mathbb{K} \backslash G^*/K^*)$, *we have*

$$(\varphi_{\mathbf{r}} * F)(\Pi_{f,\mathbf{m}}) = c \sum_{\lambda \in B(\mathbf{r})} F((\lambda, 0, 0)\Pi_{f,\mathbf{m}+\mathbf{r}}) + \sum_{h,\mathbf{s}} c_{h,\mathbf{s}} F(h\Pi_{f,\mathbf{s}})$$

where h *runs over a finite subset of* H, $\mathbf{s}$ *over* Λ_n *with* $\mathbf{s} < \mathbf{m} + \mathbf{r}$ *and* c *(resp.* $c_{h,\mathbf{s}}$*) is a constant depending only on* $\mathbf{r}$, $\mathbf{m}$ *and* f *(resp.* $\mathbf{r}$, $\mathbf{m}$, f, h *and* $\mathbf{s}$*). Furthermore we have* $c > 0$.

Proof. By Lemma 3.7,

$$(\varphi_{\mathbf{r}} * F)(\Pi_{f,\mathbf{m}}) = \sum_{\lambda \in B(\mathbf{r})} \sum_{1 \leqslant i \leqslant \deg(K\pi_{\mathbf{r}}K)} F((\lambda, 0, 0)\pi_{\mathbf{r}} k_{\mathbf{r},i} \pi_{\mathbf{m}} \Pi_f)$$

where we put $\Pi_f = \Pi_{f,0} = \mathbf{d}^*\left(\begin{pmatrix} \pi^f & 0 \\ 0 & 1_n \end{pmatrix}\right)$. It is well-known that $\pi_{\mathbf{r}} k \pi_{\mathbf{m}} \in \cup_{\mathbf{s}\in\Lambda_n, \mathbf{s}\leqslant \mathbf{r}+\mathbf{m}} K\pi_{\mathbf{s}}K$ for $k \in K$. Hence we have only to show that, if $\pi_{\mathbf{r}} k \pi_{\mathbf{m}} \in K\pi_{\mathbf{r}+\mathbf{m}}K$, then

$$\sum_{\lambda\in B(\mathbf{r})} F((\lambda, 0, 0)\pi_{\mathbf{r}} k \pi_{\mathbf{m}} \Pi_f) = \sum_{\lambda\in B(\mathbf{r})} F((\lambda, 0, 0)\Pi_{f,\mathbf{r}+\mathbf{m}}). \tag{3.4}$$

The equality (3.4) implies that $c = \#\{i \mid \pi_{\mathbf{r}} k_{\mathbf{r},i} \pi_{\mathbf{m}} \in K\pi_{\mathbf{r}+\mathbf{m}}K\}$ and hence that $c > 0$.

Let $\mathbf{r} = (r_1, \ldots, r_n) = (\rho_1^{(n_1)}, \ldots, \rho_j^{(n_j)})(\rho_1 > \cdots > \rho_j \geqslant 0)$. Applying Lemma 3.8 to $g = k\pi_{\mathbf{m}}$, we have

$$\pi_{\mathbf{r}} k \pi_{\mathbf{m}} = \mathbf{d}\left(\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_j \end{bmatrix}\right)\mathbf{d}(\tau_{\mathbf{r}} y \tau_{\mathbf{r}}^{-1})\mathbf{n}(\tau_{\mathbf{r}} x \tau_{\mathbf{r}})\pi_{\mathbf{r}+\mathbf{m}} k'$$

where $\gamma_i \in \mathrm{GL}_{n_i}(\mathfrak{o})(1 \leqslant i \leqslant a)$, $\tau_{\mathbf{r}} = \begin{bmatrix} \pi^{r_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{r_n} \end{bmatrix} = \begin{bmatrix} \pi^{\rho_1} 1_{n_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{\rho_j} 1_{n_j} \end{bmatrix}$,
$y = \begin{bmatrix} 1_{n_1} & & * \\ & \ddots & \\ 0 & & 1_{n_j} \end{bmatrix} \in \mathrm{GL}_n(\mathfrak{o})$, $x \in \mathrm{Sym}_n(\mathfrak{o})$ and $k' \in K$. Observe that

$$\begin{aligned}(\lambda, 0, 0)\mathbf{d}&\left(\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_j \end{bmatrix}\right)\mathbf{d}(\tau_{\mathbf{r}} y \tau_{\mathbf{r}}^{-1})\mathbf{n}(\tau_{\mathbf{r}} x \tau_{\mathbf{r}}) \\ &= \mathbf{d}\left(\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_j \end{bmatrix}\right)\mathbf{d}(\tau_{\mathbf{r}} y \tau_{\mathbf{r}}^{-1})\mathbf{n}(\tau_{\mathbf{r}} x \tau_{\mathbf{r}})(\lambda', \lambda'\tau_{\mathbf{r}} x \tau_{\mathbf{r}}, \lambda'\tau_{\mathbf{r}} x \tau_{\mathbf{r}}{}^t\lambda'),\end{aligned}$$

where

$$\lambda' = \lambda \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_j \end{bmatrix} \tau_{\mathbf{r}} y \tau_{\mathbf{r}}^{-1}.$$

Note that λ' runs over $\mathfrak{o}^n \cdot \mathrm{diag}(\pi^{-r_1}, \ldots, \pi^{-r_n})/\mathfrak{o}^n$ when so does λ. Since $\mathbf{d}(\tau_{\mathbf{r}} y \tau_{\mathbf{r}}^{-1})$, $\mathbf{n}(\tau_{\mathbf{r}} x \tau_{\mathbf{r}}) \in K$, $\lambda'\tau_{\mathbf{r}} x \tau_{\mathbf{r}} \in \mathfrak{o}^n$ and $\lambda'\tau_{\mathbf{r}} x \tau_{\mathbf{r}}{}^t\lambda' \in \mathfrak{o}$, we have completed the proof of (3.4). $\square$

PROPOSITION 3.10 (Step 2). *The assertion* (3.1) *in Theorem* 3.1 *holds for* $(f, \mathbf{m}; \mathbf{0}) \in \Lambda^*$ *with* $f < m_1$.

Proof. Let $W \in \mathrm{WS}_\psi(\chi, \xi)$. Let n'' be the smallest integer such that $m_{n''} > m_{n''+1}$. Put $\mathbf{r} = (1^{(n'')}, 0^{(n-n'')})$ and $\mathbf{m}' = \mathbf{m} - \mathbf{r}$ (note that $\mathbf{r}, \mathbf{m}' \in \Lambda_n$). By Lemma 3.9,

$$\overline{\xi^\wedge(\varphi_\mathbf{r})} \cdot W(f, \mathbf{m}'; 0) = (\varphi_\mathbf{r} * W)(\Pi_{f,\mathbf{m}'})$$
$$= c \sum_{\lambda' \in (\pi^{-1}\mathfrak{o}/\mathfrak{o})^{n''}} W(((\lambda', 0), 0, 0)\Pi_{f,\mathbf{m}}) + \sum_{h,\mathbf{s} < \mathbf{m}} c_{\mathbf{h},\mathbf{s}} F(h\Pi_{h,\mathbf{s}}),$$

with $c > 0$. Since $f < m_1 = \cdots = m_{n''}$, we have

$$W(((\lambda', 0), 0, 0)\Pi_{f,\mathbf{m}}) = W(\Pi_{f,\mathbf{m}}((\pi^{m_1 - f}\lambda', 0), 0, 0)) = W(f, \mathbf{m}; \mathbf{0}).$$

This proves the proposition. □

We now proceed to the last step of the proof of Theorem 3.1.

PROPOSITION 3.11 (Step 3). *The assertion* (3.1) *in Theorem* 3.1 *holds for* $(f, \mathbf{m}; \mathbf{r}) \in \Lambda^*$ *with* $\mathbf{r} \neq \mathbf{0}$.

Proof. By Lemma 3.9, we have

$$\overline{\xi^\wedge(\varphi_\mathbf{r})} \cdot W(f, \mathbf{m} - \mathbf{r}; \mathbf{0}) = (\varphi_\mathbf{r} * W)(\Pi_{f,\mathbf{m}-\mathbf{r}})$$
$$= c \sum_{\lambda \in B(\mathbf{r})} W((\lambda, 0, 0)\Pi_{f,\mathbf{m}}) + \sum_{h,\mathbf{s} < \mathbf{m}} c_{h,\mathbf{s}} W(h\Pi_{f,\mathbf{s}})$$

with $c \neq 0$. Thus it only remains to prove

$$\sum_{\lambda \in B(\mathbf{r})} W((\lambda, 0, 0)\Pi_{f,\mathbf{m}}) = c' W(f, \mathbf{m}; \mathbf{r}) + \sum_{\mathbf{r}'} c'_{\mathbf{r}'} W(f, \mathbf{m}; \mathbf{r}')$$

with $c' \neq 0$, where $\mathbf{r}'$ runs over Λ_n with $(f, \mathbf{m}; \mathbf{r}') < (f, \mathbf{m}; \mathbf{r})$. This fact immediately follows from Lemma 2.3. □

4. Global Whittaker-Shintani functions

In this section, for simplicity, we study global WS functions associated with *holomorphic* cusp forms. It is immediate to generalize our results to the non-holomorphic case.

We first recall the definition of holomorphic cusp forms. For $r \geqslant 1$, we let $\mathrm{Sp}_r(\mathbf{R})$ act on the Siegel upper half space $\mathfrak{H}_r = \{z = {}^t z \in M_r(\mathbf{C}) | \mathrm{Im}(z) > 0\}$ by $g\langle z \rangle = (az + b)(cz + d)^{-1}$ and put $j(g, z) = cz + d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_r(\mathbf{R})$, $z \in \mathfrak{H}_r$. Let l be an even natural number. Let S_l be the space of Siegel cusp forms of weight l on $G^*(\mathbf{Z}) = \mathrm{Sp}_{n+1}(\mathbf{Z})$. By definition, $F \in S_l$ is a holomorphic function on $\mathfrak{H}_{n+1}$ satisfying the following two conditions:

$$F(\gamma^*\langle z^*\rangle) = \det j(\gamma^*, z^*)^l F(z^*) \ (\gamma^* \in G^*(\mathbf{Z}), z^* \in \mathfrak{H}_{n+1}), \tag{4.1}$$

$$\sup_{g^* \in G^*(\mathbf{R})} |\det j(g^*, i1_{n+1})^{-l} F(g^*\langle i1_{n+1}\rangle)| < \infty. \tag{4.2}$$

Then $F \in S_l$ admits a Fourier-Jacobi expansion:

$$F\left(\begin{pmatrix} \tau & w \\ {}^t w & z \end{pmatrix}\right) = \sum_{m=1}^{\infty} F_m(z, w)\mathbf{e}[m\tau] \quad (\tau \in \mathfrak{H}_1, z \in \mathfrak{H}_n, w \in \mathbf{C}^n).$$

The function F_m on $\mathfrak{H}_n \times \mathbf{C}^n$ is called the m-th *Fourier-Jacobi coefficient* of F.

Let $\mathbb{S}_{l,1}$ be the space of Jacobi cusp forms of weight l and index 1 on $\mathbb{G}(\mathbf{Z})$ (see [8, §1]). Thus $f \in \mathbb{S}_{l,1}$ is a holomorphic functions on $\mathscr{D}_n = \mathfrak{H}_n \times \mathbf{C}^n$ satisfying

$$f(\gamma\langle Z\rangle) = J_{l,1}(\gamma, Z)f(Z) \quad (\gamma \in \mathbb{G}(\mathbf{Z}), Z \in \mathscr{D}_n), \tag{4.3}$$

$$\sup_{\in \mathbb{G}(\mathbf{R})} |J_{l,1}(\mathfrak{g}, Z_0)^{-1} f(\mathfrak{g}\langle Z_o\rangle)| < \infty \quad (Z_o = (i1_n, 0) \in \mathscr{D}_n). \tag{4.4}$$

Here

$$\mathfrak{g}\langle Z\rangle = (g\langle z\rangle, w\cdot j(g, z)^{-1} + \lambda\cdot g\langle z\rangle + \mu) \in \mathscr{D}_n,$$
$$J_{l,1}(\mathfrak{g}, Z) = \det j(g, z)^l \mathbf{e}[-\kappa + wj(g, z)^{-1}c{}^t w - 2\lambda j(g, z)^{-1}w - \lambda g\langle z\rangle {}^t\lambda]$$

for $\mathfrak{g} = (\lambda, \mu, \kappa)g \in \mathbb{G}(\mathbf{R})\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{R})\right)$ and $Z = (z, w) \in \mathscr{D}_n$. It is easy to see that $F_1 \in \mathbb{S}_{l,1}$ for $F \in S_l$.

As is well-known, we can lift $F \in S_l$ and $f \in \mathbb{S}_{l,1}$ to $\mathbf{C}$-valued functions on the adele groups $G^*(\mathbf{A})$ and $\mathbb{G}(\mathbf{A})$, respectively, by using the strong approximation theorems

$$G^*(\mathbf{A}) = G^*(\mathbf{Q})G^*(\mathbf{R}) \prod_{p<\infty} G^*(\mathbf{Z}_p), \tag{4.5}$$

$$\mathbb{G}(\mathbf{A}) = \mathbb{G}(\mathbf{Q})\mathbb{G}(\mathbf{R}) \prod_{p<\infty} \mathbb{G}(\mathbf{Z}_p). \tag{4.6}$$

In later discussion, we often use the same letter F (resp. f) to denote the lift of $F \in S_l$ (resp. $f \in \mathbb{S}_{l,1}$) if there is no fear of confusion.

For $F \in S_l$ and $f \in \mathbb{S}_{l,1}$, we define the global *Whittaker-Shintani function* $W_{F,f}$ *associated with* F *and* f as follows:

$$W_{F,f}(g^*) = \int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} F(\mathbb{x}g^*)\overline{f(\mathbb{x})}\,d\mathbb{x} \quad (g^* \in G^*(\mathbf{A})). \tag{4.7}$$

We easily see that

$$W_{F,f}(e) = \langle\langle F_1, f\rangle\rangle, \tag{4.8}$$

where $\langle\langle , \rangle\rangle$ is the Petersson inner product of $\mathbb{S}_{l,1}$ defined by

$$\langle\langle f, f'\rangle\rangle = \int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} f(\mathbb{x})\overline{f'(\mathbb{x})}\,d\mathbb{x}\,(f, f' \in \mathbb{S}_{l,1}). \tag{4.9}$$

Recall that $\psi_{\mathbf{A}}$ is the additive character of $\mathbf{A}$ with $\psi_{\mathbf{A}}(x_\infty) = \mathbf{e}[x_\infty]$ for $x_\infty \in \mathbf{R}$. The conductor of ψ_p, the restriction of $\psi_{\mathbf{A}}$ to $\mathbf{Q}_p$, is $\mathbf{Z}_p$. Let $\mathscr{H}_p^* = \mathscr{H}^*(G^*(\mathbf{Q}_p), G^*(\mathbf{Z}_p))$ and $\boldsymbol{\mathscr{H}}_p = \boldsymbol{\mathscr{H}}_{\psi_p}(\mathbb{G}(\mathbf{Q}_p), \mathbb{G}(\mathbf{Z}_p))$ be the Hecke algebras at p of G^* and $\mathbb{G}$, respectively. Then, for every p, $\mathscr{H}_p^*$ (resp. $\boldsymbol{\mathscr{H}}_p$) acts on S_l (resp. on $\mathbb{S}_{l,1}$) by

$$(F * \Phi_p)(g^*) = \int_{G^*(\mathbf{Q}_p)} F(g^* x_p^{*-1})\Phi(x_p^*)\,dx_p^* \quad (F \in S_l,\ \Phi_p \in \mathscr{H}_p^*,\ g^* \in G^*(\mathbf{A})), \tag{4.10}$$

$$(f * \varphi_p)(\mathbb{g}) = \int_{Z(\mathbb{G})(\mathbf{Q}_p)\backslash\mathbb{G}(\mathbf{Q}_p)} F(\mathbb{g}\mathbb{x}_p^{-1})\varphi(\mathbb{x}_p)\,d\mathbb{x}_p \quad (f \in \mathbb{S}_{l,1},\ \varphi_p \in \boldsymbol{\mathscr{H}}_p,\ \mathbb{g} \in \mathbb{G}(\mathbf{A})). \tag{4.11}$$

Assume that F and f are *Hecke eigenforms*; this means that, for every p, we have

$$F * \Phi_p = \chi_p^\wedge(\Phi_p)\cdot F \quad (\Phi_p \in \mathscr{H}_p^*) \tag{4.12}$$

$$f * \varphi_p = \xi_p^\wedge(\varphi_p)\cdot f \qquad (\varphi_p \in \boldsymbol{\mathscr{H}}_p), \tag{4.13}$$

where $\chi_p = (\chi_{p,1}, \ldots, \chi_{p,n+1})$ and $\xi_p = (\xi_{p,1}, \ldots, \xi_{p,n})$ are the Satake parameters corresponding to F and f, respectively ($\chi_{p,i}$ and $\xi_{p,j}$ are unramified characters of $\mathbf{Q}_p^\times$).

Let $W_{F,f}^{(\infty)}$ (resp. $\tilde{W}_{F,f}^{(p)}$) be the restriction of $W_{F,f}$ to $G^*(\mathbf{R})$ (resp. $G^*(\mathbf{Q}_p)$). Assume $W_{F,f} \neq 0$. Then $\tilde{W}_{F,f}^{(p)} \in \mathrm{WS}_{\psi_p}(\chi_p, \xi_p)$ is non-zero and hence $\dim \mathrm{WS}_{\psi_p}(\chi_p, \xi_p) = 1$ in view of Theorem 1.2. By Corollary 3.2, there uniquely exists $W_{F,f}^{(p)} \in \mathrm{WS}_{\psi_p}(\chi_p, \xi_p)$ such that $W_{F,f}^{(p)}(e) = 1$. This observation establishes the following Euler decomposition of $W_{F,f}$:

THEOREM 4.1. *Assume that F and f are Hecke eigenforms and $W_{F,f} \neq 0$. Then we have*

$$W_{F,f}\left(g_\infty^* \prod_{p<\infty} g_p^*\right) = W_{F,f}^{(\infty)}(g_\infty^*) \prod_{p<\infty} W_{F,f}^{(p)}(g_p^*). \tag{4.14}$$

REMARK. Assume that $W_{F,f} \equiv 0$. Then $\langle\langle F_1, f\rangle\rangle = 0$ and hence $W_{F,f}^{(\infty)} \equiv 0$ by Corollary 5.2 in the next section. Thus (4.14) is trivial in this case.

5. Explicit formula for Whittaker-Shintani functions on $G^*(\mathbf{R})$

In this section, we give an explicit formula for $W_{F,f}^{(\infty)}$ for $F \in S_l$ and $f \in \mathbb{S}_{l,1}$. To be more precise, put

$$\omega_{l,1}(\mathbb{g}) = J_{l,1}(\mathbb{g}^{-1}, Z_o)^{-1} \cdot K_{l,1}(\mathbb{g}^{-1}\langle Z_o\rangle) \qquad (\mathbb{g} \in \mathbb{G}(\mathbf{R})) \tag{5.1}$$

where

$$K_{l,1}(Z) = c_{l,1} \det\left(\frac{z + i1_n}{2i}\right)^{-l} \cdot \mathbf{e}[-w\cdot(z + i1_n)^{-1}\cdot {}^t w] \qquad (Z = (z, w) \in \mathscr{D}_n), \tag{5.2}$$

$$c_{l,1} = (2\pi)^{-n(n+1)/2} \prod_{i=0}^{n-1} \prod_{j=1}^{n-i} \left(l - \frac{i+1}{2} - j\right). \tag{5.3}$$

Let $\mathbb{K}_\infty$ be the stabilizer of $Z_o = (i1_n, 0) \in \mathscr{D}_n$ in $\mathbb{G}(\mathbf{R})$. Then $\mathbb{K}_\infty = Z(\mathbb{G})(\mathbf{R}) \cdot K_\infty$ where $K_\infty = \{g \in G(\mathbf{R}) | g\langle i1_n\rangle = i1_n\}$. It is known that

$$\omega_{l,1}(k\mathbb{g}k') = \det j(k, i1_n)^l \det j(k', i1_n)^l \omega_{l,1}(\mathbb{g}) \quad (\mathbb{g} \in \mathbb{G}(\mathbf{R}),\ k, k' \in K_\infty).$$

Let K_∞^* be the stabilizer of $i1_{n+1} \in \mathfrak{H}_{n+1}$ in $G^*(\mathbf{R})$. It is easy to see that $G^*(\mathbf{R})$ is decomposed into $\mathbb{G}(\mathbf{R}) \cdot \left\{\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right) \middle|\, t > 0\right\} \cdot K_\infty^*$.

THEOREM 5.1. *Assume that l is even and $l > 2n + 1$. Then*

$$W_{F,f}^{(\infty)}\left(\mathbb{g}\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)k^*\right)$$
$$= c_{l,1}^{-1} \cdot \det j(k^*, i1_{n+1})^{-l}\overline{\omega_{l,1}(\mathbb{g})} \cdot t^l \exp(-2\pi t^2) \cdot \langle\langle F_1, f\rangle\rangle$$

$(\mathbb{g} \in \mathbb{G}(\mathbf{R}),\ t > 0,\ k^* \in K_\infty^*)$.

Proof. For simplicity, we write W for $W^{(\infty)}_{F,f}$. It is easy to see that

$$W(g^*k^*) = \det j(k^*, i1_{n+1})^{-l} W(g^*) \quad (g^* \in G^*(\mathbf{R}),\ k^* \in K^*_\infty).$$

We first prove

$$W\left(\mathbb{g}\cdot \mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) = c_{l,1}^{-1}\overline{\omega_{l,1}(\mathbb{g})}\, W\left(\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) \qquad (\mathbb{g} \in \mathbb{G}(\mathbf{R}),\ t > 0). \tag{5.4}$$

The strong approximation theorem (4.6) implies

$$\begin{aligned} &W\left(\mathbb{g}\cdot \mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) \\ &= \int_{\mathbb{G}(\mathbf{Z})\backslash\mathbb{G}(\mathbf{R})} F\left(\mathbb{x}\mathbb{g}\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right)\overline{f(\mathbb{x})}\,\mathrm{d}\mathbb{x} \\ &= \int_{\mathbb{G}(\mathbf{Z})\backslash\mathbb{G}(\mathbf{R})} F\left(\mathbb{x}\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right)\overline{f(\mathbb{x}\mathbb{g}^{-1})}\,\mathrm{d}\mathbb{x}. \end{aligned}$$

By [8, Lemma 5.6], we have

$$f(\mathbb{g}) = \int_{\mathbb{G}(\mathbf{R})/Z(\mathbb{G})(\mathbf{R})} \omega_{l,1}(\mathbb{g}^{-1}\mathbb{y}) f(\mathbb{y})\,\mathrm{d}\mathbb{y} \quad (f \in \mathbb{S}_{l,1} \text{ and } \mathbb{g} \in \mathbb{G}(\mathbf{R})) \tag{5.5}$$

and hence

$$\begin{aligned} &W\left(\mathbb{g}\cdot \mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) \\ &= \int_{\mathbb{G}(\mathbf{Z})\backslash\mathbb{G}(\mathbf{R})} \mathrm{d}\mathbb{x}\, F\left(\mathbb{x}\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right)\overline{\int_{\mathbb{G}(\mathbf{R})/Z(\mathbb{G})(\mathbf{R})} \omega_{l,1}(\mathbb{g}\mathbb{x}^{-1}\mathbb{y}) f(\mathbb{y})\,\mathrm{d}\mathbb{y}}. \end{aligned}$$

Since K_∞ commutes with

$$\mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right) \quad \text{and} \quad F(g^*k) = F(g^*)\det j(k, i1_n)^{-l}(k \in K_\infty),$$

we have

$$W\left(\mathbb{g}\cdot\mathbf{d}^*\left(\begin{pmatrix} t & 0\\ 0 & 1_n\end{pmatrix}\right)\right)$$

$$=\int_{\mathbb{G}(\mathbf{Z})\backslash\mathbb{G}(\mathbf{R})} \mathrm{d}\mathbb{x}\, F\left(\mathbb{x}\mathbf{d}^*\left(\begin{pmatrix} t & 0\\ 0 & 1_n\end{pmatrix}\right)\right)\int_{\mathbb{G}(\mathbf{R})/Z\mathbb{G})(\mathbf{R})} \mathrm{d}\mathbb{y}\,\overline{f(\mathbb{y})}$$

$$\overline{\times\int_{K_\infty} \omega_{l,1}(\mathbb{g}k\mathbb{x}^{-1}\mathbb{y})\det j(k, i1_n)^{-l}\,\mathrm{d}k.} \tag{5.6}$$

By [8, Lemma 5.7],

$$\int_{K_\infty} \omega_{l,1}(\mathbb{g}k\mathbb{x}^{-1}\mathbb{y})\det j(k, i1_n)^{-l}\,\mathrm{d}k = c_{l,1}^{-1}\omega_{l,1}(\mathbb{g})\omega_{l,1}(\mathbb{x}^{-1}\mathbb{y}). \tag{5.7}$$

The assertion (5.4) is now an immediate consequence of (5.5), (5.6) and (5.7).
It now remains to verify

$$W\left(\mathbf{d}^*\left(\begin{pmatrix} t & 0\\ 0 & 1_n\end{pmatrix}\right)\right) = t^l\exp(-2\pi t^2)\langle\langle F_1, f\rangle\rangle. \tag{5.8}$$

For the time being, we write $F^\sim$ (resp. $f^\sim$) the lift to $G^*(\mathbf{A})$ (resp. to $\mathbb{G}(\mathbf{A})$) of F (resp. of f) to avoid confusion. We put

$$F_1^\sim(g^*) = \int_{\mathbf{Q}\backslash\mathbf{A}} F^\sim((0, 0, \kappa)g^*)\psi_{\mathbf{A}}(-\kappa)\,\mathrm{d}\kappa \qquad (g^*\in G^*(\mathbf{A})).$$

Then the restriction of $F_1^\sim$ to $\mathbb{G}(\mathbf{A})$ coincides with the lift of $F_1\in\mathbb{S}_{l,1}$. For $Z = (z, w)\in\mathscr{D}_n$, let $z = x + iy$ and $w = \lambda z + \mu$ ($x, y\in\mathrm{Sym}_n(\mathbf{R})$, $\lambda, \mu\in\mathbf{R}^n$). Then

$$\mathrm{d}\mu(Z) = (\det y)^{-n-1}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\lambda\,\mathrm{d}\mu$$

is a $\mathbb{G}(\mathbf{R})$-invariant measure on $\mathscr{D}_n$, where $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ are the usual Lebesgue measures. Let $\mathbb{g}_Z$ be any element of $\mathbb{G}(\mathbf{R})$ satisfying $\mathbb{g}_Z\langle Z_o\rangle = Z$. We may put

$$\mathbb{g}_Z = (\lambda, \mu, 0)\begin{pmatrix} 1_n & x\\ 0 & 1_n\end{pmatrix}\begin{pmatrix} y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}.$$

It is easily shown that

$$W\left(\mathbf{d}^*\left(\begin{pmatrix} t & 0\\ 0 & 1_n\end{pmatrix}\right)\right) = \int_{\mathbb{G}(\mathbf{Z})\backslash\mathscr{D}_n} F_1^\sim\left(\mathbb{g}_Z\mathbf{d}^*\left(\begin{pmatrix} t & 0\\ 0 & 1_n\end{pmatrix}\right)\right)\overline{f^\sim(\mathbb{g}_Z)}\,\mathrm{d}\mu(Z). \tag{5.9}$$

Since

$$F^{\sim}\left(\mathfrak{g}_Z \mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) = (\det y)^{l/2} t^l F\left(\begin{pmatrix} t^2 i + \lambda z^t\lambda & w \\ {}^t w & z \end{pmatrix}\right),$$

we obtain

$$F_1^{\sim}\left(\mathfrak{g}_Z \mathbf{d}^*\left(\begin{pmatrix} t & 0 \\ 0 & 1_n \end{pmatrix}\right)\right) = (\det y)^{l/2} t^l \mathbf{e}[t^2 i + \lambda z^t\lambda] F_1(Z). \tag{5.10}$$

On the other hand, we see that

$$f^{\sim}(\mathfrak{g}_Z) = (\det y)^l \mathbf{e}[\lambda z^t\lambda] f(Z). \tag{5.11}$$

The assertion (5.8) follows from (5.9), (5.10) and (5.11). □

COROLLARY 5.2. *For $F \in S_l$ and $f \in \mathbb{S}_{l,1}$, $W_{F,f} \neq 0$ if and only if $\langle\langle F_1, f \rangle\rangle \neq 0$.*

COROLLARY 5.3. *Let $F \in S_l$. If $F_1 \neq 0$ as an element of $\mathbb{S}_{l,1}$, then there exists a Hecke eigenform $f \in \mathbb{S}_{l,1}$ such that $W_{F,f} \neq 0$.*

Proof. The assertion follows from the fact that there exists a basis of $\mathbb{S}_{l,1}$ consisting of Hecke eigenforms (see [8, §6]). □

6. Integral expression of standard zeta functions

In this section, we let notation be the same as in §1. Let $|\cdot|$ be the normalized valuation of E ($|\pi| = q^{-1}$). We put $\zeta_E(s) = (1 - q^{-s})^{-1}$ and $\zeta_E^{(r)}(s) = \prod_{i=0}^{r-1} \zeta_E(s - i)$. For $\chi = (\chi_1, \ldots, \chi_{n+1}) \in X_0(E^\times)^{n+1}$ and $\xi = (\xi_1, \ldots, \xi_n) \in X_0(E^\times)^n$, we put

$$L_0(s, \chi) = \zeta_E(s) \prod_{i=1}^{n+1} \{(1 - \chi_i(\pi) q^{-s})(1 - \chi_i(\pi)^{-1} q^{-s})\}^{-1}. \tag{6.1}$$

$$L_1(s, \xi) = \prod_{i=1}^{n} \{(1 - \xi_i(\pi) q^{-s})(1 - \xi_i(\pi)^{-1} q^{-s})\}^{-1}. \tag{6.2}$$

The object of this section is to prove the following theorem.

THEOREM 6.1. *Let $W \in \mathrm{WS}_\psi(\chi, \xi)$. Then*

$$\int_{E^\times} W\left(\begin{bmatrix} t & & & \\ & 1_n & & \\ & & t^{-1} & \\ & & & 1_n \end{bmatrix}\right) |t|^{s-n-1} \mathrm{d}^* t = \frac{L_0(s, \chi)}{\zeta_E(2s) L_1(s + \frac{1}{2}, \xi)} W(e). \tag{6.3}$$

This result implies Conjecture (0.2), since $L_0(s,\chi)$ and $L_1(s,\xi)$ are the local factors of the standard zeta functions of a Siegel cusp form and a Jacobi cusp form, respectively (see Theorem 6.2 below). In the case where the forms are holomorphic, a stronger result holds in view of Theorem 5.1.

THEOREM 6.2. *Let $F \in S_l$ and $f \in \mathbb{S}_{l,1}$. Assume that F and f are Hecke eigenforms. Then*

$$\int_{\mathbf{A}^\times} W_{F,f}\left(\begin{bmatrix} t & & & \\ & 1_n & & \\ & & t^{-1} & \\ & & & 1_n \end{bmatrix}\right)|t|_{\mathbf{A}}^{s-n-1}\,\mathrm{d}^*t$$

$$=(2\pi)^{-(s+l-n-1)/2}\Gamma\left(\frac{s+l-n-1}{2}\right)$$

$$\zeta(2s)^{-1}\,\overline{\mathbf{D}(\bar{s}+\tfrac{1}{2},f)^{-1}}\,D(s,F)\langle\langle F_1, f\rangle\rangle \tag{6.4}$$

Here $|\cdot|_{\mathbf{A}}$ is the idele norm of the idele group $\mathbf{A}^\times$ of $\mathbf{Q}$ and,

$$D(s,F) = \prod_{p<\infty} L_0(s,\chi_p)$$

and

$$\mathbf{D}(s,f) = \prod_{p<\infty} L_1(s,\xi_p)$$

are the standard zeta functions attached to F and f, respectively (for more detail of the standard zeta functions, see [1], [10] *and* [9]).

For $s \in \mathbf{C}$, define a function N_s^* on G^* by

$$N_s^*(k^*\Pi_{\mathbf{f}}k^{*\prime}) = q^{-s(f_1+\cdots+f_{n+1})} \tag{6.5}$$

for $k^*, k^{*\prime} \in K^*$ and $\mathbf{f} = (f_1,\ldots,f_{n+1}) \in \Lambda_{n+1}$.

We show Theorem 6.1 by calculating the integral

$$Z(s,W) = \int_{G^*} W(g^*)N_{s+n+1}^*(g^*)\,\mathrm{d}g^* \qquad (W \in \mathrm{WS}_\psi(\chi,\xi)) \tag{6.6}$$

in two ways. First we prove the following:

PROPOSITION 6.3. *If $W \in C_\psi(\mathbb{K}\backslash G^*/K^*)$ satisfies $W * \Phi = \chi^\wedge(\Phi)W$ for every $\Phi \in \mathscr{H}^*$, then we have*

$$Z(s, W) = \frac{L_0(s, \chi)}{\zeta_E(s+n+1) \prod_{i=1}^{n+1} \zeta_E(2s+2n+2-2i)} W(e). \tag{6.7}$$

Proof. Let $\Phi_{\mathbf{f}} \in \mathscr{H}^*$ be the characteristic function of $K^*\Pi_{\mathbf{f}}K^*$ for $\mathbf{f} \in \Lambda_{n+1}$. Since W is a common eigenfunction under the action of $\mathscr{H}^*$ corresponding to χ, we easily see that

$$Z(s, W) = \sum_{\mathbf{f} \in \Lambda_{n+1}} \chi^\wedge(\Phi_{\mathbf{f}}) q^{-(s+n+1)(f_1+\cdots+f_{n+1})}.$$

Then (6.7) follows from Böcherer's result [2] (see also [10]). □

PROPOSITION 6.4. *For* $W \in C_\psi(\mathbb{K}\backslash G^*/K^*)$,

$$Z(s, W) = \frac{L_1(s+\frac{1}{2}, \bar{\xi})}{\zeta_E(s+n+1) \prod_{i=1}^{n} \zeta_E(2s+2n+2-2i)}$$

$$\times \int_{E^\times} W\left(\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) |t|^{s-n-1} \, d^*t. \tag{6.8}$$

Theorem 6.1 is a direct consequence of the above two results. To prove the proposition, we let

$$v_s^*(g^*) = \int_E \psi(\kappa) N_s^*((0, 0, \kappa)g^*) \, d\kappa \qquad (g^* \in G^*). \tag{6.9}$$

be a partial Fourier coefficient of N_s^* and let v_s be a function on $\mathbb{G}$ defined by

$$v_s((0, 0, \kappa)\Bbbk\mathbb{g}\Bbbk') = \psi(-\kappa)v_s(\mathbb{g}) \qquad (\kappa \in E, \ \Bbbk, \ \Bbbk' \in \mathbb{K}, \ \mathbb{g} \in \mathbb{G}), \tag{6.10}$$

$$v_s(\pi_{\mathbf{m}}) = q^{-(m_1+\cdots+m_n)s} \qquad (\mathbf{m} \in \Lambda_n). \tag{6.11}$$

Note that v_s is uniquely determined by the conditions (6.10) and (6.11) (see [8, Lemma 4.4]). Henceforth we denote by $\sigma_{r,r'}$ the characteristic function of $M_{r,r'}(\mathfrak{o})$. We omit the subscripts and write simply σ if there is no fear of confusion. For $\xi \in X_0(E^\times)^n$, we let ϕ_ξ be a function on $\mathbb{G}$ defined by

$$\phi_\xi((0, \mu, \kappa)\mathbb{m}\mathbf{d}(\mathrm{diag}(t_1, \ldots, t_n))(\lambda, 0, 0)\Bbbk)$$

$$= \psi(\kappa)\sigma_{1,n}(\lambda) \prod_{i=1}^{n} (\xi_i(t_i)|t|^{2n+3-2i}) \tag{6.12}$$

for $\lambda, \mu \in E^n$, $\kappa \in E$, $\mathbb{m} \in \mathbb{N}$, $t_i \in E^\times$ $(1 \leqslant i \leqslant n)$ and $\Bbbk \in \mathbb{K}$.

To demonstrate Proposition 6.4, we need the next two lemmas. We postpone their proofs until the last part of this section.

LEMMA 6.5. *For* $\xi \in X_0(E^\times)^n$, *we have*

$$L_1(s, \xi) = \prod_{i=1}^{n} \zeta_E(2s + 2n + 1 - 2i) \times \int_{Z(\mathbb{G})\backslash\mathbb{G}} v_{s+n+\frac{1}{2}}(\mathfrak{g})\phi_\xi(\mathfrak{g})\, d\mathfrak{g}. \tag{6.13}$$

LEMMA 6.6. *Let* $W \in C_\psi(\mathbb{K}\backslash G^*/K^*)$, $\mathfrak{g} \in \mathbb{G}$ *and* $t \in E^\times$. *If*

$$W\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) \neq 0,$$

then we have

$$v_s^*\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) = \zeta_E(s)^{-1}|t|^s v_s(\mathfrak{g}). \tag{6.14}$$

Proof of Proposition 6.4. By Iwasawa decomposition

$$G^* = \mathbb{G}\left\{\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\middle| t \in E^\times\right\}K^*$$

and Lemma 6.6, we have

$$\begin{aligned} Z(s, W) &= \int_{E^\times} d^*t \int_{Z(\mathbb{G})\backslash\mathbb{G}} d\mathfrak{g}\, W\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) \\ &\qquad \times v_{s+n+1}^*\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right)|t|^{-2n-2}. \\ &= \zeta_E(s+n+1)^{-1} \int_{E^\times} |t|^{s-n-1}\, d^*t \\ &\qquad \times \int_{Z(\mathbb{G})\backslash\mathbb{G}} d\mathfrak{g}\, W\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) v_{s+n+1}(\mathfrak{g}). \end{aligned}$$

Using a similar argument to the proof of Proposition 6.3, we obtain

$$\begin{aligned} &\int_{Z(\mathbb{G})\backslash\mathbb{G}} W\left(\mathfrak{g}\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right) v_{s+n+1}(\mathfrak{g})\, d\mathfrak{g} \\ &\quad = \int_{Z(\mathbb{G})\backslash\mathbb{G}} v_{s+n+1}(\mathfrak{g})\phi_{\bar{\xi}}(\mathfrak{g})\, d\mathfrak{g} \cdot W\left(\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n \end{pmatrix}\right)\right). \end{aligned}$$

By Lemma 6.5, the above integral is equal to

$$\prod_{i=1}^{n} \zeta_E(2s+2n+2-2i)^{-1} \cdot L_1(s+\tfrac{1}{2}, \bar{\xi}).$$

This proves the proposition. □

To prove Lemma 6.5, we recall some results of [9]. For $s \in \mathbf{C}$, let $J_s^{\sim}$ be a function on $\mathbb{G}$ defined by

$$J_s^{\sim}(hg) = e(s) \int_{\mathrm{GL}_{2n+1}(E)} \sigma_{2n+1,4n+2}\left(Y\begin{pmatrix}1 & \\ & g\end{pmatrix}, Y \cdot \alpha(h)\right) |\det Y|^{s+n+\frac{1}{2}} \, \mathrm{d}^* Y, \tag{6.15}$$

where

$$e(s) = \zeta_E(s+n+\tfrac{1}{2})\zeta_E^{(2n+1)}(s+n+\tfrac{1}{2})^{-1} \prod_{i=1}^{n} \zeta_E(2s+2n+1-2i), \tag{6.16}$$

$$\alpha(h) = \begin{bmatrix} \kappa - \lambda{}^t\mu & -\lambda & -\mu \\ {}^t\mu & 1_n & 0 \\ -{}^t\lambda & 0 & 1_n \end{bmatrix} \quad (h = (\lambda, \mu, \kappa) \in H). \tag{6.17}$$

Note that $J_s^{\sim}$ is denoted by $\Phi^{\sim}(\cdot\,; s)$ in [9, Lemma 2.2]. Put

$$J_s(\mathbb{g}) = \int_E \psi(\kappa) J_s^{\sim}((0, 0, \kappa)\mathbb{g}) \, \mathrm{d}\kappa. \tag{6.18}$$

LEMMA 6.7 ([9, Corollary 2.4 and Theorem 2.12])

$$L_1(s, \xi) = \int_{Z(\mathbb{G})\backslash\mathbb{G}} J_s(\mathbb{g}) \phi_\xi(\mathbb{g}) \, \mathrm{d}\mathbb{g}. \tag{6.19}$$

Lemma 6.5 immediately follows from Lemma 6.7 and the following fact.

LEMMA 6.8

$$J_s(\mathbb{g}) = \prod_{i=1}^{n} \zeta_E(2s+2n+1-2i) \times v_{s+n+\frac{1}{2}}(\mathbb{g}). \tag{6.20}$$

To prove this, we need the following elementary result that is also useful in later discussion. The proof is easy and we omit it.

LEMMA 6.9. *If* $g \in \mathrm{GL}_r(E)$ *has a Cartan decomposition* $g = k \cdot \mathrm{diag}(\pi^{m_1}, \dots, \pi^{m_r}) \cdot k'$ $(k,\ k' \in \mathrm{GL}_r(\mathfrak{o}),\ (m_1 \geqslant \cdots \geqslant m_j \geqslant 0 \geqslant m_{j+1} \geqslant \cdots \geqslant m_r)$, *we have*

$$\int_{\mathrm{GL}_r(E)} \sigma_{r,2r}(Yg^{-1}, Y)|\det Y|^s \,\mathrm{d}^* Y = \zeta_E^{(r)}(s) \cdot q^{-(m_1 + \cdots + m_j)s}. \tag{6.21}$$

Proof of Lemma 6.8. By [9, Proposition 2.3], we have

$$J_s((0, 0, \kappa)\Bbbk \mathbb{g} \Bbbk') = \psi(-\kappa) J_s(\mathbb{g})$$

for $\kappa \in E$, $\Bbbk, \Bbbk' \in \mathbb{K}$ and $\mathbb{g} \in \mathbb{G}$. This implies that the support of J_s is contained in $\cup_{\mathbf{m} \in \Lambda_n} Z(\mathbb{G})\mathbb{K}\pi_{\mathbf{m}}\mathbb{K}$ (see [8, Lemma 4.4]). Hence we have only to verify (6.20) for $\mathbb{g} = \pi_{\mathbf{m}}$ with $\mathbf{m} \in \Lambda_n$. By (6.15) and (6.18),

$$J_s(\pi_{\mathbf{m}}) = e(s) \int_E \psi(\kappa)\,\mathrm{d}\kappa$$
$$\times \int_{\mathrm{GL}_{2n+1}(E)} \mathrm{d}^* Y \sigma\left(Y\begin{pmatrix} 1 & \\ & \pi_{\mathbf{m}} \end{pmatrix}\right) \sigma\left(Y\begin{pmatrix} \kappa & \\ & 1_{2n} \end{pmatrix}\right) |\det Y|^{s+n+\frac{1}{2}}.$$

Decomposing Y into $u\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ $(u \in \mathrm{GL}_{2n+1}(\mathfrak{o}),\ x \in E^\times,\ y \in \mathrm{GL}_{2n}(E),\ z \in E^{2n})$, we have

$$J_s(\pi_{\mathbf{m}}) = e(s) \cdot I_1 I_2 I_3,$$

where

$$I_1 = \int_E \mathrm{d}\kappa \int_{E^\times} \mathrm{d}^* x |x|^{s+n+\frac{1}{2}} \sigma(x)\sigma(x\kappa)\psi(\kappa),$$

$$I_2 = \int_{E^{2n}} \sigma(z\pi_{\mathbf{m}})\sigma(z)\,\mathrm{d}z$$

and

$$I_3 = \int_{\mathrm{GL}_{2n}(E)} \sigma(y)\sigma(y\pi_{\mathbf{m}})|\det y|^{s+n-\frac{1}{2}}\,\mathrm{d}^* y.$$

We easily see that $I_1 = 1$ and $I_2 = q^{-(m_1 + \cdots + m_n)}$. Moreover we have

$$I_3 = \zeta_E^{(2n)}(s + n - \tfrac{1}{2}) \cdot q^{-(m_1 + \cdots + m_n)(s+n-\frac{1}{2})}$$

by Lemma 6.9. These prove the lemma. □

Proof of Lemma 6.6. The proof is based on a direct calculation of the left-hand side of (6.14) by using an integral expression of v_s^* (see (6.22) below).

Assume $W\left(\times\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n\end{pmatrix}\right)\right) \neq 0$. By Proposition 2.2, we may assume that $\times\mathbf{d}^*\left(\begin{pmatrix} t & \\ & 1_n\end{pmatrix}\right)$ is of the form $(\lambda, 0, 0)\Pi_{f,\mathbf{m}}$ with $f \geqslant 0$, $\mathbf{m} \in \Lambda_n$ and $\lambda \cdot \mathrm{diag}(\pi^{m_1}, \dots, \pi^{m_n}) \in \mathscr{o}^n$. The definition (6.5) of N_s^* and Lemma 6.9 imply

$$N_s^*(g^*) = \zeta_E^{(2n+2)}(s)^{-1} \int_{\mathrm{GL}_{2n+2}(E)} \sigma(Yg^*)\sigma(Y)|\det Y|^s \, \mathrm{d}^* Y.$$

Thus we obtain

$$\begin{aligned} &v_s^*((\lambda, 0, 0)\Pi_{f,\mathbf{m}}) \cdot \zeta_E^{(2n+2)}(s) \\ &= \int_E \psi(\kappa)\,\mathrm{d}\kappa \int_{\mathrm{GL}_{2n+2}(E)} \mathrm{d}^* Y \\ &\sigma\left(Y \begin{bmatrix} 1 & \lambda & \kappa & 0 \\ & 1_n & 0 & 0 \\ & & 1 & 0 \\ & & -{}^t\lambda & 1_n \end{bmatrix} \cdot \begin{bmatrix} p(f,\mathbf{m}) & 0 \\ 0 & p(f,\mathbf{m})^{-1} \end{bmatrix}\right) \sigma(Y)|\det Y|^s, \end{aligned} \tag{6.22}$$

where we write $p(f, \mathbf{m})$ for $\mathrm{diag}(\pi^f, \pi^{m_1}, \dots, \pi^{m_n})$. Decompose Y into $u\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ ($u \in \mathrm{GL}_{2n+2}(\mathscr{o})$, $\alpha, \delta \in \mathrm{GL}_{n+1}(E)$, $\beta \in M_{n+1}(E)$). Then the right hand-side of (6.22) is equal to

$$\begin{aligned} &\int_E \psi(\kappa)\,\mathrm{d}\kappa \int_{\mathrm{GL}_{n+1}(E)} \mathrm{d}^*\alpha \int_{\mathrm{GL}_{n+1}(E)} \mathrm{d}^*\delta \int_{M_{n+1}(E)} \mathrm{d}\beta |\det \alpha|^s |\det \delta|^{s-n-1} \\ &\sigma\left(\alpha\begin{pmatrix} 1 & \lambda \\ 0 & 1_n \end{pmatrix} p(f, \mathbf{m})\right) \cdot \sigma\left(\left\{\alpha\begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 \\ -{}^t\lambda & 1_n \end{pmatrix}\right\} p(f, \mathbf{m})^{-1}\right) \\ &\sigma\left(\delta\begin{pmatrix} 1 & 0 \\ -{}^t\lambda & 1_n \end{pmatrix} p(f, \mathbf{m})^{-1}\right) \cdot \sigma(\alpha)\sigma(\beta)\sigma(\delta). \end{aligned} \tag{6.23}$$

Observe that $\begin{pmatrix} 1 & \lambda \\ 0 & 1_n \end{pmatrix} p(f, \mathbf{m})$ is an integral matrix and hence that $\sigma\left(\alpha\begin{pmatrix} 1 & \lambda \\ 0 & 1_n \end{pmatrix} p(f, m)\right) \cdot \sigma(\alpha) = \sigma(\alpha)$. Changing the variable β into $\beta - \alpha\begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}$, we see that

$$v_s^*((\lambda, 0, 0)\Pi_{f,\mathbf{m}}) \cdot \zeta_E^{(2n+2)}(s) = I' \cdot I'', \tag{6.24}$$

where

$$I' = \int_E \psi(\kappa)\,\mathrm{d}\kappa \int_{\mathrm{GL}_{n+1}(E)} \mathrm{d}^*\alpha \int_{M_{n+1}(E)} \mathrm{d}\beta$$
$$\times |\det \alpha|^s \sigma\left(\beta\begin{pmatrix} 1 & 0 \\ -{}^t\lambda & 1_n \end{pmatrix} p(f, \mathbf{m})^{-1}\right) \sigma(\alpha)\sigma\left(\beta - \alpha\begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}\right), \tag{6.25}$$

$$I'' = \int_{\mathrm{GL}_{n+1}(E)} \sigma\left(\delta\begin{pmatrix} 1 & 0 \\ -{}^t\lambda & 1_n \end{pmatrix} p(f, \mathbf{m})^{-1}\right) \sigma(\delta)|\det \delta|^{s-n-1}\,\mathrm{d}^*\delta. \tag{6.26}$$

To calculate I', we decompose α into $u'\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ $(u' \in \mathrm{GL}_{n+1}(\mathcal{o}),\ x \in E^\times,$ $y \in \mathrm{GL}_n(E), z \in E^n)$ and write β as $\begin{pmatrix} \beta_1 & \beta_2 \\ {}^t\beta_3 & \beta_4 \end{pmatrix}$ $(\beta_1 \in E, \beta_2, \beta_3 \in E^n, \beta_4 \in M_n(E))$. Then

$$I' = \int_E \psi(\kappa)\,\mathrm{d}\kappa \int_{E^\times} \mathrm{d}^*x \int_{\mathrm{GL}_n(E)} \mathrm{d}^*y \int_{E^n} \mathrm{d}z \int_E \mathrm{d}\beta_1 \int_{E^n} \mathrm{d}\beta_2 \int_{E^n} \mathrm{d}\beta_3 \int_{M_n(E)} \mathrm{d}\beta_4$$
$$\times |x|^s|\det y|^{s-1}\sigma(\pi^{-f}(\beta_1 - \beta_2{}^t\lambda))\cdot\sigma(\pi^{-f}({}^t\beta_3 - \beta_4{}^t\lambda))\cdot\sigma(\beta_2 p(\mathbf{m})^{-1})$$
$$\times \sigma(\beta_4 p(\mathbf{m})^{-1})\sigma(x)\sigma(y)\sigma(z)\cdot\sigma(\beta_1 - x\kappa)\sigma(\beta_2)\sigma(\beta_3)\sigma(\beta_4),$$

where $p(\mathbf{m}) = \mathrm{diag}(\pi^{m_1}, \ldots, \pi^{m_n})$. Observe that, for $x \in \mathcal{o} - \{0\}$,

$$\int_E \sigma(\beta_1 - x\kappa)\psi(\kappa)\,\mathrm{d}\kappa = \begin{cases} \psi(x^{-1}\beta_1) & \text{if } x \in \mathcal{o}^\times \\ 0 & \text{if } x \in \pi\mathcal{o} - \{0\} \end{cases}$$

and that

$$\int_{\mathrm{GL}_n(E)} \sigma(y)|\det y|^{s-1}\,\mathrm{d}^*y = \zeta_E^{(n)}(s-1).$$

Thus

$$I' = \zeta_E^{(n)}(s-1) \int_{\mathcal{o}^\times} \mathrm{d}^*x \int_E \mathrm{d}\beta_1 \int_{\mathcal{o}^n p(\mathbf{m})} \mathrm{d}\beta_2 \int_{\mathcal{o}^n} \mathrm{d}\beta_3 \int_{M_n(\mathcal{o})\cdot p(\mathbf{m})} \mathrm{d}\beta_4$$
$$\sigma(\pi^{-f}(\beta_1 - \beta_2{}^t\lambda))\sigma(\pi^{-f}({}^t\beta_3 - \beta_4{}^t\lambda))\psi(x^{-1}\beta_1).$$

Since $\beta_2{}^t\lambda$ and $\beta_4{}^t\lambda$ are integral for $\beta_2 \in \mathcal{o}^n p(\mathbf{m})$ and $\beta_4 \in M_n(\mathcal{o})\cdot p(\mathbf{m})$, we obtain

$$I' = \zeta_E^{(n)}(s-1)\cdot q^{-(n+1)(m_1+\cdots+m_n)-fn} \int_{\mathcal{o}^\times} \mathrm{d}^*x \int_E \mathrm{d}\beta_1 \sigma(\pi^{-f}\beta_1)\psi(x^{-1}\beta_1)$$
$$= q^{-(n+1)(f+m_1+\cdots+m_n)}\zeta_E^{(n)}(s-1). \tag{6.27}$$

To calculate the integral I'', decompose δ into $u''\begin{pmatrix}\delta_1 & 0\\ {}^t\delta_2 & \delta_3\end{pmatrix}(u''\in \mathrm{GL}_{n+1}(\mathcal{O})$, $\delta_1\in E^\times$, $\delta_2\in E^n$, $\delta_3\in \mathrm{GL}_n(E))$. Then

$$I''=\int_{E^\times} d^*\delta_1\int_{E^n} d\delta_2\int_{\mathrm{GL}_n(E)} d\delta_3|\delta_1|^{s-2n-1}|\det\delta_3|^{s-n-1}$$
$$\times\,\sigma(\pi^{-f}\delta_1)\sigma(\pi^{-f}({}^t\delta_2-\delta_3{}^t\lambda))\sigma(\delta_3 p(\mathbf{m})^{-1})\cdot\sigma(\delta_1)\sigma(\delta_2)\sigma(\delta_3).$$

By a similar argument as above, we obtain

$$I''=q^{-(s-n-1)(f+m_1+\cdots+m_n)}\zeta_E(s-2n-1)\zeta_E^{(n)}(s-n-1). \tag{6.28}$$

Lemma 6.6 now follows from (6.24), (6.27) and (6.28). □

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