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Some remarks on ultrapowers and superproperties of the sum and interpolation spaces of Banach spaces

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Abstract. We give the structure of ultrapowers of sums of rearrangement invariant function spaces satisfying special equiintegrability conditions, and of ultraproducts of such spaces.

We prove that, with these conditions, superstability is conserved by real Lions-Peetre interpolation; and that Lorentz spaces $L(w, p)$ are superstable (for their natural norm).

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0. Introduction

It was proved in [2] that real interpolation spaces (for the "K-method") of two stable rearrangement invariant (r.i. in short) function spaces are stable, provided that these spaces satisfy a special equiintegrability condition for the unit ball of the one with respect to the norm of the other.

The proof of this result is based implicitly on a description of the ultrapowers of the sum $E + F$ of two r.i. spaces (when they satisfy the aforementioned condition), which we explicit here. As a consequence we obtain that (roughly speaking) when a property $\mathscr{P}$ "passes from E, F to $E + F$" (i.e. $E \in \mathscr{P}$, $F \in \mathscr{P} \Rightarrow E + F \in \mathscr{P}$) then the same is true for the corresponding superproperty: this is in particular the case for the stability property.

In order to obtain such a result for interpolation spaces, we try to give an analogous description for ultraproducts $\Pi_{t>0}(E + tF)/\mathscr{U}$. Here we are led to reinforce the "equiintegrability conditions" and to suppose a sort of uniform version of them, which is close to certain separation conditions introduced in particular in [1] or [11] in order to compute the K-functional of two r.i. spaces.

As a consequence we obtain results on the superstability of the Lions-Peetre interpolation spaces $[E, F]_{\theta,q}$ of two r.i. spaces, or of the Lorentz spaces $L_{(w,q)}$. These results give information of isometric or almost isometric nature in the case of non-decreasing weights w, in particular for the bidual of $L_{p,1}$, $p > 1$.

In the last section we give an isomorphic description of the ultraproducts $\Pi_t(E + tF)/\mathscr{U}$, under weaker equiintegrability conditions (e.g. the conditions of [1]) and a superstable renorming theorem for this case.

We refer to [10] for definition and main properties of the rearrangement invariant function spaces (which will be supposed "maximal" in the ambiguous cases); to [3] for the real interpolation method; to [8] for the main facts about stable Banach spaces; to [5] and [6] for the background on ultrapowers; to [12] for the notion of superstability.

If E is a r.i. space defined on $[0,\infty)$, its "fundamental function" is $\lambda_E(u) = \|\mathbb{1}_{[0,u]}\|_E$. Unless otherwise stated, we suppose the r.i. spaces to be normalized, i.e. $\lambda_E(1) = 1$. If f is a measurable function w.r. to the measure space $(\Omega, \mathscr{A}, \mu)$, we denote by f^* the non-decreasing rearrangement of $|f|$ (see [10]). $E(\Omega, \mathscr{A}, \mu)$ is then the space of measurable functions f defined on $(\Omega, \mathscr{A}, \mu)$ such that $f^* \in E$ (with the norm $\|f\|_{E(\Omega)} = \|f^*\|_E$).

If E is a Banach space, I an index set and $\mathscr{U}$ an ultrafilter, then every bounded family $(f_i)_{i\in I} \in \ell_\infty(I; F)$ defines an element of the ultrapower $E^I/\mathscr{U}$ which we denote by $(f_i)_i^\bullet$.

A "compatible Banach couple" is a couple (E, F) of Banach subspaces of the same topological vector space V. This is the case when E, F are Köthe function spaces, see [10]; then $V = L_0(\Omega)$.

Here we will obtain (as ultrapowers or ultraproducts) non regular Banach couples, i.e. $E \cap F$ not dense in E, F. In fact E, F will be Köthe function spaces in $L_0(\Omega')$, $L_0(\Omega'')$; Ω', Ω'' being distinct parts of the same measure space $\tilde{\Omega}$. Recall that $E + F$ (as a linear subspace of V) is equipped with the norm:

$$\|f\| = \inf_{g+h=f} (\|g\|_E + \|h\|_F)$$

In the lattice case, if $f \geqslant 0$ we may restrict the infimum to the couples (g, h) with $0 \leqslant g, h \leqslant f$.

1. Ultrapowers of $E+F$

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space; $E = E(\Omega, \mathscr{A}, \mu)$ and $F = F(\Omega, \mathscr{A}, \mu)$ two rearrangement invariant function spaces over $(\Omega, \mathscr{A}, \mu)$. We suppose:

(A1) E and F do not contain c_0.

(A2) $\sup_{\|f\|_F \leqslant 1} \|f \cdot \mathbb{1}_{|f|>A}\|_E \to 0$ when $A \to \infty$

(A3) $\sup_{\|e\|_E \leqslant 1} \|e \cdot \mathbb{1}_{|e|<\varepsilon}\|_F \to 0$ when $\varepsilon \to 0$.

Conditions (A2), (A3) may be viewed as "equiintegrability conditions" on the couple (E, F).

Let be $X = E + F$ and denote by $\tilde{E}$, $\tilde{F}$, $\tilde{X}$ the ultrapowers $E^I/\mathscr{U}$, $F^I/\mathscr{U}$, $X^I/\mathscr{U}$.

Recall that $\tilde{E}$ contains a particular band $\tilde{E}_0$, generated by the elements $\tilde{e} = (\mathbb{1}_{A_i})_i^{\bullet}$ having a representing family consisting of indicator functions of uniformly measure bounded sets. $\tilde{E}_0$ may also be characterized as the set of elements $\tilde{f} = (f_i)_i^{\bullet}$ having a representing family consisting of functions whose rearrangements f_i^* are E-equiintegrable. $\tilde{E}_0$ is lattice-isometric to the r.i. space $E(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$ where the measure space is the ultrapower $(\Omega, \mathscr{A}, \mu)^I/\mathscr{U}$ (defined for example in [5], §4); see also [16].

As E, F do not contain c_0, we have band projections in $\tilde{E}$, $\tilde{F}$ onto $E(\tilde{\Omega})$, $F(\tilde{\Omega})$ respectively; so we may write

$$\tilde{E} = E(\tilde{\Omega}) \oplus \tilde{E}_s; \;\; \tilde{F} = F(\tilde{\Omega}) \oplus \tilde{F}_s$$

where the complementary bands $\tilde{E}_s$, $\tilde{F}_s$, the "singular parts" of $\tilde{E}$, $\tilde{F}$, consist of the elements $\tilde{f}$ having representing families $(f_i)_{i\in I}$ such that $\lim_{i,\mathscr{U}} f_i^* = 0$ for the topology of convergence in measure.

Let $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) be the set of element $\tilde{e} \in \tilde{E}$ having representing families $(e_i)_{i\in I}$ with $|e_i| \geqslant \|e_i\| \cdot \mathbb{1}_{\{e_i \neq 0\}}$ (resp. $|e_i| \leqslant \|e_i\|$) for every $i \in I$, let $\tilde{E}_f = \mathscr{E}_1^{\perp}$, $\tilde{E}_p = \mathscr{E}_2^{\perp}$ the bands of elements of $\tilde{E}$ which are disjoint from $\mathscr{E}_1$ (resp. $\mathscr{E}_2$). As $\mathscr{E}_1 \cap \mathscr{E}_2$ contains the set of families $(\mathbb{1}_{A_i})_{i\in I}$ with $\mu(A_i) = 1$, $\forall i$, we have $\tilde{E}_f \subset \tilde{E}_s$, $\tilde{E}_p \subset \tilde{E}_s$. If $\tilde{f} = (f_i)_{i\in I} \in \tilde{E}_f \cap \tilde{E}_p$, set $A_{f_i} = \{|f_i| \geqslant \|f_i\|\}$; as $(\mathbb{1}_{A_{f_i}} \cdot f_i)^{\bullet} \in \mathscr{E}_1$ it is the zero element; thus we may suppose $f_i = \mathbb{1}_{A^c_{f_i}} \cdot f_i$: then $\tilde{f} \in \mathscr{E}_2$ and finally $\tilde{f} = 0$. Hence $\tilde{E}_f \cap \tilde{E}_p = \{0\}$.

Although $\tilde{E}_s$ may be not order complete, there are projection bands onto $\tilde{E}_f$ and $\tilde{E}_p$, and more precisely we have:

LEMMA 1. $\tilde{E}_s = \tilde{E}_f \oplus \tilde{E}_p$ (and similarly $\tilde{F}_s = \tilde{F}_f \oplus \tilde{F}_p$).

Proof. If $\tilde{f} = (f_i)^{\bullet} \in \tilde{E}_s$, then $\forall \varepsilon > 0$, $\forall A < \infty$, $\lim_{i,\mathscr{U}} \|\mathbb{1}_{\{\varepsilon < |f_i| < A\}} f_i\| = 0$. (for $\operatorname{Sup}_i \mu(\{|f_i| > \varepsilon\}) < \infty$, hence $(\mathbb{1}_{\{|f_i| > \varepsilon\}})^{\bullet}$ is disjoint from $\tilde{f}$, but $\mathbb{1}_{\{\varepsilon < |f_i| < A\}} |f_i| \leqslant A \mathbb{1}_{\{|f_i| > \varepsilon\}} \wedge |f_i|$).

Set $A_{f_i} = \{|f_i| \geqslant \|f_i\|\}$, $g_i = \mathbb{1}_{A^c_{f_i}} \cdot f_i$, $h_i = \mathbb{1}_{A_{f_i}} \cdot f_i$ and $\tilde{g} = (g_i)_i^{\bullet}$, $\tilde{h} = (h_i)_i^{\bullet}$.

Let $\tilde{e} = (e_i)_{i\in I} \in \mathscr{E}_1$. We have for each $\varepsilon > 0$:

$$|g_i| \wedge |e_i| \leqslant \varepsilon \mathbb{1}_{\{e_i \neq 0\}} + \mathbb{1}_{\{\varepsilon < |f_i| < \|f_i\|\}} \cdot |f_i|$$

By the preceding, the second term vanishes (when taking the $\mathscr{U}$ limit) and as $\{e_i \neq 0\} = \{|e_i| \geqslant \|e_i\|\}$ we have

$$\mathbb{1}_{\{e_i \neq 0\}} \leqslant \frac{|e_i|}{\|e_i\|} \mathbb{1}_{\{|e_i| \geqslant \|e_i\|\}} \leqslant \frac{|e_i|}{\|e_i\|}, \text{ hence } \|\mathbb{1}_{\{e_i \neq 0\}}\| \leqslant 1.$$

Hence $\| |\tilde{g}| \wedge |\tilde{e}| \| = \lim_{i,\mathscr{U}} \| |g_i| \wedge |e_i| \| \leqslant \varepsilon$, which implies (with $\varepsilon \to 0$) $\tilde{g} \perp \tilde{e}$, i.e. $\tilde{g} \in \tilde{E}_f$.

Similarly if $\tilde{e}\in\mathscr{E}_2$ we have for each $A>0$:

$$|h_i|\wedge|e_i|\leqslant \mathbb{1}_{\{\|f_i\|<|f_i|\leqslant A\}}\cdot|f_i|+\mathbb{1}_{\{|f_i|\geqslant A\}}|e_i|$$

The first term vanishes, and the second verifies:

$$\mathbb{1}_{\{|f_i|\geqslant A\}}\cdot|e_i|\leqslant \|e_i\|\mathbb{1}_{\{|f_i|\geqslant A\}}\leqslant\frac{\|e_i\|}{A}|f_i|$$

hence $\||\tilde{h}|\wedge|\tilde{e}|\|\leqslant 1/A\,\|\tilde{e}\|\,\|\tilde{f}\|\xrightarrow[A\to\infty]{}0$; and $\tilde{h}\in\tilde{E}_p$. □

REMARK. It can be easily seen that $\tilde{f}\in\tilde{E}_f$ iff it can be represented by a "flat family" $(f_i)^\bullet_i$ (with $\lim_{i,\mathscr{U}}\|f_i\|_\infty=0$) and $\tilde{f}\in\tilde{E}_p$ iff it can be represented by a "peak family" $(f_i)^\bullet_i$ (with $\lim_{i,\mathscr{U}}\mu(\operatorname{Supp}f_i)=0$).

PROPOSITION 2. *Let E, F be rearrangement invariant function spaces (over the same measure space) verifying properties (A1) to (A3) above, and $X=E+F$. Then with the previous notation:*

$$\tilde{X}=(E+F)(\tilde{\Omega})\oplus\tilde{E}_p\oplus\tilde{F}_f,$$

the norm on $\tilde{X}$ being given by:

$$\forall\tilde{h}\in(E+F)(\tilde{\Omega}),\ \forall\tilde{e}\in\tilde{E}_p,\ \forall\tilde{f}\in\tilde{F}_f,$$
$$\|\tilde{h}+\tilde{e}+\tilde{f}\|=\inf\{\|\tilde{e}'+\tilde{e}\|_{\tilde{E}}+\|\tilde{f}'+\tilde{f}\|_{\tilde{F}}/\tilde{e}'\in E(\tilde{\Omega}),\ \tilde{f}\in F(\tilde{\Omega}),\ \tilde{e}'+\tilde{f}'=\tilde{h}\}.$$

Proof. As E, F do not contain c_0, the same is true for $E+F$ (see [2]); thus:

$$\tilde{X}=(E+F)^\sim=(E+F)(\tilde{\Omega})\oplus(E+F)^\sim_s$$
$$=(E+F)(\tilde{\Omega})\oplus(E+F)^\sim_p\oplus(E+F)^\sim_f.$$

Let J: $E\to E+F$ and H: $F\to E+F$ be the natural inclusion maps; $\tilde{J}$: $\tilde{E}\to(E+F)^\sim$ and $\tilde{H}$: $\tilde{F}\to(E+F)^\sim$ their natural extensions to ultrapowers. We have $\tilde{J}\tilde{E}+\tilde{H}\tilde{F}=(E+F)^\sim$.

The hypotheses (A2), (A3) may be written:

$$\tilde{J}(\tilde{E}_f)=\{0\};\ \tilde{H}(\tilde{F}_p)=\{0\}.$$

(For example if $\tilde{f}\in\tilde{E}_f$ then for every $\varepsilon>0$ there exists a representing family (f_i) with $\|f_i\|_\infty<\varepsilon$; thus $\lim_{i,\mathscr{U}}\|f_i\|_F=0$).

On the other hand, as the image by J of an E-equiintegrable family is clearly $(E + F)$-equiintegrable:

$$\tilde{J}E(\tilde{\Omega}) \subseteq \tilde{J}\tilde{E} \cap (E + F)(\tilde{\Omega}).$$

Similarly, using the fact that $\tilde{f} \in \tilde{E}_p$ iff for each $A > 0$ there exists a representing family $(f_i)_i$ with $\forall i$, $|f_i| > A$, we see that:

$$\tilde{J}\tilde{E}_p \subseteq \tilde{J}\tilde{E} \cap (E + F)_p^{\sim}$$

thus the two preceding inclusions are equalities. Similarly

$$\tilde{H}\tilde{F}(\tilde{\Omega}) = \tilde{H}\tilde{F} \cap (E + F)(\tilde{\Omega}) \qquad \tilde{H}\tilde{F}_f = \tilde{H}\tilde{F} \cap (E + F)_f^{\sim}$$

Putting together, we obtain:

$$\tilde{J}\tilde{E}_p = (E + F)_p^{\sim} \qquad \tilde{H}\tilde{F}_f = (E + F)_f^{\sim}$$

(with equal norms).

Now let $\tilde{\varphi} = (\varphi_i)_i^{\bullet} \in (E + F)^{\sim}$, we have:

$$\begin{aligned}
\|\tilde{\varphi}\| &= \lim_{i,\mathscr{U}} \inf_{e+f=\varphi_i} (\|e\|_E + \|f\|_F) \\
&= \inf_{\substack{(e_i)_i \in E^I \\ (f_i)_i \in F^I \\ e_i + f_i = \varphi_i}} \lim_{i,\mathscr{U}} (\|e_i\|_E + \|f_i\|_F) \\
&= \inf_{\substack{\tilde{e} \in \tilde{E}, \tilde{f} \in \tilde{F} \\ \tilde{J}\tilde{e} + \tilde{H}\tilde{f} = \tilde{\varphi}}} (\|\tilde{e}\|_{\tilde{E}} + \|\tilde{f}\|_{\tilde{F}})
\end{aligned}$$

Writing $\tilde{\varphi} = \tilde{\varphi}_0 + \tilde{\varphi}_p + \tilde{\varphi}_f$, with $\tilde{\varphi}_0 \in (E + F)(\tilde{\Omega})$, $\tilde{\varphi}_p \in (E + F)_p^{\sim}$, $\tilde{\varphi}_f \in (E + F)_f^{\sim}$, and setting $\tilde{e} = J^{-1}\tilde{\varphi}_p$, $\tilde{f} = \tilde{H}^{-1}\tilde{\varphi}_f$, we obtain:

$$\|\tilde{\varphi}\| = \operatorname*{Inf}_{\substack{\tilde{e}_0 \in E(\tilde{\Omega}), \tilde{f} \in F(\tilde{\Omega}) \\ \tilde{e}_0 + \tilde{f}_0 = \tilde{\varphi}_0}} (\|\tilde{e}_0 + \tilde{e}\|_{\tilde{E}} + \|\tilde{f}_0 + \tilde{f}\|_{\tilde{F}}). \qquad \square$$

REMARK 3. In the absence of the hypotheses A2, A3, we have nevertheless that the special "equiintegrable" band of $(E + F)^{\sim}$ is yet isometric to $E(\tilde{\Omega}) + F(\tilde{\Omega})$.

REMARK 4. Generalization to Köthe function spaces.

Let us briefly indicate how to extend the preceding to Köthe function spaces (see [10] for a definition). We say (by analogy with [16]) that a Köthe function space E has Essential Subsequence Splitting Property (ESSP) iff every bounded sequence in E can be split (after extraction) in a measure-vanishing part and an "essentially E-equi-integrable" one. (A family $(x_i)_{i\in I}$ will be said "essentially E-equiintegrable" iff $\lim_{\varepsilon\to 0}\operatorname{Sup}_i \|\mathbb{1}_{\{|x_i|<\varepsilon\}}\cdot x_i\| = 0$ and $\lim_{A\to\infty}\operatorname{Sup}_i \|\mathbb{1}_{\{|x_i|>A\}}\cdot x_i\| = 0$). Replace then conditions (A1) by:

(A′1): E, F are order continuous Köthe function spaces with ESSP.

Then if E, F verify (A′1), (A2), (A3):

$$(E+F)^{\sim} = (\tilde{E}_0 + \tilde{F}_0) \oplus \tilde{E}_p \oplus \tilde{E}_f$$

where $\tilde{E}_0$, $\tilde{F}_0$ are the bands in $\tilde{E}$, $\tilde{F}$ consisting of those elements having essentially equiintegrable representants. (They are function spaces over the same measure space).

2. Application to the superproperties of the sum of two Banach spaces

Recall that given a property $(\mathscr{P})$ for Banach spaces, the corresponding superproperty $(\tilde{\mathscr{P}})$ is defined by:

$$E \in \tilde{\mathscr{P}} \Leftrightarrow \text{each ultrapower } \tilde{E} \text{ of } E \text{ verifies } (\mathscr{P}).$$

Or, equivalently, each Banach space F which is finitely representable in E has property $(\mathscr{P})$.

It is clear that if $(\mathscr{P})$ passes from E, F to $E + F$ for each compatible couple $(E\,F)$ of Banach lattices satisfying the hypotheses of Remark 4, the same is true for the corresponding superproperty.

Let us give a result of this sort for the superproperty associated to the "stability property" for Banach spaces. Recall that a Banach space E is stable (cf. [8]) iff for any bounded sequences $(x_n)_n$, $(y_m)_m$ in E and any ultrafilters $\mathscr{U}$, $\mathscr{V}$:

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|x_n + y_m\| = \lim_{m,\mathscr{V}} \lim_{n,\mathscr{U}} \|x_n + y_m\|$$

It was proved in [2] that if (E, F) is a couple of *r.i.* spaces satisfying (A_1), (A_2), (A_3), then $E + F$ is stable.

We will extend slightly this result, after giving the following definition, and deduce a corollary on superstability (see [12] for details on this last property).

DEFINITION 5. Let E be a Köthe function space between $L_1 + L_\infty$ and $L_1 \cap L_\infty$, having a band decomposition $E = E^a \oplus E^s$. We say that E^a has a r.i.

structure over E^s iff it is a r.i. space and, for any couple e', e'' of equimeasurable elements of E^a we have:

$$\forall e_1 \in E^s, \ \|e' + e_1\| = \|e'' + e_1\|.$$

In particular, if E is a r.i. space (not containing c_0), then the decomposition $\tilde{E} = E(\tilde{\Omega}) \oplus \tilde{E}_s$ of its ultrapower is of this type ($E(\tilde{\Omega})$ is r.i. over E_s).

PROPOSITION 6. *Let (E, F) be a couple of stable Köthe function spaces having band decompositions $E = E^a \oplus E^s$, $F = F^a \oplus F^s$ with E^a (resp. F^a) r.i. over E^s (resp. F^s), and the couple E^a, F^a satisfy the conditions (A_2), (A_3). Then $E + F$ (defined in the superspace $L_0(\Omega) \oplus E^s \oplus F^s$) is stable.*

Proof. It could be obtained by a slight modification of the proof of prop. 22 of [2], but we prefer to give it here using the formalism of §1.

Let $\tilde{X}$ be any ultrapower of $X = E + F$. We have (see Remark 4)

$$\tilde{X} = (E^a + F^a)(\tilde{\Omega}) \oplus \tilde{E}^a_p \oplus \tilde{F}^a_f \oplus \tilde{E}^s \oplus \tilde{E}^s = (E^a + F^a)(\tilde{\Omega}) \oplus Z$$

Any $\tilde{\xi} \in Z$ can be decomposed as $\tilde{\xi} = \tilde{e} + \tilde{f}$, where

$$\tilde{e} \in \tilde{E}^a_p \oplus \tilde{E}^s \subseteq \tilde{E}, \qquad \tilde{f} \in \tilde{F}^a_f \oplus \tilde{F}^s \subseteq \tilde{F}.$$

For every $x \in X$, $x = x^a + e^s + f^s$ ($x^a \in E^a + F^a$, $e^s \in E^s$, $f^s \in F^s$) we have:

$$\|x + \tilde{\xi}\| = \inf\{\|\tilde{y} + e^s + \tilde{e}\| + \|\tilde{z} + f^s + \tilde{f}\| / \tilde{y} \in E^a(\tilde{\Omega}), \tilde{z} \in F^a(\Omega), \tilde{y} + \tilde{z} = x^a\}$$

But there exists a natural projection P_E from $E^a(\tilde{\Omega})$ to E^a (defined by conditional expectation) which is a contraction. (See [10, th. 2a 4]). As $E^a(\tilde{\Omega})$ is rearrangement invariant over $\tilde{E}^a_p \oplus \tilde{E}^s$ we obtain, by a straightforward extension of the proof of [10, th. 2a 4] that:

$$\forall \tilde{u} \in E_a(\tilde{\Omega}), \forall \tilde{v} \in E^a_p \oplus \tilde{E}^s, \ \|P_E\tilde{u} + \tilde{v}\| \leqslant \|\tilde{u} + \tilde{v}\|$$

Hence

$$\begin{aligned} \|x + \tilde{\xi}\| &= \inf_{\substack{y \in E^a\, z \in F^a \\ y+z=x^a}} (\|y + e^s + \tilde{e}\|_E + \|z + f^s + \tilde{f}\|_{\tilde{F}}) \\ &= \inf_{\substack{y \in E\, z \in F \\ y+z=x}} (\|y + \tilde{e}\|_{\tilde{E}} + \|z + \tilde{f}\|_{\tilde{F}}) \qquad (*) \end{aligned}$$

Now let $(x_n)_n$, $(y_m)_m$ be two bounded sequences in $E + F$, whose projections (x^a_n), (y^a_m) on $E^a + F^a$ tends to 0 in measure.

The family $(x_n + y_m)_{n,m}$ defines an element $\tilde{\xi}'$ of $(E+F)^{\mathbb{N}\times\mathbb{N}}/\mathcal{U}\times\mathcal{V}$ and $(x_n + y_m)_{m,n}$ an element $\tilde{\xi}''$ of $(E+F)^{\mathbb{N}\times\mathbb{N}}/\mathcal{V}\times\mathcal{U}$.

We decompose $\tilde{\xi}' = e' + f'$, $\tilde{\xi}'' = \tilde{e}'' + \tilde{f}''$ as before. Stability of E, F implies that

$$\forall e \in E, \forall f \in F, \|e + \tilde{e}'\| = \|e + \tilde{e}''\| \text{ and } \|f + \tilde{f}'\| = \|f + \tilde{f}''\|$$

and formula (*) that $\forall x \in E + F$, $\|x + \tilde{\xi}'\| = \|x + \tilde{\xi}''\|$.

This implies that

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|x + x_n + y_m\| = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} \|x + x_n + y_m\| \qquad (**)$$

for such sequences $(x_n)_n$, $(y_m)_m$; and, as [2, prop. 17], using the norm-compactness of the set of decreasing element in $(E^a + F^a)(\mathbb{R})$ (of norm $\leqslant 1$) we remark that (**) is sufficient to test the stability of $E + F$. □

COROLLARY 7. *Let E, F be rearrangement invariant functions spaces verifying properties* (A1) *to* (A3) of §1. *If E and F are superstable, so is their sum E + F.*

3. Ultraproducts of sums of r.i. spaces.

3.1. *Ultraproduct of sums*

Let $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be two families of rearrangement invariant spaces on the same measure space $(\Omega, \mathcal{A}, \mu)$.

Let us suppose now:

(B_1) The families $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ are uniform in the sense that for every ultrafilter $\mathcal{U}$ on I, the norms not containing c_o (on $L^\infty(\Omega) \cap L^1(\Omega)$):

$$\|f\|_{\tilde{E}} = \lim_{i,\mathcal{U}} \|f\|_{E_i}, \text{ resp. } \|f\|_{\tilde{F}} = \lim_{i,\mathcal{U}} \|f\|_{F_i}$$

define (by completion) r.i. spaces not containing c_o

$$(B_2) \quad \operatorname*{Sup}_i \operatorname*{Sup}_{f\in B_{F_i}} \|f \cdot \mathbb{1}_{|f|>A}\|_{E_i} \xrightarrow[A\to\infty]{} 0$$

$$(B_3) \quad \operatorname*{Sup}_i \operatorname*{Sup}_{e\in B_{E_i}} \|e \cdot \mathbb{1}_{|e|<\varepsilon}\|_{F_i} \xrightarrow[\varepsilon\to 0]{} 0.$$

We have then the following decomposition of the ultraproducts:

$$\prod_i E_i/\mathscr{U} = \bar{E}(\tilde{\Omega}) \oplus \tilde{E}_p \oplus \tilde{E}_f, \ \prod_i F_i/\mathscr{U} = \bar{F}(\tilde{\Omega}) \oplus \tilde{F}_p \oplus \tilde{f}_f,$$

where $\tilde{\Omega}$, the «peak parts» $\tilde{E}_p$, $\tilde{F}_p$, and the «flat parts» $\tilde{E}_f$, $\tilde{F}_f$ are defined analogously to the ultrapower case.

Proposition 2 has the following analog (with plainly analogous proof):

PROPOSITION 8. *Let $(E_i)_{i\in I}$, $(F_i)_{i\in I}$ be families of r.i. spaces satisfying the conditions (B_1) to (B_3). Then:*

$$\prod_i (E_i + F_i)/\mathscr{U} = (\bar{E} + \bar{F})(\tilde{\Omega}) \oplus \tilde{E}_p \oplus \tilde{F}_f$$
$$= (\bar{E}(\tilde{\Omega}) \oplus \tilde{E}_p) + (\bar{F}(\tilde{\Omega}) \oplus \tilde{F}_f)$$

(*the last equality with equal norms*).

Note that conditions (B_2), (B_3) may be rewritten (in view of (B_1)):

$$\operatorname*{Sup}_i \ \operatorname*{Sup}_{\substack{f\in B_{F_i} \\ A\subset\Omega, \mu(A)<\varepsilon}} \|f\cdot 1\!\!1_A\|_{E_i} \xrightarrow[\varepsilon\to 0]{} 0 \text{ and}$$

$$\operatorname*{Sup}_i \ \operatorname*{Sup}_{e\in B_{E_i}} \ \inf_{\substack{A\subset\Omega \\ \mu(A)\leqslant M}} \|f\cdot 1\!\!1_{A^c}\| \xrightarrow[M\to\infty]{} 0$$

REMARK 9. Define as in [1] the space $M(E_i, F_i)$ of multipliers from E_i in F_i by

$$M_i = M(E_i, F_i) = \{f\in L^0(\Omega, \mathscr{A}, \mu)/ \text{the operator } M_f\colon g\to f\cdot g \text{ is defined and bounded } E_i\to F_i\},$$

with $\|f\|_{M_i} = \|M_f\|_{\mathscr{L}(E_i,F_i)}$: this is a r.i. space over $(\Omega, \mathscr{A}, \mu)$. Then:

(B_2) is equivalent to: (B'_2): $\operatorname*{Sup}_i \lambda_{M_i}(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$

and:

(B_3) is implied by: (B'_3) $\operatorname*{Inf}_i \lambda_{M_i}(A) \xrightarrow[A\to\infty]{} \infty$,

in the case where, moreover, $E_i = M_i\cdot F_i$ and:

$$\exists K, \forall i, \forall f\in E_i, \exists m\in M_i, g\in F_i \text{ with } f = m\cdot g \text{ and } \|m\|\,\|g\| \leqslant K\|f\|_{F_i}.$$

The first assertion is a consequence of the equality

$$\sup_{\substack{f\in B_{E_i}\\ A\in\Omega,\mu(A)\leqslant\varepsilon}} \|\mathbb{1}_A f\|_{E_i} = \|\mathbb{1}_{[0,\varepsilon]}\|_{M_i}$$

for the second, see the proof of theorem 1 of [1].

3.2. *Ultraproducts* $\Pi_t(E + tF)/\mathcal{U}$

Here E, F are two r.i. function spaces on $\mathbb{R}_+$; for each $t > 0$, $E + tF$ is $E + F$ normed with the $K(t)$ functional of Lions-Peetre (see [3]):

$$K(t, f; E, F) = \inf_{g+h=f} (\|g\| + t\|f\|).$$

To reduce to the preceding case, we define the r.i. function spaces E_t, F_t by:

$$\|f\|_{E_t} = \left\|\frac{D_{a_t}f}{\lambda_E(a_t)}\right\|_E; \qquad \|g\|_{F_t} = \left\|\frac{D_{a_t}g}{\lambda_F(a_t)}\right\|_F$$

where:

D_a is the dilation operator: $D_a(f)(s) = f\left(\frac{s}{a}\right)$

λ_E, λ_F are respectively the fundamental functions of E, F

a_t is a solution of the equation $\lambda_E(a) = t\lambda_F(A)$.

Such a solution will exist for each t if we suppose:

$$\frac{\lambda_E(u)}{\lambda_F(u)} \xrightarrow[u\to 0]{} 0 \quad \text{and} \quad \frac{\lambda_E(u)}{\lambda_F(u)} \xrightarrow[u\to\infty]{} \infty.$$

For $f \in L_0$, set $S_t(f) = \dfrac{D_{a_t}f}{\lambda_E(a_t)}$; S_t is clearly an order isometry from E_t onto E, and from F_t onto $t\cdot F$; and consequently from $E_t + F_t$ onto $E + tF$. Thus the family $(S_t)_{t>0}$ defines an order isometry $\tilde{S}$ from $\Pi_t E_t/\mathcal{U}$ onto $E = E^{\mathbb{R}_+}/\mathcal{U}$ and from $\Pi_t(E_t + F_t)/\mathcal{U}$ onto $\tilde{X} = \Pi_t(E + tF)/\mathcal{U}$. On the other hand the family $\left(\frac{1}{t} S_t\right)_{t>0}$ defines an isometry $\tilde{S}'$ from $\Pi_t F_t/\mathcal{U}$ onto $\tilde{F} = F^{\mathbb{R}_+}/\mathcal{U}$.

We set $\lambda_{E,F}(u) = \dfrac{\lambda_E(u)}{\lambda_F(u)}$.

PROPOSITION 10. *The following assertions are equivalent:*

(i) *The families of spaces $(E_t)_{t>0}$ and $(F_t)_{t>0}$ verify the condition (B_2), (uniformly for every possible choice of $(a_t)_{t>0}$).*
(ii) *They verify the condition (B_3) (uniformly w.r. to $(a_t)_{t>0}$).*
(iii) *There is an $\alpha > 0$ such that $(\lambda_{E,F}(u))/u^\alpha$ is equivalent to a non-decreasing function of u.*

Proof. (i) ⇒ (iii). In particular there exists an $\varepsilon > 0$ such that, for every $\rho < \varepsilon$ we have:

$$\operatorname{Sup}_{t>0} \frac{\lambda_{E_t}(\rho)}{\lambda_{F_t}(\rho)} \leqslant \frac{1}{2},$$

which may be written:

$$\operatorname{Sup}_{t} \frac{\lambda_E(\rho a_t)/\lambda_E(a_t)}{\lambda_F(\rho a_t)/\lambda_F(a_t)} \leqslant \frac{1}{2}.$$

as this can be obtained uniformly in $(a_t)_{t>0}$, we have in fact:

$$\operatorname{Sup}_{a>0} \frac{\lambda_{E,F}(\rho a)}{\lambda_{E,F}(a)} \leqslant \frac{1}{2}$$

On the other hand, for each $0 < \rho \leqslant 1$: $\lambda_{E,F}(\rho a) \leqslant (1/\rho)\lambda_{E,F}(a)$ because $\lambda_E(\rho a) < \lambda_E(a)$ and $\lambda_F(\rho a) \geqslant \rho\lambda_F(a)$.

Thus there exists C such that:

$$\operatorname{Sup}_{0<\rho\leqslant 1} \operatorname{Sup}_{a>0} \frac{\lambda_{E,F}(\rho a)}{\lambda_{E,F}(a)} \leqslant C,$$

which means that $\lambda_{E,F}$ is equivalent to a non-decreasing function. (Namely $h(a) = \operatorname{Sup}_{0<\rho\leqslant 1} \lambda_{E,F}(\rho a)$).

Setting now $\frac{1}{2} = \varepsilon^\alpha$ we see that $(\lambda_{E,F}(u))/u^\alpha$ is equivalent to a non-decreasing function (namely $k(a) = \operatorname{Sup}_{0<\rho<1} (\lambda_{E,F}(u))/\rho^\alpha$).

(ii) ⇒ (iii)

We have now

$$\operatorname{Sup}_t \operatorname{Sup}_f \frac{\|\mathbb{1}_{[A,\infty[} f^*\|_{F_t}}{\|f^*\|_{E_t}} \xrightarrow[A\to\infty]{} 0;$$

taking $f^* = \mathbb{1}_{[0,2A]}$ we obtain in particular:

$$\operatorname*{Sup}_t \frac{\lambda_{F_t}(A)}{\lambda_{E_t}(2A)} \xrightarrow[A\to\infty]{} 0.$$

As $\lambda_{E_t}(2A) \leqslant 2\lambda_{E_t}(A)$, we have:

$$\operatorname*{Sup}_t \frac{\lambda_{F_t}(A)}{\lambda_{E_t}(A)}, \xrightarrow[A\to\infty]{} 0,$$

which we can interpret (using the uniformity in the choice of (a_t)) as:

$$\operatorname*{Sup}_{a>0} \frac{\lambda_{E,F}(Aa)}{\lambda_{E,F}(a)} \xrightarrow[A\to\infty]{} \infty$$

which is the same as

$$\operatorname*{Sup}_{a>0} \frac{\lambda_{E,F}(\rho a)}{\lambda_{E,F}(a)} \xrightarrow[\rho\to 0]{} 0.$$

We continue as in the preceding case.

(iii) $\Rightarrow$ (i) and (ii)

We notice that

$$\lambda_{E_t,F_t}(\rho u) = \frac{\lambda_{E,F}(\rho a_t u)}{\lambda_{E,F}(a_t u)} \leqslant C\rho^\alpha \lambda_{E_t,F_t}(u)$$

Suppose that $\|f\|_{F_t} \leqslant 1$. Then for every $u > 0, f^*(u) \leqslant 1/(\lambda_{F_t}(u))$. Thus:

$$\begin{aligned}
\|\mathbb{1}_{[0,\varepsilon]} f^*\|_{E_t} &\leqslant \sum_{k=0}^{\infty} \left\| \Pi_{[(\varepsilon/2^{k+1},\varepsilon/2^k]} f^* \left(\frac{\varepsilon}{2^{k+1}}\right) \right\|_{F_t} \\
&= \sum_{k=0}^{\infty} f^* \left(\frac{\varepsilon}{2^{k+1}}\right) \lambda_{E_t} \left(\frac{\varepsilon}{2^{k+1}}\right) \\
&\leqslant \sum_{k=1}^{\infty} \frac{\lambda_{E_t}(2^{-k}\varepsilon)}{\lambda_{F_t}(2^{-k}\varepsilon)} \leqslant C \cdot \sum_{k=1}^{\infty} (2^{-k}\varepsilon)^\alpha \leqslant C'\varepsilon^\alpha \xrightarrow[\varepsilon\to 0]{} 0
\end{aligned}$$

and we obtain condition B_2.

Similarly, suppose that $\|f\|_{E_t} \leqslant 1$, thus $f^*(u) \leqslant 1/(\lambda_{E_t}(u))\,(\forall u > 0)$. Then:

$$\|\mathbb{1}_{[A,\infty[}f^*\|_{F_t} \leqslant \sum_{k=0}^{\infty} \|\mathbb{1}_{[2^kA,2^{k+1}A[}\cdot f^*\|_{F_t} \leqslant \sum_{k=0}^{\infty} \|\mathbb{1}_{[2^kA,2^{k+1}A[}f^*(2^kA)\|_{F_t}$$

$$\leqslant \sum_{k=0}^{\infty} \frac{\lambda_{F_t}(2^kA)}{\lambda_{F_t}(2^kA)} \leqslant C \sum_{k=0}^{\infty} (2^kA)^{-\alpha} \leqslant \frac{C'}{A^\alpha} \xrightarrow[A\to\infty]{} 0.$$

i.e. we obtain condition (B_3). □

In particular we have under such conditions a description of $\Pi_t(E + tF)/\mathscr{U}$.

COROLLARY 11. *Let E, F two r.i. spaces with non-trivial concavity, $\lambda_{E,F} = (\lambda_E/\lambda_F)$ their relative fundamental function. Suppose that there exists an $\alpha > 0$ such that $(\lambda_{E,F}(u))/u^\alpha$ is (equivalent to) a non decreasing function of u. Then we have band decompositions:*

(i) $\Pi_{t>0}(E + tF)/\mathscr{U} = (\bar{E} + \bar{F})(\bar{\Omega}) \oplus \bar{E}_p \oplus \bar{F}_f$
(ii) $\tilde{E} = E^{\mathbb{R}_+}/\mathscr{U} = \bar{E}(\bar{\Omega}) \oplus \bar{E}_p \oplus \bar{E}_f$
(iii) $\tilde{F} = F^{\mathbb{R}_+}/\mathscr{U} = \bar{F}(\bar{\Omega}) \oplus \bar{F}_p \oplus \bar{F}_f$

(The decompositions (ii) and (iii) are different from those described in §1).
Moreover $\Pi_{t>0}(E + tF)/\mathscr{U} = (\bar{E}(\bar{\Omega}) \oplus \bar{E}_p) + (\bar{F}(\bar{\Omega}) \oplus \bar{F}_f)$ *isometrically.*

EXAMPLES. (i) $E = L_p \quad F = L_q$ with $p < q$.
(ii) $E = L_\varphi \quad F = L_\psi$ with $\psi \sim \varphi \circ \zeta$ where ζ is a r-convex Orlicz function, for some $r > 1$.
(iii) E is p concave, F is q-convex with $p < q$. Or, more generally, E has upper Boyd index q_E, F has lower Boyd index p_F with $q_E < p_F$.

4. Application to superstability of interpolation spaces

We note that the r.i. spaces $\bar{E}(\bar{\Omega})$, $\bar{F}(\bar{\Omega})$ (of cor. 12) verify the conditions (A_1), (A_2), (A_3) of §1. Thus if E, F are superstable, we obtain by prop. 6 that $\Pi_{t>0}(E + tF)/\mathscr{U}$ is stable, for each ultrafilter $\mathscr{U}$ on $\mathbb{R}_+$. We say that the family $(X_t)_{t>0}$, $X_t = E + tF$, is "uniformly superstable". It was proved in [13] that the space $Z = (\bigoplus_{t>0} X_t)_{l^2}$ is then superstable. If $1 \leqslant p < \infty$ is a given exponent and λ a measure on $\mathbb{R}_+$, we define the space $[E, F]_{\lambda,p}$ as the space of elements f of $E + F$ such that $\int_0^\infty K(t, f; E, F)^p \, d\lambda(t) < \infty$. (For $d\lambda(t) = 1/t^{\theta p+1}\cdot dt$, we have $[E, F]_{\lambda,p} = [E, F]_{\theta,p}$, the usual Lions-Peetre interpolation space).

COROLLARY 12. *Let E, F be two superstable* r.i. spaces satisfying the *hypotheses of cor.* 11. *Then for every $1 \leqslant p < \infty$ and λ, the space $[E, F]_{\lambda,p}$ is superstable.*

For the space $[E, F]_{\lambda,p}$ is finitely representable in $l_p(Z)$, which is superstable (see [12]) as Z is. □

The Lorentz spaces $L(w, p)$, where $1 \leqslant p < \infty$ and w is a decreasing weight on $\mathbb{R}_+$ (see [10], p. 120; [9]; the sequence space version is denoted by $d(w, p)$ in [10]) are (λ, p) interpolation spaces in the preceding sense:

$$L(w, p) \approx [L^p, L^\infty]_{\lambda,p} \tag{4.1}$$

where λ is the Stieltjès measure associated to the function $-w(t^p)$. This is a consequence of the Holmstedt formula $K(f, t; L^p, L^\infty) \sim (\int_0^{t^p} f^*(s)^p \,ds)^{1/p}$ and the definition of the norm on $L(w, p)$:

$$\|f\|_{w,p} = \left(\int_0^\infty f^*(s)^p w(s)\,ds\right)^{1/p}. \tag{4.2}$$

An integration by parts shows that

$$\|f\|_{w,p}^p = \int_0^\infty \left(\int_0^t f^*(s)^p\,ds\right)(-dw)(t).$$

When $L(w, p)$ is q-concave for a certain $q < \infty$ then

$$L(w, p) \approx [L^p, L^q]_{\lambda,p} \tag{4.3}$$

where λ is now the Stieltjès measure associated to $-w(t^\alpha)$, $1/\alpha = 1/p - 1/q$. This is a consequence of the Holmstedt formula ([7] or [3])

$$K(t, f; L^p, L^q) \sim \left(\int_0^{t^\alpha} f^*(s)^p\,ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^*(s)^q\,ds\right)^{1/q} \tag{4.4}$$

and the following equivalence for the $L(w, p)$ norm, proved by [15]:

$$\|f\|_{w,p} \sim \left\{\int_0^\infty \left[\left(\int_0^t f^*(s)^p\,ds\right)^{1/p} + t^{1/p-1/q}\left(\int_t^\infty f^*(s)^q\,ds\right)^{1/q}\right]^p (-dw(t))\right\}^{1/p} \tag{4.5}$$

We obtain thus that $L(w, p)$ has a superstable equivalent norm, when it has non trivial q-concavity, which is also a consequence of the fact that it embeds in $L_p(L_q)$ (as a consequence of [15]).

We will prove in fact that the natural norm on $L(w, p)$ is superstable:

THEOREM 13. *Let w be a non increasing weight on $\mathbb{R}_+$ such that the Lorentz space $L(w, p)$ has non trivial concavity. Then $L(w, p)$ is superstable for its natural norm.*

Proof. We suppose first that $p = 1$. Denote by q' the conjugate exponent of q. We introduce a modified K-functional of interpolation between L^1 and L^q by:

$$\tilde{K}(t, f, L^1, L^q) = \|(f^* - f^*(t^{q'}))\mathbb{1}_{[0,t^{q'}]}\|_{L^1} + t\|f^*(t^{q'})\mathbb{1}_{[0,t^{q'}]} + f^*\mathbb{1}_{]t^{q'},\infty[}\|_{L^q} \tag{4.6}$$

which lies between the usual K-functional and Holmstedt estimation (4.4).

Note that

$$\tilde{K}(t, f; L^1, L^q) = t^{q'}\tilde{K}(1, D_{t^{-q'}}f; L^1, L^q). \tag{4.7}$$

LEMMA 14. $\lim_{q\to\infty} \int_0^\infty \tilde{K}(t^{1/q'}, f; L^1, L^q)(-\mathrm{d}w)(t) = \|f\|_{w,1}$

uniformly for $\|f\|_{w,1} \leqslant 1$.

Proof of Lemma 14. We have:

$$\tilde{K}(t, f) \geqslant \int_0^{t^{q'}} f^*(s)\,\mathrm{d}s - t^{q'}f^*(t^{q'}) + t\|f^*(t^{q'})\mathbb{1}_{[0,t^{q'}]}\|_q$$
$$= \int_0^{t^{q'}} f^*(s)\,\mathrm{d}s = K(t^{q'}, f; L^1, L^\infty).$$

On the other hand as the natural inclusion $L^{q,1} \hookrightarrow L^q$ is of norm 1, we have:

$$\|f^*(t^{q'})\mathbb{1}_{[0,t^{q'}]} + f^*\mathbb{1}_{]t^{q'},\infty[}\|_q \leqslant t^{q'/q}f^*(t^{q'}) + \frac{1}{q}\int_{t^{q'}}^\infty s^{1/q-1}f^*(s)\,\mathrm{d}s$$

hence:

$$\tilde{K}(t, f) \leqslant \int_0^{t^{q'}} f^*(s)\,\mathrm{d}s + \frac{t}{q}\int_{t_{q'}}^\infty s^{1/q-1}f^*(s)\,\mathrm{d}s.$$

i.e.

$$|\tilde{K}(t, f) - K(t, f; L^1, L^\infty)| \leqslant \frac{t}{q}\int_{t^{q'}}^\infty s^{1/q-1}f^*(s)\,\mathrm{d}s \tag{4.7}$$

By Fubini theorem:

$$\frac{1}{q}\int_0^\infty \left(t^{1/q'}\int_t^\infty s^{1/q-1}f^*(s)\,\mathrm{d}s\right)(-\mathrm{d}w(t))$$
$$=\frac{1}{q}\int_0^\infty f^*(s)s^{1/q-1}\left(\int_0^s t^{1/q'}\,\mathrm{d}w(t)\right)\mathrm{d}s. \tag{4.8}$$

By [15] as $L_{w,1}(\mathbb{R})$ is supposed to be q_0-concave, there exists a constant A_{q_0} such that:

$$\forall s>0 \int_0^s t^{1/q_0'}\,\mathrm{d}w(t) \leqslant A_{q_0}s^{1/q_0'+1}\,w(s) \tag{4.9}$$

Then for any $q \geqslant q_0$ the same relation remains true with a constant $A_q \leqslant A_{q_0}$.

$$\left(\text{for if } q \geqslant q_0 \text{ then } \int_0^s t^{1/q'}\,\mathrm{d}w(t) \leqslant s^{(1/q')-(1/q_0')}\int_0^s t^{1/q_0'}\mathrm{d}w(t)\right)$$

Thus the right member in 4.8 is less than:

$$\frac{A_{q_0}}{q}\int_0^\infty f^*(s)w(s)\,\mathrm{d}s = \frac{A_{q_0}}{q}\,\|f\|_{w,1} \qquad (\text{for } q \geqslant q_0).$$

i.e. we obtain

$$\left|\int_0^\infty \tilde{K}(t^{1/q'}, f; L^1, L^q)(-\mathrm{d}w)(t) - \|f\|_{w,1}\right| \leqslant \frac{A_{q_0}}{q}\,\|f\|_{w,1} \tag{4.10}$$

which proves the lemma.

The trouble with the functionals $\tilde{K}(t, .)$ is that they are not continuous for the norm of $L_1 + L_q$ (note that they are not norms). For this reason we set, for each $\varepsilon > 0$:

$$\tilde{K}^\varepsilon(t, f) = \frac{1}{\varepsilon}\int_1^{1+\varepsilon} \tilde{K}(\rho t, f)\,\mathrm{d}\rho \tag{4.11}$$

Using (4.7) we see that:

$$\tilde{K}^\varepsilon(t, f) = t^{q'}\tilde{K}^\varepsilon(1, D_{t^{-q'}}f). \tag{4.12}$$

LEMMA 15. *Each functional $\tilde{K}^\varepsilon(t, \cdot)$ is uniformly continuous on the unit ball of $L_1 + L_q$ (for the norm metric).*

Proof of Lemma 15. It suffices to prove that this is the case for the functionals

$$A\colon A(f) = \int_0^{t^{q'}} f^*(s)\,\mathrm{d}s \qquad B\colon B(f) = \left(\int_{t^{q'}}^{\infty} f^*(s)^q\,\mathrm{d}s\right)^{1/q}$$

For A it is evident, since A is a norm (majorized by the $(L_1 + L_q)$-norm). Suppose now that f_n, g_n are sequences in the unit ball of $L_1 + L_q$ with

$$\|f_n - g_n\|_{L_1 + L_q} \xrightarrow[n\to\infty]{} 0$$

Using the Subsequence Splitting Property of $L_1 + L_q$ we may suppose

$$\begin{cases} f_n = f'_n + f''_n + f'''_n \\ g_n = g'_n + g''_n + g'''_n \end{cases} \tag{4.13}$$

where: $(f'^*_n)_n$, $(g'^*_n)_n$ are $(L_1 + L_q)$-equiintegrable,

$(f''_n)_n$, $(g''_n)_n$ have supports converging to 0 in measure,

(f'''_n), $(g'''_n)_n$ converge to 0 in L_∞-norm.

Clearly $B(f''_n) \to 0$, $B(g''_n) \to 0$ and

$$B(f_n)^q \underset{n\to\infty}{\approx} B(f'_n)^q + B(f'''_n)^q \underset{n\to\infty}{\approx} B(f'_n)^q + \|f'''_n\|_q^q$$

$$B(g_n)^q \underset{n\to\infty}{\approx} B(g'_n)^q + B(g'''_n)^q \approx B(g'_n)^q + \|g'''_n\|_q^q.$$

We have also $f'^{*q}_n - g'^{*q}_n \to 0$ in measure, and by equiintegrability:

$$B(f'_n)^q - B(g'_n)^q \to 0$$

(note that the $L_1 + L_q$ equiintegrability of (f'_n), (g'_n) implies the L_q equiintegrability of $(\mathbb{1}_{[t^{q'},\infty[}f'^*_n)$, $(\mathbb{1}_{[t^{q'},\infty[}g'^*_n)$ by Holmsted formula), and on the other hand

$$\begin{aligned} \big|\|f'''_n\|_q - \|g'''_n\|_q\big| &\leqslant \|(f'''_n - g'''_n)\|_q \approx B(f'''_n - g'''_n) \\ &\leqslant \tilde{K}(t, f'''_n - g'''_n) \\ &\leqslant C\|f'''_n - g'''_n\|_{L_1 + L_q} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Finally $B(f_n) - B(g_n) \to_{n\to\infty} 0$, so we have $|B(f) - B(g)| \leqslant \varphi(t; \|f - g\|)$ where it is easy to see that the modulus of continuity φ is locally uniform for $t \in]0, \infty[$.

Hence by triangular inequality:

$$|\tilde{K}(t,f) - \tilde{K}(t,g)| \leqslant C[\|f-g\| + t\varphi(t;\ \|f-g\|)] + 2t^{q'}|f^*(t^{q'}) - g^*(t^{q'})|$$

and by integration:

$$|\tilde{K}^{\varepsilon}(t,f) - \tilde{K}^{\varepsilon}(t,g)| \leqslant C\cdot\varphi_1(t;\ \|f-g\|) + \frac{2}{\varepsilon}\int_1^{1+\varepsilon} |f^*(\rho^{q'}t^{q'}) - g^*(\rho^{q'}t^{q'})|\rho^{q'}t^{q'}\,\mathrm{d}\rho. \tag{4.14}$$

The second term in the right member of (4.14) is majorized by

$$\frac{2(1+\varepsilon)}{\varepsilon}t^{q'}\int_0^{1+\varepsilon} |f^*(\rho^{q'}t^{q'}) - g^*(\rho^{q'}t^{q'})|\rho^{q'-1}\,\mathrm{d}\rho = \frac{2(1+\varepsilon)}{2q'}\int_0^{(1+\varepsilon)^{q'}t^{q'}} |f^*(u) - g^*(u)|\,\mathrm{d}u$$

which by [17], prop. 1, is majorized by:

$$\leqslant \frac{2(1+\varepsilon)}{\varepsilon}\int_0^{(1+\varepsilon)^{q'}t^{q'}} |f-g|^*(u)\,\mathrm{d}u \leqslant C(\varepsilon, t^{q'})\|f-g\|_{L^1+L^q}.$$

Hence Lemma 15.

This lemma allows us to take the ultrapower of functional $\tilde{K}_\varepsilon$, i.e. we define for $\tilde{f} = (f_i)_i^\bullet \in (L_1 + L_q)^I/\mathscr{U}$:

$$\tilde{K}_\varepsilon(\tilde{f}) = \lim_{i,\mathscr{U}} \tilde{K}_\varepsilon(f_i) \tag{4.15}$$

Consider the decomposition of $(L_1 + L_q)^I/\mathscr{U}$ into its principal and singular parts given in §1:

$(L_1 + L_q)^I/\mathscr{U} = (L_1 + L_q)(\tilde{\Omega}) \oplus \tilde{E}_p \oplus \tilde{F}_f$. (Where $\tilde{E}_p$ is a L_1 space and $\tilde{F}_f$ a L_q-space). It is not hard to see that:

$$\tilde{K}_\varepsilon(\tilde{f}) = \tilde{K}_\varepsilon(f^*) \quad \text{when} \quad \tilde{f} \in (L_1 + L_q)(\tilde{\Omega})$$
$$\tilde{K}_\varepsilon(\tilde{f}) = \|\tilde{f}\| \quad \text{when} \quad \tilde{f} \in \tilde{E}_p \quad \text{or} \quad \tilde{F}_f.$$

and in general if $\tilde{f} = \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2$ ($\tilde{f}_0 \in (L_1 + L_q)(\tilde{\Omega})$), $\tilde{f}_1 \in \tilde{E}_p$, $\tilde{f}_2 \in \tilde{F}_f$

$$\tilde{K}_\varepsilon(t, \tilde{f}) = \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \tilde{K}(\rho t, \tilde{f})\, \mathrm{d}\rho$$

with

$$\tilde{K}(t, \tilde{f}) = \int_0^{t^{q'}} \tilde{f}_0^*(s)\, \mathrm{d}s - t^{q'} \tilde{f}_0^*(t^{q'}) + \|\tilde{f}_1\|_1 + t\left[t^{q'} \tilde{f}_0^*(t^{q'})^q + \int_{t^{q'}}^{\infty} \tilde{f}_0^*(s)^q\, \mathrm{d}s + \|f_2\|_q^q \right]^{1/q}. \tag{4.16}$$

LEMMA 16. *Functionals $\tilde{K}_\varepsilon(t, \cdot)$ are stable on the unit ball of $(L_1 + L_q)^\sim$.*

Recall that a functional Φ on a normed vector space is said to be stable if $\lim\limits_{n,\mathscr{U}} \lim\limits_{m,\mathscr{V}} \Phi(x_n - y_m) = \lim\limits_{m,\mathscr{V}} \lim\limits_{n,\mathscr{U}} \Phi(x_n - y_m)$ (for every bounded sequences (x_n), (y_m).)

Lemma 16 is proved by a reasoning analogous to the proof of prop. 6, which we will not develop anew.

As a consequence, functionals $\tilde{K}_\varepsilon(t, \cdot)$ are superstable.

LEMMA 17. *On bounded sets of $L_{w,1}$, the functional*

$$|||f|||_{w,1}^{(\varepsilon,q)} = \int_0^\infty \tilde{K}_\varepsilon(t^{1/q'}, f; L_1, L_q)(-\mathrm{d}w(t))$$

is superstable (for each $\varepsilon > 0$ and $1 < q < \infty$).

Proof. Using (4.12) and the similar equality for the usual $K(t, \cdot; L_1, L_q)$ functional we see that it suffices to prove that on $L_1(L_1 + L_q)$ the functional

$$|||f|||^{(\varepsilon,q)} = \int_0^\infty \tilde{K}_\varepsilon(1; f(t, \cdot))\, \mathrm{d}t \text{ is superstable.}$$

We have in general

$$L_1(L_1 + L_q)^I/\mathscr{U} = L_1(\tilde{\Omega})((L_1 + L_q)(\tilde{\Omega}') \oplus L_1(\tilde{\Omega}'') \oplus L_q(\tilde{\Omega}'''))$$

(see [14] for a proof of this generalization of prop. 2) and it is not too difficult to see that for $\tilde{f}$ in this ultrapower:

$$|||\tilde{f}|||^{(\varepsilon,q)} := \lim_{i,\mathscr{U}} |||f_i|||^{(\varepsilon,q)} = \int_\Omega \tilde{K}_\varepsilon(1, \tilde{f}(\tilde{\omega}, \cdot))\,d\tilde{\omega}$$

where $\tilde{K}_\varepsilon$ is defined by (4.15), (4.16), and is stable by lemma 16. Then the functional $\tilde{f} \to |||\tilde{f}|||^{\varepsilon,q}$ is stable (by an easy modification of the proof that E stable implies $L_1(E)$ stable, see e.g. [1] prop. 18).

The last step to end the proof of proposition 13 is to prove the following extension to the $\tilde{K}^\varepsilon$ functionals of the approximation result given by lemma 14.

LEMMA 18. $\lim_{\substack{q\to\infty\\ \varepsilon\to 0}} |||f|||_{w,1}^{(q,\varepsilon)} = \|f\|_{w,1}$ *uniformly on the unit ball of* $L_{w,1}$.

Proof. By integration of (4.10) we have:

$$\left|\int_0^\infty \tilde{K}_\varepsilon(t^{1/q'}, f; L^1, L^q)(-dw(t)) - |||f|||^{(\varepsilon)}\right| \infty \frac{A_0}{q}\, |||f|||^{(\varepsilon)}$$

where

$$|||f|||^{(\varepsilon)} = \frac{1}{\varepsilon}\int_1^{1+\varepsilon} K(\rho t, f; L_1, L_\infty)\,d\rho(-dw(t))$$
$$= \frac{1}{\varepsilon}\int_1^{1+\varepsilon} \rho\|D_{\rho^{-1}}f\|_{w,1}\,d\rho.$$

So we have only to prove that $|||f|||^{(\varepsilon)} \xrightarrow[\varepsilon\to 0]{} \|f\|_{w,1}$, or that

$$\|D_{\rho^{-1}}f\|_{w,1} \xrightarrow[\substack{>\\ \rho\to 1}]{} \|f\|_{w,1} \text{ (uniformly for } \|f\|_{w,1} \leqslant 1).$$

But as:

$$\|D_{\rho^{-1}}f\|_{w,1} = \int_0^\infty f^*(\rho s)w(s)\,ds \leqslant \int_0^\infty f^*(s)w(s)\,ds,$$

and:

$$\|D_{\rho^{-1}}f\|_{w,1} = \frac{1}{\rho}\int_0^\infty f^*(s)w\left(\frac{s}{\rho}\right)ds \geqslant \frac{1}{\rho}\int_0^\infty f^*(s)w(s)\,ds$$

we have

$$\|f\|_{w,1} \leqslant \rho\|D_{\rho^{-1}}f\|_{w,1} \leqslant \rho\|f\|_{w,1} \qquad (\forall \rho \geqslant 1)$$

Case $p > 1$

We have only to change in the preceding $K(t, f; L_1, L_\infty)$ in:

$$K_p(t, f; L_p, L_\infty) = \left(\int_0^{t^p} f^*(s)^p \, ds\right)^{1/p} = K(t^p, f^p; L_1, L_\infty)^{1/p}$$

and similarly

$$\tilde{K}(t, f; L_1, L_q) \text{ in } \tilde{K}_p(t, f; L_p, L_{p \cdot q}) = \tilde{K}(t^p, f^p; L_1, L_q)^{1/p}.$$

This ends the proof of Theorem 13. □

Before ending this section let us remark that Corollary 12, in the usual situation of Lions-Peetre interpolation, has interest mainly from isometric or almost isomorphic point of view. For it is not hard to see that the Lions-Peetre interpolation spaces $[E, F]_{\theta,p}$ of two "well separated" r.i. spaces E, F (i.e. with relative fundamental functions $\lambda_{E,F}$ equivalent to an α-convex function for some $\alpha > 0$) are isomorphic to Lorentz spaces $L(w, p)$ (with in general non monotone weight w).

5. Renorming interpolation spaces

We suppose here weaker separation assumptions than those of cor. 11, for example the assumptions of [1] or of [11]. Then we give an isomorphic description of the ultraproducts $\Pi_{t>0}(E + tF)/\mathscr{U}$ which enables us to give a superstable renorming theorem for interpolation spaces.

THEOREM 19. *Let E, F be two superstable r.i. spaces (on the same measure space). We suppose Arazy's conditions to be satisfied, i.e. $E = M \cdot F$ with $\lambda_M(t) \to_{t\to 0} 0$, $\lambda_M(t) \to_{t\to\infty} \infty$ ([1]). Then for every $t > 0$ there exists a norm N_t on E, F, such that:*

(i) *The family of norms $(N_t)_{t>0}$ is uniformly equivalent to the family of $K(t; E, F)$ functionals.*
(ii) *The family of spaces $(E + F, N_t)_{t>0}$ is uniformly superstable. Consequently each interpolation space $[E, F]_{\lambda,\theta}$ has an equivalent superstable renorming.*

As a corollary of the proof of th. 19, we will obtain

PROPOSITION 20. *Let E, F be two r.i. spaces, with non trivial concavity and satisfying Arazy's separation conditions. Then*

$$\prod_{t>0} (E + tF)/\mathcal{U} \approx (\bar{E}(\bar{\Omega}) + \bar{F}(\bar{\Omega})) \oplus \bar{E}_p \oplus \bar{F}_f$$

isomorphically (with constant independent of $\mathcal{U}$*), where* $\bar{E}(\bar{\Omega})$, $\tilde{E}_p$, $\bar{F}(\bar{\Omega})$, $\tilde{F}_p$ *are the bands in* $E^{\mathbb{R}_+}/\mathcal{U}$, $F^{\mathbb{R}_+}/\mathcal{U}$ *described in corollary* 11.

Proof. We know that the $K(t; E, F)$ functional is equivalent, with constants independent of t, to:

$$F(t, f; E, F) = \|f^*\mathbb{1}_{[0,\psi(t)]}\|_E + t\|f^*\mathbb{1}_{[\psi(t),\infty[}\|_F$$

(where $\psi(t) = \lambda_M^{-1}(t)$). Set:

$$N_t(f) = \|f^*\mathbb{1}_{[0,\psi(t)]}\|_E + t\|f^{**}(\psi(t)) + f^* \cdot \mathbb{1}_{[\psi(t),\infty[}\|_F.$$

(where $f^{**}(u) = \frac{1}{u}\int_0^u f^*(s)\,ds$ for every $u > 0$).

Using the r.i. structure of E, we see that:

$$f^{**}(u) \leqslant \frac{\|f^*\mathbb{1}_{[0,u]}\|_E}{\lambda_E(u)}$$

hence

$$t\|f^{**}(\psi(t))\mathbb{1}_{[0,\psi(t)]}\|_F \leqslant t\,\frac{\lambda_F(\psi(t))}{\lambda_E(\psi(t))}\,\|f^*\mathbb{1}_{[0,\psi(t)]}\|_E$$

$$\sim \|f^*\mathbb{1}_{[0,\psi(t)]}\|_E,$$

as

$$t\,\frac{\lambda_F(\psi(t))}{\lambda_E(\psi(t))} = \frac{\lambda_M(\psi(t))\lambda_F(\psi(t))}{\lambda_E(\psi(t))} \leqslant C^{st}.$$

[Note that $\lambda_M(u)\lambda_F(u) \geqslant \lambda_E(u)$ becomes an equivalence when $E = M \cdot F$. For if $\mu(A) = u$ and $\mathbb{1}_A = gh$ with $K\|\mathbb{1}_A\|_E \geqslant \|g\|_M\|h\|_F$, as in [1] prop. 3, we have:

$$K\|\mathbb{1}_A\|_E \geqslant g^{**}(u)\lambda_M(u)h^{**}(u)\lambda_F(u) \geqslant \lambda_M(u)\lambda_F(u),$$

as $\mathbb{1}_A \leqslant g^{**} \cdot h^{**}$ by Hölder inequality].

As a consequence $N_t(f) \sim F(t, f)$ (uniformly in t).

To see that N_t is a norm it suffices to check the triangle inequality for the second part of it. Set:

$$\tilde{f}(a, s) = f^{**}(a)\mathbb{1}_{[0,a]} + f^*(s)\mathbb{1}_{[a,+\infty[}$$

we see easily that

$$\forall u > 0, \int_0^u (f + g)^{\sim}(a, s)\,ds \leqslant \int_0^u \tilde{f}(a, s)\,ds + \int_0^u \tilde{g}(a, s)\,ds:$$

(as a consequence of the well-known inequality $(f + g)^{**} \leqslant f^{**} + g^{**}$) which implies that $\|(f + g)^{\sim}(a, \cdot)\|_F \leqslant \|\tilde{f}(a, \cdot)\|_F + \|\tilde{g}(a, \cdot)\|_F$. (by [10] proposition 2.a.8).

Let us give now *a description of* $\tilde{X} = \Pi_t(E + F, N_t)/\mathcal{U}$. As in §3 no. 2, we define the r.i. space E_t, F_t by

$$\|f\|_E = \frac{\|D_{\psi(t)}f\|_E}{\lambda_E(\psi(t))} \qquad \|f\|_{F_t} = \frac{t\|D_{\psi(t)}f\|_F}{\lambda_E(\psi(t))} \sim \frac{\|D_{\psi(t)}f\|_F}{\lambda_F(\psi(t))}$$

(note that the r.i. space F_t is not normalized, we have only $\lambda_{F_t}(1) \sim 1$). Then $(E + F, N_t)$ is order isometric to $(E_t + F_t, M_t)$, where:

$$M_t(f) = \|\mathbb{1}_{[0,1]}f^*\|_{E_t} + \|f^{**}(1)\cdot\mathbb{1}_{[0,1]} + \mathbb{1}_{[1,\infty]}\cdot f^*\|_{F_t}$$

$\left(\text{with } M_t(f) = \dfrac{N_t(D_{\psi(t)}\cdot f)}{\lambda_E(\psi(t))}\right)$. So $\tilde{X}$ is order isometric to $\tilde{Y} = \Pi_t(E_t + F_t, M_t)/\mathcal{U}$. We decompose $\tilde{Y} = Y_0(\tilde{\Omega}) \oplus \tilde{Y}_p \oplus \tilde{Y}_f$ (with the notations of §I). Let $\bar{E}$, $\bar{F}$ be the limit r.i. spaces (defined in §3 no. 1) $\lim_{t,\mathcal{U}} E_t$, $\lim_{t,\mathcal{U}} F_t$. We have (see Remark 3): $\tilde{Y}_0(\Omega) \approx (\bar{E} + \bar{F})(\tilde{\Omega})$ (isomorphically) but now the ultraproduct norm on $Y_0(\tilde{\Omega})$ is given by:

$$M(\tilde{f}) = \|\mathbb{1}_{[0,1]}\tilde{f}^*\|_{\bar{E}} + \|\tilde{f}^{**}(1)\mathbb{1}_{[0,1]} + \mathbb{1}_{[1,\infty]}\tilde{f}^*\|_F$$

Now if $\tilde{f} \in Y_f$ we have $\tilde{f} = (f_t)_t^\bullet$ with $\lim_{t,\mathcal{U}} \|f_t\|_\infty = 0$. Clearly $\lim_{t,\mathcal{U}} M(f_t) = \lim_{t,\mathcal{U}} \|f_t\|_F$, so $(f_t)_t^\bullet \in \tilde{F}_f$ and conversely. So (with the notations of §3 no. 1) $\tilde{Y}_f = \tilde{H}\tilde{F}_f$ (and this is an order isometry). Suppose that $(\lambda_{\bar{E}}(\varepsilon)/\varepsilon) \to_{\varepsilon\to 0} \infty$. Then if $\tilde{f} = (f_t)_t^* \in \tilde{Y}_p$ we have for every $a > 0$

$$\lim_{t,\mathcal{U}} f_t^{**}(1) \leqslant \lim_{t,\mathcal{U}} f_t^{**}(a) \leqslant \lim_{t,\mathcal{U}} \|f_t\|_{Y_t}\cdot\frac{a_t}{\lambda_{E_t}(a)} = \|\tilde{f}\|_{\tilde{Y}}\cdot\frac{a}{\lambda_{\bar{E}}(a)}$$

thus letting $a \to \infty$ we obtain $\lim_{t,\mathcal{U}} f_t^{**}(1) = 0$, and consequently

$\lim_{t,\mathcal{U}} M(f_t) = \lim_{t,\mathcal{U}} \|f_t\|_{E_t}$. In this case we obtain $\tilde{Y}_p = \tilde{J}\tilde{E}_p$ (with equal norms).
So we obtain the following representation result:

LEMMA 21. *If* $\bar{E}([0, 1])$ is not isomorphic to L_1 then

$$\tilde{X} = (\bar{E} + \bar{F})(\tilde{\Omega}) \oplus \tilde{E}_p \oplus \tilde{F}_f$$

equipped with the norm

$$\|\tilde{f}_0 + \tilde{f}_p + \tilde{f}_f\|_{\tilde{X}} = \|\tilde{f}'_0 \oplus \tilde{J}^{-1}\tilde{f}_p\|_{\tilde{E}} + \|\tilde{f}''_0 \oplus \tilde{H}^{-1}\tilde{f}_f\|_{\tilde{F}}$$

where:

$$\tilde{F}'_0 = \mathbb{1}_{\{|\tilde{f}_0| > \tilde{f}^*(1)\}} \cdot \tilde{f}_0,$$
$$\tilde{f}''_0 = \tilde{f}^{**}_0(1) \cdot \mathbb{1}_{\{|\tilde{f}_0| > \tilde{f}^*(1)\}} + \mathbb{1}_{\{|\tilde{f}_0| \leqslant \tilde{f}^*(1)\}} \cdot \tilde{f}_0.$$

As $\tilde{E}$ and $\tilde{F}$ are stable it is easy to show (by a reasoning analogous to that of §2) that $\tilde{X}$ is, ending the proof of theorem 19 (and proposition 20) in this case.

In the case where $\bar{E}([0, 1])$ is isomorphic to $L_1(\lim_{\varepsilon\to 0}(\lambda_{\bar{E}}(\varepsilon)/\varepsilon) < \infty)$ we have still $\tilde{Y}_p = \tilde{J}\tilde{E}_p$ but only with equivalent norm.

We introduce on $\tilde{E}_p$ the L_1-semi-norm $\mathcal{N}_1$: $\mathcal{N}_1(\tilde{f}) = \lim_{t,\mathcal{U}} \|f_t\|_1$. Now $\tilde{X}$ is equipped with the norm:

$$\|\tilde{f}_0 + \tilde{f}_p + \tilde{f}_p\|_{\tilde{X}} = \|\tilde{f}'_0 \oplus \tilde{f}_p\|_{\tilde{E}} + \|\tilde{f}''_0 \oplus \tilde{f}_p\|_{\tilde{F}}$$

Where now:

$$\tilde{f}''_0 = (\tilde{f}^{**}_0(1) + \mathcal{N}_1(\tilde{f}_p))\mathbb{1}_{\{|\tilde{f}_0| > \tilde{f}^*(1)\}} + \mathbb{1}_{\{|\tilde{f}_0| \leqslant \tilde{f}^*(1)\}} \cdot \tilde{f}_0.$$

(and $\tilde{f}'_0$ has the same expression then as in lemma 21).

Again the norm of $\tilde{X}$ is stable (note that the L_1 semi-norm $\mathcal{N}_1$ is stable).

□

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