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Classification of $\mathscr{A}$-simple germs from k^n to k^2

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Introduction and notation

There exist extensive classifications of map-germs from n-space ($n \geqslant 2$) into the plane up to contact equivalence ($\mathscr{K}$-equivalence), see for example [D, Da, G, M, W]. In the present paper we refine a small part of these $\mathscr{K}$-classifications by studying right-left equivalence classes ($\mathscr{A}$-classes) contained in certain $\mathscr{K}$-orbits. In particular we obtain a classification of $\mathscr{A}$-simple germs from (complex and real) n-space ($n \geqslant 2$) into the plane (the $\mathscr{A}$-simple germs of plane curves $\mathbf{C} \to \mathbf{C}^2$ have been classified in [BG]).

Let $f: k^n, 0 \to k^p, 0$ be a smooth map-germ (where $k = \mathbf{C}$ or $\mathbf{R}$, and where smooth means analytic in the former and C^∞ or analytic in the latter case). Let $\mathscr{A} = \mathrm{Diff}(k^n, 0) \times \mathrm{Diff}(k^p, 0)$ denote the group of right-left equivalences, which acts on the space of smooth terms f as follows: $(h, k) \cdot f = h \circ f \circ k^{-1}$, where $(h, k) \in \mathscr{A}$. Replacing the action on the left, i.e. the composition with elements of $\mathrm{Diff}(k^p, 0)$, by composition with elements of $Gl(p, k)$ with entries in C_n (where C_n = local ring of smooth function germs $k^n, 0 \to k, 0$) gives the group $\mathscr{K}$ of contact equivalences. A $\mathscr{G}$-orbit U (where $\mathscr{G} = \mathscr{A}$ or $\mathscr{K}$) is said to be adjacent to another $\mathscr{G}$-orbit V, denoted by $U \to V$, if any representative f of U can be embedded in an unfolding $F(u, \bar{f}(u, x))$, where $\bar{f}(0, x) = f(x)$, such that the set $\{u, x\}$ for which $\bar{f}(u, x) \in V$ contains $u = x = 0$ in its closure. A $\mathscr{G}$-orbit U is said to be $\mathscr{G}$-simple if it is adjacent to only a finite number of other $\mathscr{G}$-orbits.

Let C_n and C_p denote the local rings of function-germs in source and target whose respective maximal ideals are m_n and m_p. Let θ_f denote the C_n-module of vector fields over f, and set $\theta_n = \theta(1_{k^n})$ and $\theta_p = \theta(1_{k^p})$. One can then define the homomorphisms

$$tf: \theta_n \to \theta_f, \quad tf(\psi) = Df \cdot \psi,$$

and

$$wf: \theta_p \to \theta_f, \qquad wf(\phi) = \phi \circ f.$$

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The tangent space to the $\mathscr{A}$-orbit at f can be calculated to be $T\mathscr{A}\cdot f = tf(m_n\cdot\theta_n) + wf(m_p\cdot\theta_p)$, and the $\mathscr{A}$-codimension of f is $\mathrm{cod}(\mathscr{A}, f) = \dim_k m_n\cdot\theta_f/T\mathscr{A}\cdot f$.

Let $J^k(n,p)$ denote the space of kth order Taylor polynomials without constant terms, and write j^kf for the k-jet of f. The Lie group $\mathscr{A}^k := j^k(\mathscr{A})$ acts smoothly on $J^k(n,p)$, and we shall write $T\mathscr{A}^k\cdot f$ for the corresponding tangent space $T_{j^kf}\mathscr{A}^k\cdot j^kf$. A germ f is said to be k-determined if, for any g, $j^kf = j^kg$ implies that $f\sim g$. The calculation of $\mathscr{A}^k$-orbits (using Mather's Lemma [Ma IV, Lemma 3.1]) and of the determinacy degree of a germ f (using an estimate of duPlessis [*dP*, Corollary 3.9]) are the main tools that we use in the present classification. See [Rie] for further details on notation and techniques.

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1. Classification of $\mathscr{A}$-simple Σ^1-germs from $k^n, 0$ to $k^2, 0$, $n > 1$

We can assume that $f: k^n, 0 \to k^2, 0$ is of the form $f(x, y) = (x, f_2(x, y))$, and we denote by X, Y the coordinates in the target. The following splitting lemma is almost content-free but clarifies the subsequent discussion (see also [PW, Prop. 0.6]).

LEMMA 1.1. *Every map-germ $f: k^n, 0 \to k^2, 0$, $n > 1$, of corank 1 is $\mathscr{A}$-equivalent to a germ of the form*

$$h(x, y, z) = \left(x, g(x, y_1, \dots, y_m) + \sum_{i=1}^{n-m-1} \varepsilon z_i^2\right),$$

where $g(0, y_1, \dots, y_m) \in m_n^3$ and $\varepsilon = \pm 1$ (for $k = \mathbf{C}$, $\varepsilon = 1$); and $\mathrm{cod}(\mathscr{A}, h) = \mathrm{cod}(\mathscr{A}, (x, g)) + n - m - 1$.

Proof. Any $j^kf(x, y, z) = (x, f_2(x, y, z))$ can be reduced to h by right-coordinate changes of the form $\bar{z}_i = \phi(x, y, z)$, where the bar denotes new coordinates and not complex conjugation. It is also clear that $T\mathscr{A}^k\cdot j^kf$, $k > 1$, contains all monomials containing powers of z_i except for $z_i\cdot\partial/\partial Y$, which implies the last statement of the Lemma. □

Hence, if we take $m = 1$ the lemma above reduces the classification of germs $f: k^n, 0 \to k^2, 0$ to the classification of germs $g: k^2, 0 \to k^2, 0$ of corank 1. The $\mathscr{A}$-simple germs of the plane of corank 1 have been classified by one of the authors in [Rie]. Next, consider the case $m = 2$.

LEMMA 1.2. *For $m = 2$ (or indeed for $m \geqslant 2$) there are no $\mathscr{A}$-simple germs $f(x, y) = (x, g(x, y_1, \dots, y_m))$, where $g(0, y_1, \dots, y_m) \in m_{m+1}^3$ as in Lemma* 1.1.

Proof. The 2-jet of f is, by the hypothesis on g, given by $j^2f =$

$(x, ax^2 + bxy_1 + cxy_2)$, which can either be reduced to (x, xy_1), provided that either b or c is nonzero, or else to $(x, 0)$. First, we show that the $\mathscr{A}^3$-orbits over $j^2f = (x, xy_1)$ are at least uni-modal. Note that we can reduce any 3-jet over (x, xy_1) to $h := (x, xy_1 + ay_1^3 + by_1^2y_2 + cy_1y_2^2 + dy_2^3)$. There are exactly four generators, namely $wh(X, Y) - th(x, 0, 0)$, $th(0, y_1, 0) - th(x, 0, 0) + wh(X, 0)$, $th(0, 0, y_1)$, and $th(0, 0, y_2)$, for the subspace $V := k\{(0, y_1^i y_2^j), i + j = 3\}$ of $T\mathscr{A}^3 \cdot h$, leading to the following matrix of coefficients:

$$\begin{bmatrix} a & b & c & d \\ 3a & 2b & c & \\ b & 2c & 3d & \\ & b & 2c & 3d \end{bmatrix}$$

which doesn't have maximal rank (because (row 4)$-3\times$(row 1)$= -$(row 2)). It follows from Mather's lemma [Ma IV] that V is foliated by (at least) a 1-parameter family of $\mathscr{A}^3$-orbits.

The $\mathscr{A}^3$-orbits over $(x, 0)$ are adjacent to those over (x, xy_1), hence they are also non-simple.

For $m > 2$ the $\mathscr{A}^2$-orbits are still those of (x, xy_1) and $(x, 0)$, and the modality of the $\mathscr{A}^3$-orbits over (x, xy_1) and $(x, 0)$ is clearly greater than or equal to one. Lemma 1.2 now follows. □

Using the results of [Rie] we get the following classification.

PROPOSITION 1.3. *An $\mathscr{A}$-simple map-germ $f: k^n, 0 \to k^2, 0$ $(n > 1)$ of corank* 1 *is equivalent to one of*

Type	$f(x, y, z_1, \ldots, z_{n-2}) =$	$cod(\mathscr{A}, f)$
1	(x, y)	0
2	$(x, y^2 + \Sigma\varepsilon z_i^2)$	$n-1$
3	$(x, xy + y^3 + \Sigma\varepsilon z_i^2)$	n
4_k	$(x, y^3 + \varepsilon^{k-1}x^k y + \Sigma\varepsilon z_i^2)$, $k > 1$	$n+k-1$
5	$(x, xy + y^4 + \Sigma\varepsilon z_i^2)$	$n+1$
6	$(x, xy + y^5 + \varepsilon y^7 + \Sigma\varepsilon z_i^2)$	$n+2$
7	$(x, xy + y^5 + \Sigma\varepsilon z_i^2)$	$n+3$
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1} + \Sigma\varepsilon z_i^2)$, $k > 1$	$n+k$
12	$(x, xy^2 + y^5 + y^6 + \Sigma\varepsilon z_i^2)$	$n+3$
13	$(x, xy^2 + y^5 + \varepsilon y^9 + \Sigma\varepsilon z_i^2)$	$n+4$
14	$(x, xy^2 + y^5 + \Sigma\varepsilon z_i^2)$	$n+5$
16	$(x, x^2y + y^4 + \varepsilon y^5 + \Sigma\varepsilon z_i^2)$	$n+3$
17	$(x, x^2y + y^4 + \Sigma\varepsilon z_i^2)$	$n+4$

(*where* $\varepsilon = \pm 1$ *for* $k = \mathbf{R}$, *and* $\varepsilon = 1$ *for* $k = \mathbf{C}$).

REMARK 1.4. A deformation of a germ as in 1.1 does not increase m. The $\mathscr{A}$-classes of 1.3 are hence simple for all n, because they can only be adjacent to other $(m = 1)$-germs.

2. *Classification of $\mathscr{A}$-simple $\Sigma^2,0$ germs from k^n, 0 to k^2, 0*

The main result in this section is the following classification.

PROPOSITION 2.1. *Any simple germ $f: k^n, 0 \to k^2, 0$ (for $n > 1$) of corank 2 is $\mathscr{A}$-equivalent to some member of the following series of germs:*

(i) $k = \mathbf{R}$: $I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1})$, $l \geqslant m \geqslant 1$, *or*
$II_{2,2}^{l} = (x^2 - y^2 + x^{2l+1}, xy)$, $l \geqslant 1$;
(ii) $k = \mathbf{C}$: $I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1})$, $l \geqslant m \geqslant 1$.

The $\mathscr{A}$-codimension of $I_{2,2}^{l,m}$ and $II_{2,2}^{l}$ are $l + m + 2$ and $2(l + 1)$ respectively.

To prove this statement we classify $\mathscr{A}$-orbits contained in $\mathscr{K}$-simple orbits of germs $f: k^n, 0 \to k^2, 0$ of corank 2. Such $\mathscr{K}$-simple germs have been classified in [D, M, Da]. In the present classification of $\mathscr{A}$-simple germs we can discard all $\mathscr{K}$-orbits adjacent to some $\mathscr{K}$-orbit that doesn't contain any $\mathscr{A}$-simple orbits.

2.1. *Classification of $\mathscr{A}$-orbits in $\mathscr{K}(x^2, y^2)$, for $k = \mathbf{C}$ or $\mathbf{R}$*

PROPOSITION 2.1.1. *Any germ contained in the $\mathscr{K}$-orbit of (x^2, y^2) is $\mathscr{A}$-equivalent to some member of the series*

$$I_{2,2}^{l,m} = (x^2 + y^{2l+1}, y^2 + x^{2m+1}), \qquad l \geqslant m \geqslant 1.$$

The $I_{2,2}^{l,m}$ are $(2l + 1)$-determined, and $\operatorname{cod}(\mathscr{A}, I_{2,2}^{l,m}) = l + m + 2$.

Proof. Any k-jet $(x^2 + \Sigma a_{i,j} x^i y^j, y^2 + \Sigma b_{i,j} x^i y^j)$, where the $x^i y^j$ are of degree $k > 1$, can be reduced to $(x^2 + a_{0,k} y^k, y^2 + b_{k,0} x^k)$ by the right-coordinate change

$$(\bar{x}, \bar{y}) =$$
$$(x - \tfrac{1}{2}(a_{k,0} x^{k-1} + \cdots + a_{1,k-1} y^{k-1}), \quad y - \tfrac{1}{2}(b_{k-1,1} x^{k-1} + \cdots + b_{0,k} y^{k-1}))$$

Now, suppose $k = 2l$: the left-coordinate changes $\bar{X} = X - a_{0,2l} Y^l$ and $\bar{Y} = Y - b_{2l,0} X^l$ give (x^2, y^2), which is the single $\mathscr{A}^{2l}$-orbit over $j^{2l-1} f = (x^2, y^2)$. If $k = 2l + 1$, we have three $\mathscr{A}^{2l+1}$-orbits over $j^{2l} f = (x^2, y^2)$: (i) $(x^2 + y^{2l+1}, y^2 + x^{2l+1})$, (ii) $(x^2, y^2 + x^{2l+1})$, and (iii) (x^2, y^2). By a result of du Plessis [dP, Example 3.18], $(x^2 + y^{2l+1}, y^2 + x^{2l+1})$ is $(2l + 1)$-determined. Now, consider $\mathscr{A}^k$-orbits over $j^{2m+1} f = (x^2, y^2 + x^{2m+1})$. If $k = 2l > 2m + 1$, we find a single $\mathscr{A}^{2l}$-orbit $(x^2, y^2 + x^{2m+1})$, by the same coordinate changes as above; and if

$k = 2l + 1 > 2m + 1$, we can reduce to $(x^2 + a_{0,2l+1}y^{2l+1}, y^2 + x^{2m+1})$ leading to two $\mathscr{A}^{2l+1}$-orbits given by $a_{0,2l+1} = 0, 1$. Now, $I^{l,m}_{2,2}$ is $(2l + 1)$-determined, again by [dP, Example 3.18]. (see also [BPW, Example 6.7]). Finally, we check that

$$k\{(x, 0), (y^{2i+1}, 0), (0, y), (0, x^{2j+1}): l > i \in \mathbf{N}, m > j \in \mathbf{N}\}$$

forms a free basis for $m_n \cdot \theta_f / T\mathscr{A} \cdot f$, where $f = I^{l,m}_{2,2}$, which proves the proposition.

□

2.2. *Classification of $\mathscr{A}$-orbits contained in $\mathscr{K}(x^2 - y^2, xy)$ over* $\mathbf{R}$

PROPOSITION 2.2.1. *Any germ contained in the $\mathscr{K}$-orbit of $(x^2 - y^2, xy)$ is $\mathscr{A}$-equivalent to some member of the series*

$$II^l_{2,2} = (x^2 - y^2 + x^{2l+1}, xy), \qquad l \geqslant 1.$$

The $II^l_{2,2}$ are $(2l + 1)$-determined, and $\operatorname{cod}(\mathscr{A}, II^l_{2,2}) = 2(l + 1)$.

Proof. The calculations are entirely routine and we omit them. The $(2l + 1)$-determinacy of $II^l_{2,2}$ and its codimension follow from 2.1, since the family $II^l_{2,2}$ is equivalent over $\mathbf{C}$ to $I^{l,l}_{2,2}$.

2.3. *Other $\mathscr{K}$-orbits of type $\Sigma^{2,0}$ do not contain $\mathscr{A}$-simple orbits*

The equidimensional case ($n = p = 2$) and the non-equidimensional case ($n > 2$, $p = 2$) are considered respectively in Proposition 2.3.1 and Proposition 2.3.2.

PROPOSITION 2.3.1. *Let $f: (k^2, 0) \to (k^2, 0)$, $(k = \mathbf{R}, \mathbf{C})$ be an $\mathscr{A}$-finitely determined germ of type $\Sigma^{2,0}$. If f is $\mathscr{A}$-simple then the $\mathscr{K}$-orbit of f is of type $I_{2,2}$ or $II_{2,2}$.*

PROPOSITION 2.3.2. *Let $f: (k^n, 0) \to (k^2, 0)$, $n \geqslant 3$, $k = \mathbf{R}, \mathbf{C}$ be any $\mathscr{A}$-finitely determined germ of type $\Sigma^{2,0}$. Then f is non-simple.*

When $n = p = 2$, the $\mathscr{K}$-orbit of any Σ^2 germ not of type $I_{2,2}$ or $II_{2,2}$ is adjacent either to $I_{2,3}$ or IV_3 (see [L., Theorem 2.1], for the description of the adjacencies of real $\mathscr{K}$-orbits of types Σ^1 and $\Sigma^{2,0}$, and [G] for the complex case). Therefore, Proposition 2.3.1 will follow from Lemma 2.3.3 and 2.3.4 below, where we show that $I_{2,3}$ and IV_3 have no $\mathscr{A}$-simple orbits.

LEMMA 2.3.3. *The $\mathscr{A}$-orbits within $I_{2,3}$ are all non-simple.*

Proof. A germ f *within* $I_{2,3}$ is $\mathscr{K}$-equivalent to $(x^2 + y^3, xy)$ ([M, VI]). We show that there is no open $\mathscr{A}$-orbit within $\mathscr{K}(x^2 + y^3, xy)$, which will imply that this $\mathscr{K}$-orbit is filled up entirely with non-simple $\mathscr{A}$-orbits.

After some simple coordinate changes, we may assume that any $\mathscr{A}$-finitely determined germ in $I_{2,3}$ has the form:

$$(x^2 + y^3 + axy^2 + bxy^3 + cy^4 + \Phi(x, y), xy), \Phi \in m_2^4$$

One easily checks that $m_2^3\theta_f \subset T\mathscr{K} \cdot f$.

Now, the relevant relations in $T\mathscr{A}^4 \cdot f$ are given by $tf(x^i y^i, 0)$, $tf(0, x^i y^i)$, $i + j = 1, 2, 3$, $wf(X, 0)$, $wf(Y, 0)$, $wf(0, X)$, $wf(0, Y)$.

Thus there are only 22 generators for the vector subspace $m_2^2\theta_f/m_2^5\theta_f$, which has dimension 24. In particular, $T\mathscr{A}^4 \cdot f \not\supseteq m_2^3\theta_f/m_2^5\theta_f$, and the modality of the $\mathscr{A}$-orbit of f within $\mathscr{K}(x^2 + y^3, xy)$ is greater than one. □

LEMMA 2.3.4. *The $\mathscr{A}$-orbits within IV_3 are all non-simple.*

Proof. The proof follows immediately, since $\mathscr{K}(x^2 + y^2, x^3)$ (type IV_3) is adjacent to $\mathscr{K}(x, y^6)$ (type $\mathscr{A}_5$) ([L, Theorem 2.1]), which in turn is entirely filled up with non-simple $\mathscr{A}$-classes (This follows from Proposition 1.3, but see [Rie] for details). □

We consider now the case $n \geqslant 3$, $p = 2$.

It is well known that if $n \geqslant 4$, the $\mathscr{K}$-modality of a pair of quadrics is greater or equal to one ([W]). Therefore, we only have to consider the case $n = 3$, $p = 2$.

In the complex case, there is only one $\mathscr{K}$-orbit of type $\Sigma^{2,0}$, whose normal form is $(x^2 + y^2, y^2 + z^2)$.

PROPOSITION 2.3.5. *Any $\mathscr{A}$-orbit of a finitely determined germ within $\mathscr{K}(x^2 + y^2, y^2 + z^2)$ is at least 1-modal.*

Proof. Let $f: (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ be any $\mathscr{A}$-finitely determined germ within $\mathscr{K}(x^2 + y^2, y^2 + z^2)$. Then, with simple coordinate changes, $j^3 f$ can be reduced to:

$$(x^2 + y^2 + ax^3 + cz^3, x^2 + z^2 + by^3)$$

As before, the result follows from the information given by $T\mathscr{K}^3 \cdot f$ and $T\mathscr{A}^3 \cdot f$:

(i) $T\mathscr{K} \cdot f + m_2^4\theta_f \supset m_2^3\theta_f$.

(ii) Inspecting $T\mathscr{A}^3 \cdot f$ we see that the elements of degree three are given by $tf(x, 0, 0)$, $tf(0, y, 0)$, $tf(0, 0, z)$, $\frac{1}{3}[tf(x, y, z) - 2wf(X, 0) - 2wf(0, Y)]$ and by $J(f) \cdot m_2$ (where $J(f)$ is the Jacobian ideal) $(\operatorname{Mod} m_2^4\theta_f)$.

They generate the following subspace of $m_2^3/m_2^4\theta_f$:

$$\mathbf{C}\{\text{all mixed terms of degree three}, (x^3, x^3), (y^3, 0), (0, z^3)$$
$$\text{and } (ax^3 + cz^3, by^3)\} (\operatorname{Mod} m_2^4\theta_f).$$

Hence $T\mathscr{A}^3 \cdot f \not\supseteq m_2^3\theta_f$, and comparing with (i) we get the result. □

REMARK 2.3.6. $\mathscr{K}(x^2 + y^2, y^2 + z^2)$ splits into various real orbits. These real $\mathscr{K}$-orbits do not have open $\mathscr{A}$-orbit either. In fact, if the condition

$$tf(m_2\theta_2) + f^*(m_2)\theta_f = tf(m_2\theta_2) + wf(m_2\theta_2)$$

were true for any such real germ, we should have:

$$tf_{\mathbf{C}}(m_2\theta_2) + f^*_{\mathbf{C}}(m_2)\theta_{f_{\mathbf{C}}} = tf_{\mathbf{C}}(m_2\theta_2) + wf_{\mathbf{C}}(m_2\theta_2),$$

where $f_{\mathbf{C}}$ is its complexification. This clearly contradicts the above lemma.

Proposition 2.3.2 will follow from the above discussion.

3. Adjacencies of $\mathscr{A}$-simple $\Sigma^{2,0}$-germs $f: k^2, 0 \to k^2, 0$

The adjacencies between $\mathscr{A}$-simple Σ^1-germs from $\mathbf{C}^2$ to $\mathbf{C}^2$ are shown in [Rie]. For corank 2 terms we have the following

PROPOSITION 3.1. *Figures* 1 *and* 2 *show the adjacencies of $\mathscr{A}$-simple $\Sigma^{2,0}$-germs $f: k^2, 0 \to k^2, 0$ for $k = C$ and $k = \mathbf{R}$, respectively. (To denote $\mathscr{A}$-classes we use the notation of Propositions* 1.3 *and* 2.1).

Proof. As in [Rie] we use three invariants $m(f)$, $c(f)$, and $d(f)$, which are upper-semicontinuous under deformation, to rule out certain adjacencies. Let Σ and Δ denote the critical set and the discriminant of f, which are both germs of plane curves, and let $\delta(C)$ denote the well-known δ-invariant of a germ of a plane curve C (see [Mi]). The three invariants of f can then be calculated as follows: $m(f) = \dim_k C_n/f^*m_p$, $c(f) = \dim_k C_n/I$ (where I = ideal defined by the vanishing of 2×2 minors of $\left[\begin{matrix} Df \\ \nabla|Df| \end{matrix}\right]$), and $d(f) = \delta(\Delta) - \delta(\Sigma) - c(f)$. (For germs $f: \mathbf{C}^2$,

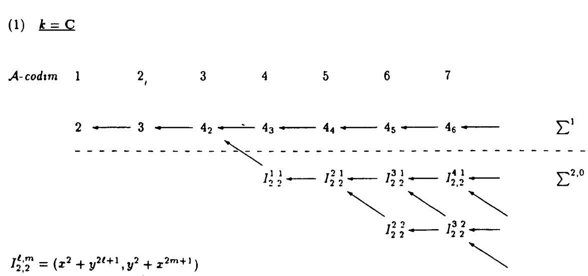

Fig. 1.

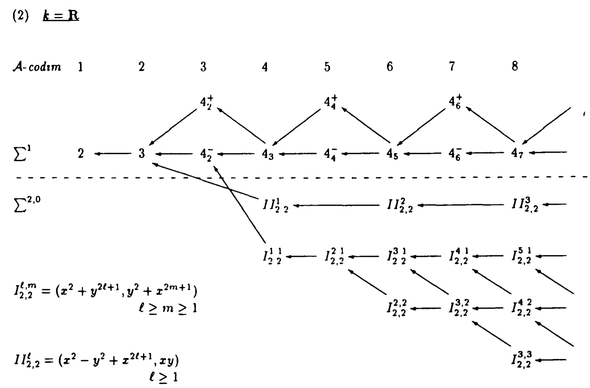

Fig. 2.

$0 \to \mathbf{C}^2$, 0 these invariants have the following geometrical meaning: $m(f)$ is the number of preimages of a target point off the discriminant Δ of f; and $c(f)$ and $d(f)$ are the numbers of cusps and transverse fold crossings of a generic deformation of f).

For Σ^1-germs these invariants have been calculated in [Rie] and for the $\mathscr{A}$-simple $\Sigma^{2,0}$-germs of the classification in §2 we have the following

LEMMA 3.2. *The invariants m, c, and d associated with the members of the series of germs* $I^{l,m}_{2,2}$ *and* $II^l_{2,2}$ *have the values:*

$$m(I^{l,m}_{2,2}) = 4, \qquad c(I^{l,m}_{2,2}) = 3, \qquad d(I^{l,m}_{2,2}) = l + m;$$

and

$$m(II^l_{2,2}) = 4, \qquad c(II^l_{2,2}) = 3, \qquad d(II^l_{2,2}) = 2l.$$

These expressions also make sense for the "stems" of these series $I^{\infty,\infty}_{2,2} = (x^2, y^2)$, $I^{\infty,m}_{2,2} = (x^2, y^2 + x^{2m+1})$ *and* $II^{\infty}_{2,2} = (x^2 - y^2, xy)$.

Proof. These are just a trivial calculation. Note that the critical sets and the discriminants of the germs $I^{l,m}_{2,2}$ consist of two branches, so that $\delta(\Sigma)$ and $\delta(\Delta)$ are sums of δ-invariants of each branch and the (local) intersection numbers of the branch pairs. Also note that, as complex-analytic germs, $II^l_{2,2} \sim I^{l,l}_{2,2}$ (and that the dimensions of the relevant local algebras are not altered by complexifying).

Also notice that the Milnor number μ of the critical sets of the germs $I^{l,m}_{2,2}$ and $II^l_{2,2}$ is equal to one. The upper semicontinuity of these invariants and the adjacencies of $\mathscr{K}$-classes described in §2.3, together with the following lemma, conclude the proof of Proposition 3.1.

LEMMA 3.3

(i) $k = \mathbf{C}$: *the degenerate mono-germs in a versal deformation of* $I^{1,1}_{2,2}$ *are of type* 4_2; *and*
(ii) $k = \mathbf{R}$: *the degenerate mono-germs in a versal deformation of* $I^{1,1}_{2,2}$ *are of type* 4_2^-, *and there are no degenerate mono-germs in a deformation of* $II^1_{2,2}$.

Proof. We consider $\mathscr{A}$-versal unfoldings $F: k^d \times k^2, 0 \to k^d \times k^2, 0$, given by $F(u, x, y) = (u, f(u, x, y))$, where $f(0, x, y) = I^{2,2}_{2,2}$ or $II^2_{2,2}$. The set $B :=$ $\{u \in (k^d, 0): c(f(u, 0, 0)) \geqslant 2\}$ gives all degenerate mono-germs in a deformation of $I^{1,1}_{2,2}$ or $II^1_{2,2}$, because $c \geqslant 2$ for any degenerate mono-germ of the plane and the origins in source and target are preserved under $\mathscr{A}$. Let $m_1(x, y)$, $m_2(x, y)$, and $m_3(x, y)$ denote the determinants of the 2×2 minors of $\begin{bmatrix} Df(u, x, y) \\ \nabla \,|\, Df(u, x, y)| \end{bmatrix}$, where D is the differential of f with respect to x and y.

Now $c(f(u, 0, 0)) = \dim_k C_n/(m_1, m_2, m_3) \geqslant 2$ if and only if

$$m_1(0, 0) = m_2(0, 0) = m_3(0, 0) = 0$$

and the 2×2 minors of

$$\begin{bmatrix} \partial m_1(0, 0)/\partial x & \partial m_1(0, 0)/\partial y \\ \partial m_2(0, 0)/\partial x & \partial m_2(0, 0)/\partial y \\ \partial m_3(0, 0)/\partial x & \partial m_3(0, 0)/\partial y \end{bmatrix}$$

vanish. The six equations define an ideal I in $k[u_1, \ldots, u_d]$.

First, consider the $\mathscr{A}$-versal deformation $f(u, x, y) = (u, x^2 + y^3 + u_1 x + u_2 y$, $y^2 + x^3 + u_3 x + u_4 y)$ of $I^{1,1}_{2,2}$. One calculates that $I = (u_1 u_3 - u_4^2,\ u_1 u_4 - u_2 u_3$, $u_1^2 - u_2 u_4,\ u_4(3u_1 u_3 + 2u_2) + u_1(4u_1 + 3u_2^2),\ u_4(3u_3^2 + 4u_4) + u_1(2u_3 + 3u_2 u_4)$, $-(4u_1 + 3u_2^2)(3u_3^2 + 4u_4) + (3u_1 u_3 + 2u_2)(2u_3 + 3u_2 u_4))$, and, calculating a standard basis for I with respect to some lexicographical ordering of the variables in $k[u_1, \ldots, u_4]$, one finds the following set of degenerate mono-germs: $B = \{u \in (k^4, 0): u_1 = u_4 = u_2 u_3 = 0\}$. Now, by direct coordinate changes, $f(0, u_2, 0, 0, x, y) \sim 4_2^-$ for $u_2 \in \mathbf{R} - \{0\}$ and $f(0, 0, u_3, 0, x, y) \sim 4_2^-$ for $u_3 \in \mathbf{R} - \{0\}$ (in the case $k = \mathbf{R}$), and $f(0, u_2, 0, 0, x, y) \sim f(0, 0, u_3, 0, x, y) \sim 4_2$ for $u_2, u_3 \in \mathbf{C} - \{0\}$ (for $k = \mathbf{C}$).

Finally, we consider the $\mathscr{A}$-versal deformation

$$f(u, x, y) = (x^2 - y^2 + x^3 + u_1 x + u_2 y,\ xy + u_3 x + u_4 y)$$

of the real germ $II^1_{2,2}$. Repeating the calculations above, one finds that $B = \{u \in (\mathbf{R}^4, 0): u_1 = u_2 = u_3 = u_4 = 0\}$. Hence there are no degenerate mono-germs in a versal deformation of $II^1_{2,2}$, and the lemma follows.

Proof of Proposition 3.1; *conclusion.* Lemma 3.3 says that $I^{1;1}_{2,2}$ and $II^1_{2,2}$ are not adjacent to the Σ^1-germ $(x, xy + y^4)$, which is the open $\mathscr{A}$-orbit in the $\mathscr{K}$-class A_3. From the adjacencies in [Rie] of Σ^1-germs it follows that none of the germs $I^{l,m}_{2,2}$ and $II^l_{2,2}$ is adjacent to some $\mathscr{A}$-orbit in A_3. Finally, one checks that $II^1_{2,2} \to 3$. □

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