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Supercuspidal representations of GSp_4 over a local field of residual characteristic 2, part I

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1. Introduction

In this paper, and in one to follow, we analyze supercuspidal representations of $G = GSp_4(F)$ where F is a local field of residual characteristic $p = 2$. We first classify these representations according to characters which occur in their restrictions to certain open compact subgroups. As in the case of $GL_n(F)$, these characters may be parametrized by 4×4 matrices over F . Accordingly, we may classify certain supercuspidal representations as occurring in the “nilpotent” case. Theorem 3.9 states that these representations are all irreducibly induced from some open compact-by-center subgroup of G . The remaining cases will be treated in the sequel.

The methods used are suggested by the construction techniques for $GL_n(F)$ developed by Howe ([H]) and exhaustion techniques developed by Kutzko and Manderscheid ([K] and [KM]). These ideas work well when no wildly ramified extensions of F are imbedded in the group in question. Moy ([M]) has used techniques involving Hecke algebra isomorphisms to classify supercuspidal representations of $GSp_4(F)$ in the case of odd primes p .

When wild ramification is present (so $p = 2$), these methods still provide a starting point. A representation (π, V) will fix a non-trivial subspace $V(L)$ when restricted to a small enough open subgroup L . We will generally take L to be a member of some normal filtration of a maximal compact-by-center subgroup of G . We then consider certain groups A for which the quotient A/L is abelian. We classify π by characters occurring in the restriction $(\pi|_A, V(L))$. The occurrence of certain characters (cuspidal) immediately forces π to be irreducibly induced. Some characters are shown to be (G, A) -principal as defined in [K]. Some cases lead us to find an open subgroup L' , larger than L , having a fixed vector. The remaining cases are analogous to the alfalfa cases of [KM]. For these we must use the constructions in [H] to distinguish between representations which are irreducibly induced and those that contain principal characters.

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Recall that the matrix coefficients of a supercuspidal representation have compact support mod the center of G . A (G, A) -principal representation of a subgroup A is one which cannot occur in the restriction to A of a supercuspidal representation of G . A representation of an open compact subgroup of G may be shown to be principal by proving that vectors in its representation space give rise to matrix coefficients of G whose support is not compact mod the center of G . The following is a restatement of some of the lemmas in [K] and gives the method we will use to demonstrate that certain representations are principal representations.

LEMMA 1.1. *Let (π, V) be an admissible representation of G . Let $v_0 \in V$ and let L be an open compact subgroup fixing v_0 . Suppose there exist elements $x_{ij} \in G$ and constants $N(i)$ and c_{ij} , for $i = 1, 2, 3 \dots$ and $1 \leq j \leq N(i)$, such that*

(a) *each $v_i \in V(L)$, where*

$$v_i = \sum_{j=1}^{N(i)} c_{ij} \pi(x_{ij}) v_{i-1}$$

(b) *No subsequence of elements of the form*

$$s_n = \prod_{i=1}^n x_{i,j}$$

with $j = j(i, n)$, $1 \leq j \leq N(i)$ for all i , and $1 \leq i \leq n$, for $n = 1, 2, 3, \dots$, is contained in a compact-by-center subset of G .

Then v_0 produces a matrix coefficient of (π, V) whose support is not compact-by-center.

Proof. Since π is admissible, the space $V(L)$ of L -fixed vectors in V is finite-dimensional. Thus there exists a linear functional v'_0 on $V(L)$ such that $\langle v_n, v'_0 \rangle \neq 0$ for infinitely many n . Extend v'_0 to be zero on the complement of $V(L)$ to define a smooth vector, also denoted v'_0 , in the contragredient of V .

Consider the matrix coefficient $g \mapsto \langle \pi(g)v_0, v'_0 \rangle$. For v_n in the subsequence above,

$$0 \neq \langle v_n, v'_0 \rangle = \sum_{j=1}^{N(n)} c_{nj} \langle \pi(x_{nj})v_{n-1}, v'_0 \rangle$$

which inductively is a linear combination of terms of the form

$$\left\langle \pi \left(\prod_{i=1}^n x_{ij} \right) v_0, v'_0 \right\rangle$$

for j with $1 \leq j \leq N(i)$.

Thus at least one such term must be non-zero. Let $s_n = \prod_{i=1}^n x_{ij}$ be such that $\langle \pi(s_n)v_0, v'_0 \rangle \neq 0$ for each such n .

The support of the above matrix coefficient of π thus contains the set of such s_n , which is not compact-mod-center by hypothesis. Hence a representation (π, V) with these properties cannot be supercuspidal.

A representation of A with such a v_0 in its space is thus a (G, A) -principal representation. In our applications of Lemma 1.1, the elements x_{ij} are generally products of some fixed translation in the affine Weyl group with appropriate unipotent elements. It will then be immediate to check that condition (b) of the Lemma holds.

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2. Structure and Notation

In this paper we will realize the group $GS\!p_4(F)$ of symplectic similitudes as those elements g of $GL_4(F)$ satisfying the relation $gJ^t g = cJ$, where c is a non-zero element of F and

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The group $GS\!p_4(F)$ over a non-archimedean local field F has three conjugacy classes of maximal compact-mod-center subgroups. See, e.g., [I]. We will define a filtration of each of the maximal compact-mod-center subgroups of G consisting of compact open normal subgroups. First, for $n > 0$, let K_n denote the n th congruence subgroup of the open maximal compact subgroup $K_0 = GS\!p_4(\mathcal{O})$. We will use the following notation to describe K_n .

$$K_n = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n & n & n & n \\ n & n & n & n \end{bmatrix},$$

where the matrix entries denote the minimum valuation of entries of the difference between an element of K_n and the identity matrix.

We define the following groups in the same manner:

$$H_{n,0} = \begin{bmatrix} n & n+1 & n & n \\ n & n & n & n-1 \\ n & n+1 & n & n \\ n+1 & n+1 & n+1 & n \end{bmatrix}$$

$$H_{n,1} = \begin{bmatrix} n+1 & n+1 & n+1 & n \\ n & n+1 & n & n \\ n+ & n+1 & n+1 & n \\ n+1 & n+2 & n+1 & n+1 \end{bmatrix}$$

$$J_{n,0} = \begin{bmatrix} n & n & n & n \\ n & n & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \end{bmatrix}$$

$$J_{n,1} = \begin{bmatrix} n+1 & n+1 & n & n \\ n+1 & n+1 & n & n \\ n+1 & n+1 & n+1 & n+1 \\ n+1 & n+1 & n+1 & n+1 \end{bmatrix}.$$

Let $K = K_0Z(G)$. Let H be the group generated by $H_{0,0}$ and z_H , and let J be the group generated by $J_{0,0}$ and z_J . Here

$$z_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix} \quad \text{and} \quad z_J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix}.$$

Then the groups K, J and H represent the three conjugacy classes of maximal compact-by-center open subgroups of G .

Let W be the Weyl group of G and W_a the affine Weyl group. An element of W_a may be expressed in the form $w t_1^r t_2^s$ where $w \in W, r$ and s are integers, and

$$t_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}, \quad t_2 = \begin{bmatrix} \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Set $t_0 = t_1 t_2$ as members of W_a . Then (modulo $Z(G)$)

$$t_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let $\{\alpha, \beta\}$ be the base for the root system Φ of G with $\beta = e_2 - e_1$ taken to be the short root and $\alpha = 2e_1$ the long root. For any root $\gamma \in \Phi$, let w_γ denote the reflection determined by γ , and let t_γ be the translation in the affine Weyl group associated to γ . Recall that t_γ acts on the standard apartment of the Bruhat-Tits building by sending a point x to $x + 2\gamma^\vee$.

Note that $t_0 = t_{\alpha+2\beta}$ is the translation associated to $\alpha + 2\beta$, and further, that $(t_1)^2 = t_\alpha t_{\alpha+2\beta} = t_{\alpha+\beta}$ and $(t_2)^2 = t_\beta$. Then we have the following double coset decompositions of $GSp_4(F)$.

LEMMA 2.1.

$$\begin{aligned} K \backslash G / K &= \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\} \\ J \backslash G / J &= \{t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geq 0\} \cup \{w_{\alpha+2\beta} t_1^{v_1} t_2^{v_2} \mid v_1, v_2 \geq 0\} \\ H \backslash G / H &= \{t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\} \cup \{w_{\alpha+\beta} t_0^{v_0} t_1^{v_1} \mid v_0, v_1 \geq 0\}. \end{aligned}$$

Let (π, V) be an irreducible admissible representation of a reductive p-adic group G . Let L be any open compact-by-center subgroup of G . Let (τ, W) be an irreducible component of (π, V) restricted to L . Suppose that τ does not induce irreducibly from L to G . Then, by Mackey theory, there exists an x in G , but not in L , which intertwines τ , i.e., there is an x outside of L for which the restrictions of τ and its conjugate $\text{Ad}(x)\tau$ to the intersection $L \cap {}^x L$ have an irreducible component in common.

Thus for some $x \in G - L$, the subgroup $x \ker(\tau)x^{-1} \cap L$ of L has a fixed vector in W under τ . Since $\ker(\tau)$ is normal in L , the product $\ker(\tau)(x \ker(\tau)x^{-1} \cap L) = \ker(\tau)x \ker(\tau)x^{-1} \cap L$ is a subgroup, which again has a fixed vector under τ .

More generally, let $M \subset \ker(\tau)$. If τ does not induce irreducibly to G , then some $x \in G - L$ intertwines τ , and we know that (the subgroup of L generated by) $MxMx^{-1} \cap L$ has a fixed vector under τ . If this (weaker) condition holds, we say that x *intertwines* τ at M .

We assign a partial ordering to the elements of G by saying that $x_1 < x_2$ when $L \cap Mx_1Mx_1^{-1} \subset L \cap Mx_2Mx_2^{-1}$. We may assume that such elements x belong to a chosen set of double coset representatives of L in G . Further, if x_2 intertwines τ at M and $x_1 < x_2$ then x_1 will also intertwine τ at M .

Given such a group M and a set of double coset representatives of L in G we

need to know which representatives are minimal with respect to this ordering. If a representation τ of L does not induce irreducibly to G , then some minimal x intertwines τ at M . We will prove that certain representations τ induce irreducibly to G by showing that no minimal representative x intertwines τ at M .

For any admissible representation (π, V) of G and any open subgroup L we let $V(L)$ denote the (necessarily) finite dimensional subspace of V fixed by L .

We now return to the case $G = GSp_4(F)$

LEMMA 2.2. *Using the double coset representatives given in Lemma 2.1 we obtain the following maximal sets of minimal (with respect to “ $<$ ”) elements.*

- (a) $\{t_0, t_1\}$ for $L = K$ and $M = K_n$.
- (b) $\{t_1, t_2, t_1 t_2, w_{\alpha+2\beta}\}$ for $L = J$ and $M = J_{n,j}$.
- (c) $\{t_0, t_1, w_{\alpha+\beta}\}$ for $L = H$ and $M = H_{n,j}$.

LEMMA 2.3. *The following are abelian quotients:*

- (a) K_m/K_n for $2m \geq n$.
- (b) $J_{m,i}/J_{n-1,0}$ for $2m + i \geq n - 1$.
- (c) $H_{m,i}/H_{n-1,0}$ for $2m + i \geq n - 1$.

Let ϕ be an additive character of F with conductor the ring of integers \mathcal{O} .

LEMMA 2.4. *Characters of the quotients in Lemma 2.4 are given by the formula*

$$\phi(I + y) = \phi(\text{trace}(x_\phi y))$$

where x_ϕ may be taken in the form

$$x_\phi = \pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

subject to the following conditions:

- (a) For $\phi \in (K_m/K_n)^\wedge$ all entries are in $\mathcal{O}/\mathcal{P}^{n-m}$,
- (b) For $\phi \in (J_{m,i}/J_{n-1,0})^\wedge$ we have

$$b, c, d \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$r, s, a, a' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$b', c', d' \in \mathcal{P}/\mathcal{P}^{n-m}$$

(c) For $\varphi \in (H_{m,i}/H_{n-1,0})^\wedge$ we have

$$a, c \in \mathcal{O}/\mathcal{P}^{n-m-1}$$

$$b \in \mathcal{O}/\mathcal{P}^{n-m-1-i}$$

$$r, s, d, d' \in \mathcal{P}/\mathcal{P}^{n-m-i}$$

$$a', c' \in \mathcal{P}/\mathcal{P}^{n-m}$$

$$b' \in \mathcal{P}^2/\mathcal{P}^{n-m+1-i}$$

Proof. Where convenient we may always tensor the original representation of G with a character of the determinant. We may also choose x_φ to be any convenient coset representative modulo the annihilator of these quotient groups which is given by elements of the form

$$\begin{bmatrix} u & w & 0 & y \\ x & v & -y & 0 \\ 0 & -z & u & x \\ z & 0 & w & v \end{bmatrix}$$

This gives the general form of x_φ . The conditions (a), (b) and (c) follow easily. We remark that we use the results of (b) extensively in the case $2m + i = n - 1$, so that $n - m - 1 = m + i$, etc.

Let $L \subset G$ and let $z \in G$. If τ is a representation of L , we denote by $\text{Ad } z(\tau)$ the representation of zLz^{-1} which for $x \in L$ is given by

$$\text{Ad } z(\tau)(zxz^{-1}) = \tau(x).$$

If φ is a character of L represented by x_φ , then $\text{Ad } z(\varphi)$ is a character of zLz^{-1} represented by $zx_\varphi z^{-1}$.

In what follows, we assume that π is an irreducible supercuspidal representation of $GS\mathcal{P}_4$ of level n . That is, n is the smallest integer such that $V(K_n)$ is non-zero. We also assume (after possibly tensoring π with a character through the determinant) that n is the minimal level among all such twists of π .

3. The Nilpotent Case

In this section, we consider the case in which π restricted to K_{n-1} contains a character φ for which $\pi^n x_\varphi$ is nilpotent mod \mathcal{P} . We may then assume that x_φ is of

the form

$$\pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where entries are taken modulo \mathcal{P} .

We seen, then, that the subgroup

$$B_n = \begin{bmatrix} n & n & n-1 & n-1 \\ n-1 & n & n-1 & n-1 \\ n & n & n & n-1 \\ n & n & n & n \end{bmatrix}$$

which is normal in the Iwahori subgroup B has a non-zero fixed vector in V . Therefore (π, V) , when restricted to B has a component (τ, Y) whose kernel contains B_n . Recall that for the rest of the paper, we assume that π is an irreducible supercuspidal representation of $GSp_4(F)$ of minimal level n .

DEFINITION 3.1. We say that such a representation (τ, Y) of nilpotent type is obtuse if for every character φ contained in the restriction of τ to B_{n-1} , both the entries a and d in x_φ are units.

Let

$$z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \pi & 0 & 0 \\ \pi & 0 & 0 & 0 \end{bmatrix}$$

LEMMA 3.2. *If (τ, V) is obtuse then (π, V) is irreducibly induced from $\langle B, z \rangle$.*

Proof. Referring to the remarks at the end of section 2., let $L = \langle B, z \rangle$ and $M = B_n$. A systematic check of double coset representatives of $\langle B, z \rangle$ in G , (i.e. elements of the affine Weyl group W_a), shows that if any such element x intertwines τ at B_n , then τ cannot be obtuse.

LEMMA 3.3. *If (τ, V) is not obtuse, then $J_{n-1,0}$ has a non-zero fixed vector in V .*

Proof. Let $v \in Y$ be an eigenvector for φ with x_φ as above. If (τ, Y) is not obtuse then there are three possibilities.

If $a = 0$ we are done.

If $a \neq 0$ and $c = 0$ then $\pi(w_\alpha)v$ is fixed by $J_{n-1,0}$.

If $a \neq 0$ and $c \neq 0$ then $\pi(u_{-\alpha}(ac^{-1}))v$ is fixed by $J_{n-1,0}$.

We may now assume that $V(J_{n-1,0})$ is a non-zero subspace.

DEFINITION 3.4. Let $L_{j,k}^{n-1} = (S_{\alpha+\beta} \cap J_{j,k})(S_{\beta} \cap J_{j,k})J_{n-1,0}$ where for any root γ , S_{γ} denotes the corresponding imbedding of $SL_2(F)$ in G . Let $\mathcal{L}_{n-1}(J)$ denote the set of those groups $L_{j,k}^{n-1}$ which are normal in J .

LEMMA 3.5. $L_{j,k}^{n-1} \in \mathcal{L}_{n-1}(J)$ if and only if $2j+k \geq n-1$ and $j \geq n-1 - \text{val}(2)$.

LEMMA 3.6. Let τ be an irreducible representation of J occurring in the restriction of π . Then one of the following statements is true.

- (a) $\pi = \text{Ind}_J^G \tau$,
- (b) $V(H_{n-1,0}) \neq \{0\}$,
- (c) $V(K_{n-1}) \neq \{0\}$, or
- (d) The restriction of π to $J_{m,i}$ contains a character φ for which we may assume that

$$x_{\varphi} = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \\ \pi & 0 & r & 0 \\ 0 & b' & 0 & s \end{bmatrix}$$

Proof. Let $L_{j,k}^{n-1}$ be the largest member of $\mathcal{L}_{n-1}(J)$ having a fixed vector. If $2j+k = n-1$ then (d) holds. We may therefore assume that $2j+k > n-1$ and that (a) does not hold. This means that some element $x = w_{\alpha+2\beta}^{\sigma} t_1^{v_1} t_2^{v_2}$ intertwines τ at $L_{j,k}^{n-1}$ where $\sigma \in \{0, 1\}$ and $v_1, v_2 \geq 0$. If neither (b) nor (c) holds, one can show that $v_1 = v_2$ so that $x = w_{\alpha+2\beta}^{\sigma} t_0^{v_0}$ with $v_0 \geq 0$.

Assume first that $k = 0$. Since x intertwines τ at $L_{j,0}^{n-1}$, it can be shown that $L_{j-1,1}^{n-1}$ has a fixed vector. Since $L_{j,0}^{n-1}$ was the largest element of $\mathcal{L}_{n-1}(J)$ having a fixed vector, it follows that $j = n-1 - \text{val}(2)$. We know that

$$x_{\varphi} = \pi^{-n} \begin{bmatrix} 0 & 0 & 1 & c \\ 0 & 0 & 0 & b \\ \pi & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

where $r, s, a, a', c \in \mathcal{P}^{n-j}$, $c' \in \mathcal{P}^{n-j+1}$, and $b, b' \in \mathcal{P}$. A Hensel's Lemma type argument shows that conjugation by elements of the form $u_{\beta}(x)$, $u_{\alpha+\beta}(x)$, $u_{-\beta}(x)$, and $u_{-(\alpha+\beta)}(x)$ in J will produce a matrix satisfying condition (d).

Now assume that $k = 1$. We know from Lemma 3.5 that $\text{val}(2) \geq j - n + 1$. If $\text{val}(r \pm s) < j - n - 1$, then the fact that x intertwines τ at $L_{j,0}^{n-1}$ would produce a vector fixed by the large group $L_{j,0}^{n-1}$. Therefore $\text{val}(r \pm s) = j - n + 1$. As in the previous case, this fact allows us to produce a character satisfying condition (d).

LEMMA 3.7. Assume that $V(H_{n-1,0}) \neq \{0\}$. Then either (π, V) is irreducibly

induced from H or (π, V) contains a K_{n-1} -fixed vector or (π, V) contains a principal character.

Proof. Assume that π is not irreducibly induced from H . From Lemma 2.2 and remarks preceding it, we see that $\pi|_H$ must have a component τ which is intertwined at $H_{n-1,0}$ by one of t_0, t_1 , or $w_{\alpha+\beta}$.

It is easy to see that if t_0 intertwines τ at $H_{n-1,0}$, then the intersection $H_{n-1,0}t_0H_{n-1,0}t_0^{-1} \cap H$ is contained in $\ker \tau$. Then, conjugating by t_0^{-1} , we see that $V(K_{n-1}) \neq \{0\}$. This contradicts the assumption that π has minimal level n .

Suppose next that t_1 or $w_{\alpha+\beta}$ intertwines τ at $H_{n-1,0}$. Then, in either case, the restriction of τ to a suitable subgroup (see Lemma 2.3(c)) will contain a character φ for which

$$x_\varphi = \pi^{-n} \begin{bmatrix} 0 & 0 & d & c \\ 0 & 0 & 0 & b \\ d' & 0 & r & a \\ c' & b' & a' & s \end{bmatrix}$$

with

$$a \in \mathcal{O} \quad r, s, b, c, a', c', d' \in \mathcal{P} \quad b', d \in \mathcal{P}^2$$

We may assume that $\text{val}(a) = 0$ and that $\text{val}(a') = 1$ or else we could easily produce a K_{n-1} -fixed vector. Applying Lemma 1.1, let $H_{n-1,0}$ play the role of L . Let $v_0 \in V(L)$ be an eigenvector of φ . Let $v'_0 = \pi(t_0)v_0$. There exist upper unipotent elements u_j and constants $c_{1,j}$ such that $v''_0 = \sum c_{1,j}\pi(u_j)v'_0$ is an eigenvector whose character is parametrized by a matrix in the form above with

$$d \in \mathcal{P} \quad b, c \in \mathcal{O} \quad c', d' \in \mathcal{P}^2 \quad b' \in \mathcal{P}^3$$

Then one can find $u \in u_\beta(\mathcal{O})u_{\alpha+2\beta}(\mathcal{P}^{-1})u_{\alpha+\beta}(\mathcal{O})$ such that $v_1 = \pi(u)v''_0$ is an eigenvector with a character satisfying the same conditions as the original φ . In the terms of Lemma 1.1, let $x_{i,j} = uu_jt_0$ for $i = 2, 3, \dots$. Thus φ is a principal character.

It therefore remains to analyze the case (d) of Lemma 3.6. Let $S_j = S_{\alpha+2\beta} \cap K_j$. Let $\tilde{S}_j = S_j J_{m,i}$. When the quotient S_h/S_j is abelian, we may parametrize its characters by matrices of the form

$$\pi^{-j} \begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

with b, b' , and $s \in \mathcal{O}$. (When $S_h \subseteq J_{m,i}$, this is consistent with our parametrization of $(J_{m,i})^\wedge$).

Recall that ϕ is a fixed additive character of F with conductor the ring of

integers \mathcal{O} . Since the residual characteristic of F is 2, we may choose $\rho \in \mathcal{O}^\times$ with the property that for $x \in \mathcal{O}$, we have $\phi(\pi^{-1}x^2) = \phi(\pi^{-1}\rho x)$.

LEMMA 3.8. *Let $k = \min\{\text{val}(2), \text{val}(r \pm s)\}$. Let j be the smallest integer such that π restricted to S_j contains a character ϕ which is trivial on S_j . There are various cases that can occur.*

(a) $j \geq 2k + 1$: ϕ may be extended to S_{j-2}/S_j . This situation may always be transformed into one of the following conditions.

- (i) If $s \in \mathcal{O}^\times$ and $b, b' \in \mathcal{P}$, then ϕ is principal.
- (ii) If $s \in \mathcal{P}$, $b \in \mathcal{O}^\times$, and $b' \in \mathcal{P} - \mathcal{P}^2$, then π is irreducibly induced from J .
- (iii) If

$$\begin{pmatrix} 0 & b \\ b' & s \end{pmatrix}$$

is irreducible mod \mathcal{P} , π is irreducibly induced from J .

(b) $j = 2k$ and $k = \text{val}(r + s) = \text{val}(r - s)$: In this case, let

$$\tilde{s} = s + \frac{\rho^2 \pi^{2k}}{(r + s)^2}$$

If

$$\begin{pmatrix} 0 & b \\ b' & \tilde{s} \end{pmatrix}$$

is irreducible mod \mathcal{P} , then π is irreducibly induced from J . Otherwise ϕ is principal and π is non-supercuspidal.

(c) $j < 2k$ or $j = 2k = \text{val}(2) < \text{val}(r \pm s)$: Here (π, V) is never supercuspidal.

Proof. In this proof we assume for convenience that $n - 1 = 2m$ and that $i = 0$. The other case is handled in a nearly identical fashion.

Since the illustration of principal characters in case (i) of (a) uses essentially the same arguments as in Section 4. of [KM], the proof is left to the reader.

Consider condition (ii) of case (a). Assume that some $x \in G$ intertwines $\text{Ind}_{S_{j-2}}^G \phi$. We need to show that $x \in J$. Let $t = t_1 t_2$ so that mod $Z(G)$ we may take t to be the translation

$$t_{\alpha+2\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}.$$

Let $\mathcal{D} = \{w_{\alpha+2\beta}t^v \mid v \geq 0\} \cup \{t^v \mid v > 0\}$. It is enough to consider elements $g \in J\mathcal{D}J$ since intertwining by elements of other double cosets of J would produce vectors fixed by subgroups already considered. We can write

$$J\mathcal{D}J = J_0\{I, z_J\}\mathcal{D}\{I, z_J\}J_0$$

where $J_0 = J_{0,0}$. The arguments for each piece of this double coset are essentially the same. We give the proof for $g \in J_0t^vJ_0$.

Let $g = j_1t^vj_2 = n_1jt^vn_2$ where

$$j \in J_0 \cap t^vJ_0t^{-v}$$

$$n_1 \in J_0/(J_0 \cap t^vJ_0t^{-v})$$

$$n_2 \in (J_0 \cap t^{-v}J_0t^v) \setminus J_0.$$

This means that

$$n_1 \in [w_\beta u_{-\beta}(\mathcal{P}/\mathcal{P}^v) \cup u_{-\beta}(\mathcal{O}/\mathcal{P}^v)]u_{-(\alpha+2\beta)}(\mathcal{P}/\mathcal{P}^{1+2v})u_{-(\alpha+\beta)}(\mathcal{P}/\mathcal{P}^{1+v})$$

and

$$n_2 \in u_{\alpha+2\beta}(\mathcal{P}/\mathcal{P}^{2v})u_{\alpha+\beta}(\mathcal{P}/\mathcal{P}^v)[u_\beta(\mathcal{P}/\mathcal{P}^v)w_\beta \cup u_\beta(\mathcal{O}/\mathcal{P}^v)]$$

We claim that neither n_1 nor n_2 may have w_β as a factor. In general, if g intertwines φ , then $g\varphi = \varphi$ on $x\tilde{S}_{j-2}x^{-1} \cap \tilde{S}_{j-2}$. Therefore $t^vn_2\varphi = j^{-1}n_1^{-1}\varphi$ on $J_{m,0} \cap t^vJ_{m,0}t^{-v}$. If n_2 has w_β as a factor, it is easy to see that $t^vn_2\varphi$ has level $n+2v$ on $u_{-(\alpha+2\beta)}$. This is a contradiction since $j^{-1}n_1^{-1}\varphi$ is trivial on $J_{n-1,0}$. Similar reasoning shows that n_1 cannot have w_β as a factor either.

Since j may be factored, we may write $x = n^-t^vn^+$ where n^+ and n^- are respectively in the intersections of J_0 with the positive and negative root Borel subgroups. Therefore, on $J_{m,0} \cap t^vJ_{m,0}t^{-v}$ we have

$$t^vn^+\varphi = n^-\varphi$$

The left hand side of this equation is trivial on the β and $\alpha + \beta$ root groups of $J_{m,0}$. We now write

$$n^-\varphi = u_{-\beta}(x)u_{-(\alpha+\beta)}(y)\varphi_1$$

where

$$\varphi_1 = u_{-\alpha}(p)u_{-(\alpha+2\beta)}(q)\varphi$$

One can verify that φ_1 is also trivial on the β and $\alpha + \beta$ root groups of $J_{m,0}$. It follows that x and y are in \mathcal{P}^{m-k+1} .

The next thing to see is that $n^- \varphi_1$ still has precisely level $j - 1$ on $u_{\alpha+2\beta}$. This is true because $j \geq 2k + 1$ forces $[u_{\alpha+2\beta}(z), u_{-\beta}(x)] \in \ker \varphi_1$ whenever $z \in \mathcal{P}^{j-2}$ and $x \in \mathcal{P}^{m-k+1}$. A similar argument applies to the effect of $u_{-(\alpha+\beta)}(y)$ on φ_1 .

We conclude that $n^- \varphi$ produces a ramified cuspidal character of S_{j-2} of the same level as that produced by φ . A similar check shows the same to be true for $n^+ \varphi$. It is evident that t^v cannot send one to the other. Similar arguments for the other kinds of double cosets of J show that the only elements that can intertwine φ are in J itself. Thus (π, V) is irreducibly induced from J . The proof for case (iii) of (a) is quite similar.

We now consider case (b). We may extend φ to $\tilde{S}_{2k-2} u_{-(\alpha+\beta)}(\mathcal{P}^m)$ so that φ is trivial on $u_{-(\alpha+\beta)}(\mathcal{P}^m)$. This is a key point in the entire problem since this extension exists precisely because F has residual characteristic 2.

For $p \in \mathcal{P}^m$ and $q \in \mathcal{P}^{2k-1}$ we may write

$$\varphi(u_{-(\alpha+\beta)}(p)u_{-(\alpha+2\beta)}(q)) = \phi(\pi^{-n}ap + \pi^{-2k}bq)$$

where a and b are integers.

Conjugation of φ by $u_{\alpha+\beta}(y)u_{\alpha+2\beta}(z)$ changes φ only on $u_{-(\alpha+\beta)}$ and $u_{-(\alpha+2\beta)}$. This action changes the coefficient of p in the formula above by $\pi^{-n}(r+s)y + \pi^{-(m+1)}\rho z^{1/2}$. The q -coefficient is changed by the amount $\pi^{-n+1}y^2 + \pi^{-2k}(b'z^2 + sz)$.

φ is a principal character when there exist elements y and z to make the q -coefficient zero without modifying the p -coefficient. This requires that

$$y^2 = \frac{\pi^{2m}\rho^2 z}{(r+s)^2}.$$

Thus the modification to the q -coefficient is

$$\pi^{-2k} \left[\left(s + \frac{\pi^{2k}\rho^2}{(r+s)^2} \right) z + b'z^2 \right].$$

Thus the new q -coefficient will be

$$\pi^{-2k}(b + \tilde{s}z + b'z^2).$$

This leads immediately to the condition of case (b). When that condition fails, an argument similar to the one given in case (a) shows that (π, V) is irreducibly induced from J .

We now consider case (c). Let

$$\gamma = \begin{cases} \beta & \text{for } n - 1 \text{ odd;} \\ \alpha + \beta & \text{for } n - 1 \text{ even.} \end{cases}$$

Then φ may be extended to $\tilde{S}_j u_{-\gamma}(\mathcal{P}^m)$ as before. As in the above cases, we give the proof for when $n - 1$ is even and $\gamma = \alpha + \beta$.

We may assume that the extended φ is trivial on $u_{-(\alpha+\beta)}(\mathcal{P}^m)$. Note that conjugating $u_{-(a+b)}(y)$ by $u_{\alpha+2\beta}(z)$ gives

$$u_{-(\alpha+\beta)}(y)u_{\beta}(zy)u_{-\alpha}(zy^2).$$

When $y \in \mathcal{P}^m$ and $z \in \mathcal{O}$, we have $\varphi(u_{\beta}(zy)) = 1$. Also (since F has residual characteristic 2) we find that the map $y \mapsto \varphi(u_{-\alpha}(zy^2))$ is an additive character of $\mathcal{P}^m/\mathcal{P}^{m+1}$. It follows that $z \in \mathcal{O}$ may be chosen so that $\varphi_1 = \text{Ad}(u_{\alpha+2\beta}(z))(\varphi)$ is a character extending the original φ which is trivial on $u_{-(\alpha+\beta)}(\mathcal{P}^m)$.

Let

$$\tilde{t} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

We will show that φ_1 is a principal character. Let φ_2 be any extension to $\tilde{S}_j u_{-(\alpha+\beta)}(\mathcal{P}^m)$ of $\text{Ad}(\tilde{t})(\varphi_1)$. Let P be the positive root Borel subgroup of $S_{\alpha+2\beta}$ and set $P_j = P \cap K_j$. Then φ_2 is trivial on $\tilde{S}_{2k} P_j$. We may choose $y \in \mathcal{P}^{m-k}$ such that $\text{Ad}(u_{\alpha+\beta}(y))(\varphi_2)$ is a character of \tilde{S}_j satisfying the initial conditions of the lemma. This demonstrates that φ is a principal character.

THEOREM 3.9. *All supercuspidal representations occurring in the nilpotent case are irreducibly induced from some open compact-by-center subgroup of G .*

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