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On the de Rham-Witt complex attached to a semi-stable family

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Introduction

Let K be a complete discrete valuation field with integer ring O_K and residue field k. We assume $\mathrm{ch}(K)=0$, $\mathrm{ch}(k)=p>0$ and k is perfect. We consider a proper semi-stable family $\mathscr{X}$ over O_K. Here "semi-stable" means

(a) $\mathscr{X}$ is regular and $X=\mathscr{X}\otimes_{O_K}K$ is smooth over K,
(b) $Y=\mathscr{X}\otimes_{O_K}k$ is a reduced divisor with normal crossings in $\mathscr{X}$.

In this note, we define a "modified de Rham-Witt complex" $W\omega_Y^{\cdot}$ on Y_{et}, whose hypercohomology $H^*(Y, W\omega_Y^{\cdot})$ is a *finitely generated* $W(k)$-module with frobenius φ. Note that, since Y may have singularities, the cohomology of the usual de Rham-Witt complex of Y does not have the finiteness property in our situation. Our theory has one more advantage that we can endow an endomorphism N on $H^*(Y, W\omega_Y^{\cdot})$ such that $N\circ\varphi=p\varphi\circ N$ and N is nilpotent on $H^*(Y, W\omega_Y^{\cdot})\otimes\mathbb{Q}$. Note that the situation is very similar to the l-adic setting: there is a nilpotent monodromy homomorphism $N\colon H^*(\bar{X}, \mathbb{Q}_l)(1)\to H^*(\bar{X}, \mathbb{Q}_l)$ on l-adic etale cohomology of the generic fiber, cf. [De] (1.7.2).

In our situation, Jannsen [Ja] §5 and Fontaine [Fo] predicted that there is a theory of p-adic limit Hodge structures. They conjectured that, to $\mathscr{X}/O_K$ as above, one can attach finite dimensional $W(k)\otimes\mathbb{Q}$-vector spaces D^q $(q\geqslant 0)$ endowed with a σ-semi-linear bijection φ, a nilpotent endomorphism N satisfying $N\circ\varphi=p\varphi\circ N$, and a decreasing filtration on $D^q\otimes_{W(k)}K$ satisfying some compatibility with φ. Moreover they predicted that there is a functorial correspondence between D^q and the p-adic etale cohomology group $H^q(\bar{X}, \mathbb{Q}_p)$. For precise conjectures, see [Fo].

Recently Kato [K1] has shown the remarkable fact that there exists a canonical isomorphism $H^q(Y, W\omega_Y^{\cdot})\otimes_{W(k)}K\simeq H^q_{\mathrm{DR}}(X/K)$. Hence the cohomology group of our modified de Rham-Witt complex is a realization of the

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space of Fontaine–Jannsen (the decreasing filtration is defined via the Hodge filtration of $H^q_{DR}(X/K)$). He has also proved a large part of the conjecture concerning the relation with p-adic etale cohomology [K2].

We give a more detailed explanation of the contents of this note.

In Section 1 we define a "modified Hodge–Witt sheaf" $W_n\omega^i_Y$, which is a sheaf of $W_n(\mathcal{O}_Y)$-module on Y_{et}. The "modification" will be similar to the one used for the "modified differential modules" in [H1]. This sheaf coincides with $W_n\Omega^i_Y$ except at the singular locus of Y. We endow $W_n\omega^i_Y$ with the usual operators π, d, F and V. Moreover we define an extension (cf. (1.4.3))

$$0 \to W_n\omega^{i-1}_Y \to W_n\tilde{\omega}^i_Y \to W_n\omega^i_Y \to 0,$$

which will supply us with N. This sheaf $W_n\tilde{\omega}^i_Y$ is also a $W_n(\mathcal{O}_Y)$-module and endowed with operators π, d, F and V. The above extension is compatible with these operators.

As in the classical case, we shall see that

$$\pi\colon W_{n+1}\omega^i_Y \to W_n\omega^i_Y \quad \text{and} \quad \pi\colon W_{n+1}\tilde{\omega}^i_Y \to W_n\tilde{\omega}^i_Y$$

are surjective and we define

$$W\omega^i_Y = \varprojlim_{\pi} W_n\omega^i_Y \quad \text{and} \quad W\tilde{\omega}^i_Y = \varprojlim_{\pi} W_n\tilde{\omega}^i_Y.$$

Then we have an extension

(c) $0 \to W\omega^{\cdot-1}_Y \to W\tilde{\omega}^{\cdot}_Y \to W\omega^{\cdot}_Y \to 0.$

In Section 2, we describe the structure of the "modified de Rham-Witt complex" $W\omega^{\cdot}_Y$. The result will be the same as for the classical de Rham-Witt complex. In particular we shall obtain that the hypercohomology group

$$D^q = H^q(Y, W\omega^{\cdot}_Y)$$

is a finitely generated $W(k)$-module (cf. (2.5)).

As in the classical case, the homomorphism of complexes

$$p^{\cdot}F\colon W\omega^{\cdot}_Y \to W\omega^{\cdot}_Y$$

induces a σ-semi-linear endomorphism φ on D^q. This φ is seen to be an isogeny from the equality $(p^{m-\cdot}V)\cdot(p^{\cdot}F) = p^{m+1}$, where $m = \dim Y$. We define N as the boundary map arising from the extension (c). Obviously we have the relation

$$N \circ \varphi = p\varphi \circ N.$$

This relation and the fact that φ is an isogeny show that N is nilpotent on $D^q \otimes \mathbb{Q}$.

In Section 3 we develop the Poincare duality theory for the "modified Hodge–Witt sheaves", following Ekedahl [Ek].

1. The definition

(1.0) Let K be a complete discrete valuation field with integer ring O_K and residue field k. We assume $\mathrm{ch}(K)=0$, $\mathrm{ch}(k)=p>0$ and k is perfect. We consider a scheme $\mathscr{X}$ finite type over O_K, satisfying the following conditions.

(1.0.1) $\mathscr{X}$ is regular and $X = \mathscr{X} \otimes_{O_K} K$ is smooth over K.

(1.0.2) $Y = \mathscr{X} \otimes_{O_K} k$ is a reduced divisor with normal crossings in $\mathscr{X}$.

In this section we define sheaves $W_n\omega_Y^i$ of $W_n(\mathcal{O}_Y)$-modules on Y_{et} for $i \geqslant 0$, which we shall call the "modified Hodge–Witt sheaves".(1) We moreover define operators

$$\pi\colon W_{n+1}\omega_Y^i \to W_n\omega_Y^i,$$
$$F\colon W_{n+1}\omega_Y^i \to W_n\omega_Y^i,$$
$$V\colon W_n\omega_Y^i \to W_{n+1}\omega_Y^i,$$
$$d\colon W_n\omega_Y^i \to W_n\omega_Y^{i+1},$$

satisfying the following relation:

$$\pi \text{ commutes with } F, V \text{ and } d; \quad FV = VF = p; \quad FdV = d.$$

These will coincide with $W_n\Omega_Y^i$ and its operators π, F, V and d except at the singular locus of Y. The idea of the "modification" is the same as for the "modified differential module" in [H1]. In this note we always work with the etale topology.

(1.1) We fix some notation. We denote $W = W(k)$ and $W_n = W_n(k)$. For a scheme S over Spec W, we denote $S_n = S \otimes_W W_n$.

As we work with the etale topology, we may assume $Y = \Sigma_i Y_i$, where each Y_i is irreducible and regular. Locally, we can choose a triple $(\mathscr{U}, \mathscr{Y} = \Sigma_{1 \leqslant i \leqslant a} \mathscr{Y}_i, f)$ with a commutative diagram of schemes

$$\begin{array}{ccc} \mathscr{X} & \to & \mathscr{U} \\ \downarrow & & \downarrow g \\ \mathrm{Spec}\, O_K & \to & \mathrm{Spec}\, W(k)\,[T], \end{array}$$

which satisfies the following properties.[2]

(1.1.1) $\mathscr{U}$ is smooth over W and $\mathscr{Y}$ is a reduced divisor in $\mathscr{U}$ with normal crossings relative to W.

(1.1.2) $\mathscr{Y}$ coincides with the inverse image of $T=0$ in $\mathscr{U}$.

(1.1.3) $\mathscr{Y} \otimes_W k = Y$.

(1.1.4) $f: \mathscr{U} \to \mathscr{U}$ is a lifting of the Frobenius satisfying $f(W[T]) \subset W[T]$ and $f^*\mathscr{Y} = p \cdot \mathscr{Y}$. (Hence $f(T) = T^p \cdot \text{unit}$.)

Let $\mathscr{D}_n$ be the structure sheaf of the divided power envelope of Y in $\mathscr{U}_n$ and $\mathscr{I}\mathscr{D}_n$ the kernel of the canonical homomorphism $\mathscr{D}_n \to \mathcal{O}_{\mathscr{Y}_n}$. We note that $\mathscr{I}\mathscr{D}_n$ is the ideal of $\mathscr{D}_n$ generated by $T^{[i]}$ for all $i > 0$. We define complexes

$$\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}): \mathscr{D}_n \to \mathscr{D}_n \otimes_{\mathcal{O}} \Omega^1(\log \mathscr{Y}) \to \mathscr{D}_n \otimes_{\mathcal{O}} \Omega^2(\log \mathscr{Y}) \to \cdots$$

$$\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(-\log \mathscr{Y}): \mathscr{I}\mathscr{D}_n \to \mathscr{I}\mathscr{D}_n \otimes_{\mathcal{O}} \Omega^1(\log \mathscr{Y}) \to \mathscr{I}\mathscr{D}_n \otimes_{\mathcal{O}} \Omega^2(\log \mathscr{Y}) \to \cdots$$

where $\mathcal{O} = \mathcal{O}_{\mathscr{U}_n}$ and $\Omega^i(\log \mathscr{Y}) = \Omega^i_{\mathscr{U}_n/W_n}(\log \mathscr{Y}_n)$ denotes the differential module with logarithmic poles along $\mathscr{Y}_n$.

(1.2) We shall prove the following lemma at the end of this section.

LEMMA (1.2.1). *In the derived category* $\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm \log \mathscr{Y})$ *are independent of the choice of lifting* $(\mathscr{U}, \mathscr{Y}, f)$.
(1.2.2) $\mathrm{d}T/T \in \mathscr{H}^1(\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}))$ *is independent of the choice of the lifting.*

Admitting the above lemma, we define[3]

$$W_n\tilde{\omega}^i_Y := \mathscr{H}^i(\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}))/\mathscr{H}^i(\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(-\log \mathscr{Y}))$$

$$W_n\omega^i_Y := W_n\tilde{\omega}^i_Y/(W_n\tilde{\omega}^{i-1}_Y \wedge (\mathrm{d}T/T)).$$

(1.3) We define the operators F, V, π and d, following the method of Illusie–Raynaud [IR] III(1.5). The definitions of F, V and d are rather elementary: F is induced from the canonical projection $\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_{n+1}}(\pm \log \mathscr{Y}) \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm \log \mathscr{Y})$, V is induced from "p": $\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm \log \mathscr{Y}) \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_{n+1}}(\pm \log \mathscr{Y})$ and d is induced by the boundary homomorphism arising from the exact sequence

$$0 \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm \log \mathscr{Y}) \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_{2n}}(\pm \log \mathscr{Y}) \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm \log \mathscr{Y}) \to 0.$$

The relations $FV = VF = p$ and $F\mathrm{d}V = d$ are verified easily.[4]
The definition of π is as follows. As

$$f^*(\mathscr{D}\Omega^i_{\mathscr{U}_M}(\pm \log \mathscr{Y})) \subset p^i \mathscr{D}\Omega^i_{\mathscr{U}_M}(\pm \log \mathscr{Y}) \quad \text{and} \quad \mathscr{D}\Omega^i_{\mathscr{U}_M}(\pm \log \mathscr{Y})$$

are flat W_M-modules for any $M > 0$, we can define f^*/p^j for $j \leqslant i$ to be the unique homomorphism such that the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{D}\Omega^i_{\mathscr{Y}_M}(\pm\log\mathscr{Y}) & \xrightarrow{f^*/p^j} & \mathscr{D}\Omega^i_{\mathscr{Y}_M}(\pm\log\mathscr{Y}) \\ \uparrow \text{can.} & & \downarrow \text{“}p^j\text{”} \\ \mathscr{D}\Omega^i_{\mathscr{Y}_{M+j}}(\pm\log\mathscr{Y}) & \xrightarrow{f^*} & \mathscr{D}\Omega^i_{\mathscr{Y}_{M+j}}(\pm\log\mathscr{Y}). \end{array}$$

Then f^*/p^{i-1} induces a homomorphism[5]

$$\mathbf{p}\colon W_n\tilde{\omega}^i_Y \to W_{n+1}\tilde{\omega}^i_Y.$$

As $f^*(\mathrm{d}T/T) = p\cdot(\mathrm{d}T/T)$ modulo $d(\mathscr{D}_{n+1})$, it induces

$$\mathbf{p}\colon W_n\omega^i_Y \to W_{n+1}\omega^i_Y.$$

The relations

$$\mathbf{p}F = F\mathbf{p}, \qquad \mathbf{p}V = V\mathbf{p} \quad \text{and} \quad \mathbf{p}d = d\mathbf{p} \tag{1.3.1}$$

are checked formally. Later in (2.2.2), we shall see that $\mathbf{p}$ is injective and its image coincides with the image of the multiplication by p in $W_{n+1}\tilde{\omega}^i_Y$ (resp. $W_{n+1}\omega^i_Y$). Then we define π to be the unique surjective homomorphism which makes the following diagram commutative.

$$\begin{array}{ccc} W_{n+1}\tilde{\omega}^i_Y & \xrightarrow{\pi} & W_n\tilde{\omega}^i_Y \\ \downarrow \mathbf{p} & \swarrow \mathbf{p} & \\ W_{n+1}\tilde{\omega}^i_Y & & \end{array} \qquad \begin{array}{ccc} W_{n+1}\omega^i_Y & \xrightarrow{\pi} & W_n\omega^i_Y \\ \downarrow \mathbf{p} & \swarrow \mathbf{p} & \\ W_{n+1}\omega^i_Y & & \end{array} \tag{1.3.2}$$

The relations $\pi F = F\pi$, $\pi V = V\pi$ and $\pi d = d\pi$ are consequences of (1.3.1).

(1.4) We give a second description of $W_n\omega^i_Y$, which is the most profitable to the calculation of the structure of $W_n\omega^i_Y$ (cf. (2.2)).

LEMMA. *Let $\tilde{\omega}^{\cdot}_{\mathscr{Y}_n}$ be the complex $\mathscr{O}_{\mathscr{Y}} \otimes_{\mathscr{O}_{\mathscr{Y}}} \Omega^{\cdot}_{\mathscr{Y}_n}(\log\mathscr{Y})$ and $\omega^{\cdot}_{\mathscr{Y}_n}$ be the complex $\tilde{\omega}^{\cdot}_{\mathscr{Y}_n}/$ (the image of $\tilde{\omega}^{\cdot -1}_{\mathscr{Y}_n} \wedge (\mathrm{d}T/T)$). We have*

$$\mathscr{H}^i(\tilde{\omega}^{\cdot}_{\mathscr{Y}_n}) \simeq W_n\tilde{\omega}^i_Y, \tag{1.4.1}$$

$$\mathscr{H}^i(\omega^{\cdot}_{\mathscr{Y}_n}) \simeq W_n\omega^i_Y. \tag{1.4.2}$$

Moreover we have an exact sequence

$$0 \to W_n\omega_Y^{i-1} \to W_n\tilde{\omega}_Y^i \to W_n\omega_Y^i \to 0, \tag{1.4.3}$$

which is compatible with the operators π, F, V and d.

(1.4.4) SUBLEMMA. *The homomorphism*

$$\mathscr{H}^i(\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(-\log \mathscr{Y})) \to \mathscr{H}^i(\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}))$$

is injective.

We may assume $\mathscr{U} = W[T_1, \dots, T_d, T]/(T_1 \cdot \cdots \cdot T_a - T)$. Then we have $\mathscr{D}_n = W_n[T_1, \dots, T_d] \otimes_{W_n[T]} W_n\langle T\rangle$, where $T = T_1 \cdot \cdots \cdot T_a$ and $W_n\langle T\rangle$ denotes the divided power algebra. We see that $\mathscr{I}\mathscr{D}_n$ is the ideal generated by $T^{[i]}$ for all $i > 0$. Now it is easily seen that

$$(d(\mathscr{D}\Omega^{i-1}_{\mathscr{U}_n}(\log \mathscr{Y})) \cap \mathscr{I}\mathscr{D}\Omega^{i}_{\mathscr{U}_n}(\log \mathscr{Y})) \subset d(\mathscr{I}\mathscr{D}\Omega^{i-1}_{\mathscr{U}_n}(\log \mathscr{Y}).$$

This proves (1.4.4).(6)

We have (1.4.1) from (1.4.4) and the exact sequence of complexes

$$0 \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(-\log \mathscr{Y}) \to \mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}) \to \tilde{\omega}^{\cdot}_{\mathscr{Y}_n} \to 0.$$

For (1.4.2) and (1.4.3), we use the exact sequence(7)

$$0 \to \omega^{\cdot-1}_{\mathscr{Y}_n} \to \tilde{\omega}^{\cdot}_{\mathscr{Y}_n} \to \omega^{\cdot}_{\mathscr{Y}_n} \to 0, \tag{1.4.5}$$

where the second arrow is induced from $\wedge(\mathrm{d}T/T)$. It suffices to show the injectivity of $\mathscr{H}^i(\omega^{\cdot-1}_{\mathscr{Y}_n}) \to \mathscr{H}^i(\tilde{\omega}^{\cdot}_{\mathscr{Y}_n})$. This follows from the fact $(d\tilde{\omega}^{i-1}_{\mathscr{Y}_n} \cap (\tilde{\omega}^{i-1}_{\mathscr{Y}_n} \wedge (\mathrm{d}T/T))) \subset d\tilde{\omega}^{i-2}_{\mathscr{Y}_n} \wedge (\mathrm{d}T/T)$.(6)

(1.5) By the following lemma, whose proof we leave to the reader, we can consider $W_n\tilde{\omega}_Y^i$ and $W_n\omega_Y^i$ as $W_n(\mathscr{O}_Y)$-modules.(8)

LEMMA. *We have an isomorphism*

$$W_n(\mathscr{O}_Y) \to \mathscr{H}^0(\omega^{\cdot}_{\mathscr{Y}_n});\ (a_0, a_1, \dots, a_{n-1}) \to \tilde{a}_0^{p^n} + p \cdot \tilde{a}_1^{p^{n-1}} + \cdots + p^{n-1} \cdot \tilde{a}_{n-1}^p.$$

where $\tilde{a}_i$ $(0 \leqslant i \leqslant n-1)$ are liftings to an element of $\mathscr{O}_{\mathscr{Y}_n}$.

(1.6) We give a third description of $W_n\tilde{\omega}_Y^i$ and $W_n\omega_Y^i$, which will be useful in developing the Poincare duality theory in §3.

LEMMA. *Let $U = \mathscr{U}_1$. Then we have isomorphisms*

$$W_n\tilde{\omega}^i_Y \simeq W_n\Omega^i_U(\log Y)/W_n\Omega^i_U(-\log Y),$$

$$W_n\omega^i_Y \simeq W_n\Omega^i_U(\log Y)/(W_n\Omega^i_U(-\log Y) + W_n\Omega^{i-1}_U(\log Y) \wedge (dT/T)).$$

Recall that $W_n\Omega^j_U(\pm\log Y)$ are defined to be the jth cohomology groups of $\Omega^{\cdot}_{\mathscr{U}_n}(\pm\log \mathscr{Y})$ (cf. [H2]). The lemma follows[(6)] from the exact sequence

$$0 \to \Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}) \to \Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y}) \to \tilde{\omega}^{\cdot}_{\mathscr{Y}_n} \to 0.$$

(1.7) We give a proof of (1.2). The following construction is due to Kato (cf. [K1]).

Take another lifting $(\mathscr{U}', \mathscr{Y}', f')$ and $g': \mathscr{U}' \to \operatorname{Spec} W[T']$. Let

$$h: \mathscr{V} \to \mathscr{U} \times_W \mathscr{U}'$$

be the blowing up defined by the product ideals of the ideals defined by $\mathscr{Y}_i \times \mathscr{Y}'_i$ $(1 \leqslant i \leqslant a)$, $\mathscr{Z}$ be the complement of the strict transform of the closed subschemes $\mathscr{Y}_i \times \mathscr{U}'$ and $\mathscr{U} \times \mathscr{Y}'_i$ $(1 \leqslant i \leqslant a)$, and let $\tilde{\mathscr{Y}}$ be the inverse image of $\Sigma\, \mathscr{Y}_i \times \mathscr{Y}'_i$. Then $\tilde{\mathscr{Y}}$ is a reduced divisor with normal crossings in $\mathscr{Z}$. The center of the blowing up is codimension one in $\mathscr{X} \subset \mathscr{U} \times \mathscr{U}'$ (diagonal image). So there is a closed immersion $i: \mathscr{X} \to \mathscr{V}$ whose image coincides with the strict transform of $\mathscr{X}$. This factors through $j: \mathscr{X} \to \mathscr{Z}$ as is seen from the construction of $\mathscr{Z}$.

Let $\mathscr{P}_n$ be the structure sheaf of the divided power envelope of $j(Y)$ in $\mathscr{Z}_n$ and let $\mathscr{I}\mathscr{P}_n$ the ideal generated by $T^{[i]}$ for all $i > 0$. As there is a unit $u \in \mathcal{O}^{\times}_{\mathscr{Z}}$ such that $T' = u \cdot T$, we see that $\mathscr{I}\mathscr{P}_n \ni T'^{[i]}$ for all $i > 0$. Hence there are canonical homomorphisms

$$\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm\log \mathscr{Y}) \to \mathscr{P}\Omega^{\cdot}_{\mathscr{Z}_n}(\pm\log \tilde{\mathscr{Y}})$$

and

$$\mathscr{D}\Omega^{\cdot}_{\mathscr{U}'_n}(\pm\log \mathscr{Y}') \to \mathscr{P}\Omega^{\cdot}_{\mathscr{Z}_n}(\pm\log \tilde{\mathscr{Y}}),$$

where $\mathscr{P}\Omega^{\cdot}_{\mathscr{Z}_n}(\pm\log \tilde{\mathscr{Y}})$ are defined as in $\mathscr{D}\Omega^{\cdot}_{\mathscr{U}_n}(\pm\log \mathscr{Y})$.

Now we show that the above homomorphisms of complexes are quasi-isomorphisms. We treat the former one. We have a diagram of schemes

$$\begin{array}{ccc} \mathscr{Z} & \to & \mathscr{U} \\ \uparrow & \square & \uparrow \\ \tilde{\mathscr{Y}} & \to & \mathscr{Y} \\ & \nwarrow & \uparrow \\ & & Y \end{array}$$

where the horizontal arrows are smooth and the others are closed immersions. We want to calculate $\mathscr{P}_n$. As the problem is etale local on Y, we may suppose that $\mathscr{X}_n = \mathscr{U}_n \otimes_W W[S_1, \dots, S_d]$ and Y is contained in the closed subscheme defined by $S_1 = \cdots = S_d = 0$.[9] Then we have

$$\mathscr{P}_n = \mathscr{D}_n \otimes_{W_n} W_n\langle S_1, \dots, S_d\rangle \quad \text{and} \quad \mathscr{I}\mathscr{P}_n = \mathscr{I}\mathscr{D}_n \otimes_{W_n} W_n\langle S_1, \dots, S_d\rangle.$$

Thus our claim is reduced to the well-known fact that

$$\Omega^{\cdot}_{W_n[T_1, \dots, T_d]} \otimes_{W_n[T_1, \dots, T_d]} W_n\langle T_1, \dots, T_d\rangle$$

is quasi-isomorphic to W_n.

To show (1.2.2), we note that the image of u in $\mathcal{O}^{\times}_{j(Y)}$ is contained in $k^{\times}$. This fact shows that $\mathrm{d}T/T - \mathrm{d}T'/T' = \mathrm{d}u/u \in \mathrm{d}\mathscr{P}_n$.

2. The structure

(2.0) In this section we shall study the structure of modified Hodge–Witt sheaves. As a result, we shall obtain the finiteness of the hypercohomology group $D^q = H^q(Y, W\omega^{\cdot}_Y)$.

(2.1) We first give the definition of differential modules in characteristic $p > 0$. Locally we fix a lifting $(\mathscr{U}, \mathscr{Y}, f)$. As in (1.4), we define

$$\tilde{\omega}^i_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_U} \Omega^i_U(\log Y)$$
$$\omega^i_Y = \tilde{\omega}^i_Y/(\tilde{\omega}^{i-1}_Y \wedge (\mathrm{d}T/T)),$$

where $U = \mathscr{U}_1$. Then $f^*/p^{\cdot}$ induces Cartier isomorphism

$$C^{-1}: \tilde{\omega}^{\cdot}_Y \simeq W_1\tilde{\omega}^{\cdot}_Y$$
$$C^{-1}: \omega^{\cdot}_Y \simeq W_1\omega^{\cdot}_Y. \tag{2.1.1}$$

To see this fact, we may assume

$$\mathscr{U} = W[T_1, \dots, T_d, T]/(T_1 \cdot \cdots \cdot T_a - T), \qquad f(T_i) = T_i^p \ (1 \leqslant i \leqslant d)$$

and $f(T) = T^p$. Then it is easy to see that the Cartier isomorphism on $\Omega^{\cdot}_U(\log Y)$ induces (2.1.1).[10]

As in Illusie [I1], we define $B_n\tilde{\omega}^i_Y$, $Z_n\tilde{\omega}^i_Y$, $B_n\omega^i_Y$ and $Z_n\omega^i_Y$ inductively by

$$B_1\tilde{\omega}^i_Y = d\tilde{\omega}^{i-1}_Y, \quad B_1\omega^i_Y = d\omega^{i-1}_Y, \quad Z_0\tilde{\omega}^i_Y = \tilde{\omega}^i_Y, \quad Z_0\omega^i_Y = \omega^i_Y,$$
$$C^{-1}: B_n\tilde{\omega}^i_Y \simeq B_{n+1}\tilde{\omega}^i_Y/B_1\tilde{\omega}^i_Y, \quad C^{-1}: B_n\omega^i_Y \simeq B_{n+1}\omega^i_Y/B_1\omega^i_Y$$
$$C^{-1}: Z_n\tilde{\omega}^i_Y \simeq Z_{n+1}\tilde{\omega}^i_Y/B_1\tilde{\omega}^j_Y, \quad C^{-1}: Z_n\omega^i_Y \simeq Z_{n+1}\omega^i_Y/B_1\omega^i_Y.$$

We note $B_n\omega_Y^i$ (resp. $Z_n\omega_Y^i$) was denoted by $\mathscr{B}_n^i$ (resp. $\mathscr{Z}_n^i$) in [H1].

(2.2) We next calculate certain subquotients of $W_n\tilde{\omega}_Y^i$ and $W_n\omega_Y^i$. The following (2.2.1) ~ (2.2.5) can be seen by direct computation as in the smooth case.[11]

LEMMA. *Consider*

$$\tilde{G}_n^m = V^m W_{n-m}\tilde{\omega}_Y^i / V^{m+1} W_{n-m-1}\tilde{\omega}_Y^i$$

(resp. $G_n^m = V^m W_{n-m}\omega_Y^i / V^{m+1} W_{n-m-1}\omega_Y^i$*) as a subquotient of*

$$\tilde{\omega}_Y^i \simeq p^m \tilde{\omega}_{\mathscr{Y}_n}^i / p^{m+1}\tilde{\omega}_{\mathscr{Y}_n}^i$$

(resp. $\omega_Y^i \simeq p^m \omega_{\mathscr{Y}_n}^i / p^{m+1}\omega_{\mathscr{Y}_n}^i$*) for* $0 \leqslant m < n$ (cf. (1.4.1)). *Then we have*

$$\tilde{G}_n^m = Z_{n-m}\tilde{\omega}_Y^i / B_{m+1}\tilde{\omega}_Y^i$$

$$G_n^m = Z_{n-m}\omega_Y^i / B_{m+1}\omega_Y^i. \qquad (2.2.1)$$

(2.2.2) $\mathbf{p}: W_n\omega_Y^i \to W_{n+1}\omega_Y^i$ *induces an injection*

$$\begin{array}{ccc} G_n^m & \to & G_{n+1}^{m+1} \\ \| \wr & & \| \wr \\ C^{-1}: Z_{n-m}\omega_Y^i / B_{m+1}\omega_Y^i & \hookrightarrow & Z_{n-m}\omega_Y^i / B_{m+2}\omega_Y^i. \end{array}$$

The same holds for $W_n\tilde{\omega}_Y^i$.

(2.2.3) *By* (2.2.1) *and* (2.2.2), π *is well-defined* (*cf.* (1.3.2)) *and induces a surjection*

$$\begin{array}{ccc} G_{n+1}^m & \to & G_n^m \\ \| \wr & & \| \wr \\ C: Z_{n+1-m}\omega_Y^i / B_{m+1}\omega_Y^i & \twoheadrightarrow & Z_{n-m}/B_{m+1}\omega_Y^i. \end{array}$$

The same holds for $W_n\tilde{\omega}_Y^i$.

(2.2.4) *The image of* $Z_{m+1}\omega_Y^{i-1}/B_1\omega_Y^{i-1} \subset W_1\omega_Y^{i-1}$ *by* $\mathrm{d}V^n$ *is contained in* $V^m W_{n-m+1}\omega_Y^i \subset W_{n+1}\omega_Y^i$. *Moreover we have a commutative diagram* (*where the upper horizontal arrow is induced from* $\mathrm{d}V^n$)

$$\begin{array}{ccc} Z_{m+1}\omega_Y^{i-1}/B_1\omega_Y^{i-1} & \to & Z_{n+1-m}\omega_Y^i / B_{m+1}\omega_Y^i = G_{n+1}^m \\ \text{can} \downarrow & & \uparrow \\ C^{-m-1}\circ d \circ C^{m+1}: Z_{m+1}\omega_Y^{i-1}/Z_{m+2}\omega_Y^{i-1} & \xrightarrow{\simeq} & B_{m+2}\omega_Y^i / B_{m+1}\omega_Y^i. \end{array}$$

The same holds for $W_n\tilde{\omega}_Y^i$.

(2.2.5) *The image of* $F^n d(V^m W_{n-m+1}\omega_Y^i)$ *in* $W_1\omega_Y^{i+1}$ *coincides with* $B_{n-m+2}\omega_Y^{i+1}/B_1\omega_Y^{i+1}$. *Moreover* $F^n d$ *induces a surjection*

$$G_{n+1}^m = Z_{n-m+1}\omega_Y^i/B_{m+1}\omega_Y^i \to B_{n-m+2}\omega_Y^{i+1}/B_{n-m+1}\omega_Y^{i+1},$$

where the homomorphism is $C^{m-n+1}\circ d\circ C^{n-m+1}$. *The same holds for* $W_n\tilde{\omega}_Y^i$.

(2.3) In the following, we consider $Z_n\tilde{\omega}_Y^i$ and $B_n\tilde{\omega}_Y^i$ (resp. $Z_n\omega_Y^i$ and $B_n\omega_Y^i$) as a subsheaf of $W_1\tilde{\omega}_Y^i$ (resp. $W_1\omega_Y^i$) via the Cartier isomorphism (2.1.1).

We define

$$\mathrm{gr}^n W\omega_Y^i = \mathrm{Ker}\ \pi\colon W_{n+1}\omega_Y^i \to W_n\omega_Y^i,$$

$$\mathrm{gr}'^n W\omega_Y^i = \mathrm{Coker}\ \mathbf{p}\colon W_n\omega_Y^i \to W_{n+1}\omega_Y^i.$$

We also define K_n^i by the exact sequence

$$0 \to K_n^i \to Z_n\omega_Y^{i-1} \oplus B_{n+1}\omega_Y^i \to B_{n+1}\omega_Y^i/B_n\omega_Y^i \to 0,$$

where the third arrow is defined by $(a,b)\to C^{-n}\circ d\circ C^n(a)+b$.

PROPOSITION (2.3.1) *We have an exact sequence*

$$0 \to K_n^i \to \omega_Y^{i-1} \oplus \omega_Y^i \to \mathrm{gr}^n W\omega_Y^i \to 0,$$

where the third arrow is defined by $(a,b)\to dV^n(a)+V^n(b)$.

(2.3.2) *There is a well-defined homomorphism*

$$\mathrm{gr}'^n W\omega_Y^i \to \omega_Y^i \oplus \omega_Y^{i+1};\quad a \to (F^n(a),\ -F^n d(a)),$$

whose image coincides with K_n^{i+1}.

We have (2.3.1) (resp. (2.3.2)) from (2.2.3) and (2.2.4) (resp. (2.2.2) and (2.2.5)). We note that the kernel of the homomorphism in (2.2.3) (resp. the image of the homomorphism in (2.2.2)) coincides with the image of the homomorphism in (2.2.4) (resp. the kernel of the homomorphism in (2.2.5).

(2.4) As a consequence of (2.3), we have

PROPOSITION (2.4.1) *The homomorphism* $\mathbf{p}$ *induces a quasi-isomorphism*

$$\mathrm{gr}^n W\omega_Y^{\cdot} \to \mathrm{gr}^{n+1} W\omega_Y^{\cdot}.$$

(2.4.2) *The canonical projection induces a quasi-isomorphism*

$$W_{n+1}\omega_Y^{\cdot}/\mathbf{p}(W_n\omega_Y^{\cdot}) \to \omega_Y^{\cdot}.$$

We prove (2.4.1). By (2.3.1) and (2.2.2) we have

$$\mathrm{gr}^{n+1} W\omega_Y^i/\mathbf{p}(\mathrm{gr}^n W\omega_Y^i) \simeq \omega_Y^{i-1}/Z_1\omega_Y^{i-1} \oplus \omega_Y^i/Z_1\omega_Y^i.$$

By the definition of the homomorphism in (2.3.1), we see that d on $\mathrm{gr}^{n+1} W\omega_Y^i/\mathbf{p}(\mathrm{gr}^n W\omega_Y^i)$ induces a homomorphism

$$\omega_Y^{i-1}/Z_1\omega_Y^{i-1} \oplus \omega_Y^i/Z_1\omega_Y^i \to \omega_Y^i/Z_1\omega_Y^i \oplus \omega_Y^{i+1}/Z_1\omega_Y^{i+1}; \quad (a, b) \to (b, 0).$$

This shows that $\mathrm{gr}^{n+1} W\omega_Y^{\cdot}/\mathbf{p}(\mathrm{gr}^n W\omega_Y^{\cdot})$ is acyclic.

To see (2.4.2), it suffices to show that

$$W_{n+1}\omega_Y^{\cdot}/\mathbf{p}(W_n\omega_Y^{\cdot}) \xrightarrow{\pi} W_n\omega_Y^{\cdot}/\mathbf{p}(W_{n-1}\omega_Y^{\cdot})$$

is a quasi-isomorphism. This is easily reduced to (2.4.1).

THEOREM (2.5) *For each n (including the case $n = \infty$), $H^q(Y, W_n\omega_Y^{\cdot})$ is a finitely generated W_n-module. Moreover there is a long exact sequence*

$$\cdots \to H^q(Y, W_n\omega_Y^{\cdot}) \xrightarrow{\mathbf{p}} H^q(Y, W_{n+1}\omega_Y^{\cdot}) \to H^q(Y, \omega_Y^{\cdot}) \to \cdots.$$

For $n < \infty$, this is a consequence of the finiteness of $H^q(Y, \omega_Y^{\cdot})$ and Proposition (2.4.2). For $n = \infty$, it suffices to note[(12)]

$$H^q(Y, W\omega_Y^{\cdot}) = \varprojlim_{\pi} H^q(Y, W_n\omega_Y^{\cdot}).$$

3. Poincaré duality

(3.0) In this section we shall define a perfect pairing

$$H^q(Y, W_n\omega_Y^{\cdot}) \times H^{m-q}(Y, W_n\omega_Y^{\cdot}) \to W_n,$$

where $m = \dim Y$. We use results and techniques of [Ek].

THEOREM (3.1) *There is a canonical isomorphism*

$$W_n\omega_Y^m \xrightarrow{\simeq} h_n^! W_n[-m].$$

where h_n denotes the canonical morphism $W_n(Y) \to \mathrm{Spec}\, W_n$.

We construct the isomorphism locally. We fix a lifting $(\mathcal{U}, \mathcal{Y}, f)$, and denote $U = \mathcal{U}_1$ and $Y = \mathcal{Y}_1$. By a result of [Ek] I, there is a canonical isomorphism

$$W_n\Omega_U^{m+1} \xrightarrow{\simeq} f_n^! W_n[-m-1]$$

where $f_n\colon W_n(U)\to \operatorname{Spec} W_n$ is the canonical morphism. It suffices to show (cf. [Ha] III, §6)

$$\mathbb{R}\mathscr{H}om_{W_n(\mathcal{O}_U)}(W_n(\mathcal{O}_Y), W_n\Omega_U^{m+1}) \simeq W_n\omega_Y^m[-1]. \tag{3.1.1}$$

By the Proposition (3.3) below, we have a canonical isomorphism

$$\mathbb{R}\mathscr{H}om_{W_n(\mathcal{O}_U)}(W_n(\mathscr{I}_Y), W_n\Omega_U^{m+1}) \simeq W_n\Omega_U^{m+1}(\log Y).$$

Now (3.1.1) follows from the exact sequence

$$0 \to W_n(\mathscr{I}_Y) \to W_n(\mathcal{O}_U) \to W_n(\mathcal{O}_Y) \to 0$$

and the fact $W_n\Omega_U^{m+1}(\log Y)/W_n\Omega_U^{m+1} \simeq W_n\omega_Y^m$ (cf. (1.6)).[13]

THEOREM (3.2) *The canonical pairing* $W_n\omega_Y^i \times W_n\omega_Y^{m-i} \to W_n\omega_Y^m$ *induces an isomorphism*

$$W_n\omega_Y^i \xrightarrow{\simeq} \mathbb{R}\mathscr{H}om_{W_n(\mathcal{O}_Y)}(W_n\omega_Y^{m-i}, W_n\omega_Y^m).$$

(3.3) We need the duality theory for Hodge–Witt sheaves with logarithmic poles.

PROPOSITION. *The canonical pairing*

$$\Omega^i_{\mathscr{U}_n}(\pm\log \mathscr{Y}) \times \Omega^{m+1-i}_{\mathscr{U}_n}(\log \mathscr{Y}) \to \Omega^{m+1}_{\mathscr{U}_n}(\pm\log \mathscr{Y})$$

induces isomorphisms

$$W_n\Omega_U^i(\pm\log Y) \to \mathbb{R}\mathscr{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega_U^{m+1-i}(\mp\log Y), W_n\Omega_U^{m+1}), \tag{3.3.1}$$

$$W_n\Omega_U^i(\log Y) \to \mathscr{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega_U^{m+1-i}(\log Y), W_n\Omega_U^{m+1}(\log Y)). \tag{3.3.2}$$

COROLLARY (3.4) *The pairing in Proposition* (3.3) *induces a perfect pairing*

$$W_n\tilde{\omega}_Y^i \times W_n\tilde{\omega}_Y^{m+1-i} \to W_n\tilde{\omega}_Y^{m+1} \simeq W_n\omega_Y^m.$$

We first deduce Corollary (3.4) from (3.3.1) and (3.3.2). Applying $\mathscr{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega_U^{m+1-i}(\log Y), *)$ to the exact sequence

$$0 \to W_n\Omega_U^{m+1} \to W_n\Omega_U^{m+1}(\log Y) \to W_n\omega_Y^m \to 0,$$

we obtain an exact sequence

$$0 \to W_n\Omega^i_U(-\log Y) \to W_n\Omega^i_U(\log Y) \to$$
$$\to \mathcal{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega^{m+1-i}_U(\log Y), W_n\omega^m_Y) \to 0.$$

(Note that $\mathcal{E}xt^1_{W_n(\mathcal{O}_U)}(W_n\Omega^{m+1-i}_U(\log Y), W_n\Omega^{m+1}_U)=0$ by (3.3.1).) Thus by (1.6), we have an isomorphism

$$W_n\tilde{\omega}^i_Y \xrightarrow{\simeq} \mathcal{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega^{m+1-i}_U(\log Y), W_n\omega^m_Y). \tag{3.4.1}$$

As the image of the pairing

$$W_n\Omega^{m+1-i}_U(-\log Y) \times W_n\Omega^i_U(\log Y)$$

is contained in $W_n\Omega^{m+1}_U$, we see that (3.4.1) factors as

$$W_n\tilde{\omega}^i_Y \to \mathcal{H}om_{W_n(\mathcal{O}_Y)}(W_n\tilde{\omega}^{m+1-i}_Y, W_n\omega^m_Y) \to$$
$$\to \mathcal{H}om_{W_n(\mathcal{O}_U)}(W_n\Omega^{m+1-i}_U(\log Y), W_n\omega^m_Y).$$

Since the second arrow is injective, we obtain (3.4).

(3.5) We next prove Proposition (3.3) by the method of [Ek]. Here we only give an outline of the proof of (3.3.2), as (3.3.1) can be proved by exactly the same way as [Ek].(14) The proof goes by induction on n, the case $n = 1$ being easy.

We consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 \to W_n\Omega^i_U(\log Y) & \xrightarrow{\mathbf{p}} & W_{n+1}\Omega^i_U(\log Y) & \to & \mathrm{gr}'^n W\Omega^i_U(\log Y) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to D^1(W_n\Omega^j_U(\log Y)) & \xrightarrow{\pi^*} & D^1(W_{n+1}\Omega^j_U(\log Y)) & \to & D^1(\mathrm{gr}^n W\Omega^j_U(\log Y)), \end{array} \tag{3.5.1}$$

where

$$j = m + 1 - i, \qquad D^1(*) = \mathcal{H}om_{W_{n+1}(\mathcal{O}_U)}(*, W_{n+1}\Omega^{m+1}_U(\log Y))$$

and the first vertical arrow is induced from the pairing

$$W_n\Omega^i_U(\log Y) \times W_n\Omega^j_U(\log Y) \to W_n\Omega^{m+1}_U(\log Y) \xrightarrow{\mathbf{p}} W_{n+1}\Omega^{m+1}_U(\log Y).$$

The third vertical arrow of (3.5.1) is an isomorphism. This can be seen as in

[Ek] by reducing it to the duality between $\Omega_U^j(\log Y)/B_n^j(\log Y)$ and $Z_n^i(\log Y)$ (resp. $\Omega_U^j(\log Y)/Z_n^j(\log Y)$ and $B_n^i(\log Y)$).[15] Thus it suffices to show that the first vertical arrow of (3.5.1) is an isomorphism. For this, we consider a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & W_n\Omega_U^i(-\log Y) & \to & W_n\Omega_U^i(\log Y) & \to & W_n\tilde{\omega}_Y^i & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & D^2(W_{n+1}\Omega_U^{m+1}) & \to & D^2(W_{n+1}\Omega_U^{m+1}(\log Y)) & \to & D^2(W_{n+1}\omega_Y^m) & \to 0 \end{array} \tag{3.5.2}$$

where $D^2(*) = \mathcal{H}om_{W_{n+1}(\mathcal{O}_U)}(W_n\Omega_U^j(\log Y), *)$ and the vertical arrows are induced from the pairing as in the first vertical arrow of (3.5.1). The third vertical arrow of (3.5.2) is an isomorphism. This fact is seen from a commutative diagram

$$\begin{array}{ccc} W_n\tilde{\omega}_Y^i & \xrightarrow{a} & \mathcal{H}om_{W_n(\mathcal{O}_Y)}(W_n\tilde{\omega}_Y^j, W_n\omega_Y^m) \\ & {\scriptstyle b}\searrow & \downarrow{\scriptstyle c} \\ & & \mathcal{H}om_{W_{n+1}(\mathcal{O}_Y)}(W_n\tilde{\omega}_Y^j, W_{n+1}\omega_Y^m). \end{array} \tag{3.5.3}$$

Where a is induced from the product structure and is an isomorphism by induction hypothesis (cf. (3.4)), b is the homomorphism in (3.5.2), and c is induced from the duality homomorphism for finite morphisms and is an isomorphism by Theorem (3.1). The commutativity of (3.5.3) is seen from the fact that c is induced from $\mathbf{p}$: $W_n\omega_Y^m \to W_{n+1}\omega_Y^m$, which fact follows from (3.1.1) and [Ek] I (3.3). By the same way, we see that the first vertical arrow of (3.5.2) is an isomorphism. Thus we have Proposition (3.3).

(3.6) We now prove Theorem (3.2). We first show

$$\mathcal{E}xt^j_{W_n(\mathcal{O}_Y)}(W_n\omega_Y^{m-i}, W_n\omega_Y^m) = 0 \quad \text{for } j > 0. \tag{3.6.1}$$

This follows from Theorem (3.1) and the fact that $W_n\omega_Y^{m-i}$ is a successive extension of locally free $\mathcal{O}_Y$-modules (cf. (2.3)), by the same argument as [Ek] II, Lemma 2.2.7.

We next prove

$$W_n\omega_Y^i \xrightarrow{\cong} \mathcal{H}om_{W_n(\mathcal{O}_Y)}(W_n\omega_Y^{m-i}, W_n\omega_Y^m) \tag{3.6.2}$$

by descending induction on i, the case $i = m$ being trivial. The commutative diagram of perfect pairings

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \uparrow & & \\
W_n\omega_Y^{i-1} & \times & W_n\omega_Y^{m+1-i} & \to & W_n\omega_Y^m \\
\downarrow & & \uparrow & & \| \\
W_n\tilde{\omega}_Y^i & \times & W_n\tilde{\omega}_Y^{m+1-i} & \to & W_n\omega_Y^m \\
\downarrow & & \uparrow & & \| \\
W_n\omega_Y^i & \times & W_n\omega_Y^{m-i} & \to & W_n\omega_Y^m \\
\downarrow & & \uparrow & & \\
0 & & 0 & &
\end{array}$$

shows that (3.6.2) for $i = k - 1$ follows from (3.6.2) for $i = k$. Thus the proof is completed.

(3.7) Now assume $\mathscr{X}/\operatorname{Spec} O_K$ is proper. By Theorems (3.1) and (3.2), we have a perfect pairing

$$H^q(Y, W_n\omega_Y^i) \times H^{m-q}(Y, W_n\omega_Y^{m-i}) \to H^m(Y, W_n\omega_Y^m) \simeq W_n.$$

Thus we obtain a perfect pairing

$$H^q(Y, W_n\omega_Y^{\cdot}) \times H^{m-q}(Y, W_n\omega_Y^{\cdot}) \to W_n.$$

By passing to the limit, we have a perfect pairing

$$H^q(Y, W\omega_Y^{\cdot}) \otimes \mathbb{Q} \times H^{m-q}(Y, W\omega_Y^{\cdot}) \otimes \mathbb{Q} \to W \otimes \mathbb{Q}.$$

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Editorial comments

(1) Despite the notations, $W_n\omega_Y^{\cdot}$ and $W_n\tilde{\omega}_Y^{\cdot}$ do not depend on Y alone, but on the semi-stable family $X/\mathcal{O}_k$. One can show, however, that the complexes depend only on $X \times_{\mathcal{O}_k} \mathcal{O}_k/(\pi_k^2)$, see [K2].

(2) One may assume that this diagram corresponds to the following diagram of rings

$$\begin{array}{ccccc} \mathcal{O}_k[T_1,\dots,T_d]/(T_1\cdots T_a-\pi_k) & \twoheadleftarrow & W[T_1,\dots,T_d] & & T_1\cdots T_a \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_k & \xleftarrow{\pi_k\leftarrow T} & W[T] & & T, \end{array}$$

where π_k is a uniformizing element in $\mathcal{O}_k$. Then f can be defined by $f(T_i)=T_i^p$.

(3) Both for the "local" definition of $W_n\omega_Y^{\cdot}$ and for the independence of $(\mathscr{U}, \mathscr{Y}, f)$ one a priori needs a more precise statement than the one shown in (1.7). In fact, one has to observe that the sheaves defined in (1.1) commute with étale base change, and one needs canonical isomorphisms which "glue", i.e., satisfy an obvious cocycle relation for three different liftings. This is avoided, if one identifies $W_n\omega_Y^i$ and $W_n\tilde{\omega}_Y^i$ with subsheaves of certain fixed sheaves on Y as in [K1] (and in [H1] (1.5) for $n=1$): e.g., $W_n\omega_Y^i$ is a subsheaf of $j_* W_n\Omega_{Y^0}^i$ for any regular dense open subscheme $j\colon Y^0 \hookrightarrow Y$. Then (1.7) shows that the subsheaves are the same for two different lifts. The maps π, d, F, V and N are independent of the choices, since the quasi-isomorphisms

$$\mathscr{D}\Omega_{\mathscr{U}_n}^{\cdot}(\pm\log \mathscr{Y}) \to \mathscr{P}\Omega_{\mathscr{Z}_n}^{\cdot}(\pm\log \tilde{\mathscr{Y}}) \leftarrow \mathscr{D}\Omega_{\mathscr{U}_n'}^{\cdot}(\pm\log \mathscr{Y}')$$

are compatible with all maps considered in (1.3) and (1.4). Note also that there is a lift of the Frobenius on $\mathscr{Z}$ which is compatible with f and f' under the projections $\mathscr{U} \leftarrow \mathscr{Z} \rightarrow \mathscr{U}'$.

(4) These relations also imply that $W_n\omega_Y^{\cdot}$ and $W_n\tilde{\omega}_Y^{\cdot}$ are in fact complexes: $d^2=F^n\mathrm{d}V^nF^n\mathrm{d}V^n=F^n\mathrm{d}^2V^np^n=0$ on $W_n\omega_Y^i$ or $W_n\tilde{\omega}_Y^i$.

(5) If $x' \in \mathscr{D}\Omega_{\mathscr{U}_{n+1}}^i(\log \mathscr{Y})$ is a lift of the local section $x \in \mathscr{D}\Omega_{\mathscr{U}_n}^i(\log \mathscr{Y})$ with $dx=0$, then $\mathbf{p}$ (class of x) = class of $f/p^{i-1}x'$.

(6)The description of cycles modulo p^n given in (11)(A) below, which is valid for $\Omega^{\cdot}_{\mathscr{U}_n}(\log \mathscr{Y})$, $\tilde{\omega}^{\cdot}_{\mathscr{Y}_n}$ and $\omega^{\cdot}_{\mathscr{Y}_n}$, proves the surjectivity of the maps

$$W_n\Omega^{i-1}_U(\log Y) \to W_n\tilde{\omega}^{i-1}_Y \to W_n\omega^{i-1}_Y,$$

which in turn proves (1.4.4) and the injectivity of the maps $W_n\omega^{i-1}_Y \to W_n\omega^i_Y$, $W_n\Omega^i_U(-\log Y) \to W_n\Omega^i_U(\log Y)$.

(7)One has the exact sequence

$$\cdots \longrightarrow \tilde{\omega}^{i-1}_{\mathscr{Y}_n} \xrightarrow{\wedge \mathrm{d}T/T} \tilde{\omega}^{i}_{\mathscr{Y}_n} \xrightarrow{\wedge \mathrm{d}T/T} \tilde{\omega}^{i+1}_{\mathscr{Y}_n} \longrightarrow \cdots$$

by the following observation: If $\mathscr{M}$ is a locally free $\mathscr{O}_Y$-module for a scheme Y and s is a section of $\mathscr{M}$ such that $\mathscr{O}_Y \xrightarrow{\cdot s} \mathscr{M}$ is injective with locally free cokernel, then the complex

$$0 \longrightarrow \mathscr{O}_Y \xrightarrow{\cdot s} \mathscr{M} \xrightarrow{\wedge s} \Lambda^2\mathscr{M} \xrightarrow{\wedge s} \Lambda^3\mathscr{M} \longrightarrow \cdots$$

is exact.

(8)(cf. [IR] III 1.5). The surjectivity of the map is an immediate consequence of (11)(A) below.

(9)Compare the more explicit description of $\mathscr{Z}$ in [H2].

(10)One may assume $d=a$. Let $V=\bigoplus_{i=1}^d \mathbb{F}_p(\mathrm{d}T_i/T_i)$, and for $v=\Sigma_{i=1}^d \alpha_i(\mathrm{d}T_i/T_i)$ let C_v be the subsheaf of $\mathscr{O}_Y$ generated over k by the monomials $T_1^{\beta_1}\cdots T_d^{\beta_d}$ with $\beta_i \in \mathbb{N}$, $\beta_i \bmod p = \alpha_i$. Then

$$\tilde{\omega}^{\cdot}_Y \cong \bigoplus_{v\in V} C_v \otimes_{\mathbb{F}_p} \Lambda^{\cdot}_{\mathbb{F}_p} V$$

with each summand respected by d. In fact

$$d(u\otimes\omega) = u\otimes(v\wedge\omega) \quad \text{for } u\in C_v, \qquad w\in\Lambda^{\cdot}_{\mathbb{F}_p}V.$$

This shows that $C_v \otimes_{\mathbb{F}_p} \Lambda^{\cdot}_{\mathbb{F}_p} V$ is acyclic for $v \neq 0$ (see (7) above) and has zero differentials for $v=0$. Since $C^{-1}(\mathrm{d}T_i/T_i)=\mathrm{d}T_i/T_i$ and $C^{-1}(f)=f^p$ for $f\in\mathscr{O}_Y$, we obtain that

$$C^{-1}: \tilde{\omega}^1_Y \xrightarrow{\sim} \mathscr{H}^1(\tilde{\omega}^{\cdot}_Y)$$

and

$$\Lambda^i_{\mathcal{O}_Y}\mathcal{H}^1(\tilde{\omega}_Y^{\cdot}) \tilde{\to} \mathcal{H}^i(\tilde{\omega}_Y^{\cdot}),$$

hence the claim for $\tilde{\omega}_Y^{\cdot}$. The Cartier isomorphism for $\omega_Y^{\cdot}$ follows from this and the fact that $C^{-1}(\mathrm{d}T/T) = \mathrm{d}T/T$.

It is needed later that the isomorphisms (2.1.1) are isomorphisms of complexes, i.e., commute with the differentials. This follows easily from the fact that d on $W_1\tilde{\omega}_Y^i = \mathcal{H}^i(\tilde{\omega}_Y^{\cdot})$ is a connecting morphism for the exact sequence

$$0 \to \tilde{\omega}_{\mathcal{Y}_1}^{\cdot} \to \tilde{\omega}_{\mathcal{Y}_2}^{\cdot} \to \tilde{\omega}_{\mathcal{Y}_1}^{\cdot} \to 0.$$

(11) Since in the smooth case the usual approach to the de Rham-Witt complex is different and the crystalline definition is only indicated on one page in [IR] III 1.5, some comments may be helpful.

Let $\tilde{\omega}_{\mathcal{Y}}^{\cdot} = \mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{U}}} \Omega_{\mathcal{U}}^{\cdot}(\log \mathcal{Y})$ and define $\omega_{\mathcal{Y}}^{\cdot}$ similarly (in the following we only treat that $\omega_{\mathcal{Y}}^{\cdot}$, the case of $\tilde{\omega}_{\mathcal{Y}}^{\cdot}$ being similar). Then the $\tilde{\omega}_{\mathcal{Y}}^i$ are flat W-modules, and $\omega_{\mathcal{Y}}^i/p^n\omega_{\mathcal{Y}}^i \cong \omega_{\mathcal{Y}_n}^i$. Define F on $\omega_{\mathcal{Y}}^i$ by $F = f/p^i$. Then $\mathrm{d}F = p\,\mathrm{d}F$, and all statements of §2 can be deduced from the formula

$$d^{-1}(p^n\omega_{\mathcal{Y}}^{i+1}) = \sum_{k=0}^{n} p^kF^{n-k}\omega_{\mathcal{Y}}^i + \sum_{k=0}^{n-1} F^k\,d\omega_{\mathcal{Y}}^{i-1}, \tag{A}$$

whose proof is literally the same as in [I] 0.2.3.13, using that $\omega_{\mathcal{Y}}^i$ is W-torsion free and that F induces the Cartier isomorphism (2.1.1) modulo p.

Let us show how to deduce the following three essential properties needed in the paper:

(i) $\mathbf{p}$ is injective, and its image coincides with $pW_{n+1}\omega_X^i$ (to define π),

(ii) $W_n\omega_Y^{\cdot}$ is a successive extension of locally free $\mathcal{O}_Y$-modules (needed in Chapter 3),

(iii) $W_{n+1}\omega_Y^{\cdot}/\mathbf{p}W_n\omega_Y^i \xrightarrow{\pi^n} W_1\omega_Y^{\cdot} \xleftarrow{C^{-1}} \omega_Y^{\cdot}$ are quasi-isomorphisms (to prove the finiteness result for the cohomology of $W\omega_Y^{\cdot}$).

Write $[x]_n$ for the class of $x \in d^{-1}(p^n\omega_{\mathcal{Y}}^{i+1})$ in

$$d^{-1}(p^n\omega_{\mathcal{Y}}^{i+1})/(p^n\omega_{\mathcal{Y}}^i + d\omega_{\mathcal{Y}}^{i-1}) \cong \mathcal{H}^i(\omega_{\mathcal{Y}_n}^{\cdot}) \cong W_n\omega_{\mathcal{Y}}^i.$$

For (i) we first observe:

$$(\alpha)\quad \mathrm{Im}(F^n\colon W_{n+1}\omega_Y^i \to W_1\omega_Y^i)$$
$$= \left\{[y]_1 \mid y \in F^{n+1}\omega_{\mathcal{Y}}^i + \sum_{k=0}^{n} F^k d\omega_{\mathcal{Y}}^{i-1} + p\omega_{\mathcal{Y}}^i\right\} \quad \text{(by (A))}$$
$$= Z_{n+1}\omega_Y^i/B_1\omega_Y^i \quad \text{(by induction)},$$

(β) $\operatorname{Ker}(V^n\colon W_1\omega_Y^i \to W_{n+1}\omega_Y^i)$

$$= \left\{[x]_1 \mid p^n x \in p^n \sum_{k=0}^{n} F^{n-k} d\omega_{\mathscr{Y}}^{i-1} + p^{n+1}\omega_{\mathscr{Y}}^i\right\} \quad \text{(by (A))}$$

$$= \left\{[x]_1 \mid x \in \sum_{k=0}^{n} F^{n-k} d\omega_{\mathscr{Y}}^{i-1} + p\omega_{\mathscr{Y}}^i\right\} \quad \text{(torsion-freeness)}$$

$$= B_{n+1}\omega_Y^i / B_1\omega_Y^i \quad \text{(by induction).}$$

Next note that $W_n\omega_Y^i \xrightarrow{V^k} W_{n+k}\omega_Y^i \xrightarrow{F^n} W_k\omega_Y^i$ is exact by definition of F and V. This shows the injectivity of the horizontal maps in the commutative diagram

$$\begin{array}{ccc} G_n^m = V^m W_{n-m}\omega_Y^i / V^{m+1} W_{n-m-1}\omega_Y^i & \xhookrightarrow{F^{n-m-1}} & V^m W_1\omega_Y^i \\ \uparrow\!\!\uparrow V^m & & \uparrow\!\!\uparrow V^m \\ W_{n-m}\omega_Y^i / V W_{n-m-1}\omega_Y^i & \xhookrightarrow{F^{n-m-1}} & W_1\omega_Y^i, \end{array}$$

inducing an isomorphism

$$G_n^m \xrightarrow[\sim]{F^{n-m-1}(V^m)^{-1}} \operatorname{Im} F^{n-m-1} / \operatorname{Ker} V^m$$

i.e., (2.2.1). Then (2.2.2) follows from the commutative diagram

$$\begin{array}{ccc} Z_2\omega_Y^i / B_2\omega_Y^i & \xrightarrow{V^n} & W_{n+1}\omega_Y^i \\ \uparrow C^{-1} & & \uparrow \mathbf{p} \\ W_1\omega_Y^i & \xrightarrow{V^{n-1}} & W_n\omega_Y^i. \end{array}$$

The injectivity of $\mathbf{p}$ is an obvious consequence of (2.2.2), but it is not clear how to deduce $\operatorname{Im}\mathbf{p} = \operatorname{Im} p$ from the information on the graded pieces. However, from (A) and the relation $dFx = pFdx$ we immediately get

$$[z]_{n+1} = \mathbf{p}[x]_n = [pFx]_{n+1} = p[Fx]_{n+1} \quad \text{for } x \in d^{-1}(p^n\omega_{\mathscr{Y}}^{i+1}) \Leftrightarrow$$

$$\Leftrightarrow [z]_{n+1} = p[y]_{n+1} \quad \text{for } y \in d^{-1}(p^{n+1}\omega_{\mathscr{Y}}^{i+1}).$$

In fact, $y \in d^{-1}(p^{n+1}\omega_{\mathscr{Y}}^{i+1})$ implies $y = Fx + p^{n+1}a + db$ with $x \in d^{-1}(p^n\omega_{\mathscr{Y}}^{i+1})$, $a \in \omega_{\mathscr{Y}}^i$ and $b \in \omega_{\mathscr{Y}}^{i-1}$.

(ii) follows from (2.2.1) and the fact that $Z_n\omega_Y^i/Z_{n+1}\omega_Y^i$, $B_{n+1}\omega_Y^i/B_n\omega_Y^i$ and $Z_n\omega_Y^i/B_n\omega_Y^i$ are locally free $\mathcal{O}_Y$-modules (use induction on n and the Cartier isomorphism, and descending induction on i for $Z_1\omega_Y^i=\mathrm{Ker}(\omega_Y^i \twoheadrightarrow B_1\omega_Y^{i+1})$ and $B_1\omega_Y^i=\mathrm{Ker}(Z_1\omega_Y^i \twoheadrightarrow W_1\omega_Y^i)$).

(iii) = (2.4.2) follows from (2.4.1) and hence from the formula (cf. (2.3.1))

$$\mathrm{gr}^n W\omega_Y^i = V^n W_1\omega_Y^i + \mathrm{d}V^n W_1\omega_Y^{i-1}, \tag{B}$$

since (B) implies (2.4.1) with the same arguments as in [I] I 3.13. For (B) note that $\mathrm{gr}^n W\omega_Y^i = \mathrm{Ker}\,\pi = \mathrm{Ker}\,p$, so that one inclusion is obvious. For the other direction write $[z]_{n+1}=[Fx]_{n+1}$ with $x\in d^{-1}(p^n\omega_{\mathscr{Y}}^{i+1})$ by (A). If $0=p[z]_{n+1}=p[Fx]_{n+1}=\mathbf{p}[x]_n$, then $x=p^na+db$ with $a\in\omega_{\mathscr{Y}}^i$ and $b\in\omega_{\mathscr{Y}}^{i-1}$ by the injectivity of $\mathbf{p}$. But then

$$\begin{aligned}[z]_{n+1} &= [Fx]_{n+1} = [p^nFa]_{n+1} + [Fdb]_{n+1} = V^n[Fa]_1 + d[p^nFb]_{n+1}\\ &= V^n[Fa]_1 + \mathrm{d}V^n[Fb]_1\end{aligned}$$

by the definitions of V, F and d on $W_\cdot\omega_Y^\cdot$ (note that $p^{n+1}F\,db=d(p^nFb)$).

(12) This holds since $\varprojlim$ is exact on the category of W-modules of finite length.

(13) Note that

$$W_n\Omega_U^{m+1}(\log Y)=W_n\Omega_U^m(\log Y)\wedge \mathrm{d}T/T \quad \text{and} \quad W_n\Omega_U^{m+1}(-\log Y)=W_n\Omega_U^{m+1},$$

by using the exactness of

$$\cdots \xrightarrow{\wedge \mathrm{d}T/T} \Omega_{\mathscr{U}_n}^\cdot(\log \mathscr{Y}) \xrightarrow{\wedge \mathrm{d}T/T} \Omega_{\mathscr{U}_n}^{\cdot+1}(\log \mathscr{Y}) \xrightarrow{\wedge \mathrm{d}T/T} \cdots,$$

the obvious identity $\Omega_{\mathscr{U}_n}^{m+1}(-\log \mathscr{Y})=\Omega_{\mathscr{U}_n}^{m+1}$, and arguments similar to those applied in (6) above.

(14) One has to use that the de Rham-Witt complex with logarithmic singularities $W_\cdot\Omega_U^\cdot(\log Y)$ satisfies a similar formalism as the usual de Rham-Witt complex $W_\cdot\Omega_V^\cdot$, including the analogues of the sequences [Ek] (0.5) and (0.6). This can be proved with the same methods as for $W_\cdot\omega_Y^\cdot$.

(15) Let j_n: $W_n(U)\to W_{n+1}(U)$ be the canonical morphism. One has to know that the morphism $W_n\Omega_V^{m+1}(\log Y)\to j_n^! W_{n+1}\Omega_U^{m+1}(\log Y)$ adjoint to $\mathbf{p}$ is an isomorphism. This follows from the corresponding facts for $W_\cdot\Omega_U^{m+1}$ and $W_\cdot\omega_Y^m$, which follow from [Ek] and (3.1). (Note that for the proof of (3.1) one only needs (3.3.1)).