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Some results on unipotent orbital integrals

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0. Introduction

Let $\mathbf{G}$ be a reductive algebraic group defined over a local field F of characteristic 0. A major question in harmonic analysis on $\mathbf{G}(F)$, as is evident from the works of Harish–Chandra, Langlands and others, is to understand orbital integrals. When F is a p-adic field, unipotent orbital integrals influence a general orbital integral through a germ expansion described by Shalika. The purpose of this paper is to present some new results on unipotent orbital integrals.

A current important problem is to stabilize the trace formula. This requires the existence of a "transfer" $f \mapsto f^H$ from a suitable space of functions on $\mathbf{G}(F)$ to the corresponding space of functions on $\mathbf{H}(F)$, where $\mathbf{H}$ is an "endoscopic" group of $\mathbf{G}$, such that certain linear combinations of orbital integrals of f, called κ-orbital integrals, are related to "stable" orbital integrals of f^H by a "transfer factor" (see [11] for a discussion of these matters). When $F = \mathbb{R}$, Shelstad [17] proved the existence of a transfer between functions belonging to the Harish–Chandra Schwartz spaces. Clozel and Delorme [4] showed that a transfer exists between compactly supported smooth functions whose left translates by elements of a maximal compact subgroup span a finite dimensional vector space. When F is a p-adic field, a transfer is known to exist only for a very few low dimensional groups. In this case, one could try to study the existence of a transfer $f \mapsto f^H$, $f \in C_c^\infty(\mathbf{G}(F))$, by writing out the Shalika germ expansions for κ-orbital integrals of f and for the stable orbital integrals of f^H, and then try to compare the terms appearing in both sides. Such a comparison requires a map between stable unipotent orbits in $\mathbf{H}(F)$ and stable unipotent orbits in $\mathbf{G}(F)$. A map defined on a subset of the set of stable unipotent orbits in $\mathbf{H}(F)$ has been introduced, independently, by both the author [1] and Hales [6]. This map, called "endoscopic induction" by Hales, is discussed in (1.3). Let $\mathcal{O}_{u_H}^{st}$ denote a stable unipotent class in the domain of endoscopic induction and let $\mathcal{O}_{u_G}^{st}$ denote its image. The existence of a transfer should entail the existence of an identity between linear combinations of orbital integrals of f over the F-classes in $\mathcal{O}_{u_G}^{st}$ and linear combinations of orbital integrals of f^H over the F-classes in $\mathcal{O}_{u_H}^{st}$. In section 1.6, Theorem 1.3, we show that such identities exist when $F=\mathbb{C}$, $\mathbf{G}$ semisimple, and $f \mapsto f^H$ is the transfer described by Clozel and Delorme.

Now, let F be a p-adic field and K be a hyperspecial maximal compact subgroup of $\mathbf{G}(F)$. Denote by $C_c^\infty(\mathbf{G}(F) /\!/ K)$ the corresponding Hecke algebra. The "fundamental lemma", as called by Langlands, conjectures the existence of a transfer between functions in the Hecke algebras. In this case one expects the transfer $f \mapsto f^H$ to be given by the Hecke algebra homomorphisms dual to an admissible embedding $\hat{H} \hookrightarrow \hat{G}$ of Langlands dual groups. In section 3 (resp. section 4), we consider the split p-adic groups $\mathbf{G}=\mathbf{SO}_{2l+1}$ (resp. $\mathbf{G}=\mathbf{Sp}_{2l}$), $\mathbf{H}=\mathbf{SO}_{2l-1} \times \mathbf{PGL}_2$ (resp. $\mathbf{H}=\mathbf{SO}_{2l}$), and prove some identities between linear combinations of orbital integrals of $f (f \in C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$, $\mathcal{O}=$the valuation ring of F) over the F-classes in the nontrivial minimal stable orbit in $\mathbf{G}(F)$ and $f^H(1)$ (see Theorem 3.2 (resp. Theorem 4.3)). We use Rao's formula [16] to compute the unipotent orbital integrals for each function f belonging to a basis of $C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$. The values $f^H(1)$ are computed by using the spherical plancherel formula and Macdonald's explicit formula for the Satake transform [14]. These results provide more evidence to the usefulness of endoscopic induction. We should mention that in [6] Hales showed that endoscopic induction satisfies some other desirable properties.

An interesting problem is to obtain a formula for $\int_{\mathcal{O}_u^{st}} f$, $f \in C_c^\infty(\mathbf{G}(F) /\!/ K)$, generalizing those already known when $\mathcal{O}_u^{st}$ is a Richardson class (see the discussion in section 5). In section 5 we conjecture such a formula when $\mathbf{G}(F)$ is a group of p-adic type in the sense of [14]. The results of sections 3 and 4 confirm the truth of this conjecture for the cases treated there. As pointed out in 5.1, this conjecture has a precise analogue for complex groups where it is true. This conjecture, if true, would lead to more identities between unipotent orbital integrals as is the case for complex groups.

Let $\mathcal{O}_u^{st} = \amalg_{i=1}^n \mathcal{O}_{u_i}$ be a decomposition of the stable class $\mathcal{O}_u^{st}$ into its F-classes. Each F-class $\mathcal{O}_{u_i}$ defines a linear functional on $C_c^\infty(\mathbf{G}(F) /\!/ K)$ by $f \mapsto \int_{\mathcal{O}_{u_i}} f$. Another interesting question is to determine the dimension of $\mathbb{C}$-space spanned by these functionals. This problem seems to play an important role in Hale's current work on "uniform germ expansions". In section 2 we consider the nontrivial minimal "special" stable class in $\mathbf{SO}_{2l+1}(F)$ ("special" is taken here in the sense of Lusztig [3]). This stable class consists of four F-classes. When $l=2$, we get all the subregular classes in $\mathbf{SO}_5(F)$. We calculate the orbital integrals of the basic functions in $C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$ using Rao's formula (the calculations here are much more complicated than those in sections 3 and 4), and we show that the $\mathbb{C}$-space spanned by these four functionals is three dimensional (see Theorem 2.17).

Most of these results appeared in the author's thesis. I would like to express my gratitude and thanks to my advisor, Professor Robert E. Kottwitz, for his help and guidance.

1. Matching of unipotent orbital integrals on complex groups

1.1. *Truncated induction and Macdonald representations*

Let V be a finite dimensional real vector space and let W be a finite Coxeter group acting as a reflection group on V. Let W_0 be a reflection subgroup of W and let $V^{W_0}=\{v\in V: w(v)=v\ \forall w\in W_0\}$. Write $V=V_0\oplus V^{W_0}$ where V_0 is a W_0-module which has no W_0-invariants. For any subspace $V'\subseteq V$ and for any integer $e\geqslant 0$ we denote by $R_e(V')$ the space of homogeneous polynomial functions on V' of degree e. $R_e(V)$ (resp. $R_e(V_0)$) is a W-module (resp. W_0-module) in the natural way.

THEOREM 1.1 [12]. *Suppose U_0 is an absolutely irreducible W_0-submodule of $R_e(V_0)$ which occurs with multiplicity* 1 *in $R_e(V_0)$ and which does not occur in $R_i(V_0)$ if $0\leqslant i<e$. Let U be the W-submodule of $R_e(V)$ generated by U_0. Then*

(i) *U is an irreducible W-module.*
(ii) *U occurs with multiplicity* 1 *in $R_e(V)$.*
(iii) *U does not occur in $R_i(V)$ if $0\leqslant i<e$.*

The process of passing from U_0 to U is called truncated induction and we write $U=J_{W_0}^{W}(U_0)$. When U_0 is the sign *representation of $W_0(\mathrm{sgn}: w\mapsto \det w)$, U is called a Macdonald representation.*

1.2. *The Springer correspondence*

Let $\mathbf{G}$ be any connected reductive linear algebraic group over $\mathbb{C}$. For each unipotent element $u\in G=\mathbf{G}(\mathbb{C})$ consider the projective variety $\mathscr{B}_u$ of all Borels containing u. Let $Z_G(u)$ denote the centralizer of u in G ($Z_G^0(u)$ is the identity component of $Z_G(u)$) and set $A(u)=Z_G(u)/Z_G^0(u)$. $\mathscr{B}_u$ has dimension $e(u)=\frac{1}{2}(\dim Z_G(u)-\operatorname{rank} G)$, and each irreducible component of $\mathscr{B}_u$ has dimension $e(u)$. The number of such components is equal to $\dim_{\mathbb{Q}} H^{2e(u)}(\mathscr{B}_u,\mathbb{Q})$ and there is a basis of this space in natural bijective correspondence with the irreducible components of $\mathscr{B}_u\cdot A(u)$ is a finite group which acts on $H^{2e(u)}(\mathscr{B}_u,\mathbb{Q})$. In fact $Z_G(u)$ acts on $\mathscr{B}_u$ by conjugation and $Z_G^0(u)$ fixes each irreducible component of $\mathscr{B}_u$. Thus $A(u)$ acts on the set of irreducible components of $\mathscr{B}_u$ and hence on $H^{2e(u)}(\mathscr{B}_u,\mathbb{Q})$. In [19], Springer defined an action of the Weyl group W of $\mathbf{G}$ on $H^i(\mathscr{B}_u,\mathbb{Q})$ $(0\leqslant i\leqslant 2e(u))$. On $H^{2e(u)}(\mathscr{B}_u,\mathbb{Q})$, this action commutes with the $A(u)$ action. Let $\chi\in\hat{A}(u)$ and let $V_{u,\chi}$ denote the χ-isotypical subspace of $H^{2e(u)}(\mathscr{B}_u,\mathbb{Q})$. If $V_{u,\chi}\neq(0)$, then it is a direct sum of equivalent irreducible W-modules. Each irreducible W-module is obtained in this way. $V_{u,\chi}$ depends (up to equivalence) only on the conjugacy class $\mathcal{O}_u$ of u. Not all pairs $(\mathcal{O}_u,\chi)$ occur, but all pairs $(\mathcal{O}_u,1)$ where 1 denotes the trivial representation of $A(u)$ occur. The injective map from all pairs $(\mathcal{O}_u, \chi)$ which occur to the corresponding irreducible W-modules is called the *Springer correspondence.*

1.3. *Endoscopic induction of unipotent conjugacy classes*

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{C}$. Let $\mathbf{H}$ be an endoscopic group of $\mathbf{G}$. Denote by $\mathcal{U}(\mathbf{G})$ (resp. $\mathcal{U}(\mathbf{H})$) the set of unipotent conjugacy classes of $\mathbf{G}$(resp. $\mathbf{H}$). Endoscopic induction is a partial map from $\mathcal{U}(\mathbf{H})$ to $\mathcal{U}(\mathbf{G})$. The domain of this map contains the conjugacy classes of all special unipotent elements (in the sense of Lusztig [3]) in $\mathbf{H}$. As mentioned in the introduction, endoscopic induction has been introduced, independently, by both the author [1] and Hales [6]. Hales defined endoscopic induction when $\mathbf{G}$ is any complex reductive group over $\mathbb{C}$ and $\mathbf{H}$ is any endoscopic group. He proved that this map satisfies some properties that are useful in the study of the Shalika germ expansion of κ-orbital integrals in p-adic groups. The author defined endoscopic induction on special unipotent classes when $\mathbf{G}$ is a classical complex group and $\mathbf{H}$ an elliptic endoscopic group, and used this map to match unipotent orbital integrals. The general definition of endoscopic induction is as follows. Let $\mathcal{O}_{u_H}$ be a unipotent class in $\mathbf{H}$. By the Springer correspondence, the pair $(\mathcal{O}_{u_H}, 1)$ is associated to an irreducible representation σ of $W(\mathbf{H})$ (the Weyl group of $\mathbf{H}$). The dual group $\hat{H}$ (see Langlands [11]) of $\mathbf{H}$ is the centralizer of a semisimple element in the dual group $\hat{G}$ of $\mathbf{G}$. $\hat{H}$ and $\hat{G}$ are both connected and the Weyl group $W(\hat{H})$ of $\hat{H}$ can be identified with a reflection subgroup of the Weyl group $W(\hat{G})$ of $\hat{G}$. On the other hand $W(\mathbf{H})$ and $W(\mathbf{G})$ can be identified with $W(\hat{H})$ and $W(\hat{G})$ respectively. Using truncated induction σ gives rise to an irreducible representation ρ of $W(\mathbf{G})$. If ρ corresponds to a pair $(\mathcal{O}_{u_G}, 1)$ for some unipotent element $u_G \in \mathbf{G}$, then (and only then) $\mathcal{O}_{u_H}$ is in the domain of endoscopic induction and $\mathcal{O}_{u_G}$ is its image.

1.4. *Unipotent orbital integrals on complex groups*

Let $\mathbf{G}$ be a complex connected semi-simple Lie group and $\mathbf{T}$ a fixed Cartan subgroup. Let $\mathfrak{g} = \text{Lie}(\mathbf{G})$ and $\mathfrak{t} = \text{Lie}(\mathbf{T})$. Fix a simple system for the root system of $(\mathfrak{g}, \mathfrak{t})$ and let D_T be the corresponding usual Weyl denominator. The Harish–Chandra transform of a function $f \in C_c^\infty(G)$ is given by: $F_f^T(t) = D_T(t) \int_{G/T} f(gtg^{-1})\, dg (t \in T_{\text{reg}})$. Regard $\mathfrak{g}$ and $\mathfrak{t}$ as real Lie algebras and consider their respective complexifications $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$. Then $\mathfrak{t}_{\mathbb{C}} \simeq \mathfrak{t} \times \mathfrak{t}$ and $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \simeq W_0 \times W_0$, where W_0 is the Weyl group of $(\mathfrak{g}, \mathfrak{t})$. Suppose $u \in G$ is a unipotent element and σ is the irreducible representation of W_0 attached to the pair $(\mathcal{O}_u, 1)$ by the Springer correspondence. Let $P_u \in S(\mathfrak{t}_{\mathbb{C}})$ be a symmetric polynomial of lowest degree satisfying:

(i) P_u is invariant by the diagonal $W_0 \subseteq W_0 \times W_0$.
(ii) P_u transforms by $\sigma \otimes \bar{\sigma}$ under $W_0 \times W_0$.

Let ∂_u be the linear differential operator on **T** dual to the polynomial P_u.

THEOREM 1.2 ([2], [5]). *For an appropriate normalization of measures we have*

$$[\partial_u F_f^T](1) = \int_{\mathcal{O}_u} f \in C_c^\infty(G).$$

1.5. *Matching of unipotent orbital integrals on complex groups*

First, we briefly review some results of Clozel–Delorme and Shelstad. Let **G** be a connected reductive algebraic group defined over $\mathbb{R}$. Let $\gamma \in \mathbf{G}(\mathbb{R})$ be strongly regular and set $\Phi(\gamma, f) = \int_{\mathcal{O}_\gamma} f, f \in \mathcal{S}(\mathbf{G}(\mathbb{R}))$ (the Harish–Chandra Schwartz space on $\mathbf{G}(\mathbb{R})$). The stable conjugacy class $\mathcal{O}_\gamma^{st}$ of $\gamma := \{g\gamma g^{-1} : g \in \mathbf{G}(\mathbb{C})\} \cap \mathbf{G}(\mathbb{R})$ is a finite union of conjugacy classes $\mathcal{O}_{\gamma_i}$ in $\mathbf{G}(\mathbb{R})$. The stable orbital integral of f over $\mathcal{O}_\gamma^{st}$ is given by: $\Phi^{st}(\gamma, f) = \Sigma_i \Phi(\gamma_i, f)$. Let **H** be an endoscopic group of **G**, defined over $\mathbb{R}$. In [17] a map is described which associates to each stable class $\mathcal{O}_{\gamma_H}^{st}$ in $\mathbf{H}(\mathbb{R})$ consisting of **G**-regular elements, a (possibly empty) stable regular class $\mathcal{O}_\gamma^{st}$ in $\mathbf{G}(\mathbb{R})$. Also, a complex valued function $\Delta(\gamma_H, \gamma)$ (called a transfer factor) is defined, where γ_H is strongly **G**-regular in $\mathbf{H}(\mathbb{R})$ and γ is strongly regular in $\mathbf{G}(\mathbb{R})$. The value of this function depends only on the conjugacy class of γ in $\mathbf{G}(\mathbb{R})$ and the stable conjugacy class of γ_H in $\mathbf{H}(\mathbb{R})$. Moreover, $\Delta(\gamma_H, \gamma) = 0$ unless $\mathcal{O}_\gamma^{st}$ is associated to $\mathcal{O}_{\gamma_H}^{st}$. Shelstad [17] proved that for each $f \in \mathcal{S}(\mathbf{G}(\mathbb{R}))$, there exists $f^H \in \mathcal{S}(\mathbf{H}(\mathbb{R}))$ such that $\Phi^{st}(\gamma_H, f^H) = \Sigma_\gamma \Delta(\gamma_H, \gamma)\Phi(\gamma, f)$, where the sum is over all conjugacy classes $\mathcal{O}_\gamma$ contained in $\mathcal{O}_\gamma^{st}$. In [4] Clozel and Delorme show that the correspondence $f \mapsto f^H$ holds between compactly supported smooth functions which are K-finite (K is a maximal compact subgroup). We are concerned with the case where **G** is a complex connected semi-simple algebraic group and **H** an endoscopic group for G. We may assume, by restriction of scalars, that both groups are defined over $\mathbb{R}$. In this case the above picture becomes quite simple. Conjugacy is the same as stable conjugacy and the transfer factor is given by: $\Delta(\gamma, \gamma_H) = D_{T_G}(\gamma)/D_{T_H}(\gamma_H)$ if $\mathcal{O}_\gamma^{st}$ is associated with $\mathcal{O}_{\gamma_H}^{st}$ and is zero otherwise ($\mathbf{T}_G$ and $\mathbf{T}_H$ are maximal tori in **G** and **H**, respectively). Moreover, we have: $F_f^{T_G}(\gamma) = F_{f^H}^{T_H}(\gamma_H)$.

THEOREM 1.3. *Assume u_H is a unipotent element in H, whose conjugacy class $\mathcal{O}_{u_H}$ is the domain of endoscopic induction. Let $\mathcal{O}_{u_G}$ denote the image of $\mathcal{O}_{u_H}$. Then*

$$\int_{\mathcal{O}_{u_H}} f^H = \text{const} \int_{\mathcal{O}_{u_G}} f$$

where $f \mapsto f^H$ is the correspondence described by Clozel and Delorme. In order to prove the theorem, we need the following lemma.

LEMMA 1.4. *Let P_{u_H} and P_{u_G} be two polynomials corresponding to $\mathcal{O}_{u_H}$ and $\mathcal{O}_{u_G}$ respectively, as described by the theorem in* (1.4). *Then there exists a non-zero constant $c \in \mathbb{C}$ such that:*

$$\sum_{w \in W(G)} w \cdot P_{u_H} = cP_{u_G}$$

Proof. Let U denote the irreducible $W(H)$-module corresponding to $(\mathcal{O}_{u_H}, 1)$ via the Springer correspondence. Then $J_{W(H)}^{W(G)}(U)$ is the irreducible $W(G)$-module corresponding to $(\mathcal{O}_{u_G}, 1)$, by the definition of endoscopic induction. By the theorem in (1.4), it is enough to check that $P := \Sigma_{w \in W(G)} w \cdot P_{u_H}$ satisfies the following conditions.

(i) P is invariant by the diagonal $W(G) \times W(G)$.
(ii) $P \in J_{W(H)}^{W(G)}(U) \otimes J_{W(H)}^{W(G)}(\bar{U})$.
(iii) P is non-zero.
(iv) P is a symmetric polynomial of least degree satisfying (i)–(iii).

(i) is clear. (ii) Follows from the definition of truncated induction. To prove (iii), observe that we may assume that $P_{u_H} \geqslant 0$. Indeed, let $Q(z_1, \dots, z_n)$ be a non-zero element of U with real coefficients. Then

$$\sum_{w \in W(H)} (w \cdot Q(z_1, \dots, z_n))(w \cdot Q(\bar{z}_1, \dots, \bar{z}_n)) = \sum_{w \in W(H)} |w \cdot Q(z_1, \dots, z_n)|^2 \geqslant 0,$$

and satisfies all the conditions of Theorem 1.2, with $W_0 = W(H)$. But then it is clear that $0 \not\equiv \Sigma_{w \in W(G)} w \cdot P_u \geqslant 0$. (iv) is a consequence of Theorem 1.1. □

Proof of Theorem. Let $\mathbf{T}_H$ and $\mathbf{T}_G$ be two maximal tori in $\mathbf{H}$ and $\mathbf{G}$ respectively. Identify $\mathbf{T}_H$ with $\mathbf{T}_G$ as in [17]. The above lemma implies that the $W(G)$ symmetrization of ∂_{u_H} is a constant multiple of ∂_{u_G}. Thus

$$\begin{aligned}
\int_{\mathcal{O}_{u_G}} f &= [\partial_{u_G}(F_f^{T_G})](1) && \text{(by Theorem 1.2.))} \\
&= c'\left[\sum_{w \in W_{(G)}} w \cdot \partial_{u_H}(F_f^{T_G})\right](1) \\
&= c[\partial_{u_H}(F_f^{T_G})](1) && \text{(since } F_f^{T_G} \text{ is } W(G)\text{-symmetric)} \\
&= c[\partial_{u_H}(F_{f^H}^{T_H})](1) && (F_f^{T_G} = F_{f^H}^{T_H}) \\
&= c\int_{\mathcal{O}_{u_H}} f^H && \text{(by Theorem 1.2)}
\end{aligned}$$

□

2. The minimal non-trivial special unipotent conjugacy classes in $\mathbf{SO}_{2l+1}(F)$

Let F be a p-adic field with odd residual characteristic, and algebraic closure $\bar{F}$. Let $\mathcal{O}$ denote the ring of integers of F, π a uniformizer of F and $\mathcal{P}=\pi\mathcal{O}$. Let $q=\#(\mathcal{O}/\mathcal{P})$ and $|\ | =$ the absolute value of F, normalized by $|\pi| = q^{-1}$. Let

$$\mathbf{G} = \mathbf{SO}_{2l+1} = \{g \in \mathbf{SL}_{2l+1} : {}^t g J_l g = J_l\} \text{ where } J_l = \begin{bmatrix} 1 & & \\ & & I_l \\ & I_l & \end{bmatrix} \text{ and } I_l = l \times l$$

identity matrix.

Let $u_G \in \mathbf{G}(F)$ be a unipotent element whose $\mathbf{G}(\bar{F})$ conjugacy class in $\mathbf{G}(\bar{F})$ is the non-trivial special class of lowest dimension. The elementary divisors of u_G correspond to the partition 31^{2l-2}. There are four $\mathbf{G}(F)$ conjugacy classes contained in the set of F-rational points on the $\mathbf{G}(\bar{F})$ conjugacy class of u_G, corresponding to the four square classes $F^\times/(F^\times)^2$. When $l=2$ we get the subregular classes. Let $K = \mathbf{G}(\mathcal{O})$. K is a maximal compact subgroup of $\mathbf{G}(F)$. $C_C^\infty(G(F) /\!/ K)$ will denote K-spherical Hecke algebra, which by definition, consists of all locally constant, compactly supported, K bi-invariant complex valued functions on $\mathbf{G}(F)$. A basis (over $\mathbb{C}$) for this algebra is given by the functions $f_{(m_1,\ldots,m_l)}$, $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_l \geqslant 0$, where $f_{(m_1,\ldots,m_l)} :=$ characteristic function of the double coset K diag $(1, \pi^{m_1},\ldots, \pi^{m_l}, \pi^{-m_1},\ldots, \pi^{-m_l})K$. In this section we compute the integrals of these basic elements over the four $\mathbf{G}(F)$ unipotent conjugacy classes mentioned above. Each of the four orbital integrals is a linear functional on the Hecke algebra. The main consequence of our computations is that the $\mathbb{C}$-space spanned by these four functionals is only three dimensional. In fact, two of the conjugacy classes will give rise to one and the same functional.

2.1. *A review of some of Rao's results*

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over F and $G=\mathbf{G}(F)$ its set of F-rational points. G is then locally compact in the Hausdorff topology it inherits as a subgroup of $\mathbf{GL}_n(F)$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $X_0 \in \mathfrak{g}$ a nilpotent element. Then by the Jacobson–Morozov theorem, there exists $H_0, Y_0 \in \mathfrak{g}$ such that

$$[H_0, X_0] = 2X_0,\ [H_0, Y_0] = -2Y_0,\ [X_0, Y_0] = H_0$$

For each integer μ, let $\mathfrak{g}_\mu = \{X \in \mathfrak{g} : [H_0, X] = \mu X\}$ and let $\mathfrak{p}_0 = \Sigma_{\mu \geqslant 0}\, \mathfrak{g}_\mu$, $\mathfrak{n}_\lambda = \Sigma_{\mu > \lambda}\, \mathfrak{g}_\mu$.

Then $\mathfrak{p}_0$ is a parabolic subalgebra and $\mathfrak{n}_0$ is its nilradical. Let $\mathbf{P}_0$ be the corresponding parabolic subgroup of $\mathbf{G}$ and $\mathbf{N}_0$ its nilradical. Set $P_0 = G \cap \mathbf{P}_0$, $\mathrm{N}_0 = G \cap \mathbf{N}_0$. Then P_0 and N_0 are closed subgroups of G and have $\mathfrak{p}_0$ and $\mathfrak{n}_0$ respectively as their Lie algebras.

Let M_0 be the centralizer of H_0 in G. Then $P_0 = M_0 \cdot N_0$. Moreover, according to a theorem of Bruhat–Tits, there exists a compact subgroup K of G such that $G = K \cdot P_0$.

Each of the subspaces $\mathfrak{g}_\mu$ is M_0-stable and the map $m \mapsto \mathrm{Ad}\, m(X_0)$ is an analytic map of M_0 into $\mathfrak{g}_2$ which is submersive.

Thus $V_0 = \mathrm{Ad}\, M_0(X_0) = \{\mathrm{Ad}\, m(X_0) : m \in M_0\}$ is open in the Hausdorff topology of $\mathfrak{g}_2$.

A lemma of Rao [16] shows that the G-orbit of X_0 is $\mathrm{Ad}\, K(V_0 + \mathfrak{n}_2)$. Let $Z_1, \ldots, Z_r$ and $Z'_1, \ldots, Z'_r$ be bases for $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ respectively, such that $B(Z_i, Z'_j) = \delta_{ij}$, where $B(\ ,\ )$ is the Killing form on $\mathfrak{g}$.

For $X \in \mathfrak{g}_2$, let $[X, Z'_i] = \Sigma_{j=1}^{r} c_{ji}(X) Z_j$ and $\varphi(X) = |\det(c_{ij})|^{1/2}$. Set G_{X_0} = centralizer of X_0 in G and let dx^* denote a G-invariant measure on G/G_{X_0}.

The following result is due to Rao [16].

THEOREM 2.1. *There exists a constant c such that for all $f \in C_c(\mathfrak{g})$ the following holds*:

$$\int_{G/G_{X_0}} f(\mathrm{Ad}\, x^*(X_0))\, \mathrm{d}x^* = c \int_{V_0 + \mathfrak{n}_2} \varphi(X) \bar{f}(X + Z)\, \mathrm{d}X\, \mathrm{d}Z$$

where $\mathrm{d}X$(resp. $\mathrm{d}Z$) *denote the usual Euclidean measure on* $\mathfrak{g}_2$(resp. $\mathfrak{n}_2$), *and* $\bar{f}(Y) = \int_K f(\mathrm{Ad}\, k(Y))\, \mathrm{d}k$.

The formula described in this theorem will be one of our main computational tools.

2.2. *Some Rao data*

2.2.1. Let $\mathbf{G} = \mathbf{SO}_{2l+1}$ and let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G}(F)) \cdot \mathfrak{g} = \{X \in \mathbf{M}_{2l+1}(F) : {}^t X J_l + J_l X = 0\}$. Consider the following Lie triple.

$$X_0 = E_{1,2} - E_{l+2,1},\ Y_0 = 2E_{2,1} - 2E_{1,l+2},\ H_0 = 2E_{l+2,l+2} - 2E_{2,2}$$

Let $u_G := \exp X_0 = I_{2l+1} + X_0 + \frac{1}{2} X_0^2$. u_G has all the properties mentioned in the introduction to this section. Elementary matrix computations give: $\mathfrak{g}_{-1} = \mathfrak{g}_1 = \mathfrak{n}_2 = (0)$ and

$$\mathfrak{g}_2 = \left\{ \begin{bmatrix} & & z & & \\ & & 0 & & \\ & & x_1 & & \\ & & \vdots & & \\ & & x_{l-1} & & \\ -z & & 0 & y_1 \cdots y_{l-1} 0 - x_1 \cdots - x_{l-1} \\ & & -y_1 & & \\ & & \vdots & & \\ & & -y_{l-1} & & \end{bmatrix} : z, x_1, \ldots, x_{l-1}, y_1, \ldots, y_{l-1} \in F \right\}$$

Thus Rao's φ-function $\equiv 1$ on $\mathfrak{g}_2$.

The open M_0-open orbits in $\mathfrak{g}_2$ have been calculated in [1]. They are denoted by $V_k (1 \leqslant k \leqslant 4)$ and are as follows. Let $\alpha_1 = 1, \alpha_2 = \varepsilon, \alpha_3 = \pi, \alpha_4 = \varepsilon\pi$, where ε is an element in $\mathcal{O}^\times$ of order $q-1$. Then $V_k = \{X \in \mathfrak{g}_2 : \Sigma_{i=1}^{l-1} 2x_i y_i - z^2 \equiv \alpha_k \operatorname{mod}(F^\times)^2\}$.

2.2.2. Let f be a spherical function with respect to K. Let $X_k \in \mathfrak{g}_2$ $(1 \leqslant k \leqslant 4)$ be representatives of the four M_0 open orbits in $\mathfrak{g}_2$ and let $u_k = \exp X_k$. The four classes $\mathcal{O}_{u_k}$ are equipped with compatible $\mathbf{G}(F)$-invariant measures as in [20]. Using Rao's formula we obtain:

$$\int_{\mathcal{O}_{u_k}} f = c_{u_k} \int_{V_k} (f \circ \exp)(X)\, \mathrm{d}X$$

where $c_{u_k} \in \mathbb{C}$ depends only on $\mathcal{O}_{u_k}$. When $f = f_{(m_1, \ldots, m_l)}$, the integral on the right is simply the measure of the set $S_{(m_1, \ldots, m_l; \alpha_k)} := V_k \cap \operatorname{supp}(f_{(m_1, \ldots, m_l)} \circ \exp)$, $(1 \leqslant k \leqslant 4,\ m_1 \geqslant \cdots m_l \geqslant 0)$.

For an element $g \in \mathbf{G}(F)$, let $\| \wedge^n g \|$ denote the maximum of the absolute values of all $n \times n$ subdeterminants of g. Notice that $\| \wedge^n (k_1 g k_2) \| = \| \wedge^n g \|$ for all k_1, $k_2 \in K$. Let $q^{-r_n(g)} := \| \wedge^n g \|$. It is clear that $g \in \operatorname{supp}(f_{(m_1, \ldots, m_l)})$ if and only if:

$$r_n(g) = -m_1 - \cdots - m_l \qquad (1 \leqslant n \leqslant l) \tag{*}$$

Thus $X \in S_{(m_1, \ldots, m_l; \alpha_k)}$ if and only if $X \in V_k$ and $\exp(X)$ satisfies (*).

The sets $S_{(m_1, \ldots, m_l; \alpha_k)}$ are computed in [1]. They are as follows. $S_{(m_1, \ldots, m_l; \alpha_k)} = \varnothing$ unless $m_3 = \cdots = m_l = 0$. To simplify the notation we write $S_{(m_1, m_2; \alpha_k)}$ instead of $S_{(m_1, m_2, 0, \ldots, 0; \alpha_k)}$. For $X \in \mathfrak{g}_2$, let $d(X) = d := 2\Sigma_{i=1}^{l-1} x_i y_i - z^2$. For $x \in F$, let $\operatorname{val}(x)$ denote the valuation of x. Consider the following cases.

(1) $m_1 = m_2 = 0$

$$S_{(0,0;\alpha_k)} = \{X \in \mathfrak{g}_2 : \operatorname{val}(x_i) \geqslant 0,\ \operatorname{val}(y_i) \geqslant 0,$$
$$\operatorname{val}(z) \geqslant 0 (1 \leqslant i \leqslant l-1),\ d \equiv \alpha_k \operatorname{mod}(F^\times)^2\}$$

(2) $m_1 = m_2 = m > 0$

$$S_{(m,m;\alpha_k)} = \{X \in \mathfrak{g}_2 : \mathrm{val}(x_i) \geqslant -m, \mathrm{val}(y_i) \geqslant -m, \mathrm{val}(z) \geqslant -m (1 \leqslant i \leqslant l-1), \text{ with at least one equality, } \mathrm{val}(d) \geqslant -m, d \equiv \alpha_k \bmod (F^\times)^2\}$$

(3) $m_1 = m > 0$, *m even and* $m_2 = 0$

$$S_{(m,0;\alpha_k)} = \left\{X \in \mathfrak{g}_2 : \mathrm{val}(x_i) \geqslant -\frac{m}{2}, \mathrm{val}(y_i) \geqslant -\frac{m}{2}, \mathrm{val}(z) \geqslant -\frac{m}{2} (1 \leqslant i \leqslant l-1), \mathrm{val}(d) = -m, d \equiv \alpha_k \bmod (F^\times)^2\right\}$$

(4) $m_1 > m_2 > 0$, *m_1 and m_2 have the same parity*

$$S_{(m_1,m_2;\alpha_k)} = \left\{X \in \mathfrak{g}_2 : \mathrm{val}(x_i) \geqslant -\frac{m_1 - m_2}{2}, \mathrm{val}(y_i) \geqslant -\frac{m_1 - m_2}{2}, \mathrm{val}(z) \geqslant -\frac{m_1 - m_2}{2} (1 \leqslant i \leqslant l-1), \text{ with at least one equality, } \mathrm{val}(d) = -m_1, d \equiv \alpha_k \bmod (F^\times)^2\right\}$$

(5) $m_1 > m_2 > 0$, *m_1 and m_2 have different parities*

$$S_{(m_1,m_2;\alpha_k)} = \varnothing.$$

2.3. *An integral formula*

Let $f(x) = f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$. The critical set C_f of f is, by definition, the set $\{x \in F^n : \nabla f(x) = 0\}$, where as usual $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

Let $i \in F$. Set $U(i) = f^{-1}(i) \backslash C_f$. Thus $a \in U(i) \Leftrightarrow f(a) = i$ and $\partial f/\partial x_k(a) \neq 0$ for some $1 \leqslant k \leqslant n$.

The $(n-1)$-form $\theta_i(x) = (-1)^{k-1}(\partial f/\partial x_k)^{-1} \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_k} \wedge \cdots \wedge \mathrm{d}x_n|_{U(i)}$ is a well defined, non-vanishing, regular form around $a \in U(i)$ and thereby giving rise to a global regular and non-vanishing $(n-1)$ form on $U(i)$ which we denote by θ_i. θ_i induces on $U(i)$ a Borel measure $|\theta_i|$ such that for every continuous function Φ on F^n with a compact support which is disjoint from C_f, we have:

$$\int_{F^n} \Phi(x) |\mathrm{d}x| = \int_F \left(\int_{U(i)} \Phi |\theta_i| \right) |\mathrm{d}i| \qquad (*)$$

Furthermore, for every such Φ, the function $F_\Phi(i) = \int_{U(i)} \Phi |\theta_i|$ is continuous on F.

Suppose now that Φ is a locally constant, compactly supported function on F^n satisfying the condition: $\operatorname{supp}\Phi \cap C_f \subset f^{-1}(0)$.

Then Theorem 1.6, p. 81 in [8] states (among other things) that F_Φ is locally constant on $F^\times$.

Now, suppose that $V \subseteq F^\times$ is a bounded open set and $Y \subseteq F^n$ a compact open set. Let $f \in \mathcal{O}[x_1, \ldots, x_n]$ be homogeneous of degree k. Set $\Phi = 1_Y$ = characteristic function of Y. We wish to obtain a "practical" formula for the integral of Φ over $f^{-1}(V)$, i.e., for $\operatorname{meas}(Y \cap f^{-1}(V))$. For an integer r, let $\pi^r Y = \{(\pi^r y_1, \ldots, \pi^r y) : (y_1, \ldots, y_n) \in Y\}$. Choose $a \in \mathbb{N}$, such that $\pi^a Y \subseteq (\mathcal{O})^n$. Next, for $i \in V$ and $e \in \mathbb{Z}^+$, define

$$N(e, Y)(i) = \#\{(z_1, \ldots, z_n) \in \overline{\pi^a Y} : \bar{f}(z_1, \ldots, z_l) = \overline{\pi^{ka} i}\}$$

where the overbars indicate reduction modulo $\mathcal{P}^{e+ka}$. $N(e, Y)(i)$ does not depend on a.

PROPOSITION 2.2

$$\int_{f^{-1}(V)} \Phi(x)|dx| = \operatorname{meas}(Y \cap f^{-1}(V))$$

$$= \int_V \left[\lim_{e \to \infty} q^{-na(k-1)} q^{-(n-1)e} N_{(e,Y)}(i)\right] |di|.$$

Proof

$$\int_{f^{-1}(V)} \Phi(x)|dx| = \int_{F^n} \Phi(x)|dx| = \int_F F_\Phi(i)|di| = \int_V F_\Phi(i)|di| \tag{1}$$

Since F_Φ is locally constant on $F^\times$, there exists a positive integer e such that:

(i) $i + \mathcal{P}^e \subseteq V$.

(ii) $F_\Phi(t) = F_\Phi(i)$, for all $t \in i + \mathcal{P}^e$.

Now, let φ denote the characteristic function of the compact open set $Y \cap f^{-1}(i + \mathcal{P}^e)$. The right-hand side of formula (*), when applied to φ gives:

$$\int_F \left(\int_{U(t)} \varphi|\theta_t|\right)|dt| = \int_{i+\mathcal{P}^e} \left(\int_{U(t) \cap Y} |\theta_t|\right)|dt|$$

$$= \int_{i+\mathcal{P}^e} \left(\int_{U(t)} \Phi|\theta_t|\right)|dt| = \int_{i+\mathcal{P}^e} F_\Phi(t)|dt|$$

$$= F_\Phi(i) q^{-e} \tag{2}$$

(The last identity follows from assumption (ii) above.) On the other hand, the left-hand side of formula (*), when applied to φ gives:

$$\int_{F^n} \varphi(x)|dx| = \text{meas}(Y \cap f^{-1}(i + \mathscr{P}^e))$$

$$= \text{meas}(\{(y_1 \ldots, y_n) \in Y : f(y_1, \ldots, y_n) \in i + \mathscr{P}^e\})$$

$$= q^{na}\,\text{meas}(\{(y_1, \ldots, y_n) \in Y_a : f(y_1, \ldots, y_n) \in \pi^{ka} i + \mathscr{P}^{e+ka}\})$$

Now, for e large enough:

$$\text{meas}(\{(y_1, \ldots, y_n) \in Y_a : f(y_1, \ldots, y_n) \in \pi^{ka} i + \mathscr{P}^{e+ka}\}) = q^{-n(e+ka)} N_{(e,Y)}(i) \quad (3)$$

(3) implies

$$\int_{F^n} \varphi(x)|dx| = q^{na} q^{-n(e+ka)} N_{(e,Y)}(i) \quad (4)$$

(2) and (4) imply that for large enough e:

$$F_\Phi(i) q^{-e} = q^{na} q^{-n(e+ka)} N_{(e,Y)}(i)$$

i.e.

$$F_\Phi(i) = \lim_{e \to \infty} [q^{-na(k-1)} q^{-(n-1)e} N_{(e,Y)}(i)] \quad (5)$$

The proposition follows from (1) and (5). □

2.4. *The integrals* $\int_{\mathcal{O}_{u_k}} f_{(m_1, \ldots, m_l)}$

2.4.1. *The functions* $N(e, Y_{(m_1,m_2)})(i)$

Let $g \in \mathcal{O}[x_1, \ldots, x_{l-1}, y_1, \ldots, y_{l-1}, z]$ be defined by:

$$g(x_1, \ldots, x_{l-1}, y_1, \ldots, y_{l-1}, z) = 2 \sum_{i=1}^{l-1} x_i y_i - z^2.$$

Notice that g is homogeneous of degree 2. For $m_1 \geqslant m_2 \geqslant 0$, set $Y_{(m_1,m_2)} = \text{supp}(f_{(m_1,m_2,0,\ldots,0)} \circ \exp) \cap \mathfrak{g}_2$.

Also, define $V_{(m_1,m_2)} \subseteq F$ as follows:

(1) $V_{(0,0)} = \mathcal{O}$.
(2) If $m > 0$, set $V_{(m,m)} = \mathscr{P}^{-m}$.

(3) If $m > 0$ and even, set $V_{(m,0)} = \mathscr{P}^{-m} - \mathscr{P}^{-m+1}$.
(4) If $m_1 > m_2 > 0$, m_1 and m_2 have the same parity; set $V_{(m_1,m_2)} = \mathscr{P}^{-m_1} - \mathscr{P}^{-m+1}$.
(5) If $m_1 > m_2 \geqslant 0$, m_1 and m_2 have different parities; set $V_{(m_1,m_2)} = \varphi$.

Then $S_{(m_1,m_2;\alpha_k)} = g^{-1}(\alpha_k(F^\times)^2 \cap V_{(m_1,m_2)}) \cap Y_{(m_1,m_2)}$ and the integral formula in (2.3) applies.

Next, we compute the functions $N(e, Y_{(m_1,m_2)})$.

Let $X = (x_1, \dots, x_{l-1}, y_1, \dots, y_{l-1}, z)$ denote a vector in F^{2l-1}. Suppose $i \in F$ and $\mathrm{val}(i) = n \geqslant 0$. For each $r \geqslant 0$, let $Y_r := \{X : \mathrm{val}(x_j) \geqslant r,\ \mathrm{val}(y_j) \geqslant r,\ \mathrm{val}(z) \geqslant r$, with at least one equality$\}$.

Let $e \geqslant n$ and consider the equation

$$g(\bar{X}) = \bar{i} \tag{*}$$

where the overbars indicate reduction modulo $\mathscr{P}^e$.

LEMMA 2.3. *The number of solutions $\bar{X}$ of* (*) *with $X \in Y_r$, i.e., $N(e, Y_r)(i)$ is equal to*

$$\begin{cases} 0 & \text{if } 2r > n \\ N(\pi^{-n}i)\, q^{n(2l-1)/2}\, q^{2(l-1)(e-n-1)} & \text{if } 2r = n \\ (N_0 - 1)\, q^{r(2l-1)}\, q^{2(l-1)(e-2r-1)} & \text{if } 2r < n \end{cases}$$

where

$$N(\pi^{-n}i) = \#\{g(\bar{X}) = \overline{\pi^{-n}i}\}$$
$$N_0 = \#\{g(\bar{X}) = 0\}$$

where the overbars, this time, indicate reduction modulo $\mathscr{P}$.

Proof. Let $X \in Y_r$ such that $\bar{X}$ is a solution of (*). We may (and do) assume that X has the following form.

$$X = \pi^r A_r + \cdots + \pi^{2r} A_{2r} + \cdots + \pi^e A_e, \quad A_r \not\equiv 0,$$

where each A_j ($j = 1, \dots, e$) is a vector in $(\mathbb{F}_q)^{2l-1}$ and $X_j := \pi^r A_r + \cdots + \pi^{2r+j} A_{2r+j}$ satisfies the congruence $g(X_j) \equiv i \bmod \mathscr{P}^{2r+j+1}$ ($j = 0, \dots, e-2r-1$). We apply the method of successive approximation. First we count the number of solutions of $g(X_0) \equiv i \bmod \mathscr{P}^{2r+1}$, then we count the number of solutions of$g(X_1) \equiv i \bmod \mathscr{P}^{2r+2}$, where $X_1 = X_0 + \pi^{2r+1} A_{2r+1}$, and X_0 is a solution of the preceding congruence. We continue in this fashion until we obtain the number of solutions of (*). Thus consider the congruence $g(\pi^r A_r + \cdots +$

$\pi^{2r}A_{2r}) \equiv i \bmod \mathscr{P}^{2r+1}$. Since g is homogeneous of degree two, we get

$$\pi^{2r}g(A_r + \cdots + \pi^r A_{2r}) \equiv i \bmod \mathscr{P}^{2r+1} \qquad (\dagger)$$

There are three cases to consider:

(1) $2r > n$ (2) $2r = n$ (3) $2r < n$

Identity (†) has clearly no solution in case (1). In case (2), (†) becomes

$$g(A_{n/2} + \pi A_{n/2+1} + \cdots + \pi^{n/2}A_n) \equiv \pi^{-n}i \bmod \mathscr{P}$$

which is equivalent to

$$g(A_{n/2}) + \nabla g(A_{n/2})\cdot(\pi A_{n/2+1} + \cdots + \pi^{n/2}A_n) \equiv \pi^{-n}i \bmod \mathscr{P}$$

or

$$g(A_{n/2}) \equiv \pi^{-n}i \bmod \mathscr{P}$$

or

$$g(A_{n/2}) \equiv \pi^{-n}i \bmod \mathscr{P}$$

where the dot denotes the standard scalar product.

Thus we obtain $N(\pi^{-n}i)$ solutions for $A_{n/2}$ and q^{2l-1} solutions for each A_j $(j = n/2+1, \ldots, n)$.

In case (3), (†) becomes

$$g(A_r + \pi A_{r+1} + \cdots + \pi^r A_{2r}) \equiv 0 \bmod \mathscr{P}$$

or

$$g(A_r) \equiv 0 \bmod \mathscr{P},\ A_r \not\equiv 0$$

Thus we obtain $(N_0 - 1)$ solutions for A_r and q^{2l-1} solutions for each A_j $(j = r+1, \ldots, 2)$.

Next, we consider the congruence

$$g(\pi^r A_r + \pi^{r+1}A_{r+1} + \cdots + \pi^{2r+1}A_{2r+1}) \equiv i \bmod \mathscr{P}^{2r+2} \qquad (\ddagger)$$

In both cases (2) and (3) we get

$$g(A_r) + \nabla g(A_r)\cdot \pi A_{r+1} \equiv \pi^{-2r} i \bmod \mathscr{P}^2$$

or

$$\nabla g(A_r)\cdot A_{r+1} \equiv \pi^{-1}(\pi^{-2r} i - g(A_r)) \bmod \mathscr{P}$$

This is the equation of a hyperplane in $\mathbb{F}_q^{2l-1}$, with A_{k+1} being the variable vector, and clearly has $q^{2(l-1)}$ solutions for each choice of A_k. Thus we obtain a total of $(N_0-1)q^{2(l-1)}q^{r(2l-1)}$ solutions of (‡) in case (3) and a total of $N(\pi^{-n}i)q^{2(l-1)}q^{r(2l-1)}$ solutions of (‡) in case (2). It should be clear now, that continuing in this fashion, we obtain the required result. □

LEMMA 2.4. *Suppose* $\operatorname{val}(i) = n \geqslant 0$. *Then* $N(e, Y_{(0,0)})(i) =$ *the number of solutions* $X (\bmod \not{p}^e)$ *with* $g(X) \equiv i \bmod \not{p}^e$, *is equal to*:

(a) $(N_0 - 1)q^{2(l-1)(e-1)}\dfrac{1 - q^{-(2l-3)(\operatorname{val}(i)+1)/2}}{1 - q^{-(2l-3)}}$ if val(i) is odd and positive

(b) $q^{2(l-1)(e-1)}\left[(N_0 - 1)\dfrac{1 - q^{-(2l-3)\operatorname{val}(i)/2}}{1 - q^{-(2l-3)}} + N(\pi^{-\operatorname{val}(i)} i)q^{-(l-3/2)\operatorname{val}(i)}\right]$ if val(i) is even and positive

(c) $q^{2(l-1)(e-1)}N(i)$ if val(i) = 0.

Proof. $N(e, Y_{(0,0)})(i) = \Sigma_{r=0}^{\infty} N(e, Y_r)(i)$. Thus

(a)
$$N(e, Y_{(0,0)})(i) = \sum_{r=0}^{(\operatorname{val}(i)-1)/2} (N_0 - 1)q^{r(2l-1)}q^{2(l-1)(e-2r-1)}$$
$$= (N_0 - 1)q^{2(l-1)(e-1)}\sum_{r=0}^{(\operatorname{val}(i)-1)/2} q^{-(2l-3)r}$$
$$= (N_0 - 1)q^{2(l-1)(e-1)}\frac{1 - q^{-(2l-3)(\operatorname{val}(i)+1)/2}}{1 - q^{-(2l-3)}}$$

(b)
$$N(e, Y_{(0,0)}(i) = \left[\sum_{r=0}^{(\operatorname{val}(i)-1)/2} (N_0 - 1)q^{2(l-1)(e-2r-1)}\right] + N(\pi^{-\operatorname{val}(i)} i)q^{n(2l-1)/2}q^{2(l-1)(e-n-1)}$$
$$= q^{2(l-1)(e-1)}\left[(N_0 - 1)\frac{1 - q^{-(2l-3)(\operatorname{val}(i)/2)}}{1 - q^{-(2l-3)}} + N(\pi^{-\operatorname{val}(i)} i)q^{-(l-3/2)\operatorname{val}(i)}\right]$$

(c) clear.

LEMMA 2.5. *Let* $m_1 \geqslant m_2 \geqslant 0$, $(m_1, m_2) \neq (0,0)$. *Assume that* $i \in V_{(m_1,m_2)}$. *Let* $e > \mathrm{val}(i)$. *Then* $N(e, Y_{(m_1,m_2)})(i)$ *is equal to*

(a) $(N_0 - 1)q^{2(l+m-1)(l-1)}$ *if* $m_1 = m_2 = m > 0$

(b) $N(\pi^m i)q^{2(l+m-1)(l-1)}$ *if* $m_1 = m > 0$, *m even and* $m_2 = 0$

(c) $(N_0 - 1)q^{2(l+m_1+m_2-1)(l-1)}$ *if* $m_1 > m_2 > 0$, m_1 *and* m_2 *have the same parity*

Proof:

(a) $N(e, Y_{(m,m)})(i) = \#\{X(\mathrm{mod}\ \mathscr{P}^{e+2m}) : 2\sum_{j=1}^{l-1} x_j y_j - z^2 \equiv \pi^{2m} i \bmod \mathscr{P}^{e+2m}$ and $X \in Y_m\} = N(e + 2m, Ym)(\pi^{2m} i) = (N_0 - 1)q^{2(e+m-1)(l-1)}$ (by Lemma 2.3).
(b) $N(e, Y_{(m,0)})(i) = \#\{X(\mathrm{mod}\ \mathscr{P}^{e+m}) : 2\sum_{j=1}^{l-1} x_j y_j - z^2 \equiv \pi^m i \bmod \mathscr{P}^{e+m}$ and $X \in \mathcal{O}^{2l-1}\} = N(e+m, \mathcal{O}^{2l-1})(\pi^m i) = N(\pi^m i)q^{2(e+m-1)(l-1)}$ (by Lemma 2.3).
(c) $N(e, Y_{(m_1,m_2)})(i) = \#\{X(\mathrm{mod}\ \mathscr{P}^{e+m_1+m_2}) : 2\sum_{j=1}^{l-1} x_j y_j - z^2 \equiv \pi^{m_1+m_2} i \bmod \mathscr{P}^{e+m_1+m_2}$ and $X \in Y_0\} = N(e + m_1 + m_2, Y_0)(\pi^{m_1+m_2} i) = (N_0 - 1)q^{2(e+m_1+m_2-1)(l-1)}$ (by Lemma 2.3). □

2.4.2. *The numbers* $N(i)$

We compute here the numbers $N(i)$ appearing in Lemma 2.3. The results are stated in Lemma 2.8. Lemma 2.6 and Lemma 2.7 are elementary results about Jacobi sums. The proofs of these lemmas are standard and we therefore omit them. The reader may consult [9] for the type of arguments used, or [1] for the proofs themselves.

Suppose χ is a non-trivial quadratic character of $\mathbb{F}_q^\times$.

Extend χ to all of $\mathbb{F}_q$ by $\chi(0) = 0$.

LEMMA 2.6. *Let l be a positive integer. Then* $\sum_{t_1+\cdots+t_{2l-1}=0} \chi(t_1) \cdots \chi(t_{2l-1}) = 0$.
Let n be a positive integer. Set $J_n(\chi) = \sum_{t_1+\cdots+t_n=1} \chi(t_1) \cdots \chi(t_n)$.

LEMMA 2.7. $J_{2l-1}(\chi) = q^{l-1}(\chi(-1))^{l-1}$.

Next, let $(\frac{\cdot}{q})$ denote the Legendre symbol. It is a quadratic character of $\mathbb{F}^\times (\cong \mathcal{O}/\mathscr{P})$, defined as follows:

$$\left(\frac{b}{q}\right) = \begin{cases} 1 & \text{if } b \in (\mathbb{F}_q^\times)^2 \\ -1 & \text{if } b \notin (\mathbb{F}_q^\times)^2 \end{cases}$$

Extend $(\frac{\cdot}{q})$ to $\mathcal{O}$ by setting $\left(\frac{i}{q}\right) = \left(\frac{\bar{i}}{q}\right)$, where $\bar{i}$ denotes the reduction of $i \bmod \mathscr{P}$ and $\left(\frac{0}{q}\right) = 0$.

Let $N(X^2 \equiv a)$ denote the number of solutions of $X^2 \equiv a$ in $\mathbb{F}_q$.

LEMMA 2.8. *Let $i \in \mathcal{O}$. Then*

$$N(i) = q^{2(l-1)} + \left(\frac{i}{q}\right)\left(\frac{-1}{q}\right)q^{l-1}.$$

Proof. Recall that for $i \in \mathcal{O}$,

$$N(i) = \#\{(x_1,\dots,x_{l-1}, y_1,\dots, y_{l-1}, z) \bmod \mathcal{P} : 2\sum_{j=1}^{l-1} x_j y_j - z^2 \equiv i \bmod \mathcal{P}\}.$$

Notice that $N_0 = N(0)$. For $1 \leqslant j \leqslant l-1$, set $x_j = \frac{1}{2}(z_j - z_{j+l-1})$, $y_j = (z_j + z_{j+l-1})$ and $z_{2l-1} = z$. Then

$$N(i) = \#\left\{(z_1,\dots,z_{2l-1}) \in (\mathbb{F}_q)^{2l-1} : \sum_{j=1}^{l} (z_j^2 - z_{j+l+1}^2) - z_{2l-1}^2 = \bar{i}\right\}.$$

(We are treating $\bar{i}$ as an element of $\mathbb{F}_q$ by identifying $\mathbb{F}_q$ with $\mathcal{O}/\mathcal{P}$.) Therefore,

$$\begin{aligned}
N(i) &= \sum_{(a_1+\cdots+a_{l-1})-(a_l+\cdots+a_{2l-1})=\bar{i}} \prod_{k=1}^{2l-1} N(z_k^2 \equiv a_k) \\
&= \sum_{(a_1+\cdots+a_{l-1})-(a_l+\cdots+a_{2l-1})=\bar{i}} \prod_{k=1}^{2l-1} \left[1+\left(\frac{a_k}{q}\right)\right] \\
&= \sum_{(a_1+\cdots+a_{l-1})-(a_l+\cdots+a_{2l-1})=\bar{i}} \left[1+\sum_{k=1}^{2l-1}\left(\frac{a_k}{q}\right)+\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{2l-1}\left(\frac{a_{k_1}}{q}\right)\left(\frac{a_{k_2}}{q}\right)\right. \\
&\qquad \left.+\cdots+\sum_{\substack{k_1,\dots,k_{2l-2}=1\\ \text{all } k_j\text{'s distinct}}}^{2l-1}\left(\frac{a_{k_1}}{q}\right)\left(\frac{a_{k_2}}{q}\right)\cdots\left(\frac{a_{k_{2l-2}}}{q}\right)+\left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right)\right] \\
&= q^{2l-2} + \sum_{(a_1+\cdots+a_{l-1})-(a_l+\cdots+a_{2l-1})=\bar{i}} \left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{(a_1+\cdots+a_{l-1})-(a_l+\cdots+a_{2l-1})=\bar{i}} \left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right) \\
&\quad = \left(\frac{-1}{q}\right)^l \sum_{a_1+\cdots+a_{2l-1}=\bar{i}} \left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right)
\end{aligned}$$

If $\bar{i}=0$, then the above sum is zero by Lemma 2.6. If $\bar{i} \neq 0$, then making the substitutions $a_k' = \bar{i} a_k$ $(1 \leqslant k \leqslant 2l-1)$, and changing the notation leads to:

$$\sum_{a_1+\cdots+a_{2l-1}=\bar{i}} \left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right) = \left(\frac{i}{q}\right)^{2l-1} \sum_{a_1+\cdots+a_{2l-1}=\bar{i}} \left(\frac{a_1}{q}\right)\cdots\left(\frac{a_{2l-1}}{q}\right)$$
$$= \left(\frac{i}{q}\right) J_{2l-1}\left(\left(\tfrac{\cdot}{q}\right)\right) = \left(\frac{i}{q}\right) q^{-l-1} \left(\frac{-1}{q}\right)^{l-1}$$

by Lemma 2.7. Thus

$$N(i) = q^{2(l-1)} + \left(\frac{-1}{q}\right)^{l}\left(\frac{i}{q}\right)\left(\frac{-1}{q}\right)^{l-1} q^{l-1}$$
$$= q^{2(l-1)} + \left(\frac{i}{q}\right)\left(\frac{-1}{q}\right) q^{l-1} \qquad \square$$

2.4.3. *The integrals* $\int_{\mathcal{O}_{u_k}} f_{(m_1,\ldots,m_l)}$

We need the following lemma to start computing the orbital integrals $\int_{\mathcal{O}_{u_k}} f_{(m_1,\ldots,m_l)}$. Let $k=1,2,3,4$ and $n \in \mathbb{Z}$. Set $A_n^{(k)} = \{i \in F^\times : \mathrm{val}(i) = n$ and $i \equiv \alpha_k \bmod (F^\times)^2\}$. For $B \subseteq F$, let $\mu(B)$ denote the Lebesgue measure of B which is normalized by $\mu(\mathcal{O}) = 1$.

LEMMA 2.9

(i) $\mu(A_{2n}^{(1)}) = \mu(A_{2n}^{(2)}) = \dfrac{q-1}{2} q^{-2n-1}$.

(ii) $\mu(A_{2n+1}^{(3)}) = \mu(A_{2n+1}^{(4)}) = \dfrac{q-1}{2} q^{-2n-2}$.

(iii) $\mu(A_{2n+1}^{(3)}) = \mu(A_{2n+1}^{(4)}) = \mu(A_{2n+1}^{(1)}) = \mu(A_{2n+1}^{(2)}) = 0$.

Proof. The proof is an immediate consequence of the following observation.

$$A_{2n}^{(1)} = \coprod_{j=1}^{(q-1)/2} \pi^{2n}\varepsilon^{2j} + \mathcal{P}^{2n+1}, \qquad A_{2n}^{(2)} = \coprod_{j=1}^{(q-1)/2} \pi^{2n}\varepsilon^{2j+1} + \mathcal{P}^{2n+1}$$
$$A_{2n+1}^{(3)} = \coprod_{j=1}^{(q-1)/2} \pi^{2n+1}\varepsilon^{2j} + \mathcal{P}^{2n+2}, \quad A_{2n+1}^{(4)} = \coprod_{j=1}^{(q-1)/2} \pi^{2n+1}\varepsilon^{2j+1} + \mathcal{P}^{2n+2}$$

(all disjoint unions)

$$A_{2n}^{(3)} = A_{2n}^{(4)} = A_{2n+1}^{(1)} = A_{2n+1}^{(2)} = \varnothing. \qquad \square$$

Recall that, since $S_{(m_1,\ldots,m_l;\alpha_k)} = \varnothing$ unless $m_3 = \cdots m_l = 0$ (2.2.2), we need only to worry about computing $\int_{\mathcal{O}_{u_k}} f_{(m_1,m_2,0,\ldots,0)}$.

PROPOSITION 2.10

(i) $$\int_{\mathcal{O}_{u_1}}(f_{(0,\ldots,0)})=c_{u_1}\frac{1-q^{-1}}{2}\left[\frac{(1-q^{-(l+1)})(1+q^{-(l-1)})}{(1-q^{-2})(1-q^{-(2l-1)})}\right]\quad \textit{if } q\equiv 1 \bmod 4$$

$$=c_{u_1}\frac{1-q^{-1}}{2}\left[\frac{(1+q^{-(l+1)})(1-q^{-(l-1)})}{(1-q^{-2})(1-q^{-2l-1})}\right]\quad \textit{if } q\equiv 3 \bmod 4$$

(ii) $$\int_{\mathcal{O}_{u_2}}(f_{(0,\ldots,0)})=c_{u_2}\frac{1-q^{-1}}{2}\left[\frac{(1+q^{-(l+1)})(1-q^{-(l-1)})}{(1-q^{-2})(1-q^{-(2l-1)})}\right]\quad \textit{if } q\equiv 1 \bmod 4$$

$$=c_{u_2}\frac{1-q^{-1}}{2}\left[\frac{(1-q^{-(l+1)})(1+q^{-(l-1)})}{(1-q^{-2})(1-q^{-(2l-1)})}\right]\quad \textit{if } q\equiv 3 \bmod 4$$

(iii) $$\int_{\mathcal{O}_{u_k}}(f_{(0,\ldots,0)})=c_{u_k}\frac{1-q^{-1}}{2}q^{-1}\frac{1-q^{-2(l-1)}}{(1-q^{-2})(1-q^{-2l-1})}\quad \textit{for } k=3,4.$$

Proof. We apply the integral formula in (2.3) with $Y=Y_{(0,0)}$ (see 2.4.1), $\Phi=1_Y$, $k=2$, $n=2l-1$, $a=0$. Let $i\in\mathcal{O}$. Using Lemma 2.4 we get

$$F_\Phi=\begin{cases} q^{-2(l-1)}(q^{(l-1)}-1)\dfrac{1-q^{-(2l-3)(\mathrm{val}(i)+1)/2}}{1-q^{-(2l-3)}} & \text{if val}(i)\text{ is odd}\\[2ex] q^{-2(l-1)}\left[q^{(l-1)}-1)\dfrac{1-q^{-(2l-3)(\mathrm{val}(i)+1)/2}}{1-q^{-(2l-3)}}\right.\\ \left.\quad+\left(q^{2(l-1)}+\left(\dfrac{\pi^{-\mathrm{val}(i)}i}{q}\right)q^{l-1}\right)q^{-(l-3/2)\mathrm{val}(i)}\right] & \text{if val}(i)\text{ is even and positive}\\[2ex] q^{-2(l-1)}\left(q^{2(l-1)}+\left(\dfrac{-i}{q}\right)q^{l-1}\right) & \text{if val}(i)=0\end{cases}$$

Let $i_{2n}\in A_{2n}^{(1)}$, $n\in\mathbb{Z}$, and suppose $q\equiv 1 \bmod 4$. Then $\int_{\mathcal{O}_{u_1}}(f_{(0,\ldots,0)})=c_{n_1}\sum_{n=0}^{\infty}\mu(A_{2n}^{(1)})F_\Phi(i_{2n})$. In what follows below, we omit writing down the constant c_{u_1}. Thus, Lemmas 2.5–2.9 imply

$$\int_{\mathcal{O}_{u_1}}(f_{(0,\ldots,0)})=q^{-1}\left(\frac{q-1}{2}\right)q^{-2(l-1)}(q^{2(l-1)}+q^{l-1})$$

$$+\frac{q-1}{2}\frac{q^{2(l-1)}-1}{1-q^{-(2l-3)}}q^{-2(l-1)}\sum_{n=1}^{\infty}q^{-(2n-1}(1-q^{-2l-3)n})$$

$$+\frac{q-1}{2}q^{-2(l-1)}(q^{2(l-1)}+q^{l-1})\sum_{n=1}^{\infty}q^{-2n-1}q^{-n(2l-3)})$$

$$=\frac{q-1}{2}q^{-1}(1+q^{-(l-1)})+\frac{q-1}{2}(1+q^{-(l-1)})\sum_{n=1}^{\infty}q^{-2n-1}q^{-n(2l-3)}$$

$$+\frac{q-1}{2}\frac{1-q^{-2(l-1)}}{1-q^{-(2l-3)}}\sum_{n=1}^{\infty} q^{-2n-1}(1-q^{-(2l-3)n})$$
$$=\frac{1-q^{-1}}{2}\frac{1+q^{-(l-1)}}{1-q^{-(2l-1)}}+\frac{1-q^{-1}}{2}\frac{1-q^{-2(l-1)}}{1-q^{-(2l-3)}}\left[\frac{q^{-2}(1-q^{-(2l-3)})}{(1-q^{-2})(1-q^{-(2l-1)})}\right]$$
$$=\frac{1-q^{-1}}{2}\left[\frac{1-q^{-(l+1)})(1+q^{-(l-1)})}{(1-q^{-2})(1-q^{-(2l-1)})}\right]$$

This proves the first statement in (i). The rest of the statements are proved in a similar way [1]. □

PROPOSITION 2.11. *Suppose $m_1=m_2=m>0$ and m even. Then*

(i) $\displaystyle\int_{\mathcal{O}_{u_k}}(f_{(m,m,0,\ldots,0)})=c_{u_k}\frac{1-q^{-1}}{2}\frac{1-q^{-2(l-1)}}{1-q^{-2}}q^{2(l-1)m}$ *for* $k=1,2.$

(ii) $\displaystyle\int_{\mathcal{O}_{u_k}}(f_{(m,m,0,\ldots,0)})=c_{u_k}\frac{1-q^{-1}}{2}q^{-1}\frac{1-q^{-2(l-1)}}{1-q^{-2}}q^{2(l-1)m}$ *for* $k=3,4.$

Proof. Apply the integral formula in (2.3) with $Y=Y_{(m,m)}$, $\Phi=1_Y$, $k=2$, $n=2l-1$, $a=m$. Lemma 2.5(a) and Lemma 2.8 imply that for $i\in V_{(m,m)}:F_\Phi(i)=(q^{2(l-1)}-1)q^{-(2l-1)m}q^{2(l-1)(2m-1)} = (1-q^{-2(l-1)})q^{(2l-3)m}$. Let $V^k = \{i\in V_{(m,m)}: i\equiv\alpha_k \bmod (F^\times)^2\}$ $(1\leqslant k\leqslant 4)$.
Now,

$$\mathrm{meas}(V^1)=\mathrm{meas}(V^2)=\frac{q-1}{2}\sum_{n=-m/2}^{\infty} q^{-(2n+1)}=\frac{q-1}{2}\frac{q^{m-1}}{1-q^{-2}}$$
$$\mathrm{meas}(V^3)=\mathrm{meas}(V^4)=\frac{q-1}{2}\sum_{n=-m-1/2}^{\infty} q^{-(2n+1)}=\frac{q-1}{2}\frac{q^{m-2}}{1-q^{-2}}$$

The rest is simple. □

PROPOSITION 2.12. *Suppose $m_1=m_2=m>0$ and m odd. Then*

(i) $\displaystyle\int_{\mathcal{O}_{u_k}}f_{(m,m,0,\ldots,0}=c_{u_k}\frac{1-q^{-1}}{2}q^{-1}\frac{1-q^{-2(l-1)}}{1-q^{-2}}q^{2(l-1)m},\quad k=1,2.$

(ii) $\displaystyle\int_{\mathcal{O}_{u_k}}f_{(m,m,0,\ldots,0)}=c_{u_k}\frac{1-q^{-1}}{2}\frac{1-q^{-2(l-1)}}{1-q^{-2}}q^{2(l-1)m},\quad k=3,4.$

Proof. The proof is the same as in the "m even" case, except for one difference, which is:

$$\mu(V^1) = \mu(V^2) = \frac{q-1}{2}\frac{q^{m-2}}{1-q^{-2}}$$

$$\mu(V^3) = \mu(V^4) = \frac{q-1}{2}\frac{q^{m-1}}{1-q^{-2}}$$ □

PROPOSITION 2.13. *Suppose $m_1 = m > 0$, m even, and $m_2 = 0$. Then*

(i) $$\int_{\mathcal{O}_{u_1}} (f_{(m,0,\ldots,0)}) = c_{u_1}\frac{1-q^{-1}}{2} q^{(2l-1)m/2}(1+q^{-(l-1)}) \quad \text{if } q \equiv 1 \bmod 4$$

$$= c_{u_1}\frac{1-q^{-1}}{2} q^{(2l-1)m/2}(1-q^{-(l-1)}) \quad \text{if } q \equiv 3 \bmod 4$$

(ii) $$\int_{\mathcal{O}_{u_2}} (f_{(m,0,\ldots,0)}) = c_{u_2}\frac{1-q^{-1}}{2} q^{(2l-1)m/2}(1-q^{-(l-1)}) \quad \text{if } q \equiv 1 \bmod 4$$

$$= c_{u_2}\frac{1-q^{-1}}{2} q^{(2l-1)m/2}(1+q^{-(l-1)}) \quad \text{if } q \equiv 3 \bmod 4$$

(iii) $$\int_{\mathcal{O}_{u_3}} (f_{(m,0,\ldots,0)}) = \int_{\mathcal{O}_{u_4}} (f_{(m,0,\ldots,0)}) = 0$$

Proof. Apply the integral formula in (2.3) with $Y = Y_{(m,0)}$, $\Phi = 1_Y$, $k=2$, $n = 2l-1$, $a = m/2$. Lemma 2.5(b) and Lemma 2.8 imply:

$$F_\Phi(i) = q^{-(2l-1)m/2} q^{-2(l-1)e} q^{2(l-1)(e+m-1)}\left(q^{2(l-1)} + \left(\frac{\pi^m i}{q}\right) q^{l-1}\right), i \in V_{(m,0)}$$

Also notice that

$$\mu(\{i \in V_{(m,0)} : i = \alpha_k \bmod (F^\times)^2\}) = \begin{cases} \frac{q-1}{2} q^{m-1} & \text{if } k = 1,2 \\ 0 & \text{if } k = 3,4 \end{cases}$$

The proposition follows easily from the above information. □

PROPOSITION 2.14. *Suppose $m_1 > m_2 > 0$, m_1 and m_2 even. Then*

(i) $$\int_{\mathcal{O}_{u_k}} (f_{(m_1,m_2,0,\ldots,0)}) = c_{u_k}\frac{1-q^{-1}}{2}(1-q^{-2(l-1)}) q^{(2(l-1)m_1 + (2l-3)m_2)/2} \quad \textit{for } k = 1,2.$$

(ii) $$\int_{\mathcal{O}_{u_k}} (f_{(m_1,m_2,0,\ldots,0)}) = 0 \quad \textit{for } k = 3,4.$$

Proof. Apply the integral formula in (2.3) with $Y=Y_{(m_1,m_2)}$, $\Phi=1_Y$, $k=2$, $n=2l-1$, $a=\dfrac{m_1-m_2}{2}$.

Lemma 2.5(c) and Lemma 2.8 imply:

$$F_\Phi(i)=q^{-(2l-1)(m_1+m_2)/2)}q^{-2(l-1)e}q^{2(l-1)(e+m_1+m_2-1)}(q^{2(l-1)}-1), \qquad i\in V_{(m_1,m_2)}.$$

Also

$$\mu(\{i\in V_{(m_1,m_2)}: i\equiv\alpha_k \operatorname{mod}(F^\times)^2\})=\begin{cases}\dfrac{q-1}{2}q^{m_1-1} & \text{if } k=1,2\\ 0 & \text{if } k=3,4\end{cases}.$$

The proposition follows from the above. □

PROPOSITION 2.15. *Suppose $m_1>m_2>0$, m_1 and m_2 odd. Then*

(i) $\displaystyle\int_{\mathcal{O}_{u_k}} f_{(m_1,m_2,0,\ldots,0)}=0$ *for* $k=1,2$.

(ii) $\displaystyle\int_{\mathcal{O}_{u_k}} f_{(m_1,m_2,0,\ldots,0)}=c_{u_k}\frac{1-q^{-1}}{2}(1-q^{-2(l-1)})q^{((2l-1)m_1+(2l-3)m_2)/2}$ *for* $k=3,4$.

Proof. The proof is similar to the one of Proposition 5. The only difference is the following:

$$\mu(\{i\in V_{(m_1,m_2)}: i=\alpha_k \operatorname{mod}(F^\times)^2\})=\begin{cases}0 & \text{if } k=1,2\\ \dfrac{q-1}{2}q^{m_1-1} & \text{if } k=3,4\end{cases}$$

PROPOSITION 2.16. *Suppose $m_1>m_2>0$, m_1 and m_2 have different parities. Then*

$$\int_{\mathcal{O}_{u_k}} f_{(m_1,m_2,0,\ldots,0)}=0 \quad \textit{for } 1\leqslant k\leqslant 4$$

Proof

$S_{(m_1,m_2;\alpha_k)}=\varnothing$ for pairs (m_1,m_2) as above. □

THEOREM 2.17. *The complex vector space spanned by the four linear functionals: $f\mapsto\int_{\mathcal{O}_{u_k}} f$, $f\in C_c^\infty(\mathbf{G}(F)\,/\!/\,K)$, is three dimensional.*

Proof. This is a consequence of Propositions (2.10–2.16). □

3. The minimal unipotent conjugacy class in $\mathbf{SO}_{2l+1}(F)$ and a matching result

Let $\mathbf{G}=\mathbf{SO}_{2l+1}(l \geqslant 2)$ and let u_G denote a representative of the lowest dimensional non-trivial $\mathbf{G}(\bar{F})$-conjugacy class in $\mathbf{G}(\bar{F})$. This class is rigid, i.e. cannot be obtained by induction from a unipotent class in some proper Levi subgroup of $\mathbf{G}(\bar{F})$ (see [18]), and consists of non-special elements whose elementary divisors are given by the partition $1^{2l-3}2\,2$. The F-rational points on this class form one $\mathbf{G}(F)$ conjugacy class $\mathcal{O}_{u_G}$.

Let $\mathbf{H}=\mathbf{SO}_{2l-1}\times\mathbf{PGL}_2$. The main result of this section is to show that there exists $c\in\mathbb{C}$ such that $\int_{\mathcal{O}_{u_G}} f = cf^H(1)$, where 1 denotes the identity element in $\mathbf{H}(F)$, and $f\mapsto f^H$ is the Hecke algebra homomorphism dual to the embedding $\mathbf{Sp}_{2(l-1)}(\mathbb{C})\times\mathbf{SL}_2(\mathbb{C})=\hat{H}\hookrightarrow\hat{G}=\mathbf{Sp}_{2l}(\mathbb{C})$.

3.1. *The unipotent orbital integral of $f_{(m_1,\ldots,m_l)}$*

Consider the following Lie triple in $\mathfrak{g}=\mathrm{Lie}(\mathbf{G}(F))$.

$$X_0 = E_{2,l+1} - E_{2l+1,l+2}$$

$$Y_0 = E_{l+1,2} - E_{l+2,2l+1}$$

$$H_0 = E_{22} + E_{2l+1,2l+1} - E_{l+1,l+1} - E_{l+2,l+2}$$

Let $u_G = \exp X_0 = I_{2l+1} + X_0$.

u_G has all the properties mentioned in the introduction to this section. Set M_0=centralizer of H_0 in $\mathbf{G}(F)$. We now describe the ingredients of Rao's formula. $\mathfrak{g}_2=\{x\cdot X_0 : x\in F\}$. The M_0-orbit of X_0 in $\mathfrak{g}_2=:V_0=\{x\cdot X_0 : x\in F^\times\}$. The spaces $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have dimension $4l-6$ and are described in [1]. Rao's φ-function is given by: $\varphi(x):=\varphi(x\cdot X_0)=|x|^{2l-3}$, $x\in F$. Set $g=\exp(x\cdot X_0)$, $x\in F$. Again, let $r_k(g)$ be given by $q^{-r_k(g)}=\|\Lambda^k(g)\|$. Then

$$r_1(g) = \min\{0, \mathrm{val}(x)\}$$

$$r_2(g) = \cdots = r_l(g) = \min\{0, \mathrm{val}(x), 2\,\mathrm{val}(x)\}$$

We wish to describe the intersection of $\exp(V_0)$ with the double coset $K\,\mathrm{diag}(1,\pi^{m_1},\ldots,\pi^{m_l},\pi^{-m_1},\ldots,\pi^{-m_l})K$. Thus we have to solve the following equations:

$$r_1(g) = -m_1,\ r_2(g) = -m_1-m_2,\ldots,r_l(g) = -m_1-\cdots-m_l$$

Observe that the solution set is empty unless $m_3=\cdots=m_l=0$, so we need to consider only the first two equations. There are three cases to consider.

(1) $m_1 = m_2 = 0$

The solution set $= \{x \in F^\times : \mathrm{val}(x) \geqslant 0\}$

(2) $m_1 = m_2 = m > 0$

The solution set $= \{x \in F^\times : \mathrm{val}(x) = -m\}$

(3) $m_1 > m_2$

The solution set is empty.

Next, we compute $\int_{\mathcal{O}_{u_G}} f_{(m_1,\ldots,m_l)}$. $\int_{\mathcal{O}_{u_G}} f_{(m_1,\ldots,m_l)} = 0$ unless $m_3 = \cdots = m_2 = 0$.

(1) $m_1 = m_2 = 0$

Set $V = \{x \in F^\times : \mathrm{val}(x) \geqslant 0\}$ and $\Phi = f_{(0,0,\ldots,0)} \circ \exp$. Then

$$\int_{\mathcal{O}_{u_G}} f_{(0,0,\ldots,0)} = c_{u_G} \int_V \Phi(x \cdot X_0)|x|^{2l-3}\,\mathrm{d}x$$

$$= c_{u_G} \sum_{k=0}^{\infty} (\mathrm{meas}\{\mathrm{val}(x) = k\}) \cdot q^{-k(2l-3)}$$

$$= c_{u_G} \sum_{k=0}^{\infty} (q^{-k} - q^{-k-1}) \cdot q^{-k(2l-3)} = c_{u_G} \frac{1 - q^{-1}}{1 - q^{-2(l-1)}}$$

where c_{u_G} depends only on $\mathcal{O}_{u_G}$ and the normalization of measure on $\mathcal{O}_{u_G}$.

(2) $m_1 = m_2 = m > 0$

Set $V = \{x \in F^\times : \mathrm{val}(x) = -m\}$, and $\Phi = f_{(m,m,0,\ldots,0)} \circ \exp$. Then

$$\int_{\mathcal{O}_{u_G}} f_{(m,m,0,\ldots,0)} = c_{u_G} \int_V \Phi(x \cdot X_0)|x|^{2l-3}\,\mathrm{d}x = c_{u_G}(1 - q^{-1})q^{2(l-1)m}$$

(3) $m_1 > m_2$

$$\int_{\mathcal{O}_{u_G}} f_{(m_1,m_2,0,\ldots,0)} = 0$$

3.2. *An explicit expression for* $f^H_{(m_1,\ldots,m_l)}(1)$

Let $g \in C_c^\infty(\mathbf{H}(F) /\!/ \mathbf{H}(\mathcal{O}))$. The spherical Plancherel theorem [14] tells us that

$$g(1) = \frac{Q(q^{-1})}{|W(H)|} \frac{1}{(2\pi i)^l} \int_{|z_l|=1} \cdots \int_{|z_2|=1} \int_{|z_1|=1} \tilde{g}(z_1,\ldots,z_l)\, |\mathbf{c}_H(z_1,\ldots,z_l)|^{-2} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_l}{z_l}$$

where

$\tilde{g}(z_1,\ldots,z_2) =$ The Satake transform of g

$|W(H)|$ = The order of the Weyl group of **H**

$\mathbf{c}_{\mathbf{H}}$ = **c**-function of $\mathbf{H}(F)$

$Q(q^{-1})$ = The Poincare polynomial of $W(H)$.

If we set $g = f^H_{(m_1,\ldots,m_l)}$, we get

$$\tilde{g}(z_1,\ldots,z_l) = \widetilde{f^H}_{(m_1,\ldots,m_l)}(z_1,\ldots,z_l) = \tilde{f}_{(m_1,\ldots,m_l)}(z_1,\ldots,z_l)$$
$$= b_{(m_1,\ldots,m_l)}(q^{-1})\cdot \sum_{\sigma\in W(G)} [c_G(z_1,\ldots,z_l)z_1^{m_1}\cdots z_l^{m_l}]^\sigma$$

where

$$\mathbf{c}_{\mathbf{G}} = \mathbf{c}\text{-function of } \mathbf{G}(F)$$

$$b_{(m_1,\ldots,m_2)}(q^{-1}) = \frac{q^{(2l-1)m_1+(2l-3)m_2+\cdots+m_l)/2}}{Q_{(m_1,\ldots,m_l)}(q^{-1})}$$

where $Q_{(m_1,\ldots,m_l)}(q^{-1})$ is the Poincaré polynomial of the subgroup of $W(G)$ consisting of the elements fixing $(m_1,\ldots,m_l)$. This subgroup is a product of Weyl groups of type B. Recall that the Poincaré polynomial Q_n of a Weyl group of type B_n is given by ([15]):

$$Q_n(q^{-1}) = \prod_{i=0}^{n-1} (1 + q^{-(i+1)})(1 + q^{-1} + \cdots + q^{-i}).$$

We shall view $(z_1,\ldots,z_l)$ as coordinates of the maximal torus shared by $\hat{\mathbf{G}}$ and $\hat{\mathbf{H}}$, and arrange things so that

$$\mathbf{c}_{\mathbf{H}}(z_1,\ldots,z_l) = \frac{1-q^{-1}z_2^2}{1-z_2^2} \prod_{\substack{1\leqslant i<j\leqslant l\\ i\neq 2, j\neq 2}} \frac{1-q^{-1}z_i^{-1}z_j}{1-z_i^{-1}z_j} \prod_{1\leqslant i<j\leqslant l} \frac{1-q^{-1}z_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}} \prod_{i=1, i\neq 2}^{i=l} \frac{1-q^{-1}z_i^{-2}}{1-z_i^{-2}}$$

$$\mathbf{c}_{\mathbf{G}}(z_1,\ldots,z_l) = \prod_{1\leqslant i<j\leqslant l} \frac{1-q^{-1}z_i^{-1}z_j}{1-z_i^{-1}z_j} \cdot \prod_{1\leqslant i<j\leqslant l} \frac{1-q^{-1}z_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}} \prod_{i=1}^{i=l} \frac{1-q^{-1}z_i^{-2}}{1-z_i^{-2}}$$

Next, we indicate how we are going to compute $f^H_{(m_1,\ldots,m_l)}(1)$. Observe first that $\tilde{f}_{(m_1,\ldots,m_l)}(z_1,\ldots,z_l)$, $\mathbf{c}_{\mathbf{H}}(z_1,\ldots,z_l)^{-1}$ and $\mathbf{c}_{\mathbf{H}}(z_1^{-1},\ldots,z_l^{-1})^{-1}$ are all holomorphic functions on $0<|z_i|<q^{1/2}$, $(1\leqslant j\leqslant l)$. Therefore we may change the contour of integration from $|z_j|=1 (1\leqslant j\leqslant l)$ into $|z_j|=1+\varepsilon_j (1\leqslant j\leqslant l)$, where $q^{1/2}-1>\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_l>1-q^{-1/2}$. Recall that $W(G)\cong S_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l$. Let $\sigma\in\langle 1\rangle \ltimes (\mathbb{Z}/2\mathbb{Z})^l$.

Assume $(z_1,\ldots,z_i,\ldots,z_l)^\sigma=(z_1,\ldots,z_i^{-1},\ldots,z_l)$. It is obvious that integrating

$$\mathbf{c}_G(z_1,\ldots,z_l)^\sigma\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})(z_1^{m_1}\cdots z_l^{m_l})^\sigma\cdot z_1^{-1}\cdots z_l^{-1}$$

over $|z_j|=1+\varepsilon_j(1\leqslant j\leqslant l)$ is the same as integrating

$$\mathbf{c}_G(z_1,\ldots,z_l)\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})z_1^{m_1-1}\cdots z_l^{m_l-1}$$

over $|z_i|=\begin{cases}1+\varepsilon_j & \text{if } j\neq i\\ 1-\varepsilon_i & \text{if } j=i\end{cases}$ (just use the substitution $z_i'=z_i^{-1}$). Since every element of $\langle 1\rangle\rtimes(\mathbb{Z}/2\mathbb{Z})^l$ is a product of elements σ as above, it follows that integrating

$$\mathbf{c}_G(z_1,\ldots,z_l)^\sigma\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})\cdot(z_1^{m_1}\cdots z_l^{m_l})^\sigma\cdot z_1^{-1}\cdots z_l^{-1}$$

over $|z_j|=1+\varepsilon_j$ $(j=1,\ldots,l)$ $(\sigma\in\langle 1\rangle\rtimes(\mathbb{Z}/2\mathbb{Z})^l)$ is the same as integrating

$$\mathbf{c}_G(z_1,\ldots,z_l)\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})z_1^{m_1-1}\cdots z_l^{m_l-1}$$

over $|z_j|=1\pm\varepsilon_j(1\leqslant j\leqslant l)$, where the sign $\pm$ depends on σ in the obvious way.

Thus we need only to compute the integrals of

$$\mathbf{c}_G(z_1,\ldots,z_l)^\sigma\mathbf{c}_H^{-1}(z_1,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},\ldots,z_l^{-1})(z_1^{m_1-1}\cdots z_l^{m_l-1})^\sigma(\sigma\in S_l\rtimes\langle 1\rangle)$$

over the 2^l tori $|z_j|=1\pm\varepsilon_j(1\leqslant j\leqslant l)$. Next, we argue that we need only to consider the functions

$$\mathbf{c}_G(z_1,\ldots,z_l)^\sigma\mathbf{c}_H^{-1}(z_1,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},\ldots,z_l^{-1})(z_1^{m_1-1}\cdots z_l^{m_l-1})^\sigma$$

where σ is a cycle of the form $(i,2)$, $i=1,\ldots,l$. (If $i=2$, then $\sigma=$identity.)

Indeed, let's consider the cycle $\sigma=(i,j)$ where $i\neq 2$, $j\neq 2$, $i>j$. The change of variables $z_i'=z_j$, $z_j'=z_i$ and the invariance of $c_H^{-1}(z_1,z_3,\ldots,z_l)c_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})$ under σ shows that integrating the above function over some torus: $|z_k|=1\pm\varepsilon_k(1\leqslant k\leqslant l)$ is the same as integrating $\mathbf{c}_G(z_1,\ldots,z_l)\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})z_1^{m_1-1}\cdots z_l^{m_l-1}$ over the same torus. Now each permutation σ can be written in the form: $\sigma=\tau\cdot(i,2)$ for some i, where τ is a permutation fixing 2. Thus we need only to compute the integral of the following l functions:

$$\Phi_k^l(m_1,\ldots,m_l):=[\mathbf{c}_G(z_1,\ldots,z_l)]^{\sigma_k}\times\mathbf{c}_H^{-1}(z_1,z_3,\ldots,z_l)$$
$$\times\,\mathbf{c}_H^{-1}(z_1^{-1},z_3^{-1},\ldots,z_l^{-1})z_1^{m_1-1}\cdots z_l^{m_l-1},\quad(1\leqslant k\leqslant l)$$

over the various 2^l tori: $|z_i|=1\pm\varepsilon_i(1\leqslant i\leqslant l)$. If $2\leqslant k\leqslant l$, then $\Phi_k^l(m_1,\ldots,m_l)$ is the product of the following functions:

(1) $\displaystyle\prod_{i=1}^{i=k-1}\frac{1-q^{-1}z_i^{-1}z_k}{1-z_i^{-1}z_k}\prod_{i=k+1}^{i=l}\frac{1-q^{-1}z_k^{-1}z_i}{1-z_k^{-1}z_i}$

(2) $\displaystyle\prod_{i=1}^{i=k-1}\frac{1-q^{-1}z_i^{-1}z_k^{-1}}{1-z_i^{-1}z_k^{-1}}\prod_{i=k+1}^{i=l}\frac{1-q^{-1}z_k^{-1}z_i^{-1}}{1-z_k^{-1}z_i^{-1}}$

(3) $\displaystyle\prod_{i=1}^{i=l}\frac{1-z_i^2}{1-q^{-1}z_i^2}$ (4) $\displaystyle\prod_{\substack{1\leqslant i<j\leqslant l\\ i\neq k,j\neq k}}\frac{1-z_iz_j^{-1}}{1-q^{-1}z_iz_j^{-1}}$

(5) $\displaystyle\prod_{\substack{1\leqslant i<j\leqslant l\\ i\neq k,j\neq k}}\frac{1-z_iz_j}{1-q^{-1}z_iz_j}$ (6) $z_1^{m_1-1}\cdots z_l^{m_l-1}$

If $k=1$, then $\Phi_1^l(m_1,\ldots,m_l)$ is equal to the product of the following functions

(1) $\displaystyle\prod_{i=2}^{i=l}\frac{1-q^{-1}z_1^{-1}z_i}{1-z_1^{-1}z_i}$ (2) $\displaystyle\prod_{i=2}^{i=l}\frac{1-q^{-1}z_i^{-1}z_1^{-1}}{1-z_i^{-1}z_1^{-1}}$

(3) $\displaystyle\prod_{i=2}^{i=l}\frac{1-z_i^2}{1-q^{-1}z_i^2}$ (4) $\displaystyle\prod_{2\leqslant i<j\leqslant l}\frac{1-z_iz_j^{-1}}{1-q^{-1}z_iz_j^{-1}}$

(5) $\displaystyle\prod_{2\leqslant i<j\leqslant l}\frac{1-z_iz_j}{1-q^{-1}z_iz_j}$ (6) $z_1^{m_1-1}z_2^{m_2-1}\cdots z_l^{m_l-1}$

Next, we are going to "formally" integrate the functions $\Phi_k^l(m_1,\ldots,m_l)$. By this we mean the following. First we freeze suitable $l-1$ variables and regard $\Phi_k^l(m_1,\ldots,m_l)$ as a function in the remaining variable (z_i say). We compute the residues of the poles of $\Phi_k^l(m_1,\ldots,m_l)$ inside the torus $|z_i|=1+\varepsilon_i$. Then, we free exactly one more variable (z_j say), and compute the residues of the poles of the residues of $\Phi_k^l(m_1,\ldots,m_l)$ which lie inside $|z_j|=1+\varepsilon_j$. We keep repeating this process while registering the results at each step. After we are done with "formal" integration, we compute the integral of $\Phi_k^l(m_1,\ldots,m_l)$ over each of the 2^l possible tori by taking into account the locations of the various poles relative to these tori. Also, we agree that only poles lying inside some $|z_i|=1+\varepsilon_i$ will be called so, and we shall ignore the factor $2\pi i$ when computing a residue. In fact we should get a product of l such factors which in the end cancel against the factor $(2\pi i)^{-l}$ appearing in the Plancherel formula.

3.3. *The integral of* $\Phi_k^l(m_1,\ldots,m_l)$ $(2\leqslant k\leqslant l)$

We start by "formally" integrating $\Phi_k^l(m_1,\ldots,m_l)$, $2\leqslant k\leqslant l$. This function has at most three z_1-poles, namely:

(a) $z_1 = z_k$, (b) $z_1 = z_k^{-1}$, (c) $z_1 = 0$(iff $m_1 = \cdots m_l = 0$)

(a) $z_1 = z_k$

Res $_{z_1=z_k}(\Phi_k^l(m_1,\ldots,m_l))$ = product of the following functions:

(1) $(1-q^{-1})z_k \prod_{i=2}^{i=k-1} \dfrac{1-q^{-1}z_i^{-1}z_k}{1-z_i^{-1}z_k} \prod_{i=k+1}^{i=l} \dfrac{1-q^{-1}z_k^{-1}z_i}{1-z_k^{-1}z_i^{-1}}$

(2) $\dfrac{1-q^{-1}z_k^{-2}}{1-z_k^{-2}} \prod_{i=2}^{i=k-1} \dfrac{1-q^{-1}z_i^{-1}z_k^{-1}}{1-z_i^{-1}z_k^{-1}} \prod_{i=k+1}^{i=l} \dfrac{1-q^{-1}z_k^{-1}z_i^{-1}}{1-z_k^{-1}z_i^{-1}}$

(3) $\dfrac{1-z_k^2}{1-q^{-1}z_k^2} \prod_{i=2}^{i=l} \dfrac{1-z_i^2}{1-q^{-1}z_i^2}$

(4) $\prod_{i=2}^{i=k-1} \dfrac{1-z_k z_i^{-1}}{1-q^{-1}z_k z_i^{-1}} \prod_{i=k+1}^{i=l} \dfrac{z_i - z_k}{z_i - q^{-1}z_k} \prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k, j\neq k}} \dfrac{1-z_i z_j^{-1}}{1-q^{-1}z_i z_j^{-1}}$

(5) $\prod_{\substack{j=2\\ j\neq k}}^{j=l} \dfrac{1-z_k z_j}{1-q^{-1}z_k z_j} \prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k, j\neq k}} \dfrac{1-z_i z_j}{1-q^{-1}z_i z_j}$

(6) $z_k^{m_1+m_k-2} z_2^{m_2-1} \cdots z_l^{m_l-1}$

It is not hard to see that this product can be simplified into the product of the following functions:

(1) $\prod_{i=k+1}^{i=l} \dfrac{z_k - q^{-1}z_i}{z_i - q^{-1}z_k} \cdot (1-1)^{l-k}$ (2) $\prod_{\substack{i\neq k\\ i=2}}^{i=l} \dfrac{z_k z_i - q^{-1}}{1-q^{-1}z_k z_i} \cdot (-1)^{l-2}$

(3) $-\dfrac{z_k^2 - q^{-1}}{1-q^{-1}z_k^2} \prod_{i=2}^{i=l} \dfrac{1-z_i^2}{1-q^{-1}z_i^2}$ (4) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k, j\neq k}} \dfrac{1-z_i z_j^{-1}}{1-q^{-1}z_i z_j^{-1}}$

(5) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k, j\neq k}} \dfrac{1-z_i z_j}{1-q^{-1}z_i z_j}$ (6) $(1-q^{-1})z_k^{m_1+m_k-1} z_2^{m_2-1} \cdots z_l^{m_l-1}$

This function has a z_k-pole only at $z_k=0$ and only when $m_1 = \cdots = m_l = 0$. Thus Res $_{z_k=0}$(Res $_{z_1=z_k}(\Phi_k^l(0,\ldots,0)))$ = product of the following functions:

(1) $(-q^{-1})^{l-k}(-1)^{l-k}$ (2) $(-q^{-1})^{l-2}(-1)^{l-2}$

(3) $q^{-1} \prod_{i=2}^{i=l} \dfrac{1-z_i^2}{1-q^{-1}z_i^2}$ (4) As above

(5) As above (6) As above

Successively taking the residues of this function at $z_2=\cdots=\hat{z}_k=\cdots=z_l=0$, we get:

$$(1-q^{-1})q^{-(2l-k-1)} \quad (2\leqslant k\leqslant l) \tag{I}$$

(b) $z_1=z_k^{-1}$

$\mathrm{Res}_{z_1=z_k^{-1}}(\Phi_k^l(m_1,\ldots,m_l))$ is the product of the following functions:

(1) $(-1)^{k-2}\prod_{i=2}^{i=k-1}\dfrac{z_i-q^{-1}z_k}{z_k-q^{-1}z_i}$ (2) $\dfrac{1-z_k^{-2}}{1-q^{-1}z_k^{-2}}\prod_{\substack{i=2\\ i\neq k}}^{i=l}\dfrac{1-z_i^2}{1-q^{-1}z_i^2}$

(3) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k\neq j}}\dfrac{1-z_i^{-1}z_j^{-1}}{1-q^{-1}z_i^{-1}z_j^{-1}}$ (4) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k\neq j}}\dfrac{1-z_i^{-1}z_j}{1-q^{-1}z_i^{-1}z_j}$

(5) $(1-q^{-1})z_k^{-m_1+m_k-1}z_2^{m_2-1}\cdots z_l^{m_l-1}$.

Changing z_k to z_k^{-1}, which amounts to changing $|z_k|=1\pm\varepsilon_k$ into $|z_k|=1\mp\varepsilon_k$, we get the product of the following functions:

(1) $(-1)^{k-2}\prod_{i=2}^{i=k-1}\dfrac{z_iz_k-q^{-1}}{1-q^{-1}z_iz_k}$ (2) $\dfrac{1-z_k^2}{1-q^{-1}z_k^2}\prod_{i=2}^{i=l}\dfrac{1-z_i^2}{1-q^{-1}z_i^2}$

(3) As above (4) As above

(5) $(1-q^{-1})z_k^{m_1-m_k-1}z_2^{m_2-1}\cdots\hat{z}_k\cdots z_l^{m_l-1}$

This function has at most one z_k-residue, namely $z_k=0$ and this happens iff $m_1=m_k$.

In this case: $\mathrm{Res}_{z_k=0}(\mathrm{Res}_{z_1=z_k^{-1}}(\Phi_k^l(m_1,\ldots,m_l))=$ product of the following functions:

(1) $(1-q^{-1})(-1)^{k-2}(-q^{-1})^{k-2}\prod_{\substack{i=2\\ i\neq k}}^{i=l}\dfrac{1-z_i^2}{1-q^{-2}z_i^2}$

(2) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k\neq j}}\dfrac{1-z_i^{-1}z_j^{-1}}{1-q^{-1}z_i^{-1}z_j^{-1}}$ (3) $\prod_{\substack{2\leqslant i<j\leqslant l\\ i\neq k\neq j}}\dfrac{1-z_i^{-1}z_j}{1-q^{-1}z_i^{-1}z_j}$

(4) $(1-q^{-1})z_2^{m_2-1}\cdots\hat{z}_k^{m_k-1}\cdots z_l^{m_l-1}$

When $k=2$, this function has at most one z_3-pole at $z_3=0$, which happens iff $m_3=\cdots=m_l=0$. When $k>2$, this function has at most one z_2-pole at $z_2=0$, which happens iff $m_1=\cdots=m_l=0$. Successively taking the residues at

$z_3 = \cdots = z_l = 0$ for $k=2$ and at $z_2 = \cdots = z_l = 0$ if $k>2$, we get

$$q^{-(k-2)}(1-q^{-1}), \quad (2 \leqslant k \leqslant l) \tag{II}$$

(c) $z_1 = 0$

$z_1 = 0$ is a pole of $\Phi_k^l(m_1, \ldots, m_l)$ $(i \leqslant k \leqslant l)$ iff $m_1 = \cdots = m_l = 0$. In this case $\mathrm{Res}_{z_1=0}(\Phi_k^l(m_1, \ldots, m_l))$ = product of the following functions:

(1) $q^{-1} \prod_{i=2}^{i=k-1} \dfrac{1-q^{-1}z_i^{-1}z_k}{1-z_i^{-1}z_k} \prod_{i=k+1}^{i=l} \dfrac{1-q^{-1}z_k^{-1}z_i}{1-z_k^{-1}z_i}$

(2) $q^{-1} \prod_{i=2}^{i=k-1} \dfrac{1-q^{-1}z_i^{-1}z_k^{-1}}{1-z_i^{-1}z_k^{-1}} \prod_{i=k+1}^{i=l} \dfrac{1-q^{-1}z_k^{-1}z_i^{-1}}{1-z_k^{-1}z_i^{-1}}$

(3) $\prod_{i=2}^{i=l} \dfrac{1-z_i^2}{1-q^{-1}z_i^2}$ (4) $\prod_{\substack{2 \leqslant i < j \leqslant l \\ i \neq k, j \neq k}} \dfrac{1-z_i z_j^{-1}}{1-q^{-1}z_i z_j^{-1}}$

(5) $\prod_{\substack{2 \leqslant i < j \leqslant l \\ i \neq k, j \neq k}} \dfrac{1-z_i z_j}{1-q^{-1}z_i z_j}$ (6) $z_2^{-1} \cdots z_l^{-1}$

$$= q^{-2}\Phi_{k-1}^{l-1}(0, \ldots, 0) \tag{III}$$

Let $\mathbb{T}^l$ denote the set consisting of the tori: $|z_i| = 1 \pm \varepsilon_i (1 \leqslant i \leqslant l)$. Let $2 \leqslant k \leqslant l$. For each l-tuple $(m_1, \ldots, m_l)$, we shall compute $\Sigma_{T \in \mathbb{T}^l} 1/(2\pi i)^l \int_T \Phi_k^l(m_1, \ldots, m_l)$.

We get zero, unless $m_1 = m_2$ and $m_3 = \cdots = m_l = 0$. Thus, there are only two interesting cases to consider:

(A) $m_1 = m_2 = 0$, (B) $m_1 = m_2 > 0$.

Fix k, $2 \leqslant k \leqslant l$, and partition $\mathbb{T}^l$ into four sets given by the following conditions:

(i) $|z_1| = 1 + \varepsilon_1$, $|z_k| = 1 + \varepsilon_k$.
(ii) $|z_1| = 1 + \varepsilon_1$, $|z_k| = 1 - \varepsilon_k$.
(iii) $|z_1| = 1 - \varepsilon_1$, $|z_k| = 1 + \varepsilon_k$.
(iv) $|z_1| = 1 - \varepsilon_1$, $|z_k| = 1 - \varepsilon_k$.

Let T_1, T_2, T_3, T_4 denote four tori, satisfying conditions (i), (ii), (iii), (iv) respectively. Set $\varphi_1^{(k)}(m_1, \ldots, m_l, T_i)$ = the contribution of the pole $z_1 = z_k$ when integrating $\Phi_k^l(m_1, \ldots, m_l)$ over T_i $(i = 1, 2, 3, 4)$. $\varphi_2^{(k)}(m_1, \ldots, m_l, T_i)$ will denote the contribution of the pole $z_1 = z_k^{-1}$ when integrating $\Phi_k^l(m_1, \ldots, m_l)$ over T_i. For $j = 1, 2$, set $\varphi_j^{(k)}(m_1, \ldots, m_l) = \Sigma_{i=1}^4 \varphi_j^{(k)}(m_1, \ldots, m_l, T_i)$.

It is clear (from (III)) that in case (A) ($m_1 = m_2 = 0$), we get:

$$\sum_{T\in\mathbb{T}^l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(0, 0, \dots, 0)$$
$$= 2^{l-2}[\varphi_1^{(k)}(0, 0, \dots, 0) + \varphi_2^{(k)}(0, 0, \dots, 0)]$$
$$+ 2q^{-2} \sum_{T\in\mathbb{T}^{l-1}} \frac{1}{(2\pi i)^{l-1}} \int_T \Phi_{k-1}^{l-1}(0, 0, \dots, 0). \qquad (*)$$

In case (B) ($m_1 = m_2 = m > 0$) we get:

$$\sum_{T\in\mathbb{T}^l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(0, 0, \dots, 0)$$
$$= 2^{l-2}[\varphi_1^{(k)}(m, m, 0, \dots, 0) + \varphi_2^{(k)}(m, m, 0, \dots, 0)]. \qquad (**)$$

Thus, we need to compute $\Sigma_{j=1}^{j=2} \varphi_j^{(k)}(m, m, 0, \dots, 0)$, with $m \geqslant 0$.

(A) $m_1 = m_2 = 0$

$\varphi_1^{(k)}(0, 0, \dots, 0) = 2(1 - q^{-1})q^{-(2l-k-1)}$ (from (I))

$\varphi_2^{(k)}(0, 0, \dots, 0) = 2q^{-(k-2)}(1 - q^{-1})$ (from (II))

Thus

$$\boxed{\begin{aligned}&\varphi_1^{(k)}(0, 0, \dots, 0) + \varphi_2^{(k)}(0, 0, \dots, 0)\\ &\quad = 2(1 - q^{-1})(q^{-(k-2)} + q^{-(2l-k-1)})\end{aligned}}$$

(B) $m_1 = m_2 > = m > 0$

$\varphi_1^{(k)}(m, m, 0, \dots, 0) = 0.$

$$\varphi_2^{(k)}(m, m, 0, \dots, 0) = \begin{cases} 0 & \text{if } k > 2 \\ 2(1-q^{-1}) & \text{if } k = 2 \end{cases}$$

Thus

$$\boxed{\begin{aligned}&\varphi_1^{(k)}(m, m, 0, \dots, 0) + \varphi_2^{(k)}(m, m, 0, \dots, 0)\\ &\quad = \begin{cases} 2(1-q^{-1}) & \text{if } k = 2 \\ 0 & \text{if } k > 2 \end{cases}\end{aligned}}$$

3.4. *The integral of* $\Phi_1^l(m_1, \ldots, m_l)$

In this section we integrate $\Phi_1^l(m_1, \ldots, m_l)$. This function has at most three z_2-poles, namely:

(a) $z_2 = z_1$, (b) $z_2 = z_1^{-1}$, (c) $z_2 = 0$(iff $m_2 = \cdots = m_l = 0$).

The details of the calculations in this section are similar to those in the previous ones. Therefore, we only give the final answers. [See [1] for the details.]

(a) $z_2 = z_1$

$$\operatorname{Res}_{z_l=0}(\cdots(\operatorname{Res}_{z_3=0}(\operatorname{Res}_{z_2=0}(\operatorname{Res}_{z_2=z_1}\Phi_1^l(m_1, \ldots, m_l))))\cdots) = \begin{cases} -(1-q^{-1})q^{-2l-3)} & \text{if } m_1 = \cdots m_l = 0 \\ 0 & \text{otherwise} \end{cases} \tag{IV}$$

(b) $z_2 = z_1^{-1}$

$$\operatorname{Res}_{z_l=0}(\cdots(\operatorname{Res}_{z_3=0}(\operatorname{Res}_{z_1=0}(\operatorname{Res}_{z_2=z_1^{-1}}\Phi_1^l(m_1, \ldots, m_l))))\cdots) = \begin{cases} 1-q^{-1} & \text{if } m_1 = m_2 \text{ and } m_3 = \cdots = m_l = 0 \\ 0 & \text{otherwise} \end{cases} \tag{V}$$

(c) $z_2 = 0$

$$\operatorname{Res}_{z_2=0}\Phi_1^l(m_1, \ldots, m_l) = \begin{cases} q^{-1}\Phi_1^{l-1}(0, \ldots, 0) & \text{if } m_2 = \cdots = m_l = 0 \\ 0 & \text{otherwise} \end{cases} \tag{VI}$$

Next, we compute $\Sigma_{T\in\mathbb{T}^l}(2\pi i)^{-l}\int_T\Phi_1^l(m_1, \ldots, m_l)$. There are two non-trivial cases to consider.

(A) $m_1 = \cdots = m_l = 0$, (B) $m_1 = m_2 = m > 0$ and $m_3 = \cdots = m_l = 0$.

Set $\varphi_1^{(1)}(m_1, \ldots, m_l, T_i)$ = the contribution of the pole $z_2 = z_1$, when integrating $\Phi_1^l(m_1, \ldots, m_l)$ over T_i $(i = 1, 2, 3, 4)$. $\varphi_2^{(1)}(m_1, \ldots, m_l, T_i)$ will denote the contribution of the pole $z_2 = z_1^{-1}$ when integrating $\Phi_1^l(m_1, \ldots, m_l)$ over T_i. For $j = 1, 2$, set $\varphi_j^{(1)}(m_1, \ldots, m_l) = \Sigma_{i=1}^4 \varphi_j^{(1)}(m_1, \ldots, m_l, T_i)$.

Thus, in case (A) we get (using (VI)):

$$\sum_{T\in\mathbb{T}^l}\frac{1}{(2\pi i)^l}\int_T \Phi_1^l(0, 0, \ldots, 0) = 2^{l-2}[\varphi_1^{(1)}(0, 0, \ldots, 0) + \varphi_2^{(1)}(0, 0, \ldots, 0)] + 2q^{-1}\sum_{T\in\mathbb{T}^{l-1}}\frac{1}{(2\pi i)^{l-1}}\int_T \Phi_1^{l-1}(0, 0, \ldots, 0) \tag{†}$$

In case (B) we get:

$$\sum_{T\in\mathbb{T}^l}\frac{1}{(2\pi i)^l}\int_T \Phi_1^l(m, m, 0, \ldots, 0)$$
$$= 2^{l-2}[\varphi_1^{(1)}(m, m, 0, \ldots, 0) + \varphi_1^{(2)}(m, m, 0, \ldots, 0)] \qquad (\ddagger)$$

We compute $\Sigma_{j=1}^{j=2}\varphi_j^{(1)}(m, m, 0, \ldots, 0)$ below, using (IV) and (V).

(A) $m_1 = m_2 = 0$

In this case we get: $\boxed{2(1-q^{-1})-2(1-q^{-1})q^{-(2l-3)}}$

(B) $m_1 = m_2 = m > 0$

In this case we get: $\boxed{2(1-q^{-1})}$

3.5. *The values* $f^H_{(m_1,\ldots,m_l)}(1)$

Using the results of (3.3) and (3.4) we can obtain formulae for

$$\sum_{T\in\mathbb{T}^l}\sum_{k=1}^{k=l}\frac{1}{(2\pi i)^l}\int_T \Phi_k^l(m, m, 0, \ldots, 0), \quad (m \geqslant 0)$$

(A) $m_1 = m_2 = 0$

Using (*), (A) in (3.3), and (†), (A) in (3.4); we get:

$$\sum_{T\in\mathbb{T}^l}\sum_{k=1}^{k=l}\frac{1}{(2\pi i)^l}\int_T \Phi_k^l(0, 0, \ldots, 0)$$
$$= 2^{l-1}[1 - q^{-2(l-1)} + (1 - q^{-1})(1 - q^{-(2l-3)})]$$
$$+ 2q^{-1}\left[\sum_{T\in\mathbb{T}^{l-1}}\sum_{k=1}^{k=l-1}\frac{1}{(2\pi i)^{l-1}}\int_T \Phi_k^{l-1}(0, 0, \ldots, 0)\right]$$
$$+ 2q^{-1}\left[\sum_{T\in\mathbb{T}^{l-1}}\frac{1}{(2\pi i)^{l-1}}\int_T \Phi_1^{l-1}(0, 0, \ldots, 0)\right]$$

(B) $m_1 = m_2 = m > 0$

Using (**), (B) in (3.3), and (‡), (B) in (3.4); we get:

$$\sum_{T\in\mathbb{T}^l}\sum_{k=1}^{k=l}\frac{1}{(2\pi i)^l}\int_T \Phi_k^l(m, m, 0, \ldots, 0) = 2^l(1 - q^{-1})$$

To obtain a closed formula in case (A), we need the following lemma.

LEMMA 3.1. *For $l \geqslant 2$ we have*

(i) $$\sum_{T \in \mathbb{T}^l} \sum_{k=1}^{l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(0, \ldots, 0) = 2^l \sum_{i=0}^{i=l-1} q^{-2i}$$

(ii) $$\sum_{T \in \mathbb{T}^l} \frac{1}{(2\pi i)^l} \int_T \Phi_1^l(0, \ldots, 0) = 2^{l-1}(1 + q^{-2(l-1)})$$

Proof. The proof is by induction on l. We omit the details. (See [1].) Using this lemma and the above formulae, we get the following expression for

$$\sum_{T \in \mathbb{T}^l} \sum_{k=1}^{k=l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(m, m, 0, \ldots, 0), \quad (m \geqslant 0)$$

(m, m)	$\sum_{T \in \mathbb{T}^l} \sum_{k=1}^{k=l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(m, m, 0, \ldots, 0)$
$m = 0$	$2^l[1 + q^{-2} + \cdots + q^{-2(l-1)}]$
$m > 0$	$2^l[1 - q^{-1}]$

3.6. *A matching result*

Recall that

$$(f^H_{(m,m,0,\ldots,0)})(1) = \frac{q^{2(l-1)m}}{Q_{(m,m,0,\ldots,0)}(q^{-1})} \frac{(1+q^{-1})Q_{l-1}(q^{-1})}{2|W_{B_{l-1}}|} \quad (l-1)!$$
$$\times \left[\sum_{T \in \mathbb{T}^l} \sum_{k=1}^{k=l} \frac{1}{(2\pi i)^l} \int_T \Phi_k^l(m, m, 0, \ldots, 0) \right]$$

where

$$Q_{(0,0,\ldots,0)}(q^{-1}) = Q_l(q^{-1}) = \sum_{i=0}^{i=l-1} (1 + q^{-(i+1)})(1 + q^{-1} + \cdots + q^{-i})$$

and that for $m > 0$

$$Q_{(m,m,0,\ldots,0)}(q^{-1}) = (1 + q^{-1}) \sum_{i=0}^{i=l-3} (1 + q^{-(i+1)})(1 + q^{-1} + \cdots + q^{-i}).$$

At last, we are ready to write down formulae for $f^H_{(m,m,0,\dots,0)}(1)$.

(A) $m=0$

$$f^H_{(0,0,\dots,0)}(1) = \frac{(1+q^{-1})\Sigma_{i=0}^{i=l-2}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})(l-1)!}{2\cdot 2^{l-1}(l-1)!\,\Sigma_{i=0}^{i=l-1}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})} = 1$$

(B) $m>0$

$$f^H_{(m,m,0,\dots,0)}(1) = \frac{q^{2(l-1)m}(1+q^{-1})\Sigma_{i=0}^{i=l-2}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})}{2\cdot 2^{l-1}(l-1)!(1+q^{-1})\Sigma_{i=0}^{i=l-3}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})}(l-1)!2^l(1-q^{-1})$$
$$= q^{2(l-1)m}(1-q^{-2(l-1)})$$

We conclude our computations with:

THEOREM 3.2. *There* $c\in\mathbb{C}$ *such that*

$$\int_{\mathcal{O}_{u_G}} f = cf^H(1) \quad \textit{for all } f\in C_c^\infty(\mathbf{G}(F)\,/\!/\,\mathbf{G}(\mathcal{O}))$$

Proof. This follows from (A) and (B) above and from the results of (3.1). □

Observe that the distribution $f^H \mapsto f^H(1)$ is stable by a result of Kottwitz [10].

4. The minimal unipotent conjugacy classes in $\mathbf{Sp}_{2l}(F)$ and a matching result

Let $G=\mathbf{Sp}_{2l}=\{g\in GL_{2l}: {}^tgJ_lg=J_l\}$, $J_l=\begin{bmatrix} 0 & I_l \\ -I_l & 0\end{bmatrix}$. Let u_G denote a representative of the lowest dimensional non-trivial $\mathbf{G}(\bar{F})$-conjugacy class in $\mathbf{G}(\bar{F})$. This class is also rigid and consists of non-special elements whose elementary divisors are given by the partition $1^{2l-2}2$. The F-rational points on this class form four $\mathbf{G}(F)$ conjugacy classes, denoted here by $\mathcal{O}_{u_i}$ $(i=1,2,3,4)$.

Let $\mathbf{H}=\mathbf{SO}_{2l}$. The main result of this section is to show that there exists $c_i\in\mathbb{C}(1\leqslant i\leqslant 4)$ such that $\Sigma_{i=1}^4 c_i\int_{\mathcal{O}_{u_i}} f = f^H(1)$, where 1 denotes the identity element in $\mathbf{H}(F)$, and $f\mapsto f^H$ is the Hecke-algebra homomorphism duel to the embedding $\mathbf{SO}_{2l}(\mathbb{C})=\hat{H}\hookrightarrow\hat{G}=\mathbf{SO}_{2l+1}(\mathbb{C})$.

4.1. *The unipotent orbital integral of* $f_{(m_1,\dots,m_l)}$

A basis for the Hecke algebra $C_c^\infty(\mathbf{G}(F)\,/\!/\,\mathbf{G}(\mathcal{O}))$ consists of the functions $f_{(m_1,\dots,m_l)}$ $(m_1\geqslant\cdots\geqslant m_l\geqslant 0)$, where $f_{(m_1,\dots,m_l)}$ = characteristic function of the double coset $K\,\mathrm{diag}(\pi^{m_1},\dots,\pi^{m_l},\pi^{-m_1},\dots,\pi^{-m_l})K$, $K=\mathbf{G}(\mathcal{O})$.

Consider the following Lie triple in $\mathfrak{g}=\mathrm{Lie}(\mathbf{G}(F))$.

$$X_0 = E_{1,l+1},\quad Y_0 = E_{l+1,1},\quad H_0 = E_{11} - E_{l+1,l+1}$$

Let $u_G=\exp X_0=I_{2l}+X_0$. u_G has all the properties mentioned in the introduction to this section. The ingredients of Rao's formula are as follows: $\mathfrak{g}_2=\{x\cdot X_0 : x\in F\}$, $\mathfrak{n}_2=0$. The $\mathbf{M}(F)$-open orbits in $\mathfrak{g}_2$ are given by: $V_i:=\{x\cdot X_0 : x\equiv\alpha_i \bmod (F^\times)^2\}$, where $\alpha_1=1$, $\alpha_2=\varepsilon$, $\alpha_3=\pi$, $\alpha_4=\varepsilon\pi$ are four representatives of $F^\times/(F^\times)^2$. The spaces $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have dimension $2l-2$. Rao's function is given by: $\varphi(x):=\varphi(x\cdot X_0)=|x|^{l-1}$. Set $g=\exp(x\cdot X_0)$, $x\in F$. As usual, we need to consider the following equations:

$$r_1(g)=-m_1,\ r_2(g)=-m_1-m_2,\dots,r_l(g)=-m_1-\cdots-m_l$$

where $r_i(g)=\min\{0,\mathrm{val}(x)\}$, $1\leqslant i\leqslant l$.

It is clear that the solution set is empty unless $m_2=\cdots=m_l=0$. There are two cases to consider:

(1) $m_1 = 0$, (2) $m_1 = m > 0$.

(1) $m_1 = 0$

The solution set is $\{x\in F^\times : \mathrm{val}(x)\geqslant 0\}$.

(2) $m_1 = m > 0$

The solution set is $\{x\in F^\times : \mathrm{val}(x) = -m\}$.

Next, we compute the orbital integrals $\int_{\mathcal{O}_{u_i}} f_{(m,0,\dots,0)}$.

(1) $m = 0$

Set $\Phi = f_{(0,\dots,0)}\circ\exp$. Then

$$\int_{\mathcal{O}_{u_i}} f_{(0,\dots,0)} = c_i\int_{V_i}\Phi(x\cdot X_0)|x|^{l-1}\,\mathrm{d}x$$

$$= c_i\sum_{k=0}^{\infty}\mu(\{x:\mathrm{val}(x)=k,\ x\equiv\alpha_i \bmod (F^\times)^2\})q^{-k(l-1)}$$

$$=\begin{cases} c_i\dfrac{1-q^{-1}}{2}\dfrac{1}{1-q^{-2l}} & \text{if } i=1,2\\[2ex] c_i\dfrac{1-q^{-1}}{2}\dfrac{q^{-l}}{1-q^{-2l}} & \text{if } i=3,4\end{cases}$$

Notice that $\Sigma_{i=1}^{4} c_i^{-1} \int_{\mathcal{O}_{u_i}} f_{(0,\dots,0)} = 1-q^{-1}/1-q^{-l}$.

(2) $m_1 = m > 0$

There are two cases to consider:

(2 − *i*)*m* is even   (2 − *ii*)*m* is odd.

In both cases write $\Phi = f_{(m,0,\dots,0)} \circ \exp$. Then

(2 − i)m is even

$$\int_{\mathcal{O}_{u_i}} f_{(m,0,\dots,0)} = \begin{cases} c_i \dfrac{1-q^{-1}}{2} q^{lm} & \text{if } i=1,2 \\ 0 & \text{otherwise} \end{cases}$$

(2 − ii)m is odd

$$\int_{\mathcal{O}_{u_i}} f_{(m,0,\dots,0)} = \begin{cases} 0 & \text{if } i=1,2 \\ c_1 \dfrac{1-q^{-1}}{2} q^{lm} & \text{if } i=3,4 \end{cases}$$

Notice that in both cases:

$$\sum_{i=1}^{4} c_i^{-1} \int_{\mathcal{O}_{u_i}} f_{(m,0,\dots,0)} = (1-q^{-1})q^{lm}$$

Thus we have the following:

PROPOSITION 4.1. *The space spanned by the four linear functionals: $f \mapsto \int_{\mathcal{O}_{u_i}} f\,(f \in C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$ is two dimensional.*

4.2. An explicit expression for $f^H_{(m_1,\dots,m_l)}(1)$

Again, we compute $f^H_{(m_1,\dots,m_l)}(1)$ using the spherical Plancherel theorem. We set up the coordinates so that

$$\tilde{f}^H_{(m_1,\dots,m_l)}(z_1,\dots,z_l) = \frac{q^{(lm_1+(l-1)m_2+\cdots+m_l)}}{Q_{(m_1,\dots,m_l)}(q^{-1})} \sum_{\sigma\in W(G)} [\mathbf{c}_G(z_1,\dots,z_l) z_1^{m_1}\cdots z_1^{m_l}]^\sigma$$

where $Q_{(m_1,\dots,m_l)}(q^{-1})$ is as in (3.2) and

$$\mathbf{c}_G(z_1,\dots,z_l) = \prod_{1\le i<j\le l} \frac{1-q^{-1}z_i^{-1}z_j}{1-z_i^{-1}z_j} \cdot \prod_{1\le i<j\le l} \frac{1-q^{-1}z_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}} \cdot \prod_{i=1}^{i=1} \frac{1-q^{-1}z_i^{-1}}{1-z_i^{-1}}.$$

We also need the expression for $c_H(z_1, \ldots, z_l)$:

$$\mathbf{c}_H(z_1, \ldots, z_l) = \prod_{1 \leqslant i < j \leqslant l} \frac{1-q^{-1}z_i^{-1}z_j}{1-z_i^{-1}z_j} \cdot \prod_{1 \leqslant i < j \leqslant l} \frac{1-q^{-1}z_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}$$

Arguing as in (3.2), it turns out that in order to evaluate $f^H_{(m_1, \ldots, m_l)}$ at 1, it is enough to integrate only one function $\Phi^l(m_1, \ldots, m_l)$ over the various 2^l tori: $|z_i| = 1 + \varepsilon_i (1 \leqslant i \leqslant l)$, where $q^{1/2} - 1 > \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_l > 1 - q^{-1/2}$. $\Phi^l(m_1, \ldots, m_l)$ is the product of the following functions

(1) $\displaystyle\prod_{1 \leqslant i < j \leqslant l} \frac{1-z_i z_j^{-1}}{1-q^{-1}z_i z_j^{-1}}$ (2) $\displaystyle\prod_{1 \leqslant i < j \leqslant l} \frac{1-z_i z_j}{1-q^{-1}z_i z_j}$

(3) $\displaystyle\prod_{i=1}^{i=l} \frac{1-q^{-1}z_i^{-1}}{1-z_i^{-1}}$ (4) $z_1^{m_1-1} \cdots z_l^{m_l-1}$.

4.3. *The integral of* $\Phi^l(m_1, \ldots, m_l)$

Observe that the only z_1-poles of $\Phi^l(m_1, \ldots, m_l)$ are located at (i) $z_1 = 1$, (ii) $z_1 = 0$ (if and only if $m_1 = 0$).

(i) $z_1 = 1$

Res $_{z_i=1}\Phi^l(m_1, \ldots, m_l)$ is the product of the following functions:

(1) $\displaystyle\prod_{1 < j \leqslant l} \frac{1-z_j^{-1}}{1-q^{-1}z_j^{-1}}$ (2) $\displaystyle\prod_{1 < j \leqslant l} \frac{1-z_j}{1-q^{-1}z_j}$

(3) $\displaystyle\prod_{2 \leqslant i < j \leqslant l} \frac{1-z_i z_j^{-1}}{1-q^{-1}z_i z_j^{-1}}$ (3) $\displaystyle\prod_{2 \leqslant i < j \leqslant l} \frac{1-z_i z_j}{1-q^{-1}z_i z_j}$

(5) $\displaystyle\prod_{i=2}^{i=l} \frac{1-q^{-1}z_i^{-1}}{1-z_i^{-1}}$ (6) $(1-q^{-1})z_2^{m_2-1} \cdots z_1^{m_l-1}$

Res $_{z_1=1}\Phi^l(m_1, \ldots, m_l)$ has no z_2-poles, except when $m_2 = \cdots = m_l = 0$ and $z_2 = 0$. In this case we get

$$\text{Res}_{z_l=0}(\cdots(\text{Res}_{z_2=0}(\text{Res}_{z_1=1}\Phi^l(m_1, 0, \ldots, 0)))\cdots) = 1 - q^{-1}$$

Notice that for half the 2^l tori: $|z_1| = 1 + \varepsilon_1$, and for the other half $|z_1| = 1 - \varepsilon_1$. Thus the contribution of the z_1-pole is $2^{l-1}l!(1-q^{-1})$. In other words

$$\frac{1}{(2\pi i)^l}\int_{|z_l|=1} \cdots \int_{|z_1|=1} \tilde{f}_{(m,0,\ldots,0)}(z_1, \ldots, z_l)|\mathbf{c}_H(z_1, \ldots, z_l)|^{-2} z_1^m \frac{dz_1}{z_1} \cdots \frac{dz_l}{z_l}$$
$$= 2^{l-1}l!(1-q^{-1}) \qquad (*)$$

(ii) $z_1 = 0$

$$\operatorname{Res}_{z_1=0}\Phi^l(0,\ldots,0) = q^{-1}\Phi^{l-1}(0,\ldots,0) \tag{**}$$

LEMMA 4.2

$$\frac{1}{(2\pi i)^l}\int_{|z_l|=1}\cdots\int_{|z_1|=1}\tilde{f}_{(0,\ldots,0)}(z_1,\ldots,z_l)|\mathbf{c}_H(z_1,\ldots,z_l)|^{-2}\frac{dz_1}{z_1}\cdots\frac{dz_l}{z_l}$$
$$= 2^{l-1}l!(1+q^{-2}) \tag{***}$$

Proof. By induction on l. □

The identities (*) and (***) imply that $f^H_{(m_1,m_2,\ldots,m_l)}(1)$ is given as follows:

4.4. *The values* $f^H_{(m_1,\ldots,m_l)}(1)$

$$f^H_{(m_1,\ldots,m_l)}(1) = \begin{cases} \dfrac{Q_l(q^{-1})2^{l-1}l!(1+q^{-l})}{|W(H)|\prod_{i=1}^{l-1}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})} & \text{if } m_1=\cdots=m_l=0 \\ \dfrac{(1-q^{-1})Q_l(q^{-1})2^{l-1}l!q^{lm_1}}{|W(H)|\prod_{i=1}^{l-2}(1+q^{-(i+1)})(1+q^{-1}+\cdots+q^{-i})} & \text{if } m_1>0,\ m_2=\cdots=m_l=0 \\ 0 & \text{otherwise} \end{cases}$$

4.5. *A matching result*

THEOREM 4.3. *There exists* $c_1, c_2, c_3, c_4 \in \mathbb{C}$ *such that*

$$\sum_{k=1}^{4} c_k\int_{\mathcal{O}_{u_k}} f = f^H(1) \quad \textit{for all } f\in C_c^\infty(\mathbf{G}(F)\,/\!/\,K).$$

Proof. It is enough to prove the above identity for all basic elements $f_{(m_1,\ldots,m_l)}$. For these functions the identity follows by comparing the expression for $f^H_{(m_1,\ldots,m_l)}(1)$ given by Lemma 2 with the results of (4.1). □

5. Concluding remarks

5.1. Let $\mathbf{G}$ be a semi-simple algebraic group over $\mathbb{C}$. Let $G=KAN$ be an Iwasawa decomposition of G, and set $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{g}=\mathrm{Lie}(A)$. Let Φ be the root system of $(\mathfrak{g},\mathfrak{a})$ and fix a positive set Φ^+. Let $u\in G$ be a unipotent element. Assume that the irreducible W-module U corresponding to the pair $(\mathcal{O}_u, 1)$ (via the Springer correspondence) is a Macdonald representation. Thus there exists a root subsystem Φ_0 of Φ such that $U = J^W_{W_0}(\mathrm{sgn})$, where W_0 is the Weyl group generated by Φ_0. By a result of Macdonald [13], the sign representation of W_0 is obtained by letting W_0 act on the space $\mathbb{R}\varphi$, where $\varphi = \Pi_{\alpha\in\Phi_0^+} H_\alpha (H_\alpha \in \mathfrak{g}^*$ is given by: $H_\alpha(\mu) = \langle \alpha, \mu \rangle$, $\mu \in \mathfrak{g}^*$). Now regard $\mathfrak{g}$ and $\mathfrak{a}$ as real Lie algebras and let $\mathbf{c}_u$ be the Harish–Chandra $\mathbf{c}$ function associated with Φ_0 (see Helgason [7]). It is not hard to show now that the polynomial P_u defined in (1.4) is given by: $P(\lambda) = \Sigma_{w\in W}\, w \cdot |\mathbf{c}_u(\lambda)|^{-2}$, $\lambda \in (-1)^{1/2}\mathfrak{a}^*$. It follows from this that if f is K-spherical function, then $\int_{\mathcal{O}_u} f = \alpha_u \int_{\mathfrak{a}^*} \tilde{f}(\lambda)|\mathbf{c}_u(\lambda)|^{-2}\,\mathrm{d}\lambda$, where $\tilde{f}(\lambda) = \int_A F^A_f(a) e^{-i\lambda(\log a)}\,\mathrm{d}a$, and α_u is a constant depending only on $\mathcal{O}_u$ and the normalization of measures. In fact, $\tilde{f}(\lambda) = \int_{\mathfrak{a}} (F^A_f \circ \exp)(y) e^{-i\lambda(y)}\,\mathrm{d}y = (F^A_f \circ \exp)^{\wedge}(\lambda)$. Thus $\int_{\mathfrak{a}^*} \tilde{f}(\lambda)|\mathbf{c}_u(\lambda)|^{-2}\,\mathrm{d}\lambda = \alpha \int_{\mathfrak{a}^*} (F^A_f \circ \exp)^{\wedge}(\lambda)\ (P_u(\lambda))\,\mathrm{d}\lambda = \alpha'[\partial_u(F^A_f \circ \exp)](0) = \alpha_u \int_{\mathcal{O}_u} f$ (by the Theorem in (1.4)).

Now, let $\mathbf{G}(F)$ be a group of p-adic type (in the sense of Macdonald [14]). Let $u \in \mathbf{G}(F)$ be a unipotent element. Let W denote the Weyl group of $\mathbf{G}$ and assume that the irreducible W-module corresponding to the $\mathbf{G}(\bar{F})$ orbit of u is a Macdonald representation. Let $\{gug^{-1} : g \in \mathbf{G}(\bar{F})\} \cap \mathbf{G}(F) = \Pi_{i=1}^n \mathcal{O}_{u_i}$, where $u_i \in \mathbf{G}(F)$ and $\mathcal{O}_{u_i}$ is the $\mathbf{G}(F)$ conjugacy class of u_i. Let $\mathbf{c}_u$ be the p-adic $\mathbf{c}$-function [14], obtained by the procedure described above. It is reasonable to conjecture the following.

CONJECTURE. *There exists constants $c_i \in \mathbb{C}(1 \leqslant i \leqslant n)$ such that*

$$\sum_{i=1}^n c_i \int_{\mathcal{O}_{u_i}} f = \int_{\hat{T}_1} \tilde{f}(z)|\mathbf{c}_u(z)|^{-2}\,\mathrm{d}^\times z \qquad (*)$$

where

$f \in C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$

$\tilde{f}$ = *Satake transform of f*

$\hat{T}_1 = \{z = (z_1, \ldots, z_n) : |z_i| = 1,\ n = \mathrm{rank}(G)\}$

The identity (*) *is true when the* $\mathbf{G}(\bar{F})$ *class of u is a Richardson class. Moreover, the results in sections* 3 *and* 4 *also prove this identity for the minimal classes in* $\mathbf{SO}_{2l+1}$ *and* $\mathbf{Sp}_{2l}$, *respectively.*

5.2. The above conjecture, if true, would lead to many identities between unipotent orbital integrals as in the case of complex groups (see section 1). Let's give an example. Let $\mathbf{G} = \mathbf{SO}_{2l+1}$, $\mathbf{H} = \mathbf{SO}_{2l-1} \times \mathbf{PGL}_2$, u_G, $u_\kappa (1 \leqslant \kappa \leqslant 4)$ as in section 2. Let $u_H = (1, \mathrm{reg}) \in \mathbf{H}$, where $\mathrm{reg} \in \mathbf{PGL}_2(F)$ denotes a regular unipotent element. Recall that $\mathbf{G}(\bar{F})$ class of u_G is a Richardson class. Thus there exists a parabolic subgroup of **P** of **G** with a Levi decomposition **P** = **MN** such that

$$f^{(P)}(1) = \sum_{k=1}^{4} c_k \int_{\mathcal{O}_{u_K}} f, \quad f \in C_c^\infty(\mathbf{G}(F) /\!/ \mathbf{G}(\mathcal{O}))$$

and $f^{(P)}$ is a constant term of f along P. Moreover, $\hat{M} \times \mathbf{SL}_2(\mathbb{C}) = \hat{H}$. Thus, since reg is a regular unipotent element in $\mathbf{PGL}_2(F)$ we have $c_{u_G} = c_{u_H}$. Let $f \mapsto f^H$ be as in section 2. Then

$$\begin{aligned} f^{(P)}(1) &= \int_{\hat{T}_1} \tilde{f}(z) |\mathbf{c}_{u_G}(z)|^{-2} \, \mathrm{d}^\times z \\ &= \int_{\hat{T}_1} \tilde{f}(z) |\mathbf{c}_{u_H}(z)|^{-2} \, \mathrm{d}^\times z \\ &= \int_{\hat{T}_1} (\widetilde{f^H})(z) |\mathbf{c}_{u_H}(z)|^{-2} \, \mathrm{d}^\times z \\ &= * \int_{\mathcal{O}_{u_H}} f^H \qquad \text{(by the Plancherel theorem for } H) \end{aligned}$$

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