COMPOSITIO MATHEMATICA

PAUL VOJTA

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Compositio Mathematica, tome 78, nº 1 (1991), p. 29-36
<http://www.numdam.org/item?id=CM_1991__78_1_29_0>

Compositio Mathematica **78**: 29–36, 1991.

On algebraic points on curves

PAUL VOJTA*
Department of Mathematics, University of California, Berkeley, CA 94720, USA

Received 22 May 1989; accepted 1 May 1990

In [V1], we conjectured that algebraic points (of bounded degree over $\mathbb{Q}$) on a fixed curve should obey a height inequality of the following form. Let P be an algebraic point on a curve C of genus $g > 1$, defined over a number field. Let K be the canonical divisor on C and let $h_K(P)$ denote the (logarithmic, absolute) height of P relative to K ([V1], Section 1.2). Also let

$$d(P) = \frac{\log D_{K(P)/\mathbb{Q}}}{[K(P):\mathbb{Q}]}$$

be the normalized logarithmic discriminant of (the field of definition of) P. Then we have

CONJECTURE 0.1 ([V1], (5.5.0.1)). *For all $\varepsilon > 0$ and all algebraic points P on C of bounded degree,*

$$h_K(P) \leqslant (1 + \varepsilon)d(P) + O(1)$$

where the constant in $O(1)$ depends on C, ε, and on the bound on the degree.

This conjecture, if true, would imply the *abc* conjecture and the asymptotic Fermat conjecture, among others (see [V1], (5.5.1)–(5.5.2) or Appendix 5.ABC and [V2], Section 5).

It is not clear whether the assumption bounding the degree is really necessary.

The purpose of this note is to prove the following weaker variant of Conjecture 0.1 in the function field case, as was promised in [V2], p. 164. (For notation, see Section 1.)

THEOREM 0.2: *Let C be a curve defined over a function field of characteristic zero, and fix $\varepsilon > 0$. Then for all algebraic points P on C,*

$$h_K(P) \leqslant (2 + \varepsilon)d(P) + O(1) \tag{0.3}$$

where the constant in $O(1)$ depends on C and ε.

*Supported by NSF Grant DMS 84-14105 and by the Miller Institute for Basic Research in Science.

Note that the degree may be unbounded in this case.

The proof uses the methods of Grauert's proof of Manin's theorem ([G], see also [D]), and therefore it is unlikely to translate into the number field case. In particular, it makes essential use of the derivative with respect to the variable on the base. On a related note, one might hope to decrease the $2 + \varepsilon$ by making use of higher derivatives as well, but this is not the case.

The method does not produce an effective bound for the constant in $O(1)$. It is plausible, however, that one might bound the number of algebraic families occurring in Corollary 4.4, and therefore get an effective bound on the number of families of exceptions to the above inequality with a given constant. It would not be a neat formula, though; I leave the details to those more interested in pursuing these questions.

To conclude this section, we note that other bounds of the form (0.3) are already known:

$$h_K(P) \leqslant 8 \cdot 3^{3g+1}(g-1)^2\left(s + 1 + \frac{d(P)}{3^g} + \frac{1}{3^{3g}}\right);$$

$$h_K(P) \leqslant 2(2g-1)^2(d(P) + s);$$

$$h_K(P) \leqslant \left(\frac{20g - 15}{6}\right) d(P) + O(1);$$

(where s is the number of bad fibres). The first two bounds above are effective; they are due, respectively, to Szpiro [Sz] and Esnault and Viehweg [E-V]. Note also that Szpiro has a variant of the first bound above in characteristic $p > 0$. The last bound follows from the function field analogue of [V2], Theorem 3.5, using Corollary 2.2 in place of Conjecture 3.3. All of the arguments carry over to the function field case directly, omitting special arguments for the fibres over 2 and ∞. For example, the Minkowski argument of Step 2 of the proof of Lemma 3.6 can be replaced by an application of the Riemann–Roch theorem. (One can also improve the above inequality slightly, as follows. Let B_6 be a smooth projective curve for which $F_6 = K(B_6)$ (notation as in [V2]); let D'' be the divisor on B_6 corresponding to the restriction of the divisor D' on X_6 to smooth fibres. We first want to make $\deg D''$ even. If X_6 has bad fibres, then this can be done by adding a point of bad reduction to D''; otherwise we must have $g(B_6) > 0$ and we can then replace B_6 with a nontrivial étale double cover to obtain the desired effect. Then, since the group $\mathrm{Jac}(B_6)(\mathbb{C})$ is divisible, D'' is linearly equivalent to a divisor divisible by two. Thus, after multiplying h by a constant in F_6, we may assume that D' is supported only on fibres of bad reduction. Then the factor 5/4 in condition (c) of the lemma disappears, and we have the bound

$$h_K(P) \leqslant \left(\frac{8g - 6}{3}\right) d(P) + O(1)$$

in place of the above bound.)

1. Notation

Let B be a smooth connected projective curve of genus q defined over an algebraically closed field k of characteristic zero. Let $\pi: X \to B$ be a flat proper morphism with connected fibres, and assume that X is a regular surface. Let g denote the genus of the generic fibre C of π.

Algebraic points P on C correspond to commutative diagrams

$$\begin{array}{ccc} & & X \\ & {}^{s}\nearrow & \downarrow{\scriptstyle \pi} \\ B' & \longrightarrow & B \end{array}$$

where B' is a (smooth, connected, projective) cover of B and $s_P: B' \to X$ is generically an injection.

Let $\omega_{X/B}$ be the relative dualizing sheaf, and let

$$h_K(P) = \frac{\deg s_P^* \omega_{X/B}}{[K(B'):K(B)]}.$$

Finally, let q' denote the genus of B' and let

$$d(P) = \frac{2q' - 2}{[K(B'):K(B)]}.$$

2. Effective divisors on ruled surfaces

Throughout this section, let $\mathscr{E}$ be a locally free sheaf of rank 2 on C; note that its degree is defined ([H], II Ex. 6.12). We recall the ruled surface $\mathbb{P}(\mathscr{E})$ ([H], Section II.7 and V 2.8) and its associated morphism $p: \mathbb{P}(\mathscr{E}) \to C$ and invertible sheaf $\mathcal{O}(1)$.

LEMMA 2.1. *Assume that* $\deg \mathscr{E} > 0$ *and let* D *be a divisor on* C *such that*

$$\deg D = \deg \mathscr{E}. \qquad (2.2)$$

Then for all rational $\varepsilon > 0$ *and all sufficiently large* $n \in \mathbb{N}$ *such that* $n\varepsilon \in \mathbb{Z}$,

$$h^0(\mathcal{O}(n(2+\varepsilon)) \otimes p^*\mathcal{O}_C(-nD)) > 0.$$

Proof. For all $n \in \mathbb{Z}$ such that $n\varepsilon \in \mathbb{Z}$, let

$$\mathscr{L}_n = \mathcal{O}(n(2+\varepsilon)) \otimes p^*\mathcal{O}_C(-nD).$$

Standard computations on ruled surfaces ([H], V 2.9) show that

$$\mathcal{O}(1)^2 = \deg \mathscr{E},$$

where the square on the left-hand side is an intersection number. It then follows from (2.2) that

$$\mathscr{L}_n^2 = n^2(2\varepsilon + \varepsilon^2)\deg \mathscr{E}$$
$$> 0.$$

Thus $\chi(\mathscr{L}_n) > 0$ for n sufficiently large. Since $\mathscr{L}_n$ meets fibres of p positively, it follows by duality that

$$h^2(\mathscr{L}_n) = 0.$$

By the Riemann–Roch theorem, it follows that

$$h^0(\mathscr{L}_n) > 0$$

for n sufficiently large. □

3. Curves in $\mathbb{P}(\Omega_X^1)$

We consider the three-fold $\mathbb{P}(\Omega_X^1)$. It is a fibering of projective lines over X, and is also provided with a canonical invertible sheaf $\mathscr{O}(1)$. Let $p\colon \mathbb{P}(\Omega_X^1) \to X$ denote the canonical morphism. Also, for a curve Y on $\mathbb{P}(\Omega_X^1)$, let $\deg Y$ denote the intersection number of Y with a generic fibre of $\pi \circ p$.

PROPOSITION 3.1. *Fix a rational number* $\varepsilon > 0$. *Then there exists a constant* c *and an effective divisor* $\bar{E}$ *on* $\mathbb{P}(\Omega_X^1)$ *(both depending on* ε*) such that for all irreducible curves* Y *on* $\mathbb{P}(\Omega_X^1)$ *not contained in* $\bar{E}$,

$$(Y \cdot p^*\omega_{X/B}) \leqslant (2 + \varepsilon)(Y \cdot \mathscr{O}(1)) + c \deg Y. \tag{3.2}$$

Proof. We first proceed by applying Lemma 2.1 to the locally free sheaf

$$\mathscr{E} = \Omega_X^1 | C = \Omega_{X/k} | C.$$

Then the generic fibre P of $\pi \circ p$ is the ruled surface of Lemma 2.1, and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ (resp. $p\colon \mathbb{P}(\mathscr{E}) \to C$) is the restriction of $\mathscr{O}_{\mathbb{P}(\Omega_X^1)}(1)$ (resp. $p\colon \mathbb{P}(\Omega_X^1) \to X$). The sequence

$$0 \to \pi^*\Omega_{B/k} \to \Omega_{X/k} \to \Omega_{X/B} \to 0$$

of sheaves on X is exact except for the left-most term. On the left it is exact wherever the fibre of π is reduced. Therefore, this gives an exact sequence of sheaves on C:

$$0 \to \mathscr{O}_C \to \mathscr{E} \to \Omega_{C/K(B)} \to 0.$$

In particular,

$$\deg \mathscr{E} = \deg \Omega_{C/K(B)} = 2g - 2$$

and, by Lemma 2.1, there exists an effective divisor E on P and a natural number n such that

$$\mathcal{O}_{\mathbb{P}(\mathscr{E})}(E) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}(n(2+\varepsilon)) \otimes p^*(\Omega_{C/K(B)})^{\otimes -n}.$$

Let $\bar{E}$ denote the closure of E in $\mathbb{P}(\Omega^1_X)$; then

$$\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(\bar{E}) \cong \mathcal{O}_{\mathbb{P}(\Omega^1_X)}(n(2+\varepsilon)) \otimes p^*\omega_{X/B}^{\otimes -n} \otimes \mathscr{L}$$

for some invertible sheaf $\mathscr{L}$ on $\mathbb{P}(\Omega^1_X)$ supported on the fibres of $\pi \circ p$. Since $(Y \cdot \bar{E}) \geqslant 0$,

$$\begin{aligned}(Y \cdot p^*\omega_{X/B}) &\leqslant (2+\varepsilon)(Y \cdot \mathcal{O}(1)) + \frac{1}{n}(Y \cdot \mathscr{L}) \\ &\leqslant (2+\varepsilon)(Y \cdot \mathcal{O}(1)) + c \deg Y. \end{aligned}$$

□

4. Proof of Theorem 0.2

It will suffice to prove Theorem 0.2 when ε is rational.

We will use the following construction, due to Grauert:

$$\begin{array}{ccc} & \nearrow^{t_P} & \mathbb{P}(\Omega^1_X) \\ & & \downarrow p \\ B' & \xrightarrow{\ s\ } & X \\ & \searrow & \downarrow \pi \\ & & B \end{array} \tag{4.1}$$

The map t_P is obtained as follows. Whenever one has a locally free sheaf $\mathscr{E}$ (e.g. Ω^1_X), giving a morphism $Y \to \mathbb{P}(\mathscr{E})$ is equivalent to giving a morphism $s\colon Y \to X$, an invertible sheaf $\mathscr{L}$ on Y, and a surjective map of sheaves $s^*\mathscr{E} \to \mathscr{L}$ on Y ([H], II 7.12). In this particular case, $Y = B'$, $\mathscr{L} = \Omega^1_{B'}$, and the map of sheaves is the natural map $s^*\Omega^1_X \to \Omega^1_{B'}$. (This map is not necessarily surjective, but one can replace $\mathscr{L}$ with the image of the map $s^*\Omega^1_X \to \Omega^1_{B'}$.) Moreover, this image is $t_P^*\mathcal{O}(1)$.

Since

$$\deg t_P^*\mathcal{O}(1) \leqslant \deg \Omega^1_{B'} = 2q' - 2,$$

we see that Theorem 0.2 follows from (3.2) for all points P such that $t_P(B') \nsubseteq \bar{E}$.

DEFINITION 4.2. Let X be a compact connected Kähler manifold, let $\mathscr{N}$ be an invertible sheaf on X, and let $\omega \in \Gamma(X, \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{N})$. Then a Pfaffian divisor with respect to ω is a divisor which, when written as a Cartier divisor $\{(U, f_U)\}$, has the property that for all U,

$$\omega \wedge \frac{\mathrm{d}f_U}{f_U} \in \Gamma(U, \Omega_X^2 \otimes \mathcal{N}).$$

We note that the above condition is independent of the choices of (U, f_U) representing D. Such divisors are also called integral submanifolds, but that is confusing terminology in the present context.

By choosing an injection $\mathcal{N} \hookrightarrow \mathcal{M}_X$ (the sheaf of meromorphic functions on X), we may think of ω as a meromorphic 1-form on X. Also, for any given point P, one can choose an injection as above so that ω is holomorphic at P.

The rest of the proof of Theorem 0.2 is a consequence of the following theorem of Jouanolou:

THEOREM 4.3. *Let X, ω, and $\mathcal{N}$ be as in the above definition. Then*:

(a) *there are infinitely many irreducible Pfaffian divisors with respect to ω if and only if ω is of the form $\omega = g\,\mathrm{d}\phi$ for meromorphic functions g and ϕ on X; and*
(b) *if there are finitely many such divisors, then their number is bounded by*

$$\dim[H^0(X, \Omega_X^2 \otimes \mathcal{N})/\omega \wedge H^0(X, \Omega_X^1)] + \rho + 1,$$

where ρ is the Picard number of X.

Proof. See [J], [D]. □

Theorem 0.2 is an immediate consequence of the following corollary of Theorem 4.3:

COROLLARY 4.4. *Let X be a smooth algebraic surface and let $\bar{E}$ be an effective divisor on $\mathbb{P}(\Omega_X^1)$. Then the set of irreducible curves Y_0 on X whose normalizations lift* (*via the construction* (4.1)) *to curves contained in $\bar{E}$ is a union of finitely many algebraic families.*

To see how this implies Theorem 0.2, we note that an algebraic family of curves $Y_0 \subseteq X$ lifts to an algebraic family of curves $Y \subseteq \mathbb{P}(\Omega_X^1)$; then the quantities $(Y \cdot p^*\omega_{X/B})$, etc. in (3.2) are all constant as Y varies over such curves. Thus there exists a value of c such that (3.2) holds for all Y_0 in that algebraic family. Since there are only finitely many algebraic families, by the above argument and Proposition 3.1 there exists a constant c such that (3.2) holds for all curves $Y \subseteq \mathbb{P}(\Omega_X^1)$ obtained as in (4.1); then Theorem 0.2 follows.

Proof of Corollary 4.4. It will suffice to prove this for each irreducible component of $\bar{E}$; hence we may assume that $\bar{E}$ is irreducible. We may also assume that $\bar{E}$ dominates X; otherwise it lies over a curve in X, and Y_0 must then equal that curve.

Now let X' be a desingularization of $\bar{E}$ and let $h: X' \to X$ be the composition of

p and the desingularization morphism. Let U be a nonempty open subset of X over which h is étale; then

$$\Omega^1_{X'}|h^{-1}(U) \cong h^*\Omega^1_X|U;$$

therefore one can lift $\bar{E}$ to an irreducible divisor E' on $\mathbb{P}(\Omega^1_{X'})$, of degree 1 over X'. After removing at most finitely many curves, we may assume that all curves Y_0 meet U and therefore lift to curves $Y'_0 \subseteq X'$ whose liftings to $\mathbb{P}(\Omega^1_{X'})$ lie in E'.

We would like to apply Theorem 4.3 to X'. Since $\bar{E}$ is a section of the morphism $p' \colon \mathbb{P}(\Omega^1_{X'}) \to X'$, it follows from the definition of $\mathbb{P}(\Omega^1_{X'})$ that $\bar{E}$ corresponds to a surjection $\Omega^1_{X'} \to \mathscr{L}$ for some invertible sheaf $\mathscr{L}$ on X'. Let ω be a meromorphic 1-form generating the kernel of this surjection. Now let Y'_0 be a curve on X' whose lifting lies in E', and let P be a smooth point on Y'_0. Then we may choose local coordinates z_1, z_2 on X' so that Y'_0 is locally defined by $z_1 = 0$. After possibly multiplying by a meromorphic function, we may assume that ω is holomorphic at P. By definition of the lifting construction, it follows that the surjection $\Omega^1_{X'} \to \mathscr{L}$ is given by

$$a_1\,dz_1 + a_2\,dz_2 \mapsto a_2\,dz_2$$

along Y'_0, i.e.,

$$\omega = g_1\,dz_1$$

on Y'_0. In a neighborhood of P, this becomes

$$\omega = g_1\,dz_1 + g_2 z_1\,dz_2.$$

Thus

$$\omega \wedge \frac{dz_1}{z_1} = g_2\,dz_2 \wedge dz_1$$

is holomorphic at P. Since this holds for almost all $P \in Y'_0$, it holds for all $P \in X'$, and thus Y'_0 is a Pfaffian divisor with respect to ω.

Thus, by Theorem 4.3, either (a) there are only finitely many irreducible curves $Y_0 \subseteq X$ whose liftings lie in $\bar{E}$, or (b) ω is of the form $g\,d\phi$ for some meromorphic functions g, ϕ on X'. In the latter case, another argument using local coordinates shows that all such curves Y'_0 are of the form ϕ = constant; therefore all such Y_0 lie in a pencil. Reducing to families of irreducible curves then gives a finite number of algebraic families. □

References

[D] Deschamps, M., Courbes de genre géométrique borné sur une surface de type général (d'après F. A. Bogomolov). Séminaire Bourbaki 30e année, 1977/78, no. 519 (Lecture Notes in Mathematics, 710), Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[E-V] Esnault, H. and Viehweg, E., Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields, *Comp. Math.*, to appear.

[G] Grauert, H., Mordells vermutung über rationale punkte auf algebraischen kurven und funktionenkörper. *Publ. Math. IHES* 25 (1967) 131–149.

[H] Hartshorne, R., *Algebraic geometry* (Graduate texts in mathematics, 52), Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[J] Jouanolou, J. P., Hypersurfaces solutions d'une equation de Pfaff analytique. *Math. Ann.* 232 (1978) 239–245.

[L] Lang, S., *Fundamentals of diophantine geometry.* Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[Sz] Szpiro, L., Propriétés numériques du faisceau dualisant relatif, in *Séminaire sur les pinceaux de courbes de genre au moins deux, Astérisque* 86 (1981), pp. 44–78.

[V1] Vojta, P., *Diophantine approximations and value distribution theory.* (Lecture notes in mathematics, 1239.) Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[V2] Vojta, P., Diophantine inequalities and Arakelov theory. In: Lang, S., *Introduction to Arakelov theory.* Springer-Verlag, Berlin-Heidelberg-New York, 1988, pp. 155–178.