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On module categories with nilpotent infinite radical

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Introduction

Throughout the paper k denotes a fixed algebraically closed field. By an algebra A is meant an associative, finite-dimensional k-algebra with an identity, which we shall assume to be basic and connected. By an A-module is meant a finite dimensional right A-module. We shall denote by mod A the category of finite-dimensional right A-modules, by rad(mod A) the radical of mod A, and by rad$^\infty$(mod A) the intersection of all powers rad^i(mod A), $i \geqslant 0$, of rad(mod A).

From the existence of Auslander–Reiten sequences in mod A we know that rad(mod A) is generated by the irreducible maps as a left and as a right ideal. We shall show here in (1.8) that an algebra A is representation-finite if and only if rad$^\infty$(mod A) = 0. On the other hand, it is well-known that, if A is a representation-infinite tame hereditary algebra, then rad$^\infty$(mod A) is nilpotent. In contrast, if A is wild hereditary, then rad$^\infty$(mod A) is neither left nor right T-nilpotent. We are interested in studying the algebras A for which rad$^\infty$(mod A) is left or right T-nilpotent (resp. nilpotent). We shall show that all such algebras are tame (see (1.7)).

The main result of the paper gives a complete description of standard selfinjective algebras A for which rad$^\infty$(mod A) is nilpotent. Recall that an algebra A is called standard (cf. [S1]) if it admits a simply connected Galois covering. Moreover, an algebra A is called domestic [R2] if there exists a finite number of $k[X]$-A-bimodules Q_i which are finitely generated free left $k[X]$-modules satisfying the following condition: for each dimension d, all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $V \otimes_{k[X]} Q_i$ for some i and some indecomposable finite-dimensional $k[X]$-module V (here $k[X]$ denotes the polynomial algebra in one variable). The following theorem is the main result of this paper.

THEOREM. *Let A be a standard representation-infinite selfinjective algebra. The following conditions are equivalent*

  (i) rad$^\infty$(mod A) *is right T-nilpotent.*
  (ii) rad$^\infty$(mod A) *is left T-nilpotent.*

(iii) $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ *is nilpotent.*
(iv) *A is domestic.*
 (v) *A is isomorphic to an algebra $\hat{B}/G$ where $\hat{B}$ is the repetitive algebra of a representation-infinite tilted algebra B of Euclidean type and G is an admissible infinite cyclic group of k-linear automorphism of $\hat{B}$.*

The equivalence of the last two conditions has been shown in [S1, (1.5)]. The assumption that A is selfinjective is essential for the equivalence of the remaining conditions. We shall show in (1.4) examples of standard domestic algebras A for which $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ is not right or left T-nilpotent. On the other hand, the infinite radical $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ is not right or left T-nilpotent for the known nondomestic tame algebras A. We conjecture that all algebras A with $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ T-nilpotent are domestic.

In the paper we shall identify an algebra A with the associated finite category whose objects are formed by a complete set of its primitive orthogonal idempotents. Morphisms in $\mathrm{mod}\,A$ are written on the opposite side of the scalars. We shall also agree to identify the vertices of the Auslander–Reiten quiver $\Gamma(A)$ of A with the corresponding indecomposable A-modules. We denote by τ and τ^{-1} the Auslander–Reiten translations DTr and TrD, respectively. For a background used in this paper we refer the reader to [R3] and [S2].
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1. Examples

We shall present here several examples with totally different behavior of the infinite radical. Recall that $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$ is called left (right) T-nilpotent if, for each sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathrm{rad}^{\infty}(\mathrm{mod}\,A)$, there exists a natural number m such that $f_m \cdots f_1 = 0$ $(f_1 \cdots f_m = 0)$.

(1.1) If A is representation-finite, then by the lemma of Harada and Sai, see [R1, (2.2)], $\mathrm{rad}^{\infty}(\mathrm{mod}\,A) = 0$.

(1.2) Let H be the path-algebra of the wild quiver

$$\circ \rightrightarrows \circ$$
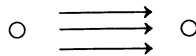
and let X be an indecomposable module of dimension-type $(1, 1)$. Then X is regular without nontrivial regular submodules and factormodules, thus each

nonzero map $f: U \to X$ with U regular is surjective and each nonzero map $g: X \to V$ with V regular is injective.

Since $\text{Ext}(X, X) \neq 0$, by the Auslander–Reiten formula there exist a nonzero map $f_0: X \to \tau X$, which is injective. Since A is hereditary τ is a left exact functor, thus all maps $f_i = \tau^i f: \tau^i X \to \tau^{i+1} X$ are injective for $i \geq 0$. Similarly for $j < 0$ all maps $\tau^j f: \tau^j X \to \tau^{j+1} X$ are surjective. Thus for all integers i and j with $j < i$ the composition $f_i \cdots f_j$ is nonzero. As all f_i are in $\text{rad}^\infty(\text{mod } A)$ the ideal $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T-nilpotent.

REMARK. Let A be wild hereditary and $(X_i)_{i \in \mathbb{Z}}$ be any family of indecomposable regular A-modules. Then one can show that there exist $n_i \in \mathbb{Z}$ and maps $f_i: \tau^{n_i} X_i \to \tau^{n_i+1} X_{i+1}$ with $f_i \in \text{rad}^\infty(\text{mod } A)$ such that for all integers j and l with $j < l$ the composition $f_l \cdots f_j$ is nonzero.

(1.3) Let A be any tilted algebra. Then it follows from [R3] and [K] that the following conditions are equivalent

  (i) A is tame.
 (ii) A is domestic.
(iii) $\text{rad}^\infty(\text{mod } A)$ is nilpotent.

(1.4) Let A be the algebra given by the quiver
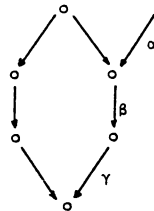
bound by the relation $\alpha\beta\gamma = 0$. Thus A is a one-point extension of a hereditary algebra of type $\tilde{\mathbb{A}}_{3,3}$ by an indecomposable nonhomogenous regular module of regular length 2. Then by [R2] A is domestic but by Lemma (4.2) $\text{rad}^\infty(\text{mod } A)$ is not right T-nilpotent. Considering the new Auslander–Reiten component one can show that $\text{rad}^\infty(\text{mod } A)$ is left T-nilpotent.

If B is the algebra given by the quiver
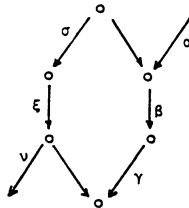
bound by the relations $\alpha\beta\gamma = 0$ and $\sigma\xi\nu = 0$, then again B is domestic, but $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T-nilpotent.

We don't know any algebra A for which $\text{rad}^\infty(\text{mod } A)$ is not nilpotent but is T-nilpotent on both sides.

(1.5) Let A be a tubular algebra (in the sense of [R3]). We claim that $\text{rad}^\infty(\text{mod } A)$ is neither left nor right T-nilpotent and $\text{rad}^\infty(\text{mod } A) \neq (\text{rad}^\infty(\text{mod } A))^2$ holds:

By [R3, (5.2)] we have

$$\Gamma(A) = \mathscr{P}_0 \vee \left(\bigvee_{\gamma \in \mathbb{Q}_0^\infty} \mathscr{T}_\gamma \right) \vee \mathscr{Q}_\infty$$

where $\mathscr{P}_0$ is a preprojective component, $\mathscr{Q}_\infty$ is a preinjective component and for any $\gamma \in \mathbb{Q}_0^\infty \mathscr{T}_\gamma$ is a separating tubular family.

Let P be a projective module, E be an injective module with $\text{Hom}(P, E) \neq 0$ and let $f : P \to E$ be a nonzero map. Then f factors through $\mathscr{T}_1$, that is, there exists a module $X \in \text{add}(\mathscr{T}_1)$ such that $f = h_0 g_0$ with $g_0 : P \to X$ and $h_0 : X \to E$.

Let $1 < \gamma_1 < \gamma_2 < \ldots$ be a strictly increasing sequence of rational numbers. Inductively we define the maps β_i and h_i by: h_0 factors through $\mathscr{T}_{\gamma_1}$, that is $h_0 = h_1 \beta_1$ with $\beta_1 : X \to X_1$, $h_1 : X_1 \to E$ with $X_1 \in \text{add}(\mathscr{T}_{\gamma_1})$. For $i > 1$ there exists $X_i \in \text{add}(\mathscr{T}_{\gamma_i})$ such that $h_{i-1} = h_i \beta_i$ with $\beta_i : X_{i-1} \to X_i$ and $h_i : X_i \to E$.

For all $l \in \mathbb{N}$ we have

$$h_0 = h_l \beta_l \cdots \beta_1 \quad \text{with } \beta_i \in \text{rad}^\infty(\text{mod } A).$$

Dually, choosing a sequence

$$1 > \delta_1 > \delta_2 > \ldots \text{ in } \mathbb{Q}_0^\infty$$

we get a family of maps $\alpha_i \in \text{rad}^\infty(\text{mod } A)$ and $g_i \in \text{rad}^\infty(\text{mod } A)$ such that

$$g_0 = \alpha_1 \ldots \alpha_l g_l \quad \text{for all } l \in \mathbb{N}.$$

Therefore $\text{rad}^\infty(\text{mod } A)$ is not T-nilpotent on either side.

Finally, observe that, for $P \in \mathscr{P}_0$ and $Y \in \mathscr{T}_0$ with $\text{Hom}(P, Y) \neq 0$, a nonzero map $f : P \to Y$ is in $\text{rad}^\infty(\text{mod } A)$ but not in $(\text{rad}^\infty(\text{mod } A))^2$.

(1.6) Let A be a standard nondomestic selfinjective algebra of polynomial growth. Then by [S2, (1.5)] A is isomorphic to $\hat{B}/G$ where $\hat{B}$ is the repetitive algebra of a tubular algebra B and G an admissible infinite cyclic group of k-linear automorphisms of $\hat{B}$. Using the structure of $\text{mod } \hat{B}$, see [NS, §3] one can show, similarly as in §2, that $\text{rad}^\infty(\text{mod } A) = (\text{rad}^\infty(\text{mod } A))^2$.

(1.7) PROPOSITION. *Let A be an algebra such that* rad$^\infty$(mod A) *is left or right T-nilpotent. Then A is tame.*

Proof. Suppose A is not tame. Then by [D] (see also [CB]), A is wild, that is, denoting by $k\langle X, Y\rangle$ the free algebra in two noncommuting variables X and Y, there exists a $k\langle X, Y\rangle$—A-bimodule M, free and finitely generated as a $k\langle X, Y\rangle$—module such that the functor $F = -\otimes M$: mod $k\langle X, Y\rangle \to$ mod A preserves indecomposability and isomorphism classes; in particular F is faithful.

Let now H be as in Example (1.2). There exists a family of maps $(f_i: X_i \to X_{i+1})_{i\in\mathbb{Z}}$ with $f_i \in$ rad$^\infty$(mod A) such that for all $j < i$ the composition $f_i \ldots f_j$ is nonzero.

On the other hand, it is well known that there exists a full exact embedding G: mod $H \to$ mod $k\langle X, Y\rangle$.

Applying the faithful functor FG: mod $H \to$ mod A we get a family $(FG(f_i))_{i\in\mathbb{Z}}$ of maps in rad$^\infty$(mod A) with arbitrary long nonzero compositions. Thus rad$^\infty$(mod A) is neither left nor right T-nilpotent.

(1.8) COROLLARY. *Let A be an algebra. Then A is representation-finite if and only if* rad$^\infty$(mod A) $= 0$.

Proof. By (1.1) we have only to show that for representation-infinite algebras rad$^\infty$(mod A) $\neq 0$; so assume A is representation-infinite. For A wild by (1.7) rad$^\infty$(mod A) $\neq 0$. For A tame, by [CB] and the validity of the Brauer–Thrall II conjecture (for a proof, see, for instance [F]) there exist (infinitely many) stable tubes of rank 1. Let M be an indecomposable module from one of these tubes and P be an indecomposable projective module with Hom$(P, M) \neq 0$. Then P and M are in different components, and therefore Hom$(P, M) =$ rad$^\infty(P, M)$.

2. Standard domestic selfinjective algebras

The aim of this section is to prove the following proposition which gives the implication (iv) $\Rightarrow$ (iii) of the theorem.

(2.1) PROPOSITION. *Let A be a representation-infinite domestic standard selfinjective algebra. Then* rad$^\infty$(mod A) *is nilpotent.*

Proof. We know from [S1, (1.5)] that $A \cong \hat{B}/G$ where B is a representation-infinite tilted algebra of Euclidean type $\Delta_B = \tilde{\mathbb{A}}_{p,q}, \tilde{D}_{r+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ with a complete slice in its preinjective component and G is an admissible infinite cyclic group of k-linear automorphism of $\hat{B}$. Let $F: \hat{B} \to \hat{B}/G = A$ be the canonical Galois covering with group G and F_λ: mod $\hat{B} \to$ mod A the associated push-down functor, see [BG]. By [ANS, (4.3)] we have

$$\Gamma(\hat{B}) = \bigvee_{p\in\mathbb{Z}} (\mathcal{X}_p \vee \mathcal{T}_p)$$

where the components $\mathscr{X}_p$ and $\mathscr{T}_p$ have the following properties: If v_B denotes the Nakayama-automorphism, we have $\mathscr{X}_{2q} = v_B^q(\mathscr{X}_0)$, $\mathscr{X}_{2q+1} = v_B^q(\mathscr{X}_1)$, $\mathscr{T}_{2q} = v_B^q(\mathscr{T}_0)$, $\mathscr{T}_{2q+1} = v_B^q(\mathscr{T}_1)$.

Here $\mathscr{X}_0(\mathscr{X}_1$, respectively) denotes a component whose stable part is isomorphic to the translation quiver $\mathbb{Z}\Delta_B$, whereas $\mathscr{T}_0(\mathscr{T}_1)$ is a $\mathbb{P}_1(k)$-family of quasi-tubes, whose stable part is a tubular $\mathbb{P}_1(k)$-family of tubular type

$$n_B = (p, q), (2, 2, r), (2, 3, 3), (2, 3, 4) \quad \text{or} \quad (2, 3, 5)$$

respectively.

Recall that a translation-quiver $\mathscr{T}$ is called a *quasi-tube* (see [S2]), if the full translation subquiver $\mathscr{T}_s$ of $\mathscr{T}$ consisting of all points of $\mathscr{T}$ which are not projective-injective is a tube in the sense of [R3].

The components $\mathscr{X}_p$, $\mathscr{T}_p (p \in \mathbb{Z})$ are standard with $\text{Hom}(\mathscr{T}_p, \mathscr{X}_p) = 0$ and $\text{Hom}(\mathscr{X}_q \vee \mathscr{T}_q, \mathscr{X}_p \vee \mathscr{T}_p) = 0$ for $p, q \in \mathbb{Z}$, $q > p$.

Moreover, it was shown in [S1, (2.13)] that G is generated by an element g such that

$$g(\mathscr{X}_p) = \mathscr{X}_{p+m}, \quad g(\mathscr{T}_p) = \mathscr{T}_{p+m} \text{ for some } m > 0 \text{ and all } p \in \mathbb{Z}.$$

Further, [ANS, (4.3)] implies that $(\text{rad}^\infty(\text{mod } \hat{B}))^5 = 0$ Finally, [ANS, (4.3)] also states that $\hat{B}$ is locally support-finite. Therefore, by [DS, (2.5)], the push-down functor $F_\lambda : \text{mod } \hat{B} \to \text{mod } \hat{B}/G$ is a Galois covering, so F_λ is dense and induces isomorphisms

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\hat{B}}(M, {}^{g^n}N) \to \text{Hom}_A(F_\lambda M, F_\lambda N)$$

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\hat{B}}({}^{g^n}M, N) \to \text{Hom}_A(F_\lambda M, F_\lambda N) \quad \text{for all } M, N \in \text{mod } \hat{B}.$$

Let M and N be indecomposable $\hat{B}$-modules and $f : F_\lambda M \to F_\lambda N$ a nonzero map. Then there exists a (uniquely determined) family of maps $(h_n : M \to {}^{g^n}N)_{n \in \mathbb{Z}}$ in mod $\hat{B}$ such that $f = \Sigma F_\lambda(h_n)$.

We claim that for $f \in \text{rad}^\infty(F_\lambda M, F_\lambda N)$ all h_n are in $\text{rad}^\infty(M, {}^{g^n}N)$. It is clear if M and ${}^{g^n}N$ belong to different components of $\Gamma(\hat{B})$. Consider now the case that M and ${}^{g^n}N$ belong to the same component $\mathscr{C}$. Since $F_\lambda(N) = F_\lambda({}^{g^n}N)$ we may assume that M and N are in the same component $\mathscr{C}$. Then we have $f = \Sigma_{n \geq 0} F_\lambda(h_n)$ with $h_n \in \text{rad}^\infty(M, {}^{g^n}N)$ for all $n \geq 1$. We shall show that $h_0 = 0$. Suppose $h := h_0 \neq 0$. Then we get $F_\lambda(h) = f - \Sigma_{n \geq 1} F_\lambda(h_n) \in \text{rad}^\infty(F_\lambda M, F_\lambda N)$. Now there exists a natural number m such that $\text{rad}^m(M, N) = 0$. This is trivial if $\mathscr{C}$ is one of the components $\mathscr{X}_i$ and will be shown in (2.2) if $\mathscr{C}$ is a quasi-tube. On the other hand, we have $F_\lambda(h) \in \text{rad}^\infty(F_\lambda M, F_\lambda N) \subset \text{rad}^m(F_\lambda M, F_\lambda N)$. Hence there

exist in $\mathscr{C}$ paths of irreducible maps

$$M \xrightarrow{\;f_{1,i}\;} M_{1,i} \xrightarrow{\;f_{2,i}\;} \cdots \xrightarrow{\;f_{m,i}\;} M_{m,i} = M_i, \quad i = 1 \ldots r$$

and maps $u_i \in \operatorname{Hom}_A(F_\lambda M_i, F_\lambda N)$ such that

$$F_\lambda(h) = \sum_{i=1}^r u_i F_\lambda(f_{m,i}) \ldots F_\lambda(f_{1,i})$$

where all summands $u_i F_\lambda(f_{m,i}) \ldots F_\lambda(f_{m,1})$ are nonzero.

Further, since $F_\lambda \colon \operatorname{mod} \hat{B} \to \operatorname{mod} A$ is a Galois covering, for each i, there exist maps $v_{n,i} \colon M_i \to {}^{g^n}N$, $n \geqslant 0$ such that

$$u_i = \sum_{n \geqslant 0} F_\lambda(v_{n,i}).$$

So we have

$$F_\lambda(h) = \sum_{i=1}^r \sum_{n \geqslant 0} F_\lambda(v_{n,i} f_{m,i} \cdots f_{1,i})$$

$$= F_\lambda \left(\sum_{i=1}^r \sum_{n \geqslant 0} v_{n,i} f_{m,i} \cdots f_{1,i} \right)$$

with $v_{n,i} f_{m,i} \cdots f_{1,i} \in \operatorname{rad}^m(M, {}^{g^n}N)$

Since F_λ induces an isomorphism

$$\bigoplus_{n \geqslant 0} \operatorname{Hom}(M, {}^{g^n}N) \to \operatorname{Hom}(F_\lambda M, F_\lambda N)$$

we obtain

$$h = \sum_{i=1}^r \sum_{n \geqslant 0} v_{n,i} f_{m,i} \cdots f_{1,i}$$

and consequently

$$h = \sum_{i=1}^r v_{0,i} f_{m,i} \cdots f_{1,i}$$

By our choice of m the group $\operatorname{rad}^m(M, N)$ is zero. Thus all summands $v_{0,i} f_{m,i} \cdots f_{1,i}$ are zero and so h, a contradiction.

Let us finally show that $(\text{rad}^\infty(\text{mod } A))^5 = 0$.

Take a chain of maps between indecomposable A-modules

$$X_0 \xrightarrow{\;f_1\;} X_1 \xrightarrow{\;f_2\;} X_2 \xrightarrow{\;f_3\;} X_3 \xrightarrow{\;f_4\;} X_4 \xrightarrow{\;f_5\;} X_5$$

such that $f_i \in \text{rad}^\infty(X_{i-1}, X_i)$.

Let $X_i = F_\lambda(M_i)$ for some indecomposable $\hat{B}$-module M_i. For each i there exists a family of maps $f_{n,i} \in \text{rad}^\infty(M_{i-1}, g^n M_i)$ such that $f_i = \Sigma_{n \in \mathbb{Z}} F_\lambda(f_{n,i})$.

Fix some family $(n_1, \ldots, n_5) \in \mathbb{Z}^5$ and take

$$\varphi_1 = f_{n_1,1}, \; \varphi_2 = g^{n_1} f_{n_2,2}, \; \varphi_3 = g^{n_1+n_2} f_{n_3,3},$$

$$\varphi_4 = g^{n_1+n_2+n_3} f_{n_4,4} \quad \text{and} \quad \varphi_5 = g^{n_1+n_2+n_3+n_4} f_{n_5,5}$$

Then we have

$$F_\lambda(f_{n_5,5}) \circ \cdots \circ F_\lambda(f_{n_1,1}) = F_\lambda(\varphi_5) \circ \cdots \circ F_\lambda(\varphi_1)$$

$$= F_\lambda(\varphi_5 \ldots \varphi_1).$$

Since $\varphi_i \in \text{rad}^\infty(\text{mod } \hat{B})$ and $(\text{rad}^\infty(\text{mod } \hat{B}))^5 = 0$ we have $\varphi_5 \ldots \varphi_1 = 0$ and therefore $f_5 \ldots f_1 = 0$. This shows that $(\text{rad}^\infty(\text{mod } A))^5 = 0$.

(2.2) **LEMMA.** *Let A be a selfinjective algebra, $\mathcal{T}$ be a standard quasi-tube in $\Gamma(A)$ and X, Y be indecomposables in $\mathcal{T}$. Then there exists a number m such that $\text{rad}^m(X, Y) = 0$.*

Proof. Let E_i ($i = 1, \ldots, r$) be the indecomposable projective–injective modules in $\mathcal{T}$ and take $p \in \mathbb{N}$ such that $\text{rad}^p(\text{End}(E_i)) = 0$ for all $i = 1, \ldots, r$.

Notice that $\mathcal{T}_s$ is a stable tube, since A is selfinjective. The modules $X_i = E_i/\text{Soc } E_i$ and $Y_i = \text{rad } E_i$ are in $\mathcal{T}_s$.

By the standardness of $\mathcal{T}$ it is enough to show that all irreducible paths from X to Y of length at least m are zero.

First we consider the case X and Y not projective–injective. Since $\mathcal{T}_s$ is a stable tube the notion of the regular length $l(M)$ of a point M in $\mathcal{T}_s$ is defined. Let

$$l = \max\{l(X), l(Y), l(X_i), l(Y_i) \,|\, i = 1, \ldots, r\}.$$

Let $\tilde{\mathcal{T}}$ be the universal covering of $\mathcal{T}$, see [BG]. Then the stable part $\tilde{\mathcal{T}}_s$ of $\tilde{\mathcal{T}}$ is of type $\mathbb{Z}A_\infty$ and we have

$$\begin{array}{ccc} \tilde{\mathcal{T}}_s & \to & \tilde{\mathcal{T}} \\ \downarrow & & \downarrow F \\ \mathcal{T}_s & \to & \mathcal{T} \;\; = \tilde{\mathcal{T}}/H \end{array}$$

where H is an infinite cyclic group $H = \langle h \rangle$ and F is the covering map, see [BG].

In the mesh-category $k(\tilde{\mathcal{T}}_s)$ every nonzero path between vertices X', $Y' \in \tilde{\mathcal{T}}_s$ with $l(X') \leqslant l$ and $l(Y') \leqslant l$ has length at most $2l$.

Let $N > 2l$ and w be a nonzero irreducible path in $k(\tilde{\mathcal{T}})$ of length N from X' to Y', where X' and Y' are in $\tilde{\mathcal{T}}_s$ with $l(X') \leqslant l$ and $l(Y') \leqslant l$. We claim that in the mesh-category $k(\tilde{\mathcal{T}})$ we have $w = \Sigma\, w_i$, where w_i are paths from X' to Y' of length N passing through at least $(N - 2l)/2(l+1)$ projective–injective vertices. Indeed, in $k(\tilde{\mathcal{T}})$ we get $w = \Sigma\, v_i$, where each v_i is a path from X' to Y' passing through an injective–projective vertex $E(i)$, that is, $v_i = n_1 \alpha \beta n_2$ with

$$X' \xrightarrow{\;n_1\;} Y(i) \xrightarrow{\;\;\alpha\;\;} E(i) \xrightarrow{\;\;\beta\;\;} X(i) \xrightarrow{\;n_2\;} Y'$$

Since $l(X(i)) \leqslant l$, $l(Y(i)) \leqslant l$ we can repeat the argument for n_1 and n_2, if necessary and get the assertion.

Take now $m = 4(l+1)^2(p+1)r$ and again X and Y in $\mathcal{T}_s$ with $l(X) \leqslant l$, $l(Y) \leqslant l$. If w is an irreducible path of length $N \geqslant m$ in $\mathcal{T}$ from X to Y, there exists a path $\tilde{w}$ from $^1 X$ to $^{h^i} Y$, for some i, in $\tilde{\mathcal{T}}$ such that $F(\tilde{w}) = w$.

In $k(\tilde{\mathcal{T}})$ we have $\tilde{w} = \Sigma\, w_i$, where each w_i passes at least through $r(p+1)$ projective–injective vertices. But then $F(\tilde{w}) = w = \Sigma\, F(w_i)$ and each path $F(w_i)$ passes through at least $r(p+1)$ projective–injective modules. Since there are only r projective–injective modules in $\mathcal{T}$, for each i, we find a fixed projective–injective module $E(i)$ such that $F(w_i)$ meets $E(i)$ at least $p+1$ times. But $\mathrm{rad}^p(\mathrm{End}(E_i)) = 0$, so $F(w_i) = 0$.

If X or Y is projective–injective then there exists only one irreducible map $X \to X_1$ with source X and one irreducible map $Y_1 \to Y$ with target Y. Since X_1 or Y_1 are in $\mathcal{T}_s$, we can apply the first case.

3. Special biserial algebras

(3.1) Recall that a locally bounded category R is *special biserial*, if it is isomorphic to a bound quiver category $k\mathcal{Q}/I$, where the bound quiver $(\mathcal{Q}, I)$ satisfies the following conditions:

(1) The number of arrows in $\mathcal{Q}$ with a prescribed source or target is at most two,
(2) for any arrow α of $\mathcal{Q}$, there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

A triangular locally bounded category R is called *gentle* if it is isomorphic to $k\mathcal{Q}/I$, where the bound quiver $(\mathcal{Q}, I)$ satisfies (1), (2) and the following two conditions:

(3) I is generated by a set of paths of length two,

(4) For any arrow α of $\mathcal{Q}$ there is at most one arrow ξ and at most one arrow η such that $\alpha\xi$ and $\eta\alpha$ belong to I.

By a *tree category* we mean a locally bounded category whose ordinary quiver is a tree.

It was shown in [PS] that a special biserial selfinjective algebra A admits a Galois covering $R \to A = R/G$ where R is a simply connected locally bounded category $R = \hat{B}$, for some gentle tree B, and G is an admissible torsion-free group of k-linear automorphisms of R. Here, $\hat{B}$ denotes the repetitive algebra of B (see [HW, S1]).

(3.2) PROPOSITION. *Let R be a special biserial simply connected selfinjective locally bounded category, G an admissible group of k-linear automorphisms of R and assume additionally that $A = R/G$ is nondomestic. Then $\operatorname{rad}^{\infty}(\operatorname{mod} R)$ and $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ are neither left nor right T-nilpotent.*

Proof. We first consider $\operatorname{rad}^{\infty}(\operatorname{mod} R)$. From the proof of [ES, (2.4)] we see that R contains a full convex subcategory D of the following form

bound only by $\alpha_i\beta_i$, $\sigma_i\gamma_i$ $(i \in \mathbb{Z})$, where may be $v_i = x_i$. We may also assume that the number of objects on the line

$$x_0 \to \cdots \leftarrow x_1$$

is greater than $\dim_k A$.

By M_n we denote the indecomposable R-module whose support is the line

$$v_0 \leftarrow \cdots \leftarrow y_0 \leftarrow x_0 \to z_0 - \cdots \to v_n \leftarrow \cdots \leftarrow y_n \leftarrow x_n.$$

It was shown in [ES, (2.4)] that the modules M_n, $n \geqslant 1$ belong to pairwise different Auslander–Reiten component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. Hence the canonical projections $f_n: M_{n+1} \to M_n$ are in $\operatorname{rad}^{\infty}(M_{n+1}, M_n)$. Obviously for all $l \in \mathbb{N}$ $f_1 \circ \cdots \circ f_l \neq 0$ holds, that is, $\operatorname{rad}^{\infty}(\operatorname{mod} R)$ is not right T-nilpotent.

Considering the sequence $(X_n)_{n \in \mathbb{N}}$ of indecomposable R-modules whose supports are the lines

$$v_{-n} \leftarrow \cdots \leftarrow y_{-n} \leftarrow x_{-n} \to \cdots \to v_0 \leftarrow \cdots \leftarrow x_0$$

we get a chain of canonical monomorphisms

$$X_1 \xrightarrow{\ g_1\ } X_2 \xrightarrow{\ g_2\ } X_3 \xrightarrow{\ g_3\ } \cdots$$

Similarly, as for the modules M_n, we can also show that the modules X_n, $n \geqslant 1$, belong to pairwise different Auslander–Reiten components of type $\mathbb{Z}A_\infty^\infty$. Thus we have $g_i \in \mathrm{rad}^\infty(X_n, X_{n+1})$ and therefore $\mathrm{rad}^\infty(\mathrm{mod}\ R)$ is not left T-nilpotent.

The assertion for $\mathrm{rad}^\infty(\mathrm{mod}\ A)$ then follows from the following lemma.

(3.3) LEMMA. *Let R be a locally bounded category and G be a group of k-linear automorphisms of R acting freely on the objects of R. Assume that $\mathrm{rad}^\infty(\mathrm{mod}\ R/G)$ is right (left) T-nilpotent. Then $\mathrm{rad}^\infty(\mathrm{mod}\ R)$ is right (left) T-nilpotent.*

Proof. If $\mathrm{rad}^\infty(\mathrm{mod}\ R)$ is not right T-nilpotent then there exists a chain

$$M_0 \xleftarrow{\ f_0\ } M_1 \xleftarrow{\ f_1\ } M_2 \xleftarrow{\ f_2\ } \cdots$$

in $\mathrm{mod}\ R$ with $f_i \in \mathrm{rad}^\infty(M_{i+1}, M_i)$ such that $f_0 \ldots f_l \neq 0$ for all $l \geqslant 0$. Applying the push down functor $F_\lambda \colon \mathrm{mod}\ R \to \mathrm{mod}\ R/G$ we see that $F_\lambda(f_i) \in \mathrm{rad}^\infty(F_\lambda M_{i+1}, F_\lambda M_i)$ with $0 \neq F_\lambda(f_0) \ldots F_\lambda(f_l) = F_\lambda(f_0 \ldots f_l)$ for all l.

Thus $\mathrm{rad}^\infty(\mathrm{mod}\ R/G)$ is not right T-nilpotent.

(3.4) The following is a direct consequence of (2.1), (3.1) and the structure theorem [S, (1.5)] for standard domestic selfinjective algebras.

COROLLARY. *Let A be a special biserial selfinjective algebra. Then A is domestic if and only if $\mathrm{rad}^\infty(\mathrm{mod}\ A)$ is nilpotent.*

4. Preparatory lemmas

We will present here several lemmas, needed for the remaining parts of the proof of our main result.

(4.1) LEMMA. *Let A be a category, B a full subcategory of A and assume that $\mathrm{rad}^\infty(\mathrm{mod}\ A)$ is right (left) T-nilpotent. Then $\mathrm{rad}^\infty(\mathrm{mod}\ B)$ is right (left) T-nilpotent.*

Proof. Let $E_0 \colon \mathrm{mod}\ A \to \mathrm{mod}\ B$ be the restriction functor associated with the full embedding $E \colon B \to A$. Let $E_\lambda \colon \mathrm{mod}\ B \to \mathrm{mod}\ A$ be a left adjoint functor to E_0 such that $E_0 E_\lambda \xrightarrow{\sim} \mathbb{1}_{\mathrm{mod}\ B}$. Then the lemma follows from the fact of E_λ being faithful.

(4.2) LEMMA. *Let B be an algebra whose Auslander–Reiten quiver contains a tube $\mathcal{T}$. Assume that M is an indecomposable module in $\mathcal{T}$ such that the full subcategory $\mathcal{X}$ of the vector space category $\mathrm{Hom}(M, \mathrm{mod}\ B)$ consisting of the*

objects $\{\operatorname{Hom}(M, X) \neq 0 \mid X \in \mathscr{T}\}$ *has a cofinite, full, successor-closed sub-category of the form*

$$\operatorname{Hom}(M, N_1) \to \operatorname{Hom}(M, N_2) \to \operatorname{Hom}(M, N_3) \to \cdots$$
$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$\operatorname{Hom}(M, M_1) \to \operatorname{Hom}(M, M_2) \to \operatorname{Hom}(M, M_3) \to \cdots$$

where the modules M_i, N_i *are neither projective nor injective.*

If $B[M]$ *denotes the one-point extension of* B *by* M, *then* $\operatorname{rad}^\infty(\operatorname{mod} B[M])$ *is not right T-nilpotent.*

Proof. Let X be a point from the mouth of $\mathscr{T}$ such that there exists an infinite sectional path

$$\Sigma: \cdots \to X_i \to X_{i-1} \to \cdots \to X.$$

This path meets the sectional path pointing to infinity

$$N_1 \to N_2 \to N_3 \to \cdots$$

at the modules $N_i' = N_{p_0 + ir}$ for some fixed p_0 and r.

Let $\psi_i \colon N_{i+1}' \to N_i'$ be the composition of the canonical maps lying on the sectional path Σ. Then we have $\psi_1 \circ \cdots \circ \psi_l \neq 0$ for all $l \geqslant 1$. We claim that $\psi_i \colon (0, N_{i+1}', 0) \to (0, N_i', 0)$ considered as map in mod $B[M]$ is in $\operatorname{rad}^\infty(\operatorname{mod} B[M])$ and thus $\operatorname{rad}^\infty(\operatorname{mod} B[M])$ is not right T-nilpotent.

Since $\operatorname{Hom}(M, \psi_i) \colon \operatorname{Hom}(M, N_{i+1}') \to \operatorname{Hom}(M, N_i')$ is zero, we get a nonzero map

$$(0, \psi_i) \colon (k, N_{i+1}', 1) \to (0, N_i', 0).$$

Thus we have the following diagram of indecomposable modules

$$
\begin{array}{ccc}
(0, N_{i+1}', 0) & \xrightarrow{\ \psi_i\ } & (0, N_i', 0) \\
\downarrow{\scriptstyle\alpha_1} & & \uparrow{\scriptstyle(0,\,\psi_i)} \\
(k, N_{i+1}' \oplus M_{p_0 + ir + 1}, \Delta) & \xrightarrow{\ \beta_1\ } & (k, N_{i+1}', 1) \\
\downarrow{\scriptstyle\alpha_2} & & \uparrow{\scriptstyle\text{id}} \\
(k, N_{i+1}' \oplus M_{p_0 + ir + 2}, \Delta) & \xrightarrow{\ \beta_2\ } & (k, N_{i+1}', 1) \\
\downarrow{\scriptstyle\alpha_3} & & \uparrow{\scriptstyle\text{id}} \\
(k, N_{i+1}' \oplus M_{p_0 + ir + 3}, \Delta) & \xrightarrow{\ \beta_3\ } & (k, N_{i+1}', 1) \\
\downarrow{\scriptstyle\alpha_4} & & \uparrow{\scriptstyle\text{id}} \\
\vdots & & \vdots
\end{array}
$$

where $\Delta: k \rightarrow \mathrm{Hom}(M, N'_{i+1} \oplus M_{p_0+ir+l})$ denotes the diagonal map, β_l the canonical projections, α_1 the canonical monomorphism and $\alpha_l (l \geqslant 2)$ the obvious maps induced by $M_{p_0+ir+l} \rightarrow M_{p_0+ir+l+1}$. Clearly we have

$$\psi_i = (0, \psi_i) \circ \beta_l \circ \alpha_l \circ \cdots \circ \alpha_1$$

for all $l \geqslant 1$, that is, $\psi_i \in \mathrm{rad}^\infty(\mathrm{mod}\ B[M])$.

(4.3) LEMMA. *Let B be an algebra whose Auslander–Reiten quiver contains* (*not necessarily different*) *tubes* $\mathcal{T}_1$ *and* $\mathcal{T}_2$. *Assume* M_1 *is an indecomposable module in* $\mathcal{T}_1$ *and* N_1 *is an indecomposable module in* $\mathcal{T}_2$ *such that the full subcategory* $\mathcal{Y}$ *of the vector space category* $\mathrm{Hom}(M_1 \oplus N_1\ B\text{-mod})$ *defined by the objects* $\{\mathrm{Hom}(M_1 \oplus N_1, Y) \neq 0 \mid Y \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ *is of the form*

$$\mathrm{Hom}(M_1, M_1) \rightarrow \mathrm{Hom}(M_1, M_2) \rightarrow \mathrm{Hom}(M_1, M_3) \rightarrow \cdots$$
$$\mathrm{Hom}(N_1, N_1) \rightarrow \mathrm{Hom}(N_1, N_2) \rightarrow \mathrm{Hom}(N_1, N_3) \rightarrow \cdots$$

If $B[M_1 \oplus N_1]$ *denotes the one-point extension of B by* $M_1 \oplus N_1$, *then* $\mathrm{rad}^\infty(\mathrm{mod}\ B[M_1 \oplus N_1])$ *is not right T-nilpotent.*

Proof. Keeping the notations of (4.2) we analogously get a commutative diagram of indecomposable modules

$$
\begin{array}{ccc}
(0, N'_{i+1}, 0) & \xrightarrow{\psi_i} & (0, N'_i, 0) \\
\downarrow{\scriptstyle\alpha_1} & & \uparrow{\scriptstyle(0,\psi_i)} \\
(k, N'_{i+1} \oplus M_1, \Delta) & \xrightarrow{\beta_1} & (k, N'_{i+1}, 1) \\
\downarrow{\scriptstyle\alpha_2} & & \uparrow{\scriptstyle\mathrm{id}} \\
(k, N'_{i+1} \oplus M_2, \Delta) & \xrightarrow{\beta_2} & (k, N'_{i+1}, 1) \\
\downarrow{\scriptstyle\alpha_3} & & \uparrow{\scriptstyle\mathrm{id}} \\
\vdots & & \vdots
\end{array}
$$

Again we have

$$\psi_i = (0, \psi_i)\beta_l \alpha_l \ldots \alpha_1 \quad \text{for all } l \in \mathbb{N},$$

that is,

$$\psi_i \in \mathrm{rad}^\infty(\mathrm{mod}\ B[M_1 \oplus N_1])$$

with

$$\psi_1 \circ \cdots \psi_t \neq 0 \quad \text{for all } t \in \mathbb{N}.$$

(4.4) LEMMA. *Let* $B = C[M]$ *be a one-point extension of a tame concealed algebra C with extension vertex* a_0 *by a simple regular module M and let D be the category obtained from B by rooting the infinite branch L*

$$\circ \rightarrow \circ \leftarrow \circ \rightarrow \cdots \rightarrow \quad \circ \quad \leftarrow \circ \rightarrow \quad \circ \quad \leftarrow \cdots$$
$$a_0 \quad a_1 \quad a_2 \qquad\qquad a_{2i-1} \quad a_{2i} \quad a_{2i+1}$$

at the vertex a_0. *Then* $\mathrm{rad}^\infty(\mathrm{mod}\, D)$ *is not right T-nilpotent.*

Proof. Let $\mathscr{T}$ be the tube of $\Gamma(B)$ containing the module M. Consider in $\mathscr{T}$ the sectional paths

$$\Sigma: M = M_0 \xrightarrow{\;g_1\;} M_1 \xrightarrow{\;g_2\;} M_2 \rightarrow \cdots$$

starting at M and pointing to infinity and

$$\Omega: N_0 \xleftarrow{\;\;h_1\;\;} N_1 = M_1 \xleftarrow{\;\;h_2\;\;} N_2 \xleftarrow{\;\;h_3\;\;} N_3 \leftarrow \cdots$$

containing M_1 and pointing to the mouth of $\mathscr{T}$.

Then there is a sequence $1 = i_1 < i_2 < i_3 < \ldots$ such that the modules $X_r = N_{i_r}, r \geqslant 1$ form the intersection of both paths Σ and Ω.

Denote by $f_r: X_{r+1} \rightarrow X_r$ the composition $f_r = h_{i_r+1} h_{i_r+2} \circ \cdots \circ h_{i_{r+1}}$. Obviously all f_r are surjective and thus $f_1 \circ \cdots \circ f_l \neq 0$ for all $l \in \mathbb{N}$.

We claim that $f_r \in \mathrm{rad}_D^\infty(X_{r+1}, X_r)$ for all $r \geqslant 1$ which shows that $\mathrm{rad}^\infty(\mathrm{mod}\, D)$ is not right T-nilpotent.

Fix some $r \geqslant 1$. Let Y_i (Z_i, respectively) for $i \geqslant 1$ be the indecomposable D-module such that the restriction of Y_i (Z_i) to C is X_{r+1} (M, respectively) and the restriction of Y_i and Z_i to L is the unique indecomposable module whose support is the finite branch

$$a_0 \rightarrow a_1 \leftarrow \cdots \rightarrow a_{2i-1}.$$

For all $i \geqslant 1$ we get the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 \to & M & \overset{\alpha}{\longrightarrow} & X_{r+1} & \overset{\beta}{\longrightarrow} & U_r & \to 0 \\
 & \downarrow & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \mathrm{id}} & \\
0 \to & Z_1 & \longrightarrow & Y_1 & \overset{p_1}{\longrightarrow} & U_r & \to 0 \\
 & \downarrow & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \mathrm{id}} & \\
0 \to & Z_2 & \longrightarrow & Y_2 & \overset{p_2}{\longrightarrow} & U_r & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
0 \to & Z_{i-1} & \longrightarrow & Y_{i-1} & \overset{p_{i-1}}{\longrightarrow} & U_r & \to 0 \\
 & \downarrow & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \mathrm{id}} & \\
0 \to & Z_i & \longrightarrow & Y_i & \overset{p_i}{\longrightarrow} & U_r & \to 0
\end{array}
$$

where $U_r = N_{i_{r+1}-1}$, $\beta = h_{i_{r+1}}$, α is the composition
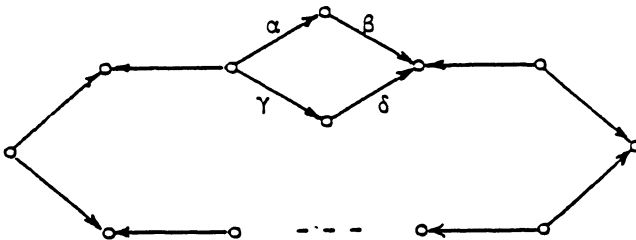
$$M \xrightarrow{\;g_1\;} M_1 \xrightarrow{\;g_2\;} M_2 \to \cdots \xrightarrow{\;g_j\;} M_j = X_{r+1},$$

and the maps $p_i(\gamma_i)$ are the obvious epimorphisms (monomorphisms). Then for any $i > 1$ we have

$$\beta = p_i \gamma_i \ldots \gamma_1,$$

that is, $\beta \in \mathrm{rad}_D^\infty(X_{r+1}, U_r)$. But $f_r = f_r' \beta$ then also is contained in $\mathrm{rad}^\infty(\mathrm{mod}\, D)$ and we are done.
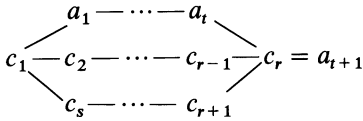
(4.5) LEMMA. *Let D be an algebra given by the quiver*

bound by $\alpha\beta = a\gamma\delta$, $a \in k^*$, and these are the only paths of length at least too. Then rad$^\infty$(mod A) *is neither left nor right T-nilpotent.*

Proof. Imitate the proof of [AS1, Lemma 3.3(III)] using (3.3), (4.1) and (4.2).

(4.6) LEMMA. *Let B an algebra given by the quiver*

$$
\begin{array}{c}
a_1 \,\text{---}\, \cdots \,\text{---}\, a_t \\
c_1\text{---}c_2\text{---}\cdots\text{---}c_{r-1}\text{---}c_r = a_{t+1} \\
c_s \,\text{---}\, \cdots \,\text{---}\, c_{r+1}
\end{array}
$$

bound only by zero-relations and such that the category C formed by the edges $c_1 \text{---} c_2 \text{---} \cdots \text{---} c_s \text{---} c_1$ *is hereditary of type* $\tilde{\mathbb{A}}_{s-1}$. *Then* rad$^\infty$(mod $\hat{B}$) *is neither left nor right T-nilpotent.*

Proof. Taking a suitable full subcategory of B, by (4.1), we may assume that the walks $c_1 \text{---} a_1 \text{---} \cdots \text{---} a_t \text{---} a_{t+1} = c_r$, $c_1 \text{---} c_2 \text{---} \cdots \text{---} c_r$ and $c_1 \text{---} c_s \text{---} \cdots \text{---} c_r$ have radical square zero.

We may also assume that B is gentle. Indeed, if one of the subcategories $\mathscr{C}$ formed by the objects a_1, c_1, c_2, c_s or $a_t, c_{r-1}, c_r, c_{r+1}$ is not gentle, then mod $\hat{B}$ contains a subcategory mod H, where H is a wild hereditary algebra obtained from C by a one-point extension or coextension, and we are done.

If B is gentle, then $T(B)$ is special biserial and nondomestic (see [ANS, (5.2)]). Observe that there are Galois coverings $R \xrightarrow{\ F'\ } \hat{B}$, $\hat{B} \xrightarrow{\ F''\ } T(B)$ and $R \xrightarrow{\ F''F'\ } T(B)$ with R special biserial selfinjective and simply connected. Then, by (3.2), rad$^\infty$(mod R) is neither left nor right T-nilpotent and consequently rad$^\infty$(mod $\hat{B}$) is neither left nor right T-nilpotent, by (3.3).

5. Repetitive algebras

For the proof of the implications (i) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (v) we need the description of some representation-infinite algebra B for which rad$^\infty$(mod $\hat{B}$) is left or right T-nilpotent. Again $\hat{B}$ denotes the repetitive algebra of B, see [HW, S1]. The Nakayama-automorphism of $\hat{B}$ is denoted by ν_B.

(5.1) PROPOSITION. *Let Λ be a finite category which contains a tame concealed full convex subcategory C. Assume that* rad$^\infty$(mod $\hat{\Lambda}$) *is right (left) T-nilpotent. Then Λ is a domestic branch enlargement of C (in the sense of [AS, (2.2)]) and $\hat{\Lambda} \cong \hat{B}$ for some representation-infinite tilted algebra B of Euclidean type.*

Proof. Assume that rad$^\infty$(mod $\hat{\Lambda}$) is right T-nilpotent. The proof is done by the following series of lemmas:

(5.2) LEMMA. *Let* $B = C[M]$ *be a one-point extension of* C *which is a full subcategory of* $\hat{\Lambda}$. *Then* M *is a simple regular* C-*module.*

Proof. Imitate the proof of [AS2, Lemma 4.4] using (1.7), (4.2) and (4.3).

(5.3) LEMMA. *Let* $B = C[M]$ *be a one-point extension of* C *by a simple regular* C-*module* M. *Then* $A = B[M]$ *is not a full subcategory of* $\hat{\Lambda}$.

Proof. The assertion follows from (4.2).

(5.4) LEMMA. *Let* $B = C[M]$ *be a one-point extension of* C *by a simple regular* C-*module* M, *with extension vertex* a, *and* $D = B[X]$ *be a one-point extension of* B *with extension vertex* b. *Suppose* D *is a full subcategory of* Λ *and let* N *be an indecomposable direct summand of* X, *containing the simple module* $S(a)$ *in its top. Then either* $N \cong P(a)_B$ *or* $N \cong S(a)_B$ *holds.*

Proof. Follow the proof of [AS2, Lemma 4.6] using (5.2).

(5.5) LEMMA. Λ *does not contain a full subcategory* K *of the form*
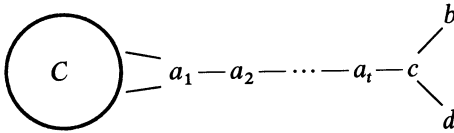
where the full subcategory formed by the objects a_t, b, c, d *is hereditary.*

Proof. Suppose A contains a full subcategory of the above type. We can additionally assume that the walk $a_1 - a_2 - \cdots - a_t - c$ has radical-square zero. Applying, if necessary, the Nakayama-automorphism ν_Λ on K, we may assume that a_1 is an extension vertex.

By (5.2) the largest C-submodule of $P(a_1)_K$ is a simple regular C-module. Thus, if $a_2 \to a_1$, by (5.4), we can assume that $S(a_1)$ is a direct summand of rad $P(a_2)$. Hence $\hat{\Lambda}$ contains a full subcategory $\hat{L}$, where L has the same form as K but in which the radical-square zero walk $a_1 - a_2 - \cdots - a_t$ is not bound and the edge $a_1 - a_2$ is oriented as $a_1 \to a_2$.

Since L is a full subcategory of $\hat{\Lambda}$, also rad$^\infty$(mod L) is right T-nilpotent. Additionally we may assume that the edge $c - d$ is oriented as follows $c \leftarrow d$. Otherwise we apply an APR-tilting module to the vertex d, and the endomorphism algebra L' has the property that rad$^\infty$(mod L') is right T-nilpotent if and only if rad$^\infty$(mod L) is right T-nilpotent.

Let E be the full subcategory of L consisting of all objects of L except d. Then $L = E[P(c)_E]$ and E is a truncated branch extension of C (that is, a tubular extension in the sense of [R3, (4.7)]). Since rad$^\infty$(mod E) is T-nilpotent by (1.5), (1.7), [NS, (2.1)] and [R3, (4.9)] E is a representation-infinite tilted algebra of Euclidean type with a complete slice in its preinjective component.

But the vector space category $\text{Hom}_E(P(c)_E, \text{mod } E)$ satisfies the conditions of

Lemma (4.2) and therefore $\mathrm{rad}^\infty(\mathrm{mod}\ L)$, and hence $\mathrm{rad}^\infty(\mathrm{mod}\ \hat{\Lambda})$ is not right T-nilpotent, a contradiction.

(5.6) LEMMA. Λ *does not contain a full subcategory K of the form*
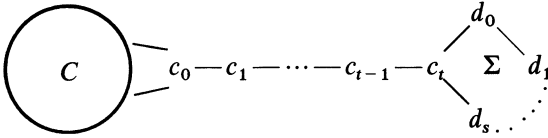
where Σ is a noncommutative cycle.

   Proof. Taking, if necessary, value of ν_Λ on K we may assume that c_0 is an extension vertex. Then imitate the proof of [AS2, Lemma 4.8] using (4.4), (5.2), (5.4) and (5.5).

(5.7) LEMMA. *Let a and b be two objects of Λ outside C each of them connected to C by an edge. Then any walk in Λ connecting a and b must intersect C.*

   Proof. Follow the proof of [AS2, Lemma 4.9] using (4.4), (5.2), (5.6) and, if necessary, the reflection ν_Λ.

(5.8) LEMMA. Λ *is a branch enlargement of C.*

   Proof. Repeat the proof of [AS2, Lemma 4.10] using reflection operators and (5.2)–(5.7).

(5.9) LEMMA. Λ *is a domestic branch enlargement of C. Then we have $\hat{\Lambda} \cong \hat{B}$ for some representation-infinite tilted algebra B of Euclidean type with a complete slice in its preinjective component.*

   Proof. By [ANS, (2.6)] there exists a truncated branch extension B of C, obtained from Λ by a sequence of reflections, with the same tubular type, such that $\hat{\Lambda} \cong \hat{B}$.

   Since $\mathrm{rad}^\infty(\mathrm{mod}\ \hat{\Lambda}) \cong \mathrm{rad}^\infty(\mathrm{mod}\ \hat{B})$ is right T-nilpotent by (1.5), (1.7) and [NS, (2.1)] B is a domestic truncated branch extension of C, and hence, by [R3, (4.9)], B is the required tilted algebra.

6. The proof of the theorem

The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are obvious. The equivalence of (iv) and (v) has been shown in [S2, (1.5)]. The implication (iv) $\Rightarrow$ (iii) was proved in Section 2.

   We shall now prove that (i) $\Rightarrow$ (v), the implication (ii) $\Rightarrow$ (v) is dual.

   Let A be a standard representation-infinite selfinjective algebra such that $\mathrm{rad}^\infty(\mathrm{mod}\ A)$ is right T-nilpotent. Then $A \cong R/G$ where R is a selfinjective simply

connected locally bounded category and G is an admissible group of k-linear automorphism of R.

We denote by v the Nakayama-automorphism of R: For each object x of R the object $v(x)$ is defined by the property that the top of $P(x)$ is the socle of $P(v(x))$, where $P(y)$ denotes the indecomposable projective R-module at y. A v-slice of R is a full subcategory of R which is connected and does not contain two objects from the same v-orbit.

Since $\mathrm{rad}^{\infty}(\mathrm{mod}\ A)$ is right T-nilpotent, by (3.3) and (1.7) also $\mathrm{rad}^{\infty}(\mathrm{mod}\ R)$ is right T-nilpotent and R is tame. We have to consider two cases:

(i) Every full finite subcategory of R is representation-finite. Then by [PS] R is a special biserial selfinjective category and, by (3.2), A is domestic. Consequently, by [S2, (1.5)] we have $A \cong \hat{B}/H$ where B is a representation-infinite tilted algebra of type $\hat{\mathbb{A}}_n$ and H is an admissible infinite cyclic group of k-linear automorphisms of $\hat{B}$.

(ii) R contains a representation-infinite full finite subcategory. Using the Lemmas (4.5) and (4.6) and the tameness of R we see, as in [S2, (4.3)], that R contains a tame concealed full convex subcategory C. Obviously C is a convex v-slice of R.

For an arbitrary convex v-slice D of R we denote by D^{+} (resp. D^{-}) the full subcategory of R consisting of the objects of D and all objects x of R satisfying the following two conditions:

(a) the v-orbit of x does not intersect D.
(b) $R(x, y) \neq 0$ ($R(y, x) \neq 0$, respectively) for some objects y of D.

Similarly as in [S2, (4.4)], using (5.1) we see that $D^{+}, (D^{-}$, respectively) is again a convex v-slice of R. Consider the following sequence of v-slices

$$C_0 \subset C_2 \subset C_3 \subset \cdots$$

where $C_0 = C$, $C_{2n-1} = C_{2n-2}^{+}$, $C_{2n} = C_{2n-1}^{-}$ for $n \geqslant 1$ and let $B = \bigcup_{n \in \mathbb{N}} C_n$. Obviously B is a convex v-slice of R.

We claim that $R = \hat{B}$. Suppose $\hat{B}$ is a proper subcategory of R. Since R is connected, then there is an object $x \in R$, $x \notin \hat{B}$ connected by an arrow to B. Then $v^r x$, for some $r \in \mathbb{Z}$, is connected by an arrow to B and, since $B = \bigcup_{n \in \mathbb{N}} C_n$, the object $v^r x$ belongs to B. Thus $R = \hat{B}$ holds.

For each $n \in \mathbb{N}$, $\hat{C}_n$ is a full subcategory of $\hat{B} = R$ and so, by (4.1), $\mathrm{rad}^{\infty}(\mathrm{mod}\ \hat{C}_n)$ is right T-nilpotent. Hence C_n is a domestic branch enlargement of C, by (5.1). Moreover, by (4.4) B cannot contain an infinite line, that is, $B = C_n$ for some n. Finally, it follows from the construction of C_n that $B = C_n$ is in fact a truncated branch extension of C. Then B is a representation-infinite tilted algebra of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$ ($n \geqslant 4$, $p = 6$, 7, 8, $\tilde{A}_n$ is excluded since R is simply connected) with a complete slice in its preprojective component.

By [S2, (2.13), (2.14)] G is an infinite cyclic group. Therefore we have proved that A has the required form $A \cong \hat{B}/G$.

Note added in proof

Recently K. Bongartz informed us about a more general version of Corollary 1.8:

PROPOSITION. *Let R be an artinian ring, such that all indecomposable injective modules have finite length. Then R is representation-finite if and only if* $\mathrm{rad}^{\infty}(\mathrm{mod}\ R) = 0$.

The proof he gave, is surprisingly simple: If R is of finite type, as above the assertion follows from the Harada-Sai lemma.

If R is not representation-finite, by [1, 3.1] there exists an infinite sequence in mod R

$$U_0 \xrightarrow{f_0} U_1 \xrightarrow{f_1} U_2 \xrightarrow{f_2} \cdots$$

with U_i indecomposable and f_i nontrivial monomorphisms, or an infinite sequence

$$V_0 \xleftarrow{g_0} V_1 \xleftarrow{g_1} V_2 \xleftarrow{g_2} \cdots$$

with V_i indecomposable in mod R and g_i nontrivial epimorphisms, for all i.

In the first case for the injective hull

$$\varepsilon: U_0 \to E(U_0)$$

we get

$$\varepsilon = \varepsilon_i f_i \cdots f_0,$$

for some $\varepsilon_i \in \mathrm{Hom}(U_{i+1}, E(U_0))$, that is

$$\varepsilon \in \mathrm{rad}^{\infty}(\mathrm{mod}\ R).$$

In the latter case, analogously we consider the projective cover $\pi: P(V_0) \to V_0$.

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