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Maximal rank curves and singular points of the Hilbert scheme

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0. Introduction and preliminaries

Let X be a curve in $\mathbf{P}^3$, i.e. a closed one-dimensional and equidimensional, locally Cohen-Macaulay subscheme of $\mathbf{P}^3$. We say that X is non-obstructed if the corresponding point of the Hilbert scheme is smooth, otherwise we say that X is obstructed. A geometrical characterization of non-obstructedness is not known even for smooth space curves, and several examples of obstructed smooth space curves are known (also in generically non-reduced components of the Hilbert scheme), see for instance [M1], [S], [EF], [K1], [K3], [K4], [E]. We want to point out that all these examples are curves which have not maximal rank, with the exception of a non-reduced curve in [K4], 3.22.

On the other side, in 1975 Ellinsgrud proved that arithmetically Cohen-Macaulay curves are non-obstructed [E1]. Recall that a curve C in $\mathbf{P}^3$ is arithmetically Cohen-Macaulay if and only if its deficiency module is zero (hence it has maximal rank). Trying to generalize this result we will prove that a *general* (in a sense that will be made more precise) maximal rank curve whose deficiency module is concentrated in one degree is non-obstructed. This is the best that one can hope, since we will give a criterion for constructing obstructed curves in these liaison classes (from which one clearly sees why these curves are 'particular'). This criterion will give a smooth, obstructed maximal rank curve. We will give a minimal set of generators for the homogeneous ideal of this curve (by using Macaulay, [BS]) and then we will check smoothness.

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During the Trieste Conference on Projective Varieties (Trieste, June 1989) we met Charles Walter, who had constructed an example of smooth obstructed maximal rank curve, and in fact it turned out that his example has the same cohomological properties of our example. His elegant technique for proving smoothness is different ([W]).

We need some preliminaries. The field $\mathbf{k}$ is an algebraically closed field of characteristic zero. We set $R = \mathbf{k}[x, y, z, t]$, $\mathfrak{m} = (x, y, z, t)$ and $\mathbf{P}^3 = \mathrm{Proj}(R)$. Given a curve X in $\mathbf{P}^3$, we denote by $\mathscr{I}_X$ (resp. $I(X)$) the ideal sheaf (resp. the homogeneous ideal) of X. If X sits in some closed subscheme Y of $\mathbf{P}^3$, then $\mathscr{I}_{X/Y}$ denotes the ideal sheaf of X in Y. We set

$d =$ degree of X,

$p_a =$ arithmetic genus of X,

$s = \min\{t \mid H^0(\mathbf{P}^3, \mathscr{I}_X(t)) \neq 0\}$,

$\sigma = \min\{t \mid H^0(H, \mathscr{I}_{X \cap H}(t)) \neq 0\}$ where H is a general plane,

$e = \max\{t \mid H^1(\mathbf{P}^3, \mathscr{O}_X(t)) \neq 0\}$,

$c = \max\{t \mid H^1(\mathbf{P}^3, \mathscr{I}_X(t)) \neq 0\}$,

$b = \min\{t \mid H^1(\mathbf{P}^3, \mathscr{I}_X(t)) \neq 0\}$, and

$M(X) = \bigoplus_t H^1(\mathbf{P}^3, \mathscr{I}_X(t))$, deficiency module of X. If $M(X) = 0$, then let $c = -\infty$ and $b = +\infty$.

Recall that a curve X in $\mathbf{P}^3$ is said to be arithmetically Buchsbaum (or, more briefly, Buchsbaum) if and only if $\mathfrak{m} \cdot M(X) = 0$, and X is said to have maximal rank if and only if the natural restriction maps

$$H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(t)) \to H^0(X, \mathscr{O}_X(t))$$

have maximal rank for every $t \in \mathbf{Z}$.

We consider liaison classes of curves in $\mathbf{P}^3$ whose deficiency module is concentrated in at most one degree. If n is a non-negative integer, we denote by

$$\mathbf{L}_n = \left\{ Y \text{ in } \mathbf{P}^3 \mid M(Y) = \bigoplus_{t \in \mathbf{Z}} H^1(\mathbf{P}^3, \mathscr{I}_Y(t)) = \mathbf{k}^n \text{ concentrated in at most one degree} \right\}$$

the liaison class of curves with at most one deficiency group different from zero and of dimension n. These classes are studied in [BM].

If Y is a curve, we denote by F_Y its cohomology function, that is to say

$$F_Y\colon \{0, 1, 2\} \times \mathbf{Z} \to N$$

$$F_Y(i, t) = h^i(\mathbf{P}^3, \mathscr{I}_Y(t)) = \dim H^i(\mathbf{P}^3, \mathscr{I}_Y(t)),$$

and if F: $\{0, 1, 2\} \times \mathbf{Z} \to N$ is any function, we will denote $H_F = \{C$ curve in $\mathbf{P}^3 \mid F_C = F\}$.

If two curves X and Y have only one deficiency group different from zero and $F_X = F_Y$, then they are trivially in the same liaison class; moreover, if $F = F_X$, then from [B], 2.2, we have that H_F is a locally closed irreducible subset of the Hilbert scheme. In this sense we will be able to speak about a 'general' element H_F. So we will use the sentence 'a generic Buchsbaum curve C of diameter 1' with the meaning 'a general element of H_{F_C}', for every $C \in \mathbf{L}_n$, $\forall n$.

For results about obstructedness and deformation theory in the Hilbert scheme we will refer mainly to [K4]; indeed, many proofs here are generalizations of results there contained, hence we recall some results about obstructedness and deformations theory in the Hilbert scheme $H(d, p_a)$ of curves of degree d and arithmetic genus p_a used in the sequel. We denote by $D(d, p_a; f_1, f_2)$, resp. $D(d, p_a; f)$, the Hilbert scheme of nests (or flags) parametrizing curves X from $H(d, p_a)$ *and* complete intersections Y of type or bidegree (f_1, f_2), resp. surfaces Y of degree f, such that $Y \supseteq X$. Forgetting $Y = V(f_1, f_2)$, we get a natural projection

$$\mathrm{pr}_1\colon D(d, p_a; f_1, f_2) \to H(d, p_a),$$

inducing a tangent map

$$p_1\colon A^1(X \subseteq Y) \to H^0(X, \mathcal{N}_X)$$

between the tangent spaces of $D = D(d, p_a; f_1, f_2)$ and $H(d, p_a)$ at the closed points $(X \subseteq Y \subseteq \mathbf{P}^3)$ and $(X \subseteq \mathbf{P}^3)$ respectively.

By [K4], (1.11) there is an exact sequence

$$\text{(i1)}\quad 0 \to \bigoplus_{i=1}^{2} H^0(Y, \mathcal{I}_{X/Y}(f_i)) \to A^1(X \subseteq Y) \xrightarrow{p_1} H^0(X, \mathcal{N}_X)$$
$$\xrightarrow{\beta} \bigoplus_{i=1}^{2} H^1(\mathbf{P}^3, \mathcal{I}_X(f_i))$$
$$\to A^2(X \subseteq Y) \to H^1(X, \mathcal{N}_X) \to \bigoplus_{i=1}^{2} H^1(X, \mathcal{O}_X(f_i))$$

where $\beta = \beta_{X/Y}$ is defined by sending a global section ϕ of the normal sheaf $\mathcal{N}_X = \mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^3}}(\mathcal{I}_X, \mathcal{O}_X)$ to $(t\phi_1(F_1), t\phi_2(F_2))$, where

$$t\colon H^0(X, \mathcal{O}_X(f_i)) \to H^1(\mathbf{P}^3, \mathcal{I}_X(f_i))$$

is the natural map and

$$\phi_i\colon H^0(\mathbf{P}^3, \mathcal{I}_X(f_i)) \to H^0(X, \mathcal{O}_X(f_i))$$

is induced by $\phi: \mathscr{I}_X \to \mathcal{O}_X$. Then

(i2) the fibers of pr_1 are smooth and irreducible. Moreover pr_1 is smooth at $(X \subseteq V(F_1, F_2) \subseteq \mathbf{P}^3)$ provided $H^1(\mathbf{P}^3, \mathscr{I}_X(f_i)) = 0$ for $i = 1, 2$, cf. [K4], th. 1.16.

(i3) More generally, let S be a local artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$ and let $X_S \subseteq \mathbf{P}^3 \times S$ (with ideal sheaf $\mathscr{I}_{X_S}$) be any deformation of $(X \subseteq \mathbf{P}^3)$ to S. If the natural map

$$H^0(\mathbf{P}^3 \times S, \mathscr{I}_{X_S(f_i)}) \otimes_S \mathbf{k} \to H^0(\mathbf{P}^3, \mathscr{I}_X(f_i))$$

is surjective for $i = 1, 2$, then pr_1 is smooth at any $t \in \mathrm{pr}_1^{-1}((X \subseteq \mathbf{P}^3))$, cf. [K4], lemma 1.17.

(i4) If $\beta_{X/Y}$ is surjective and $H(d, p_a)$ is smooth at $(X \subseteq \mathbf{P}^3)$, then D is smooth at $(X \subseteq Y \subseteq \mathbf{P}^3)$, cf. [K4], prop. 3.12.

(i5) Furthermore by [K4], th. 2.6 there exists an isomorphism

$$\tau: D(d, p_a; f_1, f_2) \to D(d', p_a'; f_1, f_2)$$

of schemes which on the underlying sets of closed points is defined by sending $(X \subseteq Y \subseteq \mathbf{P}^3)$ to $(X' \subseteq Y \subseteq \mathbf{P}^3)$ where X' is linked to X by Y. Of course τ induces an isomorphism between the corresponding tangent spaces and in fact we have an isomorphism between the 'obstruction spaces' $A^2(X \subseteq Y)$ and $A^2(X' \subseteq Y)$ as well, provided the linkage is geometric, cf. [K4], cor. 2.14.

(i6) Let X and X' be linked via some complete intersection Y of type (f, g). Then one knows that $\mathscr{I}_{X/Y}(f + g - 4) \cong \omega_{X'}$ is the dualizing sheaf of X' from which one easily proves the duality

$$H^1(\mathbf{P}^3, \mathscr{I}_X(v))^v \cong H^1(\mathbf{P}^3, \mathscr{I}_{X'}(f + g - 4 - v)), \; v \in \mathbf{Z}.$$

So if $n = s(X/Y)$ is the least integer satisfying $H^0(Y, \mathscr{I}_{X/Y}(n)) \neq 0$, we get

$$c(X') = f + g - 4 - b(X)$$
$$e(X') = f + g - 4 - s(X/Y)$$

and two corresponding formulas interchanging X and X'.

Of course if $D = D(d, p_a; f)$, then (i1), (i2), (i3) and (i4) are still true, slightly reformulated.

1. General non-obstructedness

A maximal rank curve of diameter 1 (i.e. whose deficiency module has diameter 1) is known to be non-obstructed under some conditions on its numerical character ([MR]). In this section we shall prove that, without this assumption on the numerical character, any such curve admits a non-obstructed generization with the same cohomology. By the irreducibility of H_F, this is the equivalent to the following

PROPOSITION 1.1. *A generic Buchsbaum curve X of maximal rank and of diameter* 1 *is non-obstructed.*

Proof. We link X via some complete intersection Y of type (f, g) satisfying $H^1(\mathbf{P}^3, \mathscr{I}_X(v)) = 0$ for $v = f, g, f-4, g-4$, and we get a curve C with $e(C) < c(C)$ and $H^1(\mathbf{P}^3, \mathscr{I}_C(v)) = 0$ for $v = f, g$ by (i6). Since (i2) implies that pr_1 and pr'_1: $D(d', p'_a; f_1, f_2) \to H(d', p'_a)$ are smooth at $(X \subseteq Y \subseteq \mathbf{P}^3)$ and $(C \subseteq Y \subseteq \mathbf{P}^3)$ respectively, X is non-obstructed provided C is non-obstructed. So the proposition will follow from

PROPOSITION 1.2. *A generic Buchsbaum curve C* of *diameter* 1 *satisfying $e(C) < c(C)$ is non-obstructed.*

In the proof of (1.2) and in several other places in this paper we need to determine $h^1(X, \mathscr{N}_X)$ and related objects in terms of some of the invariants of a minimal resolution

$$0 \to P_3 \to P_2 \to P_1 \to I(X) \to 0, \quad P_j = \bigoplus_1 R(-n_{j_i})$$

of $I(X)$.

Note that $\mathrm{pd}_R I(X) \leqslant 2$, since $I(X) \cong \bigoplus_v \Gamma(\mathbf{P}^3, \tilde{I(X)}(v))$, i.e. $\mathrm{depth}_{\mathfrak{m}} I(X) \geqslant 2$.

Moreover by non-obstructedness, lemma 1.3 also gives us the dimension of the Hilbert schemes involved in proposition 1.1 and 1.2. We sketch a proof since the version in [K4], section 4, is somewhat more restrictive.

In the sequel, ${}_v\mathrm{Hom}_R(M, \text{-})$ denotes homomorphisms of graded R-modules of degree v. If $\Gamma_{\mathfrak{m}}(M)$ is the group of sections of $\tilde{M}$ with support in $V(\mathfrak{m}) \subseteq \mathrm{Spec}(R)$, i.e.

$$\Gamma(M)_v = \mathrm{Ker}(M_v \to \Gamma(\mathbf{P}^3, \tilde{M}(v))),$$

we denote by ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(M, -)$, resp. $H^i_{\mathfrak{m}}(-)$, the right derived functor of $\Gamma_{\mathfrak{m}}(\mathrm{Hom}_R(M, -))_v$, resp. of $\Gamma_{\mathfrak{m}}(-)$.

LEMMA 1.3. *Let X be a Buchsbaum curve in $\mathbf{P}^3$ and let $H(X) = \bigoplus_v H^1(X, \mathcal{O}_X(v))$. Then there is an exact sequence*

$$H^0(X, \mathcal{N}_X) \xrightarrow{\beta} \bigoplus_1 H^1(\mathbf{P}^3, \mathcal{I}_X(n_{1_i})) \to \bigoplus_1 H^1(\mathbf{P}^3, \mathcal{I}_X(n_{1_i} - 4))^v$$
$$\to H^1(X, \mathcal{N}_X) \to E \to 0$$

where β is correspondingly defined as in (i1) *and where E is determined by the exact sequence*

$$0 \to \bigoplus_1 H^1(\mathbf{P}^3, \mathcal{I}_X(n_{2_i})) \to E \to {}_0\mathrm{Hom}_R(I(X), H(X))$$
$$\to \bigoplus_1 H^1(\mathbf{P}^3, \mathcal{I}_X(n_{3_i})) \to 0$$

In particular, if $\operatorname{diam} X \leqslant 2$ *and $e(X) \leqslant b(X)$, then β is surjective. If in addition $e(X) < s(X)$ and $c(X) < n_{2_i}$ for any i, then $H^1(2, \mathcal{N}_X) = 0$.*

COROLLARY 1.4. *If X is a Buchsbaum curve with* $1 \leqslant \operatorname{diam} X \leqslant 2$ *and $e(X) < c(X) \leqslant s(X)$, or if X is an arithmetically Cohen-Macaulay curve with $e(X) < s(X)$, then $H^1(X, \mathcal{N}_X) = 0$.*

Proof of 1.3.

Let M be a graded R-module and put $I = I(X)$. By [SGA2], exp. VI, there is an exact sequence

$$(*)\quad \mathrm{Ext}^1_{\mathcal{O}_{\mathbf{P}^3}}(\tilde{I}, \tilde{M}) \to {}_0\mathrm{Ext}^2{}_{\mathfrak{m}}(I, M) \to {}_0\mathrm{Ext}^2_R(I, M) \to \mathrm{Ext}^2_{\mathcal{O}_{\mathbf{P}^3}}(\tilde{I}, \tilde{M})$$
$$\to {}_0\mathrm{Ext}^3_{\mathfrak{m}}(I, M) \to 0$$

and a spectral sequence $E_2^{p,q} = {}_0\mathrm{Ext}^p_R(I, H^q_{\mathfrak{m}}(M))$ converging to ${}_0\mathrm{Ext}^{p+q}_{\mathfrak{m}}(I, M)$.

First let $M = I$ and observe that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbf{P}^3}}(\tilde{I}, \tilde{I}) \cong H^{i-1}(X, \mathcal{N}_X)$ for $i = 1, 2$, by [K1], remark 2.2.6.

Since $\mathrm{depth}_{\mathfrak{m}} I \geqslant 2$, i.e. $H^i{}_{\mathfrak{m}}(\mathrm{I}) = 0$ for $i \leqslant 1$, the spectral sequence for $p + q = 2$ and the triviality of the R-module structure of $M(X) = H^2{}_{\mathfrak{m}}(I)$ imply

$${}_0\mathrm{Ext}^2{}_{\mathfrak{m}}(I, I) \cong {}_0\mathrm{Hom}(I, H^2{}_{\mathfrak{m}}(I)) \cong \bigoplus_1 H^1(\mathbf{P}^3, \mathcal{I}_X(n_{1_i})).$$

The same arguments imply the second exact sequence of (1.3) provided we let $E = {}_0\mathrm{Ext}^3{}_{\mathfrak{m}}(I, I))$. In fact the spectral sequence for $p + q = 3$ gives us the exact sequence

$$0 \to {}_0\mathrm{Ext}^1{}_R(I,\ H^2{}_{\mathfrak{m}}(I)) \to E \to {}_0\mathrm{Hom}(I,\ H^3{}_{\mathfrak{m}}(I))$$

$$\xrightarrow{d_{2,-1}} {}_0\mathrm{Ext}^2{}_R(I,\ H^2{}_{\mathfrak{m}}(I)),$$

and $H^3{}_{\mathfrak{m}}(I)) = H(X)$. Moreover, the surjectivity of $d_{2,-1}$ follows from the spectral sequence for $p + q = 4$ because ${}_0\mathrm{Ext}^4{}_{\mathfrak{m}}(I, I) = 0$ by continuing the long exact sequence(*).

The proof of the existence of the two exact sequences is now complete if we show that

$${}_0\mathrm{Ext}^2{}_R(I,\ I) \cong \oplus\ H^1(\mathbf{P}^3,\ \mathscr{I}_X(n_{1_i} - 4))^v.$$

To prove it, we observe that ${}_0\mathrm{Ext}^2{}_R(I,\ -)$ is right-exact, and this implies the exactness of

$${}_0\mathrm{Ext}^2{}_R(I,\ \oplus\ R(-n_{2_i})) \to {}_0\mathrm{Ext}^2{}_R(I,\ \oplus\ R(-n_{1_i})) \to {}_0\mathrm{Ext}^2{}_R(I,\ I) \to 0.$$

Since the module structure of $M(X)$ is trivial, it is enough to prove that

$${}_0\mathrm{Ext}^2{}_R(I,\ R(-v)) \cong H^1(\mathbf{P}^3,\ \mathscr{I}_X(v-4))^v.$$

However by Serre duality and $\mathrm{depth}_{\mathfrak{m}} R(-v) = 4$,

$$H^1(\mathbf{P}^3,\ \mathscr{I}_X(v-4))^v \cong \mathrm{Ext}^2_{\mathcal{O}_{\mathbf{P}^3}}(\mathscr{I}_X,\ \mathcal{O}_{\mathbf{P}^3}(-v)) \xleftarrow{\approx} {}_0\mathrm{Ext}^2{}_R(I,\ R(-v)).$$

Indeed the isomorphism on the right side follows directly from the exact sequence (*) with $M = R(-v)$, since the spectral sequence implies ${}_0\mathrm{Ext}^i{}_{\mathfrak{m}}(I, R(-v)) = 0$ for $i \leqslant 3$ because $\mathrm{depth}_{\mathfrak{m}} R(-v) = 4$.

Finally suppose $e(X) \leqslant b(X)$. Since it is well known that $n_{1_i} \leqslant \max\{c(X) + 2,\ e(X) + 3\}$ and since the diameter of $M(X)$ is at most 2, we get $n_{1_i} - 4 < b(X)$. So β is surjective and $H^1(X,\ \mathscr{N}_X) \cong E$. However $e(X) < s(X)$ implies immediately ${}_0\mathrm{Hom}(I,\ H(X)) = 0$ and so $c(X) < n_{2_i}$ implies $E = 0$ by the second exact sequence of (1.3) and the proof of (1.3) is complete.

REMARK 1.5. In general one may prove that the dimension of the $\mathbf{k}$-vector space ${}_0\mathrm{Hom}_R(I(X),\ H(X))$ of lemma 1.3 is

$$\sum_i h^1(X,\ \mathcal{O}_X(n_{1_i})) - \sum_i h^1(X,\ \mathcal{O}_X(n_{2_i})) + \sum_i h^1(X,\ \mathcal{O}_X(n_{3_i})).$$

In fact, using the spectral sequence $E_2^{p,q}$ (with $M = I$) in the proof above and ${}_0\mathrm{Ext}^i{}_{\mathfrak{m}}(I, I) = 0$ for $i = 4, 5$ we get $E_2^{p,3} = 0$ for $p = 1, 2$ and we conclude easily by applying ${}_0\mathrm{Hom}_R(-, H^3{}_{\mathfrak{m}}(I))$ to the minimal resolution of I.

COROLLARY 1.6. *Let X be a maximal rank Buchsbaum curve, with $e(X) < s(X)$. Then $H^1(X, \mathcal{N}_X) \cong \bigoplus_i H^1(\mathbf{P}^3, \mathcal{I}_X(n_i - 4))^v$, where the n_i's are the degrees of a minimal set of generators of the homogeneous ideal $I(X)$.*

Proof. Since X is of maximal rank, $H^1(\mathbf{P}^3, \mathcal{I}_X(X_{j_i}) = 0$ for $j = 1, 2, 3$ and any i. Moreover $e(X) < s(X)$ implies ${}_0\mathrm{Hom}(I(X), H(X)) = 0$, and the conclusion follows immediately from the two exact sequences of lemma 1.3.

LEMMA 1.7. *Let C be a curve of diameter 1 with $s = s(C)$, $e = e(C)$, $c = c(C)$, and suppose $e < c$. Then there exists a complete intersection $Y \supseteq C$ of type $(s, c + 2)$ linking C to C' such that*

$$s - 1 \leqslant s(C') \leqslant s, \quad c(C') = s - 2, \quad e(C') \leqslant c - 2.$$

It follows that either

(i) $c(C') \leqslant c - 1$, *so there exists a complete intersection $Y' \supseteq C'$ of type $(s, c + 1)$ linking C' to C'' such that $e(C'') = e - 1$, $c(C'') = c - 1$, $s(C'') \geqslant s - 1$*

or

(ii) $c \leqslant s - 2$ *(in fact we have equality).*

Proof. The existence of Y follows from the general fact that $\mathcal{I}_C(v)$ is globally generated for $v \geqslant \max\{e(C) + 3, c(C) + 2\}$, and now the inequalities for $s(C')$, $c(C')$ and $e(C')$ follows from (i6). By the same arguments and by $e(C') + 3 \leqslant c + 1$, (i) is true provided $c(C') + 2 \leqslant c + 1$. So if (i) is not true, then $c(C') \geqslant c$, and we conclude by $c(C') = s - 2$.

The key lemma in the proof of proposition 1.2 is the following:

LEMMA 1.8. *Let C be a generic Buchsbaum curve in $\mathbf{P}^3$ of diameter 1 and suppose $e < c$ and $s + 1 \leqslant c$. Then link C to a curve C' using a sufficiently general complete intersection $Y = V(F, G)$ of type $(s, c + 2)$ containing C, and link C' to a curve X using a general $Y' = V(F', G')$ of type $(s, c + 1)$ containing C'. Then X is a generic Buchsbaum curve, and Y' is a general complete intersection containing X. Moreover if X is non-obstructed, then C is non-obstructed.*

Proof. C' must be a generic Buchsbaum curve and Y is general with respect to C' because for any generization $\tilde{C}'$ of C' with the same cohomology, there exists a complete intersection $\tilde{Y} \supseteq \tilde{C}'$ of type (s, c + 2) such that the curve linked to $\tilde{C}'$ by $\tilde{Y}$ is a generization of C, cf. [K4], prop. 3.7. The same argument for genericness works for X and Y' as well.

Now observe that C is non-obstructed if and only if C' is non-obstructed thanks to (i2) and the isomorphism τ of (i5) of the introduction and

$$H^1(\mathbf{P}^3, \mathscr{I}_C(s)) = 0, \quad H^1(\mathbf{P}^3, \mathscr{I}_{C'}(s))^v = H^1(\mathbf{P}^3, \mathscr{I}_C(c-2)) = 0,$$
$$H^1(\mathbf{P}^3, \mathscr{I}_C(c+2)) = 0, \quad H^1(\mathbf{P}^3, \mathscr{I}_{C'}(c+2))^v = H^1(\mathbf{P}^3, \mathscr{I}_C(s-4)) = 0.$$

In the same way

$$\mathrm{pr}_1'\colon D(d', p_a'; s, c+1) \to H(d', p_a')$$

is seen to be smooth at $(C' \subseteq Y' \subseteq \mathbf{P}^3)$. However the projection

$$\mathrm{pr}_1\colon D(d, p_a; s, c+1) \to H(d, p_a)$$

is not necessarily smooth at $t = (X \subseteq Y' \subseteq \mathbf{P}^3)$, since $h^1(\mathbf{P}^3, \mathscr{I}_X(s)) = h^1(\mathbf{P}^3, \mathscr{I}_{C'}(c-3)) = h^1(\mathbf{P}^3, \mathscr{I}_C(s+1))$ is not zero in the case $c = s+1$. If $c > s+1$, then pr_1 is smooth at t, and so C' and C are non-obstructed since X is non-obstructed.

Now suppose $c = s+1$ and X is non-obstructed. To complete the proof it is enough to prove that $D(d, p_a; s, c+1)$ is smooth at t. First suppose that F' is a minimal generator of $I(X)$. In this case the map β of (1.3) is surjective, and so is the map

$$\beta_{X/V(F')}\colon H^0(X, \mathscr{N}_X) \to H^1(\mathbf{P}^3, \mathscr{I}_X(s))$$

of (i4), and hence we conclude by (i4).

Finally suppose that F' is not a minimal generator of $I(X)$. Since we can assume that $Y' = V(F', G')$ is a *general* complete intersection of type $(s, c+1)$ containing X by the first part of the proof, $I(X)$ contains no minimal generators of degree $s = c(X)$.

(In fact if F' is not minimal and $I(X)$ contains a minimal generator of degree s, then one may also use the following proposition 2.1 to see that C' – and so C — is not a generic Buchsbaum curve).

It follows that $H^1(\mathbf{P}^3, \mathscr{I}_X(n_i)) = 0$ for any i, where the n_i's are the degrees of a minimal set of generators of $I(X)$. By [K2], remark 3.7 or by [PiS], the graded deformations of $R \to A = R/I(X)$ (in the category of local artinian $\mathbf{k}$-algebras S with residue field $\mathbf{k}$) and the deformations of $X \subseteq \mathbf{P}^3$ correspond uniquely. We will use this correspondence and the criterion (i3) to prove that pr_1 is smooth at t. So let $X_S \subseteq \mathbf{P}^3_S$, $\mathbf{P}^3_S = \mathbf{P}^3 \times \mathrm{Spec}(S)$, be a deformation of $X \subseteq \mathbf{P}^3$ to a local artinian $\mathbf{k}$-algebra S. Then there exists a graded deformation $R_S \to A_S$ of $R \to A = R/I(X)$ to S such that, applying the Proj-functor, we get back $X_S \subseteq \mathbf{P}^3_S$.

Put $I_S = \ker(R_S \to A_S)$ and $I(X_S) = \bigoplus_v H^0(\mathbf{P}^3_S, \mathscr{I}_{X_S}(v))$ and observe that there exists a morphism $I_S \to I(X_S)$ since $I(X_S)$ is the largest homogeneous ideal in R_S which defines $X_S \subseteq \mathbf{P}^3_S$. Since A_S is S-flat and $A_S \otimes_S \mathbf{k} \xrightarrow{\approx} A$, it follows that $I_S \otimes_S \mathbf{k} \xrightarrow{\approx} I(X)$, i.e. the composition

$$I_S \otimes_S \mathbf{k} \to I(X_S) \otimes_S \mathbf{k} \to I(X)$$

is an isomorphism. In particular the map $I(X_S) \otimes_S \mathbf{k} \to I(X)$ is surjective, and so

$$H^0(\mathbf{P}^3_S, \mathscr{I}_{X_S}(v)) \otimes_S \mathbf{k} \to H^0(\mathbf{P}^3, \mathscr{I}_X(v))$$

is surjective for every v, and we conclude by (i3).

Proof of proposition 1.2. By lemma 1.7 there exists a finite sequence of double linkages

$$C = C_1 \sim C_2 \sim \cdot \quad \cdots \sim C_k$$

where each double linkage $C_i \sim C_{i+1}$ is performed using complete intersections of type $(s, c+2)$ and $(s, c+1)$, where $s = s(C_i)$ and $c = c(C_i)$, and at the end we have $c(C_k) \leqslant s(C_k) - 2$. Moreover by 1.7 we have $e(C_i) < c(C_i)$ and $c(C_i) - s(C_i) \leqslant c(C_{i-1}) - s(C_{i-1})$.

In particular there exists an integer n such that $c(C_i) \geqslant s(C_i) + 1$ for $i < n$ and $c(C_i) \leqslant s(C_i)$ for $n \leqslant i \leqslant k$. By corollary 1.4, $H^1(C_n, \mathscr{N}_{C_n}) = 0$ and by lemma 1.8 C_i, for $i < n$, is non-obstructed provided C_{i+1} is non-obstructed. This completes the proof.

2. Techniques for constructing obstructed curves

In this section we consider different techniques for constructing obstructed curves (of maximal rank). Inspired by Sernesi's example [S], Ellia and Fiorentini (cf [EF]) succeeded in giving a criterion for obstructedness using the Liaison-invariance of $A^2(X \subseteq Y)$. Their result was generalized in [K4], leading to a simple criterion, cf. [K4] th. 3.18, for the obstructedness of a bilinked curve. We were not able to find obstructed curves of maximal rank using these previous results. The following proposition provides, however, a tool for constructing smooth obstructed maximal rank curves, and can be used to complete the proof of proposition 1.2.

PROPOSITION 2.1. *Let X be a Buchsbaum curve with $1 \leqslant \operatorname{diam} X \leqslant 2$ and $e(X) \leqslant b(X)$, let f be an integer satisfying $H^1(\mathbf{P}^3, \mathscr{I}_X(f)) \neq 0$ and suppose $s(X) < f$ and that $I(X)$ contains a minimal generator F of degree f. Let F_0 be a non-minimal generator (i.e. $F_0 = \sum_{i=1}^r H_i F_i$, $F_i \in I(X)$ and $\deg F_i < \deg F_0$, $1 \leqslant i \leqslant r$) of degree*

f, and link X to a curve C, resp. to a curve C_0, using a complete intersection of the form $Y = V(F, G)$, resp. $Y_0 = V(F_0, G)$, where $G \in I(X)$ is of degree $g \geqslant c(X) + 1$. If $H^1(\mathbf{P}^3, \mathscr{I}_X(g-4)) = 0$, then

(i) *C_0 is an obstructed curve, and if F is sufficiently general, then C is a generization of C_0 in the Hilbert scheme;*
(ii) *C is non-obstructed provided X is non-obstructed.*

Furthermore if $g \leqslant b(X) + 3$ and $e(X) < b(X)$, then C and C_0 are of maximal rank.

Proof. To prove that C_0 is obstructed, we first claim that

$$\dim A^1(X \subseteq Y_0) > \dim A^1(X \subseteq Y).$$

For this we consider the exact sequence of (i1) of the introduction and the corresponding sequence involving $A^i(X \subseteq Y_0)$. We observe that $\beta_{X/Y}$ is surjective by lemma 1.3 since F is a minimal generator of $I(X)$ and $H^1(\mathbf{P}^3, \mathscr{I}_X(g)) = 0$. The corresponding map β_{X/Y_0} of $X \subseteq Y_0$ is however zero since, by definition of β_{X/Y_0}, we have

$$\beta_{X/Y_0} = \beta_{X/V(F_0)} = \sum_{i=1}^{r} \psi_{H_i} \circ \beta_{X/V(F_i)}$$

where

$$\psi_{H_i}\colon H^1(\mathbf{P}^3, \mathscr{I}_X(f-h_i)) \to H^1(\mathbf{P}^3, \mathscr{I}_X(f))$$

is the map induced by 'multiplication' with $H_i \in R$. So β_{X/Y_0} since $\psi_{H_i} = 0$ for Buchsbaum curves.

Now recall the exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{I}_X \to \mathscr{I}_{X/Y} \to 0$$

and the similar one for $\mathscr{I}_{X/Y_0}$. Since Y and Y_0 are complete intersection of the same type,

$$h^0(Y, \mathscr{I}_{X/Y}(v)) = h^0(Y_0, \mathscr{I}_{X/Y_0}(v))$$

for every v. The exact sequence of (i1) therefore implies

$$\dim A^1(X \subseteq Y) + h^1(\mathbf{P}^3, \mathscr{I}_X(f)) = \dim A^1(X \subseteq Y_0)$$

and the claim follows from the assumption $H^1(\mathbf{P}^3, \mathscr{I}_X(f)) \neq 0$.

Next we claim that $t_0 = (X \subseteq Y_0 \subseteq \mathbf{P}^3)$ is obstructed, i.e. that the Hilbert-flag scheme $D = D(d, p_a; f, g)$ is singular at the closed point t_0. Suppose D is non-singular at t_0, and let $W \subseteq D$ be the unique irreducible component containing t_0. Since $t = (X \subseteq Y \subseteq \mathbf{P}^3)$ and t_0 are in the same fiber of pr_1: $D \to H(d, p_a)$ and since the fibers of pr_1 are irreducible by (i2), then $t \in W$. Hence

$$\dim A^1(X \subseteq Y_0) = \dim \mathcal{O}_{D,t_0} = \dim W \leqslant \dim \mathcal{O}_{D,t} \leqslant \dim A^1(X \leqslant Y)$$

and we have a contradiction.

We now conclude quickly. Indeed by the liaison isomorphism

$$\tau \colon D \xrightarrow{\sim} D' = D(d', p'_a; f, g)$$

of (i5), D' is singular at $t'_0 = (C_0 \subseteq Y_0 \subseteq \mathbf{P}^3)$. Moreover $\mathrm{pr}'_1 \colon D' \to H(d', p'_a)$ is smooth at t'_0 since

$$H^1(\mathbf{P}^3, \mathscr{I}_{C_0}(f)) \cong H^1(\mathbf{P}^3, \mathscr{I}_X(g-4))^v = 0$$

$$H^1(\mathbf{P}^3, \mathscr{I}_{C_0}(g)) \cong H^1(\mathbf{P}^3, \mathscr{I}_X(f-4))^v = 0.$$

It follows that $H(d', p'_a)$ is singular at $(C_0 \subseteq \mathbf{P}^3)$, i.e. C_0 is obstructed. Finally the irreducibility of the fibers of pr_1 implies that t is a generization of t_0 provided Y is general enough, and so C is a generization of C_0 in $H(d', p'_a)$.

Now (ii) follows from (i4) since we have already proved that $\beta_{X/Y}$ is surjective. Furthermore by (i6) C_0 and C are of maximal rank and we are done.

REMARK 2.2. The proof of proposition 2.1 admits the following generalization: Let X be a curve in $\mathbf{P}^3$ and suppose that there are two complete intersections Y and Y_0 of the same type (f, g) such that

$$\dim \operatorname{coker} \beta_{X/Y} < \dim \operatorname{coker} \beta_{X/Y_0}.$$

If $H^1(\mathbf{P}^3, \mathscr{I}_X(f-4)) = H^1(\mathbf{P}^3, \mathscr{I}_X(g-4)) = 0$, then the curve C_0 linked to X by Y_0 is obstructed.

At least under some more extra assumption on the curve X of (2.1) we can prove that the obstructed curve C_0 is in the intersection of two irreducible components of the Hilbert scheme.

PROPOSITION 2.3. *Let X be a generic Buchsbaum curve in $\mathbf{L}_n$ with $e(X) < c(X)$, $s(X) < c(X)$, and suppose that the number m of minimal generators of $I(X)$ of degree $f = c(X)$ satisfies $1 \leqslant m \leqslant n$. If F_0, F, G, C and C_0 are as in proposition* 2.1, *then C_0 sits in the intersection of two irreducible components V_1 and V_2 of the Hilbert scheme. Moreover $(C \subseteq \mathbf{P}^3) \in V_1 - V_2$, and the generic curve of V_2 belongs to the liaison class $\mathbf{L}_{n-m}$.*

Proof. By prop. 1.2, X is non-obstructed, and so it belongs to a unique irreducible component V of $H(d, p_a)$, say with generic point $(\tilde{X} \subseteq \mathbf{P}^3)$. We claim that the number $m(\tilde{X})$ of minimal generators of $I(\tilde{X})$ of degree f is zero. Indeed suppose $m(\tilde{X}) > 0$, i.e. that $I(\tilde{X})$ has a minimal generator $\tilde{F}$ of degree f. Since by semicontinuity $e(\tilde{X}) = e(X)$ and $c(\tilde{X}) = c(X)$ (or $c(\tilde{X}) = -\infty$) it follows from lemma 1.3 that

$$\beta_{\tilde{X}/V(\tilde{F})}: H^0(\tilde{X}, \mathcal{N}_{\tilde{X}}) \to H^1(\mathbf{P}^3, \mathcal{I}_{\tilde{X}}(f))$$

is surjective.

In general, however, $\beta_{\tilde{X}/V(\tilde{F})} = 0$ for non-obstructed generic curves. In fact, since the fibers of $\mathrm{pr}_1: D(d, p_a; f) \to H(d, p_a)$ are smooth, we have by generic flatness ([M2], page 57) the existence of an open subscheme U of $H(d, p_a)$, $(\tilde{X} \subseteq \mathbf{P}^3) \in U$, such that the restriction of pr_1 to $\mathrm{pr}_1^{-1}(U)$, $\mathrm{pr}_1^{-1}(U) \to U$, is smooth. Hence its tangent map at $(\tilde{X} \subseteq V(\tilde{F}) \subseteq \mathbf{P}^3)$,

$$p_1: A^1(\tilde{X} \subseteq V(\tilde{F})) \to H^0(\tilde{X}, \mathcal{N}_{\tilde{X}})$$

is surjective, and so $\beta_{\tilde{X}/V(\tilde{F})} = 0$ by (i1) of the introduction.

Now combining this with the surjectivity of $\beta_{\tilde{X}/V(\tilde{F})}$, we get $H^1(\mathbf{P}^3, \mathcal{I}_{\tilde{X}}(f)) = 0$. Hence by Riemann-Roch we have

$$h^0(\mathbf{P}^3, \mathcal{I}_{\tilde{X}}(f)) = h^0(\mathbf{P}^3, \mathcal{I}_X(f)) - n.$$

Since $h^0(\mathbf{P}^3, \mathcal{I}_{\tilde{X}}(v)) = h^0(\mathbf{P}^3, \mathcal{I}_X(v))$ for $v < f$ by semicontinuity, we deduce $m(\tilde{X}) = m - n$, and now $m(\tilde{X}) = 0$ by the assumption $m \leqslant n$. So we have a contradiction and the claim is proved.

Now observe that the arguments above also lead to $n - m = h^1(\mathbf{P}^3, \mathcal{I}_{\tilde{X}}(f))$, i.e. that $\tilde{X}$ belongs to $\mathbf{L}_{n-m}$. Moreover by the proven claim above it is clear that there is no complete intersection $V(F_1, G_1)$ containing $\tilde{X}$ such that $(\tilde{X} \subseteq V(F_1, G_1) \subseteq \mathbf{P}^3)$ is a generization of $t = (X \subseteq V(F, G) \subseteq \mathbf{P}^3)$ in $D(d, p_a; f, g)$ because if such a complete intersection exists, then F_1 must be a non-minimal element by the proven claim, i.e. $F_1 \in \mathfrak{m} I(\tilde{X})$, and so the specialization F must sit in $\mathfrak{m} I(X)$, i.e. we have a contradiction.

Next we claim that there is a complete intersection $V(\tilde{F}_0, \tilde{G})$ of type (f, g) containing the generic curve $\tilde{X}$ above such that $\tilde{t} = (\tilde{X} \subseteq V(\tilde{F}_0, \tilde{G}) \subseteq \mathbf{P}^3)$ is a generization of $t_0 = (X \subseteq V(F_0, G) \subseteq \mathbf{P}^3)$ in $D = D(d, p_a; f, g)$.

Indeed $F_0 = \sum_{i=1}^r H_i F_i$ where $\deg F_i = f_i < f$, $1 \leqslant i \leqslant r$, and $\mathrm{pr}_1: D(d, p_a; f_i) \to H(d, p_a)$ is smooth at $(X \subseteq V(F_i) \subseteq \mathbf{P}^3)$ because $H^1(\mathbf{P}^3, \mathcal{I}_X(f_i)) = 0$. So for any i we have a generization $(\tilde{X} \subseteq V(\tilde{F}_i) \subseteq \mathbf{P}^3)$ of $(X \subseteq V(F_i) \subseteq \mathbf{P}^3)$ in $D(d, p_a; f_i)$, and one may take $\tilde{F}_0$ to be $\sum_{i=1}^r H_i \tilde{F}_i$. In the same way, $H^1(\mathbf{P}^3, \mathcal{I}_X(g)) = 0$

implies the existence of a generization ($\tilde{X} \subseteq V(\tilde{G}) \subseteq \mathbf{P}^3$) of ($X \subseteq V(G) \subseteq \mathbf{P}^3$), and the claim follows easily from this. Observe that choosing $\tilde{F}_0$ and $\tilde{G}$ general enough, one may suppose that $\tilde{t}$ is a generic point of D since $\tilde{X}$ is a generic curve of $H(d, p_a)$.

In conclusion, we have proved that in $D = D(d, p_a; f, g)$, t_0 admits two generizations t and $\tilde{t}$, that $\tilde{t}$ is a generic point of D and that $\tilde{t}$ is not a generization of t. This shows that t_0 sits in the intersection of two different irreducible components of D, and using the liaison isomorphism

$$\tau\colon D \xrightarrow{\approx} D' = D(d', p_a'; f, g)$$

and the smoothness of

$$\mathrm{pr}_1'\colon D' \to H(d', p_a')$$

in some open subset $U' \subset D'$ containing t_0, t and $\tilde{t}$ (for instance, define U' to consist of points ($X' \subseteq V(F', G') \subseteq \mathbf{P}^3$) for which $H^1(\mathbf{P}^3, \mathscr{I}_{X'}(f)) = H^1(\mathbf{P}^3, \mathscr{I}_{X'}(g)) = 0$), we have exactly a similar situation in $H(d', p_a')$. This completes the proof since the smoothness of pr_1' in U' also shows that the curve linked to $\tilde{X}$ by $V(\tilde{F}_0, \tilde{G})$ is a generic curve of $H(d', p_a')$ and so it belongs to $\mathbf{L}_{n-m}$ since $\tilde{X}$ does.

REMARK 2.4. (a) By the arguments in the first part of the proof we can see that a curve X in $\mathbf{L}_n$ with $e(X) < c(X)$, $s(X) < c(X)$ must have a generization $\tilde{X}$ in the Hilbert scheme which belongs to the liaison class $\mathbf{L}_r$ where $r = \max(0, n - m)$.

(b) If in addition to the assumption of (2.3) we suppose $m = 1$, then one may prove that C is a generic curve of the component V_1 of the Hilbert scheme.

Now we use corollaries 1.4 and 1.6 for giving another technique of construction of obstructed curves: they provide an easy way for computing $H^1(C, \mathscr{N}_C)$, if $e(C) < s(C)$, when C is arithmetically Cohen-Macaulay (and in that case $H^1(C, \mathscr{N}_C) = 0$), or when $C \in \mathbf{L}_n$, $n > 0$. Then the idea is to construct a flat family $\{Y_t\}$ of curves, satisfying $e < s$, such that Y_t is arithmetically Cohen-Macaulay if $t \neq 0$, and $Y_0 \in \mathbf{L}_n$, and then to force Y_0 to have minimal generators of degree $c(Y_0) + 4$. In this situation one has

$$H^1(C, \mathscr{N}_C) \begin{cases} = 0 \text{ if } t \neq 0 \\ \neq 0 \text{ if } t = 0 \end{cases}$$

and therefore Y_0 must be obstructed.

In order to produce examples of applications of those statements and of the above argument we need a technique for deforming arithmetically Cohen-Macaulay curves to Buchsbaum curves (or, more generally, to curves which are not arithmetically Cohen-Macaulay).

This is essentially done in [BB]; but now we want to construct curves with maximal rank, and since there is described the case of families of curves with fixed speciality (and postulation jumping), we now sketch the proof for the case of a family of curves with fixed postulation.

PROPOSITION 2.5. *Let $\mathscr{L}$ be an even liaison class. Then there exists an irreducible flat family of curves $\{Y_t\}_{t\in T}$ such that*

(i) $Y_0 \in \mathscr{L}$
(ii) Y_t *is arithmetically Cohen-Macaulay if* $t \neq 0$.

Moreover, this family is with fixed postulation, i.e.

$t \to h^0(\mathbf{P}^3, \mathscr{I}_{Y_t}(n))$ *is constant on* T *for every* n.

Proof. Let F be a locally free sheaf satisfying $H^1(\mathbf{P}^3, F(t)) = 0$ for every t, corresponding to the liaison class $\mathscr{L}$. This means that

$$\bigoplus_i H^2(\mathbf{P}^3, F(i)) \cong M \text{ up to shift,}$$

where M is the Hartshorne-Rao module of a curve of $\mathscr{L}$. Let now G be a direct sum (of enough large rank) of line bundles (of enough large degrees) such that there exists an injective morphism of vector bundles

$$0 \to F \xrightarrow{\sigma} G.$$

Now consider the morphisms

$$\Phi_1 = 1_F \oplus 0\colon F \to F \oplus G$$
$$\Phi_0 = 0 \oplus \sigma\colon F \to F \oplus G.$$

Thanks to [BB], 2.2, there exist a direct sum of line bundles P, with $\operatorname{rank} P = p = \operatorname{rank} G - \operatorname{rank} F - 1$, and morphisms

$$\Psi_1\colon P \to F \oplus G$$
$$\Psi_0\colon P \to F \oplus G$$

such that

$$\Lambda_1 = (\Phi_1, \Psi_1)\colon F \oplus P \to F \oplus G$$
$$\Lambda_0 = (\Phi_0, \Psi_0)\colon F \oplus P \to F \oplus G$$

drop rank in codimension two (in fact, one lifts $p = \operatorname{rank} G - \operatorname{rank} F - 1$ general sections of $[(F \oplus G)/\Phi_i(F)](t)$, where t is an integer such that $[(F \oplus G)/\Phi_i(F)](t)$ is globally generated; P is then $p\mathcal{O}_{\mathbf{P}^3}(-t)$).

Then there exists an open non void subset T of $\mathbf{k}$ for which the morphism

$$\Lambda_t = t\Lambda_1 + (1 - t)\Lambda_0 \colon F \oplus P \to F \oplus G$$

drops rank in codimension two, hence along a curve Y_t, $t \in T$.

Hence we have a flat family of curves $\{Y_t\}_{t \in T}$ and exact sequences

$$0 \to F \oplus P \xrightarrow{\Lambda_t} F \oplus G \to \mathscr{I}_{T_t}(r) \to 0$$

(and r does not depend on t).

Since $H^1(\mathbf{P}^3, F(n)) = 0$ for every n, we get from the associated long exact sequences that

$$t \to h^0(\mathbf{P}^3, \mathscr{I}_{Y_t}(n))$$

is constant on T for every n.

Moreover, the exact sequences

$$0 \to H^1(\mathbf{P}^3, \mathscr{I}_{Y_t}(r + n)) \to H^2(\mathbf{P}^3, F(n)) \xrightarrow{H^2(\Lambda_t(n))} H^2(\mathbf{P}^3, F(n))$$

show that

$$H^1(\mathbf{P}^3, \mathscr{I}_{Y_t}(r + n)) = 0 \text{ for every } n \text{ if } t \neq 0$$

since $H^2(\Lambda_t(n))$ is an isomorphism if $t \neq 0$, and that

$$\bigoplus_n H^1(\mathbf{P}^3, \mathscr{I}_{Y_0}(r + n)) \cong \bigoplus_n H^2(\mathbf{P}^3, F(n)) \neq 0$$

since $H^2(\Lambda_0(n))$ is zero.

Hence Y_t is arithmetically Cohen-Macaulay if $t \neq 0$, and $Y_0 \in \mathscr{L}$.

Example 2.6.

Let us apply this proposition, performing explicitly its steps, in order to produce examples of Buchsbaum curves which are specializations of arithmetically

Cohen-Macaulay curves. We concentrate on the liaison class $\mathbf{L}_n$. In this case, needed vector bundle is the tangent bundle to $\mathbf{P}^3$, let us denote it by T.

There exists an injective morphism

$$0 \to T \xrightarrow{\sigma^*} 6\mathcal{O}_{\mathbf{P}^3}(2)$$

(in fact, there is a surjection $6\mathcal{O}_{\mathbf{P}^3}(-2) \to \Omega_{\mathbf{P}^3}$);

let a be an integer >0, and let us consider the injective morphism

$$\sigma = \sigma^* \oplus 0\colon\ T \to 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}$$

(this will force the existence of minimal generators in the needed degree).

Now consider, as in proposition 2.5, the morphisms

$$\Phi_1 = 1_T \oplus 0\colon\ T \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}$$

$$\Phi_0 = 0 \oplus \sigma\colon\ T \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}.$$

Thus we have two exact sequences

$$0 \to T \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3} \to 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3} \to 0$$

$$0 \to T \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3} \to T \oplus [(6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3})/\operatorname{Im}\sigma] \to 0.$$

Note that $H^1(\mathbf{P}^3, T(t)) = 0\ \forall t$, that $T(t)$ is globally generated if $t \geqslant -1$ (thanks to the Castelnuovo-Mumford lemma) and that $[6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}](t)$ (and his quotient too) is globally generated if $t \geqslant 0$.

Hence we can choose $P = (5 + a)\mathcal{O}_{\mathbf{P}^3}(-1)$, and find two morphisms which drop rank along a curve

$$\Lambda_1 = (\Phi_1, \Psi_1)\colon\ T \oplus (5 + a)\mathcal{O}_{\mathbf{P}^3}(-1) \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}$$

$$\Lambda_0 = (\Phi_0, \Psi_0)\colon\ T \oplus (5 + a)\mathcal{O}_{\mathbf{P}^3}(-1) \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus a\mathcal{O}_{\mathbf{P}^3}.$$

As in proposition 2.5, we have a family of curves $\{Y_t\}_{t \in T}$ defined by exact sequences of the form

$$0 \to T \oplus (5 + a)\mathcal{O}_{\mathbf{P}^3}(-1) \to T \oplus 6\mathcal{O}_{\mathbf{P}^3}(2) \oplus \mathcal{O}_{\mathbf{P}^3} \to \mathcal{I}_{Y_t}(\delta) \to 0$$

where $Y_0 \in \mathbf{L}_1$ and Y_t is arithmetically Cohen-Macaulay if $t \neq 0$.

One can compute the cohomology functions of these curves and find

$t\neq 0$	$\mathscr{I}_{Y_t}$	$\delta-5$	$\delta-4$	$\delta-3$	$\delta-2$	$\delta-1$	δ
	h^0	0	0	0	6	24	$60+a$
	h_1	0	0	0	0	0	0
	h^2	*	$20+3a$	$5+a$	0	0	0

	$\mathscr{I}_{Y_0}$	$\delta-5$	$\delta-4$	$\delta-3$	$\delta-2$	$\delta-1$	δ
	h^0	0	0	0	6	24	$60+a$
	h^1	0	1	0	0	0	0
	h^2	*	$21+3a$	$5+a$	0	0	0

This forces Y_0 to have generators in degree $\delta = c(Y_0) + 4$, and therefore $H^1(Y_0, \mathscr{N}_{Y_0}) \neq 0$. Since $e(Y_t) < s(Y_t)$ $\forall t$, we have $H^1(Y_t, \mathscr{N}_{Y_t})=0$ if $t \neq 0$. Hence Y_0 cannot correspond to a smooth point of the Hilbert scheme.

REMARK 2.7. Starting with $T \oplus H$, where H is a direct sum of line bundles, or changing the twists of the line bundles in this construction, one gets different examples, always in $\mathbf{L}_1$.

REMARK 2.8. Note that Y_0 has maximal rank. Note moreover that in this way we are not able to check that there is a *smooth* obstructed maximal rank curve in $\mathbf{L}_1$, even if there are smooth curves in $\mathbf{L}_1$ with the same cohomology function of Y_0.

3. An example of obstructed smooth space curve of maximal rank

In this section, using Prop. 2.5, we will produce concrete examples of obstructed smooth connected maximal rank curves.

Let Y be a smooth connected maximal rank curve of degree $d(Y) = 24$ and arithmetic genus $p_a(Y) = 66$ with an Ω-resolution of the following kind (see [C], th. 2.3 for the existence of such curves):

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-9) \oplus \mathscr{O}_{\mathbf{P}^3}(-8) \oplus \mathscr{O}_{\mathbf{P}^3}(-6) \to \mathscr{O}_{\mathbf{P}^3}(-7) \oplus \Omega_{\mathbf{P}^3}(-4) \to \mathscr{I}_Y \to 0.$$

Note that Y is an arithmetically Buchsbaum curve with the following invariants: $\sigma(Y) = e(Y) = 5$, $s(Y) = 6$, $c(Y) = 4$ and $M(Y) \cong \mathbf{k}$ (i.e., $Y \in \mathbf{L}_1$). Its numerical character ([BM]) is (8, 7, 7, 6, 6), and hence the existence of such a smooth curve follows also from [BM], th. 5.3.

We link Y to an arithmetically Buchsbaum curve X by means of two general surfaces S_6 and S_9 of degrees 6 and 9 respectively containing Y (this is possible since $\mathscr{I}_Y(8)$ is globally generated). Thanks to [PS], prop. 2.5, the ideal sheaf $\mathscr{I}_X$ of X has a locally free resolution of the following kind:

$$0 \to \mathbf{T}_{\mathbf{P}^3}(-11) \oplus \mathcal{O}_{\mathbf{P}^3}(-8) \to \mathcal{O}_{\mathbf{P}^3}(-9)^2 \oplus \mathcal{O}_{\mathbf{P}^3}(-7) \oplus \mathcal{O}_{\mathbf{P}^3}(-6)^2$$
$$\to \mathscr{I}_X \to 0.$$

In particular, $e(X) = 5 < s(X) = 6 < c(X) = 7$, and the homogeneous ideal $I(X)$ of X has two minimal generators in degree 6 and at least one minimal generator in degree 7, let us call them F_6, G_6 and F_7.

Let now C_0 be a curve linked to X by means of $H \cdot F_6 + H' \cdot G_6$ and F_9, where F_9 is a general surface of degree 9 containing C_0 and H, H' are general planes. Invoking [PS], prop. 2.5 again we get a locally free resolution of $\mathscr{I}_{C_0}$

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-10)^2 \oplus \mathcal{O}_{\mathbf{P}^3}(-9) \oplus \mathcal{O}_{\mathbf{P}^3}(-7)^2$$
$$\to \mathcal{O}_{\mathbf{P}^3}(-9) \oplus \mathcal{O}_{\mathbf{P}^3}(-8) \oplus \mathcal{O}_{\mathbf{P}^3}(-7) \oplus \Omega_{\mathbf{P}^3}(-5) \to \mathscr{I}_{C_0} \to 0.$$

Furthermore, $\deg(C_0) = 33$, $p_a(C_0) = 117$, $s(C_0) = 7$, $c(C_0) = 5$, $\sigma(C_0) = e(C_0) = 6$. So, C_0 has maximal rank and, by prop. 2.3, it is obstructed (more, precisely, it is in the intersection of two irreducible components of the Hilbert scheme).

It remains to prove that between the C_0's constructed in such a way there is a smooth connected curve; that is to say, choosing suitably Y and the surfaces we can construct a smooth curve with the required properties.

To this end, we use *Macaulay* ([BS]). We construct explicitly Y, K and C_0, performing a sequence of direct linkages starting from the homogeneous ideal of two skew lines (xz, xt, yz, yt). Macaulay gives us a minimal system of generators of $I(Y)$, $I(X)$ and then of $I(C_0)$, and then we check that Y, X and C_0 are smooth connected space curves.

The computations are rather lengthy, so we omit them. We only write here the minimal system of generators of $I(C_0)$; note that it is generated by 7 elements (5 of degree 7, 1 of degree 8 and 1 of degree 9); unfortunately these polynomials are huge. We list the 7 polynomials from the computer printing, where the curve is called bkm.

REMARK 3.2. Note that the obstructed smooth maximal rank curve C_0 constructed previously has not natural cohomology.

Appendix. Equations of the curve C_0 of §3

```
bkm:
; codimension = 2
; degree      = 33
; genus       = 117

% type bkm
; x7+6671x6y-12589x5y2-97/78x4y3-12023x3y4+59/50x2y5-43/63xy6-6913x6z \
;     -125/78x5yz-10/77x4y2z-12041x3y3z+22/97x2y4z-54/31xy5z-67/48x5z2 \
;     +3792x4yz2-20/3x3y2z2+116/19x2y3z2+37/103xy4z2+86/119x4z3-13097x3yz3 \
;     -14067x2y2z3-20/19xy3z3-114/29x3z4+4934x2yz4+2336xy2z4-9121x2z5+2242xyz5 \
;     +51/122xz6+2772x6t-1906x5yt-97/122x4y2t+13/8x3y3t-4275x2y4t-8382xy5t \
;     +35/107y6t-124/29x5zt+6549x4yzt+14291x3y2zt+206x2y3zt-35/9xy4zt \
;     +61/41y5zt+11/5x4z2t+2324x3yz2t-8473x2y2z2t-663xy3z2t-13/121y4z2t \
;     +71/124x3z3t-61/77x2yz3t-9781xy2z3t-39/61y3z3t-15/13x2z4t-11437xyz4t \
;     -28/57y2z4t-43/5xz5t-103/79yz5t-119/8z6t-13/112x5t2+121/49x4yt2 \
;     -81/71x3y2t2-12922x2y3t2+41/11xy4t2-7094y5t2+64/3x4zt2-65/37x3yzt2 \
;     +63/103x2y2zt2-11/125xy3zt2+8262y4zt2+26/95x3z2t2-113/70x2yz2t2 \
;     -71/31xy2z2t2-105/16y3z2t2+30/53x2z3t2+3856xyz3t2+14346y2z3t2+9164xz4t2 \
;     -5063yz4t2-12212z5t2-8757x4t3+63/95x3yt3+10962x2y2t3+2019xy3t3-48/77y4t3 \
;     -29/39x3zt3+32/31x2yzt3-8324xy2zt3-14914y3zt3-1/25x2z2t3-817xyz2t3 \
;     +122/115y2z2t3-91/110xz3t3+12068yz3t3-8698z4t3-38/63x3t4-10/23x2yt4 \
;     -81/26xy2t4-63/86y3t4-13/95x2zt4-17/26xyzt4-53/42y2zt4-1430xz2t4 \
;     +12548yz2t4+15/89z3t4+37/112x2t5+32/17xyt5+5456y2t5+92/105xzt5-12974yzt5 \
;     -120/103z2t5-8121xt6+76/23yt6-2890zt6-64/115t7

; x6y+6671x5y2-12589x4y3-97/78x3y4-12023x2y5+59/50xy6-43/63y7-6913x5yz \
;     -125/78x4y2z-10/77x3y3z-12041x2y4z+22/97xy5z-54/31y6z-67/48x4yz2 \
;     +3792x3y2z2-20/3x2y3z2+116/19xy4z2+37/103y5z2+86/119x3yz3-13097x2y2z3 \
;     -14067xy3z3-20/19y4z3-114/29x2yz4+4934xy2z4+2336y3z4-9121xyz5+2242y2z5 \
;     +51/122yz6+11599x6t+11384x5yt-82/7x4y2t+59/112x3y3t+111/106x2y4t \
;     -6424xy5t+10114y6t+43/17x5zt-37/63x4yzt+109/85x3y2zt-2008x2y3zt \
;     +9324xy4zt+32y5zt+6770x4z2t-7215x3yz2t-109/95x2y2z2t-7014xy3z2t \
;     +12882y4z2t-45/71x3z3t+3089x2yz3t+8691xy2z3t-27/65y3z3t-15326x2z4t \
;     -73/105xyz4t-11121y2z4t+29/20xz5t+116/21yz5t+77/101z6t+13746x5t2 \
;     -14831x4yt2-98/103x3y2t2+94/67x2y3t2+11618xy4t2-11259y5t2-103/19x4zt2 \
;     +73/16x3yzt2-43/76x2y2zt2+22/7xy3zt2-32/101y4zt2-56/103x3z2t2 \
;     -21/85x2yz2t2+106/29xy2z2t2-3539y3z2t2+80/29x2z3t2+75/68xyz3t2 \
;     +61/113y2z3t2+6494xz4t2+113/82yz4t2+5301z5t2+14/111x4t3+63/22x3yt3 \
;     -4544x2y2t3+50xy3t3-108/101y4t3-7186x3zt3+8924x2yzt3+12268xy2zt3 \
;     +79/49y3zt3-7/71x2z2t3+5/64xyz2t3-111/23y2z2t3-99/31xz3t3-111/20yz3t3 \
;     +17/91z4t3-14/121x3t4+3652x2yt4-125/64xy2t4-19/35y3t4-5/103x2zt4 \
;     -13343xyzt4-93/8y2zt4-5309xz2t4-55/37yz2t4+3931z3t4-1340x2t5+88/35xyt5 \
;     -6841y2t5+93xzt5-9/77yzt5+49/6z2t5+74/65xt6-51/29yt6+2/7zt6-4694t7

; x6z+6671x5yz-12589x4y2z-97/78x3y3z-12023x2y4z+59/50xy5z-43/63y6z-6913x5z2 \
;     -125/78x4yz2-10/77x3y2z2-12041x2y3z2+22/97xy4z2-54/31y5z2-67/48x4z3 \
;     +3792x3yz3-20/3x2y2z3+116/19xy3z3+37/103y4z3+86/119x3z4-13097x2yz4 \
;     -14067xy2z4-20/19y3z4-114/29x2z5+4934xyz5+2336y2z5-9121xz6+2242yz6 \
;     +51/122z7+93/125x6t-15659x5yt-5854x4y2t-624x3y3t-4871x2y4t+2726xy5t \
;     -1366y6t+27/38x5zt+3576x4yzt-55/21x3y2zt+10902x2y3zt-1191xy4zt \
;     -109/46y5zt-49/64x4z2t-107/41x3yz2t+1004x2y2z2t-39/25xy3z2t+46/123y4z2t \
;     +93/100x3z3t-23/52x2yz3t+1592xy2z3t+76/111y3z3t-13777x2z4t-37/27xyz4t \
;     +103/41y2z4t+115/97xz5t-6158yz5t-103/119z6t+1/36x5t2+73/54x4yt2 \
;     +9824x3y2t2+5622x2y3t2+78/79xy4t2-5638y5t2+86/37x4zt2+65/121x3yzt2 \
;     +109/106x2y2zt2+2260xy3zt2+12143y4zt2+1443x3z2t2-31/78x2yz2t2 \
;     -89/90xy2z2t2+61/124y3z2t2-29/65x2z3t2-119/4xyz3t2-39/86y2z3t2-73/8xz4t2 \
;     +13/44yz4t2+23/63z5t2-7/44x4t3+1/66x3yt3-1/84x2y2t3+48/77xy3t3+39/2y4t3 \
;     -51/91x3zt3+9401x2yzt3-6626xy2zt3+25/99y3zt3-45/8x2z2t3+4/61xyz2t3 \
;     -121/91y2z2t3+9862xz3t3+5390yz3t3-39/23z4t3+64/67x3t4-8/85x2yt4 \
;     +93/56xy2t4+107/59y3t4+124/123x2zt4-11327xyzt4-4304y2zt4+116/9xz2t4 \
;     +123/44yz2t4-10305z3t4-76/109x2t5-11279xyt5-53/80y2t5-5/37xzt5+65/71yzt5 \
;     -9/25z2t5-4245xt6-37/116yt6+85/32zt6+3969t7

; x5y2-9294x4y3-123/106x3y4-47/10x2y5-79/55xy6+9751y7+1191x5yz+3517x4y2z \
;     +22/35x3y3z-10/9x2y4z+12960xy5z-1285y6z+119/100x5z2+9281x4yz2 \
;     -118/101x3y2z2-67/95x2y3z2-4028xy4z2+86/63y5z2+5763x4z3-8207x3yz3 \
;     -61/82x2y2z3-46/101xy3z3-125/118y4z3+6162x3z4+118/103x2yz4-51/11xy2z4 \
;     -30/7y3z4-115/79x2z5+73/126xyz5+91/38y2z5+14780xz6+11114yz6-37/26z7 \
;     +76/7x6t-12379x5yt-67/62x4y2t+62/63x3y3t-74/75x2y4t-7458xy5t+218y6t \
;     +24/89x5zt+117/49x4yzt-107/90x3y2zt+68/55x2y3zt-31/17xy4zt+97/91y5zt \
;     -45/109x4z2t+20/61x3yz2t+8260x2y2z2t+11921xy3z2t+67/13y4z2t-56/83x3z3t \
;     -40/121x2yz3t+69/119xy2z3t+12293y3z3t-3504x2z4t-1544xyz4t+49/46y2z4t \
```

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;       +3053xz5t+95/28yz5t+11841z6t+6039x5t2-31/98x4yt2-8277x3y2t2+1324x2y3t2 \
;       +76/93xy4t2+11268y5t2-15396x4zt2+113/96x3yzt2-14920x2y2zt2-25xy3zt2 \
;       +71/107y4zt2+33/76x3z2t2+4/115x2yz2t2+17/9xy2z2t2+61/114y3z2t2 \
;       +14896x2z3t2-100/23xyz3t2-85/97y2z3t2+121/114xz4t2-8052yz4t2+596z5t2 \
;       +11/35x4t3+82/115x3yt3+79/50x2y2t3-11656xy3t3-2071y4t3+89/35x3zt3 \
;       +4/75x2yzt3+10710xy2zt3-1/23y3zt3-72/95x2z2t3+67/122xyz2t3+91/12y2z2t3 \
;       +4/15xz3t3-41/42yz3t3-23/105z4t3-5943x3t4-67/126x2yt4+5/106xy2t4 \
;       +43/48y3t4+51/115x2zt4+105/71xyzt4-196y2zt4-5177xz2t4+77/12yz2t4 \
;       +89/111z3t4+9684x2t5-69/29xyt5-105/37y2t5+37/94xzt5+13244yzt5+11003z2t5 \
;       -11151xt6-37/62yt6+9973zt6+107/44t7

; x4y3-51/47x3y4-6700x2y5-83/34xy6+113/117y7-41/126x5yz+88/87x4y2z+80/89x3y3z \
;       -1418x2y4z-76/3xy5z-103/116y6z-41/100x5z2+7/116x4yz2-87/11x3y2z2 \
;       -90/59x2y3z2+109/92xy4z2+36/49y5z2-15064x4z3-36/37x3yz3+4105x2y2z3 \
;       +8/3xy3z3-29/51y4z3-103/72x3z4+14049x2yz4+6975xy2z4+10588y3z4+5231x2z5 \
;       +71/82xyz5+6723y2z5-11048xz6-2159yz6+14243z7-12068x6t+111/64x5yt \
;       -14385x4y2t+88/51x3y3t+6/31x2y4t-4047xy5t-60/103y6t+37/15x5zt-8287x4yzt \
;       -121/36x3y2zt-103/47x2y3zt-30/37xy4zt+73/68y5zt-125/104x4z2t+32/59x3yz2t \
;       -35/22x2y2z2t-63/124xy3z2t-87/65y4z2t-113/97x3z3t-8304x2yz3t+6844xy2z3t \
;       -2541y3z3t-49/123x2z4t+108/115xyz4t+83/69y2z4t-61/97xz5t-3926yz5t \
;       -43/23z6t+15200x5t2-101/126x4yt2-55/47x3y2t2+6801x2y3t2+80/113xy4t2 \
;       -63/64y5t2+103/106x4zt2-2626x3yzt2-94/23x2y2zt2+9280xy3zt2-27/92y4zt2 \
;       -39/80x3z2t2+10795x2yz2t2+71/81xy2z2t2+10948y3z2t2+9730x2z3t2 \
;       -70/13xyz3t2+100/101y2z3t2-69/55xz4t2+5548yz4t2-25/106z5t2+102/47x4t3 \
;       -116/103x3yt3-12402x2y2t3-36/17xy3t3-63/37y4t3-92/81x3zt3+5575x2yzt3 \
;       +89/93xy2zt3+2684y3zt3-29/36x2z2t3-29/20xyz2t3-14747y2z2t3+4081xz3t3 \
;       -49/38yz3t3+70/19z4t3+4232x3t4-19/93x2yt4-7429xy2t4+27/125y3t4 \
;       -14107x2zt4-113/91xyzt4+41/23y2zt4-13957xz2t4+8187yz2t4-33/67z3t4 \
;       -9575x2t5+38/93xyt5+72/71y2t5+8171xzt5-5763yzt5+69/50z2t5+30/121xt6 \
;       -77/29yt6-107/66zt6-103/72t7

; x2y6+54/121xy7+54/121y8-11/90x3y4z+5194x2y5z+1046xy6z-11277y7z+12211x5yz2 \
;       +28/37x4y2z2+13181x3y3z2+59/102x2y4z2+47/124xy5z2-15253y6z2+48/85x5z3 \
;       +27/20x4yz3+11/116x3y2z3-8276x2y3z3-7/106xy4z3-2275y5z3-14106x4z4 \
;       +14751x3yz4-92/25x2y2z4+11975xy3z4-71/116y4z4-12/107x3z5+41/54x2yz5 \
;       -12038xy2z5+3779y3z5+110/19x2z6+82/47xyz6+16/53y2z6-71/113xz7-6370yz7 \
;       +1577z8+52/85x3y4t-63/11x2y5t+63/107xy6t-12257y7t-29/78x5yzt+6239x4y2zt \
;       -151x3y3zt+5103x2y4zt-619xy5zt+1503y6zt-2323x5z2t+30x4yz2t-4/79x3y2z2t \
;       +34/69x2y3z2t-111/110xy4z2t-53/72y5z2t+11/100x4z3t-1273x3yz3t \
;       +101/63x2y2z3t-1808xy3z3t+78/23y4z3t-64/125x3z4t-6445x2yz4t-33/38xy2z4t \
;       +97/5y3z4t+125/3x2z5t+113/27xyz5t-27/16y2z5t+49/55xz6t+2480yz6t-15/67z7t \
;       -25/26x6t2-3322x5yt2-15460x4y2t2-76/97x3y3t2-57/109x2y4t2-12205xy5t2 \
;       +1676y6t2-7799x5zt2-76/55x4yzt2+4547x3y2zt2+119/38x2y3zt2-105/29xy4zt2 \
;       -108/115y5zt2+13/35x4z2t2+6113x3yz2t2+28/107x2y2z2t2-76/113xy3z2t2 \
;       -63/103y4z2t2-76/53x3z3t2-10545x2yz3t2+52/121xy2z3t2+8315y3z3t2 \
;       -31/74x2z4t2-15084xyz4t2-85/114y2z4t2+14467xz5t2+2543yz5t2+10616z6t2 \
;       -10858x5t3-6796x4yt3-3906x3y2t3+738x2y3t3-3907xy4t3-80/67y5t3 \
;       +17/118x4zt3-3080x3yzt3+25/68x2y2zt3+9224xy3zt3-9/35y4zt3+6991x3z2t3 \
;       -3/61x2yz2t3-11841xy2z2t3+8706y3z2t3+23/63x2z3t3+44/89xyz3t3+38/21y2z3t3 \
;       -50/41xz4t3-117/88yz4t3-53/14z5t3-31/102x4t4+3443x3yt4+69/10x2y2t4 \
;       +123/106xy3t4+12542y4t4+83/49x3zt4-10838x2yzt4-74/113xy2zt4-43/72y3zt4 \
;       -57/64x2z2t4+125/19xyz2t4+19/77y2z2t4-13348xz3t4-75/91yz3t4+14141z4t4 \
;       +123/56x3t5-28/109x2yt5-6550xy2t5-7955y3t5+58/19x2zt5-82/85xyzt5 \
;       -95/29y2zt5+111/4xz2t5+103/44yz2t5+41/2z3t5-4687x2t6-10025xyt6-1127y2t6 \
;       +7397xzt6+25/86yzt6+69/116z2t6-91/92xt7-45/82yt7+4/75zt7+4121t8

; y9+2039xy7z+9797y8z-26/125x2y5z2+32/49xy6z2+15907y7z2-6489x4y2z3+167x3y3z3 \
;       -5604x2y4z3+43/30xy5z3-76/29y6z3-15013x5z4-29/39x4yz4+3849x3y2z4 \
;       +119/66x2y3z4-3251xy4z4+81/109y5z4-920x4z5+69/16x3yz5+58/109x2y2z5 \
;       +89/63xy3z5-9/122y4z5+5753x3z6+63/32x2yz6-11383xy2z6+1/65y3z6+56/61x2z7 \
;       +7795xyz7+75/58y2z7+5172xz8-43/26yz8+85/44z9-97/9xy7t+10527y8t \
;       -13608x2y5zt-73/32xy6zt-25/56y7zt-49/6x4y2z2t+29/97x3y3z2t-3216x2y4z2t \
;       +3540xy5z2t-12169y6z2t-104/57x5z3t+48/109x4yz3t-41/93x3y2z3t+3596x2y3z3t \
;       +75/119xy4z3t+55/63y5z3t+7434x4z4t+13448x3yz4t+23/58x2y2z4t-31/12xy3z4t \
;       +123/100y4z4t-4592x3z5t+31/18x2yz5t-38/3xy2z5t-45/44y3z5t-5226x2z6t \
;       -67/95xyz6t+21/80y2z6t-101xz7t-8983yz7t+79/124z8t-4904x2y5t2-42/85xy6t2 \
;       +8754y7t2+117/119x4y2zt2-6681x3y3zt2-9018x2y4zt2-14195xy5zt2+107/33y6zt2 \
;       -13879x5z2t2-6827x4yz2t2-12319x3y2z2t2-10991x2y3z2t2+86/93xy4z2t2 \
;       +5178y5z2t2+119/45x4z3t2+94/45x3yz3t2+8299x2y2z3t2-76/67xy3z3t2 \
;       -71/94y4z3t2+93/91x3z4t2+13985x2yz4t2+14/79xy2z4t2-12018y3z4t2 \
;       +101/11x2z5t2+71/101xyz5t2-6176y2z5t2-20/97xz6t2+121/100yz6t2+1/75z7t2 \
;       +3/17x4y2t3+62/63x3y3t3+29/68x2y4t3+6845xy5t3-45/101y6t3-126/125x5zt3 \
;       +8913x4yzt3-92/47x3y2zt3+5/109x2y3zt3+77/60xy4zt3+101/79y5zt3 \
;       +28/33x4z2t3+71/112x3yz2t3+3921x2y2z2t3-2924xy3z2t3-2/121y4z2t3 \
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;       -12589x3z3t3-41/119x2yz3t3+88/41xy2z3t3-8919y3z3t3-15674x2z4t3 \
;       +69/14xyz4t3-5761y2z4t3+97xz5t3-34/67yz5t3-75/19z6t3-19/83x5t4+3926x4yt4 \
;       +2238x3y2t4-4535x2y3t4+33/49xy4t4+26/47y5t4-50/109x4zt4-15002x3yzt4 \
;       -114/23x2y2zt4+19/12xy3zt4+1/109y4zt4+9727x3z2t4-37/12x2yz2t4 \
;       +117/55xy2z2t4+83/51y3z2t4-75/112x2z3t4-1151xyz3t4+35/36y2z3t4 \
;       +29/98xz4t4-103/38yz4t4+15225z5t4-113/98x4t5-5/88x3yt5+7513x2y2t5 \
;       -23/26xy3t5+2646y4t5+37/91x3zt5+47/78x2yzt5-29/109xy2zt5+13462y3zt5 \
;       -72/83x2z2t5+29/12xyz2t5+10620y2z2t5-1072xz3t5+45/77yz3t5-50/11z4t5 \
;       +12965x3t6+113/76x2yt6-10617xy2t6-9744y3t6+111/20x2zt6-31/84xyzt6 \
;       -11/114y2zt6+12075xz2t6+35yz2t6-25/106z3t6+17/29x2t7+10580xyt7 \
;       +48/125y2t7-25/42xzt7-107/41yzt7+10366z2t7-32/125xt8+88/81yt8+2840zt8 \
;       +8248t9

; NOW, WE WILL PROVE THAT THE CURVE DEFINED BY bkm IS SMOOTH

% jacob bkm b
[126k]
% wedge b 2 k
[189k][252k][315k][378k][441k][504k][567k][630k][692k][755k][818k][881k][944k]
; [1007k][1070k]

% flatten k k
[1133k][1196k][1259k][1321k][1384k][1395k][1431k]

% std k k

% codim k
; codimension : 4

% exit
```

References

[B] G. Bolondi: *Irreducible families of curves with fixed cohomology. Arch. der Math.* 53 (1989), 300–305

[BB] E. Ballico and G. Bolondi: *Deficiency modules and specialization.* To appear in Proc. A.M.S.

[BM] G. Bolondi and J. C. Migliore: *Classification of maximal rank curves in the liaison class* $\mathbf{L}_n$, *Math. Ann.* 277 (1987), 585–603.

[BS] D. Bayer and M. Stillman: *Macaulay*, version 3.0 (1989).

[C] M. C. Chang: *Characterization of arithmetically Buchsbaum subschemes of codimension* 2 *in* $\mathbf{P}^n$. Preprint (1987).

[E] Ph. Ellia: D'autres composantes non réduites de Hilb $\mathbf{P}^3$. *Math. Ann.* 277 (1987), 433–446.

[EF] Ph. Ellia and M. Fiorentini: Défaut de postulation et singularités du Schéma de Hilbert. *Annali Univ. di Ferrara* 30 (1984) 185–198.

[El] G. Ellinsgrud: Sur le schéma d'Hilbert des variétés de codimension 2 dans $\mathbf{P}^e$ à cône de Cohen-Macaulay. *Ann. Sc. Ec. Norm. Sup.* 2 (1975), 423–432.

[K1] J. O. Kleppe: *The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curve in the* 3*-space*, Thesis, Univ. of Oslo (1981).

[K2] J. O. Kleppe: Deformations of graded algebras, *Math. Scand.* 45 (1979, 205–231.

[K3] J. O. Kleppe: *Non-reduced components of the Hilbert scheme of smooth space curves*, in Space Curves, Proceedings Rocca di Papa 1985, Springer Lecture Notes in Math. 1266. Springer 1987.

[K4] J. O. Kleppe: *Liaison of families of subschemes , in* $\mathbf{P}^n$, In: Algebraic Curves and Projective Geometry, Proceedings Trento 1988, Springer Lecture Notes in Math. 1389, 128–173. Springer, 1989.

[M1] D. Mumford: Further pathologies in algebraic geometry, *Amer. J. Math.* 89 (1962), 642–648.

[PS] Chr. Peskine and L. Szpiro: Liaison des variétés algébriques, *Inv. Math.* 26 (1974), 271–302.
[PiS] R. Piene and D. Schlessinger: On the Hilbert scheme compactification of the space of twisted cubics, *Amer. J. Math.* 107 (1985), 761–774.
[S] E. Sernesi: *Un esempio di curva ostruita in* $\mathbf{P}^3$. Sem. di variabili Complesse, Bologna (1981), 223–231.
[SGA2] A. Grothendieck: *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux*, North-Holland, Amsterdam, 1968.
[W] Ch. Walter: *Some examples of obstructed curves in* $\mathbb{P}^3$.