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Hypersurface sections and obstructions (rational surface singularities)

KURT BEHNKE[1] and JAN ARTHUR CHRISTOPHERSEN[2]
[1]*Mathematisches Seminar, Universitat Hamburg, Bundestrasse 55, 2000 Hamburg 13, F.R.G.,*
[2]*Matematisk Institutt, PB 1053 Blindern, 0316 Oslo 3, Norway*

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Introduction

This paper came about from a wish to get examples of, and thereby hopefully a better understanding of, obstruction spaces for isolated singularities. The state of the art best approximation to a true obstruction space is the so called T^2 and it is this module we will examine. We first give a general lemma relating T^2 to deformations of hypersurface sections. The bulk of the rest of the paper is devoted to studying T^2 for rational surface singularities. As a means to partially compute T^2 we prove some results on curves of minimal δ invariant that may be of general interest.

By an isolated singularity we will mean the germ of an analytic space at a point (always denoted 0) with a representative that is smooth away from 0. All spaces will be local unless otherwise stated. We briefly recall some of the definitions and results of deformation theory for isolated singularities.

A deformation of X is a flat map of local spaces, $\mathscr{X} \to S$ with X as the fiber over 0. It is versal if for any other deformation $\mathscr{Y} \to T$, there exists a morphism $T \to S$ such that $\mathscr{Y}$ is the pull back of $\mathscr{X}$. An isolated singularity X has a versal deformation ([Schlessinger], [Grauert]). We will assume that the tangent space T_0S of S at the special point has minimal dimension. (Such a deformation is usually called mini-versal or semi-universal). Then T_0S is isomorphic as a vector space to the so called T^1_X, the space of isomorphism classes of first order infinitesimal deformations of X. The T^i will be defined in §2.

If $\dim_{\mathbb{C}} T^1_X = \tau$, then we may think of S lying in $(\mathbb{C}^\tau, 0)$. Let I_S be its ideal, $\mathbf{m}_\tau$ the maximal ideal at 0 in $\mathbb{C}^\tau$ and let $*$ denote the $\mathbb{C}$ dual of a vector space. Then there is an injective 'obstruction map'

$$(I_S/\mathbf{m}_\tau I_S)^* \hookrightarrow T^2_X.$$

The space on the left is the true obstruction space, and the map is not in general

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bijective. Nevertheless, T_X^2 contains the obstructions, and there is an obstruction calculus involving T_X^2 (see e.g. [Laudal]) that has proven to be important in calculating versal deformations. In the case of cyclic quotient singularities for example the obstruction map is also surjective. (See [Arndt], [Christophersen 2], and [Stevens 3].)

There is by now a good knowledge of infinitesimal deformations. For isolated quotient singularities e.g. $T^1 = 0$ in dimensions at least 3 [Schlessinger 2], and for quotient surface singularities [Behnke-Kahn-Riemenschneider] can serve as a reference. The overall picture of infinitesimal deformations of rational surface singularities has been clarified a bit in [Behnke-Knörrer]: if the fundamental cycle Z on the resolution $\tilde{X}$ of X is reduced and sufficiently negative then $\dim_{\mathbb{C}} T_X^1 = \dim_{\mathbb{C}} H^1(\tilde{X}, \Theta_{\tilde{X}}) + n - 3$, $n + 1$ the embedding dimension. The cohomology group on the right hand side can be computed in many cases. For T^1 of non-rational surface singularities see e.g. [Pinkham 1], [Behnke], [Wahl 4] and [Wahl 5].

The basis for our work is a lemma which, in its simplest form, states that for an isolated singularity X and a hypersurface section Y with defining equation f, which also is an isolated singularity;

$$\tau_Y - e_f = \dim_{\mathbb{C}} T_X^2 / f T_X^2$$

where τ_Y is the Tjurina number of Y and e_f is the dimension of a smoothing component in the versal base space of Y. This lemma was first shown in [Christophersen 1] where it was used to compute T^2 for cyclic quotient singularities. In Section 1.3 we give a generalized version and a complete proof of it. The proof uses standard properties of the cotangent complex which we recall in Section 1.1 and a result of Greuel and Looijenga relating dimensions of smoothing components to a subspace of T_Y^1. This was first used to prove Wahl's conjecture on the dimension of smoothing components [Greuel-Looijenga].

The lemma has several applications, e.g. to the question of un-obstructedness, and we give examples in section 1.4 and section 6. If one knows something about T_X^2 then one gets information about the deformations of hypersurface sections and vice versa. Our main application though is to the study of T^2 for rational surface singularities.

In [Christophersen 1] the lemma was applied to special monomial curves known to annihilate T_X^2 for cyclic quotient singularities. Our (more powerful) approach is to study a general hypersurface section of any rational singularity. These curves turn out to have minimal δ invariant with respect to their embedding dimension. We classify such curve singularities in section 3. They are wedges of certain monomial curves and we call them partition curves.

It turns out that all partition curves of embedding dimension n are special hypersurface sections of the cone over the rational normal curve of degree $n-$

call it X_n. On the other hand we prove in section 2 a useful proposition telling us that for any Cohen-Macaulay singularity defined by the vanishing of the 2×2 minors of a $2 \times n$ matrix, the entries annihilate T^2. This proves that the maximal ideal of X_n annihilates $T^2_{X_n}$ and that $\tau_Y - e_Y$ is constant for all partition curves with the same embedding dimension. In particular one gets this number from the special case of the ordinary n-tuple point in $\mathbb{C}^n$ which was computed in [Greuel].

A little more work gives us the module structure for T^2_X for any rational surface singularity. For surprisingly many the maximal ideal annihilates T^2_X and our main result is:

THEOREM. *Let X be a rational surface singularity of embedding dimension $n + 1 \geqslant 4$ with exceptional divisor $E = \bigcup E_i$ and fundamental cycle Z. Let f be the defining equation of a generic hypersurface section of X. Then*

(1) *The minimal number of generators of T^2_X as $\mathcal{O}_X$ module is $(n-1)(n-3)$; moreover there exists a minimal set of generators $z_1, \ldots, z_{n+1}$ of the maximal ideal, starting with $z_1 = f$, such that $z_2, \ldots, z_{n+1}$ annihilate T^2_X.*
(2) *If the fundamental cycle Z is reduced and if for any connected subgraph $E' \subset E$ with $Z \cdot E' = 0$ the self-intersection number $E' \cdot E' = -2$ or -3 then* $\dim_{\mathbb{C}} T^2_X = (n-1)(n-3)$.
(3) *If X is a quotient singularity then* $\dim_{\mathbb{C}} T^2_X = (n-1)(n-3)$.

The hypotheses of (2) of course mean that the projective tangent cone is reduced, and on the first blow up there are at most triple points.

This result was achieved in the case of cyclic quotient singularities by J. Arndt in [Arndt]. There are similar results for the T^2 of minimally elliptic singularities which we will present in a forthcoming paper.

If assertion (2) were true without the condition that Z be reduced then statement (3) would be immediate. But we shall give an example of a rational quintuple point with nonreduced Z where (2) does not hold. So we must check the quotient singularities with non-reduced Z and $n + 1 \geqslant 5$. Five of the exceptional quotient singularities were computed, with the help of Dave Bayer, using the computer program *Macaulay* by Dave Bayer and Mike Stillman.

We would like to think of this as a 'stability' result for the deformations of rational surface singularities. The recent progress in [Kollár-Shepherd-Barron], [Arndt], [de Jong-van Straten], [Stevens 3] and [Christophersen 2] on versal deformations of certain rational singularities seems to indicate stability in the following sense: If one fixes certain invariants of the singularity such as the embedding dimension and/or the dual graph, then the versal base spaces for the general singularities in this class have a common singular factor and the rest is just adding a smooth factor as T^1 varies. Our result adds to this by saying that T^2 and therefore hopefully the obstructions depend on very few and coarse invariants of the singularity.

We shall describe briefly what we do in subsequent sections. The first two are of a rather general nature, and they set up the machinery. In section 1 we recall some of the basic properties of the cotangent complex, and we deduce the Main Lemma explained above. Section 2 is about the annihilation properties of T^2 for some determinantal singularities.

Sections 3–5 deal with rational surface singularities. In section 3 we classify reduced curve singularities which have minimal delta invariant $n-1$ with respect to their embedding dimension n. In section 4 we prove that partition curves are exactly the general hypersurface sections of rational surface singularities, and section 5 is devoted to the proof of our main result.

Finally in section 6 we discuss obstruction spaces and/or smoothing components of cones over rational normal scrolls, and 'fat' points. For example we show that smoothing components of the Artinian $\mathbb{C}$-algebra of embedding dimension r and minimal multiplicity $r+1$ have dimension r^2.

There is an appendix, written by Jan Stevens, on infinitesimal deformations of wedges of curve singularities.

We have benefitted greatly from Jan Stevens ideas, suggestions and calculations throughout this project. We would also like to thank Gert-Martin Greuel, Theo de Jong, Ragni Piene, Duco van Straten and Prabhakar Rao for their helpful ideas.

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1. Hypersurface sections and the cotangent complex

1.1. We shall briefly recall some of the properties of the cotangent complex that we will need later. For definitions, proofs and details on deformations of singularities and the cotangent complex see [Artin 2], [André], [Buchweitz], [Grauert], [Illusie], [Laudal], [Lichtenbaum-Schlessinger], [Rim], [Schlessinger 2] and [Tjurina].

For simplicity we assume all rings are commutative and noetherian. For a ring A and an A algebra B, there exists a complex of B modules; the *cotangent complex* $C^{\boldsymbol{\cdot}}_{B/A}$. For any B module M the complex gives the *cotangent homology* modules

$$T_i(B/A;\ M) := H_i(C^{\boldsymbol{\cdot}}_{B/A} \otimes_B M)$$

and the *cotangent cohomology* modules

$$T^i(B/A; M) := H^i(\mathrm{Hom}_B(C^{\bullet}_{B/A}, M)).$$

As mentioned in the introduction the two modules $T^i_X := T^i(\mathcal{O}_X/\mathbb{C}; \mathcal{O}_X)$ for $i = 1, 2$, are of special importance for the deformation theory of isolated singularities X. This cohomology theory has among others, the following properties.

1.1.1. There exists a natural *spectral sequence* with E^2 term

$$E^2_{p,q} = \mathrm{Ext}^p_B(T_q(B/A; B), M) \Rightarrow T^{p+q}(B/A; M).$$

1.1.2. *Base change.* Given a cocartesian diagram

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u, "\phi"] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

with ϕ flat, and B' module M', there is a natural isomorphism

$$T^i(B'/A'; M') \cong T^i(B/A; M').$$

1.1.3. In the situation of 1.1.2, assume also that A' is a flat A module. Then there is a natural isomorphism

$$T^i(B'/A'; M \otimes_A A') \cong T^i(B/A; M) \otimes_A A'$$

for any B module M.

1.1.4. If $A \to B \to C$ are ring homomorphisms and M is a C module, then there is a *Zariski-Jacobi* long exact sequence

$$\cdots \to T^i(C/B; M) \to T^i(C/A; M) \to T^i(B/A; M)$$
$$\to T^{i+1}(C/B; M) \to \cdots.$$

1.1.5. A short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of B modules induces a long exact sequence in cohomology

$$\cdots \to T^i(B/A;\ M') \to T^i(B/A;\ M) \to T^i(B/A;\ M'')$$
$$\to T^{i+1}(B/A;\ M') \to \cdots.$$

1.1.6. If B is a *smooth* A algebra then $T^i(B/A; M) = 0$ for $i \geqslant 1$ and all B modules M.

1.1.7. *Semicontinuity.* Let k be a field, A a k algebra and B a flat A algebra. Assume that A is a discrete valuation ring with residue field k and quotient field K. Thus $B \otimes_A k$ is the ring of the special fiber and $B \otimes_A K$ the ring of the generic fiber in the one parameter deformation $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Then

$$\dim_k T^i(B \otimes k/k;\ B \otimes k) \geqslant \dim_K T^i(B \otimes K/K;\ B \otimes K).$$

We give a proof for lack of reference.

Proof. We have

$$T^i(B \otimes K/K;\ B \otimes K) \cong T^i(B/A;\ B) \otimes_A K$$

by 1.1.3, so its dimension over K equals $\mathrm{rk}_A\, T^i(B/A; B)$. This number is obviously no larger than $\dim_k T^i(B/A; B) \otimes_A k$. If t is a parameter for A, then by 1.1.5 and 1.1.2 the exact sequence

$$0 \to B \xrightarrow{\cdot t} B \to B \otimes_A k \to 0$$

induces injections

$$T^i(B/A;\ B) \otimes_A k \to T^i(B \otimes k/k;\ B \otimes k).$$

This gives the result. □

Notice that this means we have semicontinuity for all T^i in any deformation where there is a smooth curve from the special point of the base space to any other point.

1.1.8. Let $Y = X \times (\mathbb{C}^n, 0)$ so that $\mathcal{O}_Y = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}$. Set $T^i_X\{x_1, \ldots, x_n\} := T^i_X \otimes_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}$. Then $T^i_Y \cong T^i_X\{x_1, \ldots, x_n\}$ when $i \geqslant 1$. Again we include a proof.

Proof. By induction it is enough to prove the statement for $n = 1$. Since $\mathcal{O}_Y$ is a flat $\mathcal{O}_X$ module, 1.1.3 says that $T^i_X\{x\} \cong T^i(\mathcal{O}_X/\mathbb{C};\ \mathcal{O}_Y)$ and this is again isomorphic to $T^i(\mathcal{O}_Y/\mathbb{C}\{x\};\ \mathcal{O}_Y)$ by 1.1.2. The long exact sequence 1.1.4 induced from $\mathbb{C} \to \mathbb{C}\{x\} \to \mathcal{O}_Y$ and the fact that $\mathbb{C}\{x\}$ is regular give immediately that $T^i(\mathcal{O}_Y/\mathbb{C}\{x\}; \mathcal{O}_Y) \cong T^i_Y$ for $i \geqslant 2$ and an exact sequence

$$T^0(\mathbb{C}\{x\}/\mathbb{C}; \mathcal{O}_Y) \to T^1(\mathcal{O}_Y/\mathbb{C}\{x\}; \mathcal{O}_Y) \to T^1_Y \to 0.$$

From the explicit description of the cotangent modules given below it is easy to see that the leftmost map is the zero map by our construction of Y. □

1.1.9. There is a *global* counterpart to this theory. We will only need the fact that for a scheme X there are sheaves $\mathcal{T}^i_X$ with stalks $\mathcal{T}^i_{X,x} \cong T^i(\mathcal{O}_{X,x}/\mathbb{C}; \mathcal{O}_{X,x})$.

1.2. The first three cohomology modules have an explicit description in terms of more familiar modules.

1.2.1. $T^0(B/A; M) = \mathrm{Der}_A(B, M)$, the module of A derivations into M.

1.2.2. Let P_A be a free A algebra such that $B = P_A/I$ for an ideal I. There is a standard exact sequence

$$I/I^2 \to \Omega^1_{P_A/A} \otimes_{P_A} B \to \Omega^1_{B/A} \to 0.$$

Applying $\mathrm{Hom}_B(-, M)$ we get a map

$$\mathrm{Der}_A(P_A, M) \to \mathrm{Hom}_B(I/I^2, M)$$

and $T^1(B/A; M)$ is the cokernel.

1.2.3. Consider now an exact sequence

$$0 \to R \to F \xrightarrow{j} I \to 0$$

of P_A modules with $F \cong P_A^m$ free. Let R_0 be the sub-module of R generated by the trivial relations; i.e. those of the form $j(x)y - j(y)x$. Then R/R_0 is a B module and we have an induced map

$$\mathrm{Hom}_B(F/R_0 \otimes_{P_A} B, M) \to \mathrm{Hom}_B(R/R_0, M)$$

and $T^2(B/A; M)$ is the cokernel.

The module on the left is just the sum of m copies of M and the map is

$$(x_1, \ldots, x_m) \mapsto \left[\bar{r} \mapsto \sum r_i x_i\right]$$

where $r \in F$ represents $\bar{r} \in R/R_0$.

1.3. Let X be the germ of an analytic space of positive dimension and

$$f\colon X \to (\mathbb{C}, 0)$$

a germ of an analytic function, i.e. $f \in \mathcal{O}_X$ the local ring of X at 0. We will say that $Y = f^{-1}(0)$ is a *hypersurface section* of X if $f \in \mathbf{m}_X$ (the maximal ideal of germs vanishing at 0) and f is a non-zero divisor in $\mathcal{O}_X$. We will from now on assume that Y is an isolated singularity.

1.3.1. Since $f\colon X \to (\mathbb{C}, 0)$ is a flat deformation of Y, there is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{Y} \arrow[d, "\pi"] \\ (\mathbb{C}, 0) \arrow[r, "j"'] & S \end{tikzcd}$$

where π is the versal deformation of Y. Let $\mathcal{O}_1 = \mathbb{C}\{t\}$ be the local ring of $(\mathbb{C}, 0)$ at 0. Then $\mathcal{O}_X$ is a $\mathcal{O}_1$ module via f^*, that is the map $t \mapsto f$.

The short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\cdot f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

induces as in 1.1.4 a long exact sequence of which the most interesting part reads

$$\begin{aligned} \cdots \to T^1(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) &\xrightarrow{\cdot f} T^1(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) \xrightarrow{\alpha} T^1(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_Y) \\ \to T^2(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) &\xrightarrow{\cdot f} T^2(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) \to T^2(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_Y) \to \cdots \end{aligned}$$

By 1.1.2 $T^i(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_Y) \cong T^i_Y$. In the proof of Wahl's conjecture on the dimension of smoothing components, Greuel and Looijenga ([Greuel-Looijenga, Corollary 2.2]) show that $\dim_{\mathbb{C}} \operatorname{Im}(\alpha)$ is the dimension of the Zariski tangent space of S at the generic point of the curve $j(\mathbb{C})$. Denote this dimension by e_f.

Consider now the Zariski-Jacobi sequence

$$\begin{aligned} \cdots \to T^i(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) \to T^i_X \to T^i(\mathcal{O}_1/\mathbb{C};\ \mathcal{O}_X) \\ \to T^{i+1}(\mathcal{O}_X/\mathcal{O}_1;\ \mathcal{O}_X) \to \cdots \end{aligned}$$

associated as in 1.1.4 with the homomorphisms

$$\mathbb{C} \to \mathcal{O}_1 \to \mathcal{O}_X.$$

Since $\mathcal{O}_1$ is regular $T^i(\mathcal{O}_1/\mathbb{C}; \mathcal{O}_X) = 0$ for $i \geqslant 1$ by 1.1.6 and $T^i(\mathcal{O}_X/\mathcal{O}_1; \mathcal{O}_X) \cong T^i_X$ for $i \geqslant 2$.

If X was an isolated singularity and 0 was an isolated critical point for f, then $f: X \to (\mathbb{C}, 0)$ is a *smoothing*; i.e. the generic fiber is smooth. In this case a general point $j(t)$ is smooth and e_f is the dimension of the irreducible smoothing component containing $j(\mathbb{C}\ 0)$. Also T^2_X has finite length.

1.3.2. Putting this together we get the

MAIN LEMMA. *If $Y = f^{-1}(0)$ is a hypersurface section of X and an isolated singularity then*

(1) *there is a long exact sequence*

$$T^1(\mathcal{O}_X/\mathcal{O}_1; \mathcal{O}_X) \to T^1_Y \to T^2_X \xrightarrow{\cdot f} T^2_X \to T^2_X \to \cdots$$

$$\cdots \to T^i_X \xrightarrow{\cdot f} T^i_X \to T^i_Y \to \cdots$$

(2) $\dim_{\mathbb{C}}(T^2_X/fT^2_X) - \mathrm{rk}_{\mathcal{O}_1} T^2_X = \dim_{\mathbb{C}} T^1_Y - e_f$

(3) *if f is a smoothing of Y then e_f is the dimension of the smoothing component of f and*

$$\dim_{\mathbb{C}}(T^2_X/fT^2_X) = \dim_{\mathbb{C}} T^1_Y - e_f \qquad \square$$

1.4. A singularity is *unobstructed* if the versal base space is smooth. We state now an immediate corollary of the Main Lemma. The discussion here should be compared to [Buchweitz, §5].

1.4.1. It is well known that in general the obstruction space is only a subspace of T^2. The following statement indicates what T^2 actually measures.

COROLLARY. (1) (Buchweitz) *For an isolated singularity X of positive dimension, $T^2_X = 0$ if and only if X is unobstructed and every hypersurface section is unobstructed.*

(2) *A smoothable singularity is unobstructed if and only if it is a hypersurface section of an isolated singularity X with $T^2_X = 0$.*

Proof. (1) follows from the Main Lemma and Nakayama's lemma. In (2) take X to be the total space over a curve in the base space with smooth generic fiber. $\square$

REMARK. If an unobstructed singularity is not smoothable the above method shows that it is a hypersurface section f of a germ X with T^2_X free $\mathcal{O}_1$ module.

1.4.2. We call a singularity X *k-unobstructed* if $T^{i+1}_X = 0$ for $1 \leqslant i \leqslant k$. It is 0-unobstructed if it is unobstructed. A singularity Y is a *complete intersection* in X

if the ideal of Y in X is generated by a regular sequence of length $\dim X - \dim Y$. It follows from the Main Lemma (1) that a hypersurface section of a k-unobstructed singularity is $(k-1)$-unobstructed. Therefore, a complete intersection Y of codimension l in a k-unobstructed singularity X is $(k-l)$-unobstructed if $k \geqslant l$.

1.4.3. EXAMPLE. To illustrate these notions we look at cones over scrolls. By a rational normal scroll (see e.g. [Eisenbud-Harris]) of type $S(a_0, \dots, a_d)$, we will, for simplicity, just mean the $d+1$ dimensional projective manifold in $\mathbb{P}^{(\Sigma a_i)+d}$ defined by the vanishing of the 2×2 minors in the matrix

$$\begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,a_0-1} & x_{1,0} & \cdots & x_{1,a_1-1} & \cdots\cdots & x_{d,a_d-1} \\ x_{0,1} & x_{0,2} & \cdots & x_{0,a_0} & x_{1,1} & \cdots & x_{1,a_1} & \cdots\cdots & x_{d,a_d} \end{pmatrix}.$$

Let $X(a_0, \dots, a_d)$ be the singularity of the affine cone over $S(a_0, \dots, a_d)$ at the vertex. The special case $X_N = X(1, \dots, 1)$, $d = N-1$, is the generic determinantal singularity given by a $2 \times N$ matrix and is rigid for $N \geqslant 3$. None of the others are rigid as can be seen by perturbing the matrix to make it generic.

Not only is X_N rigid, it is $(N-2)$-unobstructed, as follows from [Svanes]. On the other hand every $X(a_0, \dots, a_d)$ is a complete intersection in X_N for $N = \sum_{i=0}^{d} a_i$, of codimension $\sum_{i=0}^{d} (a_i - 1) = N - (d+1)$. This shows the

PROPOSITION. *If $X = X(a_0, \dots, a_d)$ is the singularity at the vertex of the cone over the rational normal scroll of type $S(a_0, \dots, a_d)$ with a_i, $d \geqslant 1$, then X is $(d-1)$-unobstructed.* □

REMARK. The interesting case is when $d = 1$, i.e. X is a three-fold. Then in fact, as we shall see, T_X^2 is non-trivial and comes from obstructed hypersurface sections.

2. Annihilation of T^2 for some determinantal singularities

2.1. We shall show that for Cohen-Macaulay singularities (of any dimension) defined by the minors of a $2 \times n$ matrix, the entries of the matrix annihilate T^2. Examples of such singularities are the cones over rational normal scrolls discussed in example 1.4.3.

2.1.1. PROPOSITION. *Let $f_1, \dots, f_n, g_1, \dots, g_n$ be elements of the maximal ideal of the local ring $\mathcal{O}_e$ of $\mathbb{C}^e$ at 0. Let X be the space defined by*

$$\operatorname{rk}\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ g_1 & g_2 & \cdots & g_n \end{pmatrix} \leqslant 1.$$

Assume that f_i, g_i are such that *X is Cohen-Macaulay of codimension $n-1$ in* $\mathbb{C}^e$. *Then the $\mathcal{O}_X$ module T_X^2 is annihilated by the ideal*

$$(f_1, \ldots, f_n, g_1, \ldots, g_n)\mathcal{O}_X.$$

Proof. Let $F_{i,j} = f_i g_j - f_j g_i$ for $1 \leqslant i < j \leqslant n$ be the defining equations. From our assumptions the Eagon-Northcott complex ([Eagon-Northcott]) gives a resolution of $\mathcal{O}_X$. In particular the module of relations R among the $F_{i,j}$ is generated by the $2\binom{n}{3}$ relations

$$R_{i,j,k} = f_i F_{j,k} - f_j F_{i,k} + f_k F_{i,j}$$
$$S_{i,j,k} = g_i F_{j,k} - g_j F_{i,k} + g_k F_{i,j}$$

for $1 \leqslant i < j < k \leqslant n$.

Since interchanging rows and columns will not change the space it is enough to show that $f_1 T_X^2 = 0$. Let $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(R/R_0, \mathcal{O}_X)$ and set $\phi(R_{i,j,k}) = \phi_{i,j,k}^1$, $\varphi(S_{i,j,k}) = \phi_{i,j,k}^2$. We want to show that

$$f_1 \phi \in \mathrm{Im}\,(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_e^{\binom{n}{2}}/R_0 \otimes_{\mathcal{O}_e} \mathcal{O}_X, \mathcal{O}_X)).$$

More precisely we need to show the existence of $\binom{n}{2}$ elements $h_{i,j}$ of $\mathcal{O}_X$, $(1 \leqslant i < j \leqslant n)$, such that

$$f_1 \phi_{i,j,k}^1 = f_i h_{j,k} - f_j h_{i,k} + f_k h_{i,j}$$
$$f_1 \phi_{i,j,k}^2 = g_i h_{j,k} - g_j h_{i,k} + g_k h_{i,j} \tag{1}$$

for all $1 \leqslant i < j \leqslant n$.

Again from the Eagon-Northcott complex we get relations among relations

$$f_1 R_{i,j,k} = f_i R_{1,j,k} - f_j R_{1,i,k} + f_k R_{1,i,j} \tag{2}$$
$$f_1 S_{i,j,k} + g_1 R_{i,j,k} = f_i S_{1,j,k} + g_i R_{1,j,k} - (f_j S_{1,i,k} + g_j R_{1,i,k}) + f_k S_{1,i,j} + g_k R_{1,i,j} \tag{3}$$

for all $1 < i < j < k \leqslant n$. Also, for any $\alpha = 1, \ldots, n$

$$f_\alpha S_{i,j,k} - g_\alpha R_{i,j,k} \in R_0. \tag{4}$$

From (3) and (4) we obtain

$$f_1 S_{i,j,k} \equiv g_i R_{1,j,k} - g_j R_{1,i,k} + g_k R_{1,i,j} \mod R_0. \tag{5}$$

Now set

$$h_{i,j} = \begin{cases} 0 & \text{if } i = 1 \\ \phi^1_{1,i,j} & \text{if } i > 1 \end{cases}$$

and use (2), (4) and (5) to show that the equations (1) are satisfied. □

2.1.2. REMARK. There should be a generalization to spaces defined by maximal minors of an arbitrarily large matrix. The relations among relations coming from dividing out the trivial relations will have coefficients that are minors of order one less than maximal. Possibly it is these minors that kill T^2 in general.

2.1.3. COROLLARY. *If X is as in the proposition, the entries of the matrix generate $\mathbf{m}_X$ and $T^2_X \neq 0$ then* $\operatorname{Ann} T^2_X = \mathbf{m}_X$.

3. Curves with minimal delta invariant

3.1. In this section we define a set of very simple curve singularities, which we call *partition curves*, and we show that they are exactly the reduced curve singularities which have minimal possible value $n - 1$ of their delta invariant with respect to their embedding dimension n. We begin by recalling a number of basic invariants which are needed throughout this section. An excellent reference for curve singularities is [Buchweitz-Greuel].

3.1.1. For a Cohen-Macaulay singularity $Y \subset (\mathbb{C}^n, 0)$ the *Cohen-Macaulay type* t is the minimal number of generators of the dualizing module ω_Y. By local duality t is also the last non-vanishing Betti number of a minimal free resolution of the local ring $\mathcal{O}_Y$ over a regular local ring $\mathcal{O}_{n,0}$, of which it is a quotient. In particular t is invariant under hypersurface sections.

3.1.2. From now on we will assume that Y is a reduced curve singularity. In particular $\mathcal{O}_Y$ is Cohen-Macaulay. Let $v\colon \tilde{Y} \to Y$ be the normalization map. The length of the Artinian module $(v_* \mathcal{O}_{\tilde{Y}})_0/\mathcal{O}_Y$ is the *delta invariant* δ of Y. The *Milnor number* μ is the length of the quotient $(\omega_Y/d\mathcal{O}_Y)$, where $d\colon \mathcal{O}_Y \to \Omega^1_Y \to \omega_Y$ is the composite of exterior differentiation and the natural map $\Omega_Y \to \omega_Y$ [Buchweitz-Greuel]. Milnor's formula [Buchweitz-Greuel, Proposition 1.2.1] says that if Y has r branches $\mu = 2\delta - r + 1$.

3.1.3. If Y is the union of Y_1 and Y_2 with no component in common, and I_1 is the ideal of Y_1 and I_2 is the ideal of Y_2 in an ambient $\mathbb{C}^n$ the *intersection number* is $i(Y_1, Y_2) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/(I_1 + I_2)$. The reader should note that this intersection number is *not* the one used in intersection theory. The curve Y is called *decomposable* (or Y is the *wedge* of Y_1 and Y_2, denoted by $Y = Y_1 \vee Y_2$) if the Zariski tangent spaces of Y_1 and Y_2 have only the point 0 in common.

LEMMA. [Hironaka] *For* Y, Y_1, Y_2 *as above* $Y = Y_1 \vee Y_2$ *if and only if* $i(Y_1, Y_2) = 1$. □

With the help of intersection numbers the delta invariant of a curve singularity can easily be computed from the delta invariants of its components.

LEMMA. [Buchweitz-Greuel, Lemma 1.2.2]
(1) *Let* δ, δ_1, δ_2 *be the delta invariants of* Y, Y_1, *and* Y_2. *Then* $\delta = \delta_1 + \delta_2 + i(Y_1, Y_2)$.
(2) *Let* $Y = Y_1 \cup \cdots \cup Y_r$ *be a union of* r *curves with* δ-*invariants* $\delta_1, \dots, \delta_r$. *Then* $\delta \geqslant \delta_1 + \cdots + \delta_r + r - 1$ *with equality if and only if* $Y = Y_1 \vee \cdots \vee Y_r$. □

Similarly the Cohen-Macaulay type of a wedge of curves can be calculated.

LEMMA. *Let* $Y = Y_1 \vee Y_2$, *and assume that for* $i = 1, 2$ Y_i *has Cohen-Macaulay type* t_i. (*If* Y_i *is a smooth branch of* Y *we set* $t_i = 0$ *formally.*) *Then the Cohen-Macaulay type* t *of* Y *is* $t = t_1 + t_2 + 1$. □

In an appendix to our paper Jan Stevens proves a formula for the dimension of T^1_Y of decomposable curves.

3.1.4. LEMMA. [Greuel, Theorem 2.5.(3)] *If a curve singularity* Y *is quasi-homogeneous and smoothable then the dimension of the smoothing components in the base space of the semi-universal deformation is*

$$e = \mu + t - 1. \qquad \square$$

3.2. For $m \in \mathbb{N}$ let $Y(m) \subset \mathbb{C}^m$ be the monomial curve [Pinkham 1, §12] with semigroup generated by $m, m+1, \dots, 2m-1$ (the ordinary semigroup of genus $m-1$). In parametric form $Y(m)$ is given by the map $v\colon \mathbb{C} \to \mathbb{C}^m$ sending t to $v(t) = (t^m, \dots, t^{2m-1})$. The ideal of $Y(m)$ in $\mathbb{C}[z_1, \dots, z_m]$ is minimally generated by the 2×2-minors of the $2 \times m$-matrix

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_{m-1} & z_m \\ z_2 & z_3 & \cdots & z_m & z_1^2 \end{pmatrix}.$$

Let $m = m_1 + \cdots + m_r$ be a partition of m. The *partition curve* associated with $m_1, \dots, m_r$ is the wedge of monomial curves

$$Y(m_1, \dots, m_r) = Y(m_1) \vee \cdots \vee Y(m_r),$$

where

$$Y(m_i) \subset \mathbb{C}^{m_i} \subset \mathbb{C}^m = \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_r}.$$

We illustrate this definition by a list of all partition curves with $m \leqslant 3$

$m = 1$	(1)	a line
$m = 2$	$(1, 1)$	a node
	(2)	a cusp
$m = 3$	$(1, 1, 1)$	three coordinate axes in $\mathbb{C}^3$
	$(2, 1)$	a cusp in (z_1, z_2)-plane and z_3-axis
	(3)	the monomial curve $Y(3)$

REMARK. In his thesis [van Straten] D. van Straten has defined a class of nonisolated weakly normal surface singularities, which he calls partition singularities. Looking at either his construction which proceeds by gluing copies of $\mathbb{C}^2$ or at the equations he gives it is easy to see that partition curves are hypersurface sections of partition singularities.

3.2.1. Equations of partition curves are easy to get. Let

$$z_1^{(1)}, \dots, z_{m_1}^{(1)}, z_1^{(2)}, \dots, z_{m_2}^{(2)}, \dots, z_1^{(r)}, \dots, z_{m_r}^{(r)}$$

be coordinates of $\mathbb{C}^m$, $m = m_1 + \cdots + m_r$, and consider the matrices

$$M_i = \begin{pmatrix} z_1^{(i)} & z_2^{(i)} & \cdots & z_{m_i - 1}^{(i)} & z_{m_i}^{(i)} \\ z_2^{(i)} & z_3^{(i)} & \cdots & z_{m_i}^{(i)} & (z_1^{(i)})^2 \end{pmatrix}, \ i = 1, \dots, r, \quad m_i > 1,$$

whose 2×2-minors are equations for $Y(m_i)$. Then $Y(m_1, \dots, m_r)$ is defined by

$$\operatorname{rk} M_i \leqslant 1, \ i = 1, \dots, r, \ m_i > 1$$

$$z_i^{(k)} z_j^{(l)} = 0, \ k \neq l, \ 1 \leqslant i \leqslant m_k, \ 1 \leqslant j \leqslant m_l.$$

3.2.2. These equations are quasihomogeneous, and they are the natural set of equations to work with. But they are hiding the following important fact:

PROPOSITION. *Partition curves are determinantal, i.e. there is a $2 \times m$-matrix M with entries in the maximal ideal of $\mathcal{O}_{\mathbb{C}^m,0}$ whose 2×2 minors generate the ideal of $Y(m_1, \dots, m_r)$.*

Proof. We copy van Straten's argument in [van Straten, 1.3.10]. For $i = 1, \dots, r$ let

$$M_i = \begin{cases} \text{as above} & \text{if } m_i > 1 \\ \begin{pmatrix} z_1^{(i)} \\ 0 \end{pmatrix} & \text{if } m_i = 1 \end{cases}$$

Let

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 1, \ldots, r,$$

be complex 2×2-matrices. Consider the matrix

$$\tilde{M} = (A_1 \cdot M_1 \cdots A_r \cdot M_r).$$

It is now easy to show that if the A_i are nonsingular and sufficiently general then the 2×2-minors of $\tilde{M}$ generate the ideal of $Y(m_1, \ldots, m_r)$ locally around the origin in $\mathbb{C}^m$. □

REMARK. To our knowledge the partition curves appear in the literature for the first time in [Aleksandrov]. Theorem 1 of that article asserts that all reduced curve singularities which have $c = \delta + 1$, c the colength of the conductor, actually are the partition curves, but no proof is given. The author also writes down equations, and claims the assertion of our Proposition 3.2.2, but his equations are not correct. In particular the determinantal representation in his Lemma 2 is wrong.

3.2.3. From the definitions and lemmas in 3.1 we arrive at the following list of invariants for partition curves

PROPOSITION. *Let $m = m_1 + \cdots + m_r$ be a partition of m. Then the partition curve $Y(m_1, \ldots, m_r)$ has the following invariants:*

(1) *The delta invariant is $\delta = m - 1$.*
(2) *The Cohen-Macaulay type is $t = m - 1$.*
(3) *The Milnor number is $\mu = 2m - r - 1$.*
(4) *The smoothing components have dimension $3(m - 1) - r$. (We shall see below that all partition curves are smoothable.)* □

3.3. Our main result in this section is

PROPOSITION. *Let X be a reduced curve singularity of embedding dimension n.*
(1) *The delta invariant of X is at least $n - 1$.*
(2) *$\delta = n - 1$ if and only if X is a partition curve.*

REMARK. See also [Buchweitz-Greuel, 1.2.4]. The authors prove there that if $\mu = m - 1$ then X is the union of the coordinate axes in $\mathbb{C}^m$. This is of course a special case of our result, since $\mu \geqslant \delta$, and the only partition curve with $r = m$ is $Y(1, \ldots, 1)$.

3.3.1. *Proof.* (1) The first part is very easy. Let us consider the irreducible case first. The normalization map $\nu: \tilde{X} \to X$ exhibits $\mathcal{O}_X$ as a subring of $\mathcal{O}_{\tilde{X}} \cong \mathbb{C}\{t\}$. The *value semigroup* $\Gamma_X \subset \mathbb{N}$ of X is the set of orders p_i of the power series in $\mathbb{C}\{t\}$ belonging to $\mathcal{O}_X$. In a more conceptual way it is the image $v(\mathcal{O}_X)$ of the valuation v of the field of fractions $K(\mathcal{O}_X) \cong K(\mathcal{O}_{\tilde{X}})$ associated with the discrete valuation ring $\mathcal{O}_{\tilde{X}}$.

Clearly the multiplicity m_X of $\mathcal{O}_X$ equals the smallest nonzero element of Γ_X, and δ_X is the number of gaps, i.e. the number of elements in $\mathbb{N}\backslash\Gamma_X$. Hence $\delta_X \geqslant m_X - 1$, and by a standard result in multiplicity theory for a curve $m_X \geqslant n$, whence the claim.

If now $X = X_1 \cup \cdots \cup X_r$ is the decomposition of X into irreducible components X_i, and X_i has embedding dimension n_i, then

$$(*) \quad \delta_X \geqslant \sum_{i=1}^{r} \delta_{X_i} + r - 1 \geqslant \sum_{i=1}^{r} (n_i - 1) + r - 1 \geqslant n - 1$$

by 3.1 and the estimate in the irreducible case.

(2) Let us assume now that $\delta_X = n - 1$ is minimal. Then all the estimates in (*) are actual equalities. In particular $X = X_1 \vee \cdots \vee X_r$ is a wedge by 3.1.3, and all the X_i have minimal delta invariant. Hence we can assume X to be irreducible.

The semigroup of X is $\{0, n, n+1, n+2, \dots\}$ with no gaps following n: Since X has multiplicity n the smallest nonzero element in Γ_X is n, and there are already $n - 1$ gaps between 0 and n.

By a result of Teissier [Zariski, Appendice, Théorème 1.3.] there is a one parameter deformation $\mathcal{X} \xrightarrow{\phi} (D, 0)$ with a section $\sigma: D \to \mathcal{X}$ such that for $t \in D\backslash\{0\}$ the fibre $(\mathcal{X}_t, \sigma(t)) \cong X$, and $\mathcal{X}_0$ is the monomial curve singularity with the same semigroup. From the proof [loc. cit., 1.10] it can be seen that ϕ is in the space of deformations of positive weight of the monomial curve. According to [Pinkham 1, Lemma 12.5 and Lemma 12.6] for singularities with at most one gap following the smallest nonzero element of their value semigroup no such deformations exist. □

4. Hypersurface sections of rational surface singularities

4.1. We shall prove in this section that the general hypersurface section of a rational surface singularity is a partition curve, and moreover that every partition curve actually occurs as a general hypersurface section of some rational surface singularity.

4.1.1. Let $f \in \mathbf{m} \subset \mathcal{O}_X$ be an element of the local ring of X which vanishes at 0 and projects onto a sufficiently general element of $\mathbf{m}/\mathbf{m}^2$. Then $Y = f^{-1}(0)$ is a general hypersurface section of X [Reid, Definition 2.5]. In particular the embedding dimension of Y is one less than the embedding dimension of X and the multiplicity stays the same.

REMARK. We stick to the notion hyper*surface* section rather than hyperplane section since we work locally and we do not speak of a particular embedding.

Though it is an easy application of the Bertini theorem the following observation is of fundamental importance. Let $\tilde{X} \xrightarrow{\pi} X$ be a resolution of the surface singularity X. Assume that π factors through the blowing up of the maximal ideal $\mathbf{m}$ of X at 0, i.e. $\pi^{-1}\mathbf{m}\cdot\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z)$ is locally free, and that the (reduced) exceptional divisor is a union of smooth projective curves. Then for Y a general hypersurface section the preimage $\pi^{-1}(Y) = Z + \tilde{Y}$, and $\tilde{Y} = \overline{\pi^{-1}(Y\setminus\{0\})}$ is a disjoint union of smooth curves which meet the exceptional divisor transversely in smooth points.

What we need below is that the vertical part of the divisor (f) is exactly Z, and that the horizontal part is smooth.

4.1.2. Our main result here is

THEOREM. *A general hypersurface section Y of a rational surface singularity of embedding dimension $n+1 \geqslant 3$ is a partition curve for some partition $n = n_1 + \cdots + n_r$.*

REMARK. We shall see below that we can read off the partition from the resolution data.

4.2. For the proof we need to show that Y has delta invariant $n-1$. A result of Morales [Morales, Corollary 2.1.4], stated below, asserts exactly this. We include a rather elementary topological argument.

4.2.1. Throughout this subsection let X be more generally a normal surface singularity, with a resolution $\pi\colon \tilde{X} \to X$ such that π factors through the blow up of the maximal ideal, and that moreover the exceptional divisor E has nonsingular components E_i with normal crossings. Denote the genus of E_i by g_i and the self-intersection number by $-b_i$. Let Y be a hypersurface section of X given by $f = 0$, general enough so that the strict preimage $\tilde{Y}$ of Y on $\tilde{X}$ consists of smooth curves intersecting the exceptional divisor transversely in smooth points.

PROPOSITION. [Morales, Corollary 2.1.4] *If f is as above, and if the divisor of $f\circ\pi$ is $A + \tilde{Y}$ with $A = \sum_{i=1}^{k} a_i E_i$ and $\tilde{Y}$ with no compact component then the delta invariant of Y is*

$$\delta = \tfrac{1}{2} A\cdot(K - A),$$

K the canonical divisor of $\tilde{X}$, and also

$$\delta = p_a(A) - A\cdot A - 1.$$

Proof. By [Buchweitz-Greuel, Corollary 4.2.3.(1)] the Milnor number of Y is the first Betti number of a smooth nearby fibre. Since π is biholomorphic outside the exceptional set the level set $f \circ \pi(x) = t$ on $\tilde{X}$ for a nice representative X is a good candidate. The Euler characteristic of this level set is computed in [Brieskorn-Knörrer, §8.5, Lemma 3] as

$$\chi = \sum_{i=1}^{k} a_i(2 - 2g_i - r_i),$$

where r_i is the total number of components of $(f \circ \pi)$, compact and non-compact ones, different from E_i, which meet E_i. (In fact the authors work out the case $g_i = 0$ but their arguments apply word by word to the general case.)

Using $\chi = 1 - \mu$ and Milnor's formula 3.1.2 we get

$$2\delta = r - \chi,$$

r the number of branches of Y. Let now $r_i = s_i + t_i$ for $i = 1, \ldots, k$, where s_i is the number of noncompact components meeting E_i. Then $A \cdot E_i + s_i = 0$ and $A \cdot E = -\sum_{i=1}^{k} s_i = -r$. Hence

$$\begin{aligned} 2\delta &= r - \sum_{i=1}^{k} a_i(2 - 2g_i - s_i - t_i) \\ &= r - A \cdot A - \sum_{i=1}^{k} a_i((b_i - t_i) - (-2 + 2g_i + b_i)) \\ &= r - A \cdot A + A \cdot E + A \cdot K \quad \text{by adjunction} \\ &= -A \cdot A + A \cdot K. \end{aligned}$$

The second expression is computed using the formula

$$p_a(A) = \tfrac{1}{2}(A \cdot A + A \cdot K) + 1$$

for the arithmetic genus p_a of an exceptional cycle. □

4.2.2. COROLLARY. (1) *If X is rational, and f is general then* $\delta = -Z^2 - 1 = n - 1$.

(2) *if X is minimally elliptic* [Laufer] *with $-Z^2 \geqslant 2$ and again f is generic then* $\delta = -Z^2 = n$.

Proof. In both cases A is the fundamental cycle Z. In the rational case $p_a(Z) = 0$, and for X minimally elliptic $p_a(Z) = 1$. (Note also that for most minimally elliptic singularities $K = -Z$. □

REMARK. A complete classification of Gorenstein curve singularities with $\delta = n$ as in 4.2.2(2) will be provided in a forthcoming paper.

4.3. As we announced above it is not difficult to find the partition.

4.3.1. PROPOSITION. *Let X, $\tilde{X}$, and Y be as before. Let $E = \bigcup_{i=1}^{k} E_i$ be the decomposition of the exceptional set into irreducible components, and let $Z = \sum_{i=1}^{k} n_i E_i$ be the fundamental cycle. If $Z \cdot E_i = -r_i \leqslant 0$ the partition for the general hypersurface section Y is*

$$(\underbrace{n_1, \dots, n_1}_{r_1}, \underbrace{n_2, \dots, n_2}_{r_2}, \dots, \underbrace{n_k, \dots, n_k}_{r_k}).$$

(Of course it is understood that n_i's with $r_i = 0$ are omitted.)

Proof. If $Z \cdot E_i = -r_i$ then there are r_i components of $\tilde{Y}$ passing through E_i. A global holomorphic function on $\tilde{X}$ which vanishes on E vanishes at least to order n_i along E_i. This shows that the delta invariant of these components is at least $n_i - 1$. Hence it is equal to $n_i - 1$. (Note that $\sum_{i=1}^{k} r_i n_i = -Z^2 = n$.) □

The case of reduced fundamental cycle seems to have been well known before (cf. [Stevens 1], [Kollar, Section 3.4]).

COROLLARY. *If X is a rational surface singularity of embedding dimension $n + 1$, such that the fundamental cycle on the minimal resolution is reduced, the general hypersurface section Y is isomorphic to the singularity of the n coordinate axes in $\mathbb{C}^n$.* □

REMARK. In [van Straten] one finds that even more generally for a weakly rational weakly normal Cohen-Macaulay surface singularity the delta invariant of a general hypersurface section is one less than the multiplicity. The proof is omitted there, so we have included it here for our case.

EXAMPLES. (a) $Y(1, 1)$ is the general hypersurface section of the A_n singularities.

(b) For D_n, E_6, E_7, E_8 the general hypersurface section is the ordinary cusp $Y(2)$.

(c) If $n = 3$ we have $Y(1, 1, 1)$, $Y(2, 1)$, and $Y(3)$. For example $Y(1, 1, 1)$ is the general hypersurface section of the cone over the rational normal curve of degree three. We get $Y(2, 1)$ in case of the tetrahedral quotient singularity $\mathbb{T}_5$, and $Y(3)$ is the general hypersurface section of the rational surface singularity with dual resolution graph as shown in Figure 1. □

4.3.2. This motivates the following result which we include for the sake of completeness.

PROPOSITION. *Each partition curve actually occurs as a general hypersurface section of a suitable rational surface singularity.*

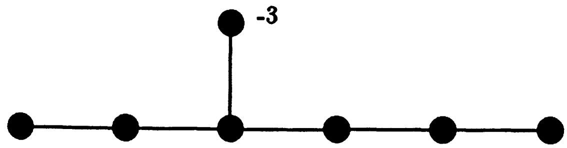

Fig. 1. ● $-b$ represents an exceptional $\mathbb{P}^1$ of self-intersection number $-b$. (-2)'s are omitted.

Proof. Let $n = n_1 + \cdots n_r$ be a partition of n. We display in Figure 2 the dual resolution graph of a rational surface singularity which has $Y(n_1, \dots, n_r)$ as a general hypersurface section, and leave the verification to the reader.

These graphs have been suggested to us by [van Straten, Theorem 1.3.12]. □

REMARK. In particular we have shown that partition curves are smoothable.

4.4. To apply the results of 2.3 to rational surface singularities we show that partition curves are also *special hypersurface sections* of the cones over rational normal curves of corresponding degree.

PROPOSITION. *Let $n = n_1 + \cdots + n_r$ be a partition of n, and let X_n be the singularity at the vertex of the cone over the rational normal curve of degree n.*

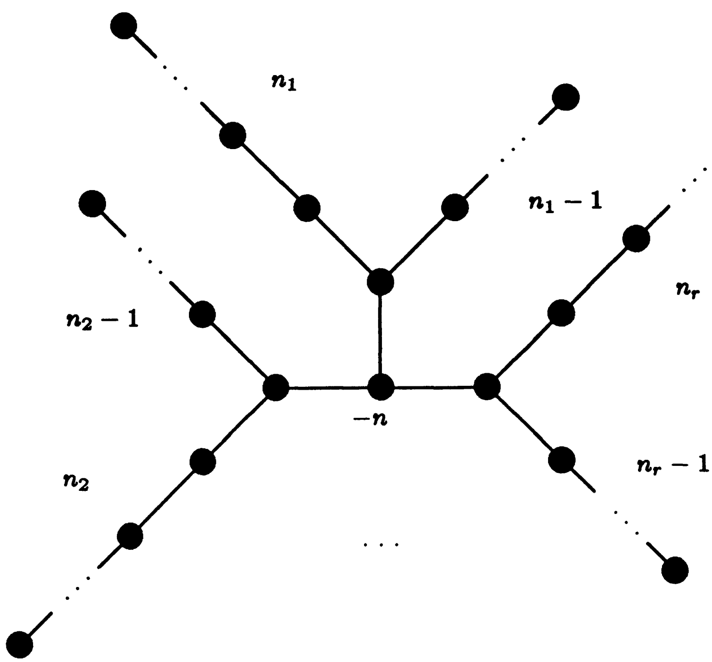

Figure 2.

Then there exists a hypersurface section $\{f = 0\}$ *of* X_n *which is isomorphic to the partition curve* $Y(n_1, \ldots, n_r)$.

Proof. The resolution $\pi: \tilde{X}_n \to X_n$ is the contraction of a smooth rational curve E of self-intersection number $E^2 = -n$. After possibly restricting to a smaller neighborhood of E we can construct r disjoint smooth curves $\tilde{Y}_1, \ldots, \tilde{Y}_r$ which intersect E in r points $p_1, \ldots, p_r$ with intersection numbers $n_1, \ldots, n_r$. The intersection number $E \cdot (E + \tilde{Y}_1 + \cdots \tilde{Y}_r) = 0$ hence by [Artin 1, proof of Theorem 4] there exists a global holomorphic function $\tilde{f}$ on $\tilde{X}_n$ with divisor $(\tilde{f}) = E + \tilde{Y}_1 + \cdots + \tilde{Y}_r$. The function $\tilde{f}$ descends to a holomorphic function on X_n and cuts out an isolated curve singularity Y with r irreducible components. Clearly the delta invariant of the component Y_i corresponding to $\tilde{Y}_i$ is at least $n_i - 1$, and the delta invariant of Y can be computed as in 4.1 to be $n - 1$. □

4.5. The following result is the outcome of this section with respect to computation of T^2.

PROPOSITION. *Let* $Y = Y(n_1, \ldots, n_r)$ *be a partition curve with* $n = n_1 + \cdots n_r$. *Let* $\tau = \dim T^1_Y$ *and* e *be the dimension of the smoothing components. Then*

$$\tau - e = \max(0, (n-1)(n-3))$$

and depends only on the embedding dimension n and not on the particular partition curve.

Proof. By the preceding Proposition $Y(n_1, \ldots, n_r)$ is a hypersurface section of the cone X_n over the rational normal curve of degree n. The singularity X_n is determinantal, and the maximal ideal annihilates $T^2_{X_n}$ by 2.1.3. Hence by part 3 of our Main Lemma we have

$$\dim T^2_{X_n} = \tau - e.$$

So the right hand side doesn't depend on the particular partition. That it is equal to $(n-1)(n-3)$ follows from the results about the deformation theory of the *generic* hypersurface section of X_n, i.e. the singularity of the n lines in $\mathbb{C}^n$, which we recall below. □

Just to ease the formulation of the following results let an ordinary n-tuple point be the singularity of $Y(1, \ldots, 1)$ of embedding dimension n.

4.5.1. PROPOSITION. [Buchweitz-Greuel, Proposition 7.2.6] *Let* Y *be an ordinary n-tuple point.*

(1) *Let* $f: \mathcal{Y} \to D$ *be any small representative of a deformation of* Y. *For* $t \in D \setminus \{0\}$ *the fibre* $\mathcal{Y}_t$ *has at most ordinary m-tuple points for* $m \leqslant n$ *as singularities.*
(2) *Let* $n_1, \ldots, n_p$ *be any integers such that* $n_i \geqslant 2$ *and* $\sum_{i=1}^{p} (n_i - 1) \leqslant n$. *Then*

there exists a deformation $f: \mathcal{Y} \to (D, 0)$ of Y such that for a small representative the fibre $\mathcal{Y}_t$ for $t \neq 0$ has exactly p singular points $y_1, \dots, y_p$, and $(\mathcal{Y}_t, y_i)$ is an ordinary n_i-tuple point.

(3) *The parameter space S of the semi-universal deformation of Y is of pure dimension $2n - 3$.* □

In addition there is

PROPOSITION. [Greuel, Proposition 3.5.(2)] *The n-tuple point $Y = Y(1, \dots, 1)$ has*

$$\dim T_Y^1 = \begin{cases} 1, & n = 2 \\ n(n-2), & n \geqslant 3 \end{cases} \qquad \square$$

REMARK. We could as well have referred to [Arndt] or [Christophersen 2] for the computation of $T_{X_n}^2$. But we thought it would be nicer to have a proof which doesn't use many computations, and relies only on work about curves. Also these Propositions will be of help in subsequent sections.

4.5.2. To close this section we write down what we get for the dimension of the T^1 of a partition curve.

PROPOSITION. *For a partition curve $Y = Y(n_1, \dots, n_r)$ of embedding dimension $n = n_1 + \cdots + n_r$*

$$\dim T_Y^1 = n(n-1) - r.$$

Proof. We have $\tau - e = (n-1)(n-3)$ by 4.5, and $e = 3(n-1) - r$ by 3.2.3(4), whence the claim. □

5. T^2 for rational surface singularities

5.1. In this section we state and prove the main application of our main lemma. If M is a module over a local ring, denote $\mathrm{cg}\, M = \dim M/\mathbf{m}M$, i.e. the minimal number of generators of M. For a singularity X, let $n + 1 = \dim \mathbf{m}/\mathbf{m}^2$, i.e. the embedding dimension of the singularity.

5.1.1. For a rational singularity let $E = \bigcup E_i$ be the exceptional divisor in the minimal resolution and Z the fundamental cycle.

THEOREM. *For a rational surface singularity X of embedding dimension $n + 1 \geqslant 4$ and f the defining equation of a generic hypersurface section*

(1) $\mathrm{cg}\, T_X^2 = (n-1)(n-3)$; *moreover there exists a minimal set of generators $z_1, \dots, z_{n+1}$ of the maximal ideal, starting with $z_1 = f$, such that $z_2, \dots, z_{n+1}$ annihilate T_X^2.*

(2) *If the fundamental cycle Z is reduced and if for any connected subgraph $E' \subset E$ with $Z \cdot E' = 0$ the self-intersection number $E' \cdot E' = -2$ or -3 then* $\dim_{\mathbb{C}} T_X^2 = (n-1)(n-3)$

(3) *if X is a quotient singularity then* $\dim_{\mathbb{C}} T_X^2 = (n-1)(n-3)$.

5.1.2. There are examples of quotient surface singularities where Z is not reduced, so that (3) is not automatically implied by (2). It is of course natural to ask if the condition in (2) that Z be reduced is necessary, but in 5.4 below we shall give an example of a rational quintuple point with non-reduced Z where (2) does not hold.

5.1.3. Subsections 5.1 up to 5.3 are devoted to the proof of this theorem. Part (1) follows easily from Section 3 and the Main Lemma. To prove (2) we actually compute T^2 of the (non-isolated) singularity at the vertex of the tangent cone of X. There is a subclass of the dihedral quotient singularities and seven exceptional quotient singularities with $n + 1 \geqslant 5$ and nonreduced fundamental cycle. We check these case by case to prove (3).

Proof of Theorem 5.1.1 (1). From the Main Lemma and Proposition 4.4, we know that

$$\dim T_X^2/fT_X^2 = (n-1)(n-3).$$

On the other hand one checks easily that $\operatorname{cg} T_X^2/fT_X^2 = \operatorname{cg} T_X^2$. Since $\mathbf{m}_X T_C^2 = 0$ by 2.1.3 and 3.2.2 for $C = f^{-1}(0)$ and $T_X^2/fT_X^2 \hookrightarrow T_C^2$ we finally get

$$\operatorname{cg} T_X^2 = \operatorname{cg} T_X^2/fT_X^2 = \dim T_X^2/fT_X^2 = (n-1)(n-3)$$

and the second statement. □

5.2. Let X now satisfy the conditions of Theorem 5.1.1 (2) and denote the singularity of the tangent cone of X at the vertex by $\bar{X}$. Let $C = \operatorname{Proj} \mathcal{O}_{\bar{X}}$ be the projectivized tangent cone.

5.2.1. Now C is also the exceptional divisor of the first blow up of X. Our assumptions imply that C is a (in general singular) reduced, arithmetically Cohen-Macaulay, rational curve of degree n in $\mathbb{P}^n$, ([Wahl 2, Proof of 2.1 and proof of 3.6]). By [Xambo] the singularities of C are ordinary k-fold points.

The singularities on the blow up $\hat{X}$ of X are contractions of subgraphs $E' \subset E$ on the minimal resolution with $Z \cdot E' = 0$. By our assumption they are rational double or triple points, and in particular they are hypersurfaces or Cohen-Macaulay of embedding codimension two.

The singularities of C are hypersurface sections of singularities of $\hat{X}$ cut out by a local equation of the exceptional curve. In particular they are hypersurfaces or

Cohen-Macaulay of codimension two as well. Hence the sheaf $\mathscr{T}^2_C$ vanishes (1.1.9) and $T^2_{\bar{X}}$ is supported at the vertex.

5.2.2. For any graded ring A, T^2_A inherits a grading from $\mathrm{Hom}(R/R_0, A)$. To compute $T^2_{\bar{X},k}$ we shall need a slightly modified version of a result of Schlessinger.

THEOREM. *Let $Y \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay projective scheme of positive dimension. Assume $\mathscr{T}^2_Y = 0$. Let $\mathscr{N}_Y$ be the normal sheaf of Y* in $\mathbb{P}^n$. *If $\bar{X}$ is the germ of the cone over Y at the vertex, then there is an injective map*

$$T^2_{\bar{X},k} \hookrightarrow H^1(Y, \mathscr{N}_Y(k))$$

for all k.

Proof. The proof of [Schlessinger 2, Theorem 1] applies, since $T^2_{\bar{X}}$ is supported at the vertex. □

5.2.3. The computation of $T^2_{\bar{X}}$ yields:

THEOREM. *If X is a rational surface singularity satisfying the conditions in Theorem* (2), *them* $\dim_{\mathbb{C}} T^2_{\bar{X}} = (n-1)(n-3)$.

Proof. The proof is in three steps.

STEP 1. $H^1(C, \mathscr{N}_C(k)) = 0$ for $k \geqslant -1$.

Consider the exact sequence

$$0 \to \Theta_C(k) \to \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_C(k) \to \mathscr{N}_C(k) \to \mathscr{T}^1_C(k) \to 0.$$

Since C is reduced, $\mathscr{T}^1_C$ has support at points. Therefore $H^1(C, \mathscr{T}^1_C(k)) = 0$ for all k, and $H^1(C, \mathscr{N}_C(k)) = 0$ if $H^1(C, \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_C(k)) = 0$. To compute the latter cohomology group, twist and apply cohomology to the standard restricted Euler sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{n+1} \to \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_C \to 0.$$

By the rationality of C, $H^1(C, \mathcal{O}_C(k)) = 0$ for $k \geqslant 0$ and the result follows.

STEP 2. $T^2_{\bar{X},k} = 0$ for $k \leqslant -3$.

We know by [Wahl 2] that the resolution of $\mathcal{O}_C$ has the form

$$\mathcal{O}_{\mathbb{P}^n}(-4)^{b_3} \to \mathcal{O}_{\mathbb{P}^n}(-3)^{b_2} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{b_1} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_C \to 0.$$

Obviously $T^2_{\bar{X},k} = 0$ for $k < -3$, since $T^2_{\bar{X}}$ has the induced grading from $\mathrm{Hom}(R/R_0, \mathcal{O}_{\bar{X}})$. In [Wahl 2, Corollary 2.10] it is shown that for smooth C, also

$T^2_{\bar{X},-3} = 0$. The proof can be suitably modified to show this in general though, because if $\phi \in \mathrm{Hom}(R/R_0, \mathcal{O}_{\bar{X}})_{-3}$ and $r_1, \ldots, r_{b_2}$ generate R, then $\phi(r_i) \in \mathbb{C}$. We leave this to the reader.

STEP 3. Application of the Main Lemma.

By 5.2.2 and Step 1 and 2 we know that $T^2_{\bar{X}}$ is concentrated in degree -2. In particular $\mathbf{m}_{\bar{X}} T^2_{\bar{X}} = 0$. The generic hypersurface section Y of $\bar{X}$ is n lines in general position in $\mathbb{C}^n$. If f is its defining equation, then the singuarities of the nearby fibers in the deformation $f\colon \bar{X} \to \mathbb{C}$ are just the singularities of C ('sweeping out the cone'). By assumption these are unobstructed singularities, and they are smoothable, and so by openness of versality the generic point of the corresponding curve in the versal base space of Y is a smooth point of a smoothing component. From Proposition 4.5 we know that the difference $\tau - e = (n-1)(n-3)$ for a generic hypersurface section Y. The Main Lemma 1.3.2 (3) together with $\mathbf{m} T^2{}_{\bar{X}} = 0$ gives

$$\dim T^2_{\bar{X}} = \dim T^2_{\bar{X}}/fT^2_{\bar{X}} = \tau - e = (n-1)(n-3) \qquad \square$$

5.2.4. *Proof of Theorem* 5.1.1 (2). There is a normally flat deformation $\pi\colon \mathscr{X} \to (D, 0)$ with a section $\sigma\colon (D, 0) \to \mathscr{X}$ and a good representative $\pi\colon \mathscr{X} \to D$ such that for $t \in D\backslash\{0\}$ the singularity $(\mathscr{X}_t, \sigma(t)) \cong X$ and $\mathscr{X}_0 \cong \bar{X}$. (See [Gerstenhaber] and [Fulton, Chapter 5] for additional references).

Hence semicontinuity 1.1.7 establishes the inequality $\dim T^2_X \leqslant (n-1)(n-3)$. $\square$

5.3. We wish to show that all quotient surface singularities have $\dim T^2 = (n-1)(n-3)$. Quotient surface singularities are listed in [Brieskorn], [Riemenschneider] and [Kahn] according to their group, dual graph, invariant polynomials and equations. All quotient surface singularities satisfy the second condition of Theorem 5.1.1 (2), so we only have to check those with non-reduced fundamental cycle.

5.3.1. Before going through the list we include here a short discussion on how to compute the cotangent modules for quotient singularities. There are several different ways described in the literature for computing T^1, see for example [Pinkham 4], [Behnke-Kahn-Riemenschneider] and [Arndt]. We will add to this list a new method that also handles T^2. It turns out that the idea behind this is already in [Buchweitz].

Let first in general $X = \mathbb{C}^n/G$ with $n \geqslant 2$ and G a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ acting freely outside the origin. The essential fact is that, since the group is finite and the characteristic is 0, the functor 'take invariants' is exact. Thus, by definition of the cotangent complex, $T^i_X \cong T^i_X(\mathcal{O}_{\mathbb{C}^n})^G$. Using the properties of the cotangent complex described in 1.1 one shows

$$T^i_X \cong \mathrm{Ext}^n_{\mathcal{O}_{\mathbb{C}^n}}(T_{i+1-n}(\mathcal{O}_{\mathbb{C}^n}/\mathcal{O}_X;\ \mathcal{O}_{\mathbb{C}^n}), \mathcal{O}_{\mathbb{C}^n})^G$$

for $i \geqslant n-1$ and $T^i_X = 0$ for $1 \leqslant i \leqslant n-2$. (See also [Buchweitz, 5.3.5].)

By local duality on $\mathbb{C}^n$,

$$\operatorname{Ext}^n_{\mathcal{O}_{\mathbb{C}^n}}(T_{i+1-n}(\mathcal{O}_{\mathbb{C}^n}/\mathcal{O}_X;\, \mathcal{O}_{\mathbb{C}^n}),\, \mathcal{O}_{\mathbb{C}^n}) \cong T_{i+1-n}(\mathcal{O}_{\mathbb{C}^n}/\mathcal{O}_X;\, \mathcal{O}_{\mathbb{C}^n})^*$$

where $*$ stands for $\mathbb{C}$ dual, since the cotangent homology module is torsion. To make this duality G-equivariant one has to keep track of the natural G-action on the dualizing module $\Omega^n_{\mathbb{C}^n}$ on $\mathbb{C}^n$. See Christophersen 2, CQS for an illustration.

When $n = 2$ we get

$$T^1_X \cong \operatorname{Ext}^2_{\mathcal{O}_{\mathbb{C}^2}}(\Omega_{\mathbb{C}^2/X},\, \mathcal{O}_{\mathbb{C}^2})^G$$
$$T^2_X \cong \operatorname{Ext}^2_{\mathcal{O}_{\mathbb{C}^2}}(T_1^{\mathbb{C}^2/X},\, \mathcal{O}_{\mathbb{C}^2})^G.$$

Here $T_1^{\mathbb{C}^2/X}$ is the kernel of the natural map $\Omega_X \otimes \mathcal{O}_{\mathbb{C}^2} \to \Omega_{\mathbb{C}^2}$.

The benefit of this method is that the free G equivariant $\mathcal{O}_{\mathbb{C}^2}$ resolution of $\Omega_{\mathbb{C}^2/X}$ and $T_1^{\mathbb{C}^2/X}$ are relatively easier (compared to working on X) to set up once one knows the invariant polynomials and the equations. For example one does not need to use the relations to compute T^2.

5.3.2. The cyclic quotients have reduced fundamental cycle, so the result in this case follows from Theorem 5.1.1 (2). It is also quite easy to check that $\mathbf{m}T^2 = 0$ using the method above.

In the dihedral case, the fundamental divisor is reduced if the self-intersection number of the central curve is not -2. In either case, using the above method one calculates that $\mathbf{m}T^2 = 0$. The actual calculations are too long and complicated to be written here though. See [Christophersen 2].

We are now left with 7 quotient singularities with non-reduced fundamental cycle and $n + 1 \geqslant 5$. They are in the notation of [Riemenschneider]

$$\mathbb{T}_5,\ \mathbb{O}_7,\ \mathbb{O}_{11},\ \mathbb{I}_{17},\ \mathbb{I}_{19},\ \mathbb{I}_{23},\ \text{and}\ \mathbb{I}_{29}.$$

From their equations (as e.g. described in [Kahn]) one sees that $\mathbb{O}_7$ and $\mathbb{I}_{19}$ are determinantal and the entries generate the maximal ideal.

The other 5 were treated fast and easily by the computer program *Macaulay* (version of February 1989) written by Dave Bayer and Mike Stillman. See [Bayer-Christophersen] on how to use *Macaulay* to compute T^1 and T^2 for any quasi-homogeneous singularity.

5.4. In this subsection we shall give an example of a rational surface singularity with nonreduced fundamental cycle where the maximal ideal does not annihilate T^2. It was suggested to us by Theo de Jong who has computed the admissible deformations (cf. [de Jong-van Straten]) of the projection into $\mathbb{C}^3$ of a closely related singularity, finding that he needed 9 equations to describe the base space of the semiuniversal deformation. This example was known for a long time as the first counter-example to the general conjecture about the dimension of T^1 for rational surface singularities (see [Behnke-Kahn-Riemenschneider]).

Let X be the quasihomogeneous rational surface singularity of multiplicity $n = 5$ with dual graph of the minimal resolution as shown in Figure 3.

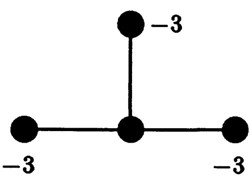

Figure 3.

Its canonical Gorenstein cover Y is the simply elliptic singularity $\tilde{E}_6$ given by

$$z^3 = x^3 + y^3$$

The projection π: $Y \rightarrow X$ is induced by the $\mathbb{Z}_3$-action

$$\zeta(x, y, z) = (\zeta x, \zeta^2 y, \zeta z),$$

ζ a primitive third root of unity (cf. [Wahl 2]). For the generating invariants, the equations, and a computation of T^1 we refer to [Behnke-Kahn-Riemenschneider].

Using the method of Section 5.3.1 we find that T_X^2 has corank 8 but dimension 9 with graded parts of dimensions

$$\dim_{\mathbb{C}}(T_X^2)_k = \begin{cases} 2, & k = -6 \\ 6, & k = -5 \\ 1, & k = -4 \\ 0, & \text{otherwise} \end{cases}$$

5.5. There is an interesting series of rational surface singularities of embedding dimension 5 which recently has been studied by Th. de Jong and D. van Straten. Their objective was to find the base spaces of semi-universal deformations of rational quadruple points. We add this as an example, because on the basis of their work our methods give complete information about the structure of T^2 although T^2 is not annihilated by the maximal ideal.

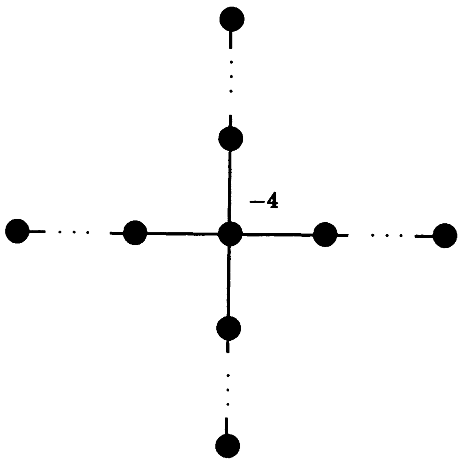

Figure 4. All arms are of length $p - 1$.

EXAMPLE. Let X be the rational quadruple point with dual graph of the minimal resolution as shown in Figure 4.

Equations for such a singularity are (in case that the points of intersection of the branches with the central curve are $\{1, -1, i, -i\}$)

$$\operatorname{rk}\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5^p + x_1 \end{pmatrix} \leqslant 1$$

In [de Jong-van Straten, 6.1] the following results about the semi-universal deformation of these singularities are stated as a conjecture. (The authors tell us that meanwhile proofs are available):

(1) The base space is of the form $B(p) \times S$ with a smooth factor S, and embedding dimension $5p - 1$ for $B(p)$.
(2) $B(p)$ has $p + 1$ irreducible components Y_k, $k = 0, \ldots, p$ of dimensions $\dim Y_k = 2p - 1 + 2k$.
(3) The multiplicity of Y_k at the origin is $\binom{p}{k}$, in particular Y_0 and Y_p are smooth.
(4) Y_k has smooth normalization for all k.

In any case they display a minimal set of $3p$ equations for the base space of the miniversal deformation. This shows that T_X^2 is at least $3p$-dimensional. Recall that the module T_X^2 is minimally generated by 3 elements, because X has embedding dimension 5. By Proposition 2.1.1 T_X^2 is annihilated by the elements $x_1, \ldots, x_4, x_5^p$. So if we denote a suitable set of generators by e_1, e_2, e_2 then a vector space basis of T_X^2 is given by

$$x_5^i e_j, \quad j \in \{1, 2, 3\}, 0 \leqslant i \leqslant p - 1$$

and we have the obvious multiplication table.

In particular $\dim T_X^2 = 3p$ and T_X^2 is the *true* obstruction space, i.e. the obstruction homomorphism is an isomorphism.

6. Examples and applications

We discuss two examples that illuminate different ways T^2 reflects the obstructions of singularities. The examples are rational normal scrolls and 'fat' points.

6.1. We take up again our example of Section 1. We have shown there that the singularity at the affine cone over the two dimensional rational normal scroll $S(a_0, a_1)$ is unobstructed. Here we are going to compute the cotangent cohomology T_X^i, $i = 1, 2$.

6.1.1. PROPOSITION. *Let $X = X(a_0, a_1)$ be the singularity at the vertex of the affine cone over the two dimensional rational normal scroll $S(a_0, a_1)$ with a_0, $a_1 \geqslant 1$. Then T_X^2 has dimension $a_0 + a_1 - 3$, and is annihilated by the maximal ideal of the local ring.*

Proof. The last assertion is clear from the determinantal equations given in 1.4.3 and by 2.1.3. Let us consider the hyperplane section Y of X given by $x_{1,0} = x_{0,a_0}$. Clearly Y is the affine cone over the rational normal curve of degree $n = a_0 + a_1$. From [Mumford] we know that T_Y^1 has dimension $2n - 4$ ($n \geqslant 3$), and by [Pinkham 2, appendix] for $n \neq 4$ there is exactly one smoothing component, of dimension $(n - 1)$.

For $n = 4$ there are two components, one of dimension three and one of dimension one. In [Wahl 3, Example 3.21] we find that the second Betti number of the nonsingular fibres over the three dimensional component is one, and over the one dimensional component it is zero. On the other hand a smooth fibre of our one parameter family is isomorphic to $S(a_0, a_1)\backslash H$, H a hyperplane, and has $\mathbb{P}^1$ as a deformation retract. Hence the second Betti number is one, and the smoothing is on the three dimensional component (cf. [Kollár-Shepherd-Barron, Example 2.8]).

Thus in both cases the Main Lemma gives us $\dim T_X^2 = 2n - 4 - (n - 3) = a_0 + a_1 - 3$. $\square$

6.1.2. For the convenience of the reader we shall say a bit more about the deformation theory of the cone $X(a_0, a_1)$.

PROPOSITION. *The homogeneous parts of T_X^1 of $X = X(a_0, a_1)$, $a_0 \geqslant a_1$ have dimensions*

$$\dim(T_X^1)_k = \begin{cases} a_0 + a_1 - 2, & k = -1, \\ a_0 - a_1 - 1, & k = 0, \\ a_0 - (k+1)a_1 - 1, & a_0 - (k+1)a_1 \geqslant 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. By [Wahl 1, Theorem 3.7. and Corollary 3.8.] the graded pieces of T^1_X can be computed as the kernels of natural maps

$$T^1_X(k) = ker[H^1(S(a_0, a_1), M(k)) \\ \to \oplus H^1(S(a_0, a_1), \mathcal{O}_{S(a_0,a_1)}(k+1))].$$

where M is a locally free sheaf of rank 3 defined by the extension

$$0 \to \mathcal{O}_{S(a_0,a_1)} \to M \to \Theta_{S(a_0,a_1)}(k) \to 0$$

with extension class the class of $\mathcal{O}_{S(a_0,a_1)}(1)$ in $H^1(S(a_0, a_1), \Omega^1_{S(a_0,a_1)})$.

Cohomology groups for scrolls can easily be computed using the structure of $S(a_0, a_1)$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$$S(a_0, a_1) = \mathbb{P}(\mathscr{E}), \quad \mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1).$$

The Leray spectral sequence for the projection map $\pi: S(a_0, a_1) \to \mathbb{P}^1$ gives us

$$H^1(S(a_0, a_1), \mathcal{O}_{S(a_0,a_1)}(l)) = 0, \quad l \geqslant -1$$

(use that $\pi_* \mathcal{O}_{S(a_0,a_1)}(l) = S_l(\mathscr{E})$, and $R^i\pi_* \mathcal{O}_{S(a_0,a_1)}(l) = 0$, $i > 0$, $l \geqslant 0$). Hence for $k \geqslant -1$ we see that $T^1(k) \cong H^1(S(a_0, a_1), \Theta_{S(a_0,a_1)}(k))$. One can use the exact sequences ($S = S(a_0, a_1)$)

$$0 \to \Theta_{S/\mathbb{P}^1} \to \Theta_S \to \pi^*\Theta_{\mathbb{P}^1} \to 0$$

$$0 \to \mathcal{O}_S \to (\pi^*\mathscr{E}^*)(1) \to \Theta_{S/\mathbb{P}^1} \to 0$$

together with the Leray spectral sequence and the projection formula to arrive at the isomorphism

$$H^1(S(a_0, a_1), \Theta_{S(a_0,a_1)}(k)) \cong H^1(\mathbb{P}^1, (S_{k+1}\mathscr{E}) \otimes \mathscr{E}^*), \quad k \geqslant -1,$$

which gives the result, since X is defined by quadratic equations with linear syzygies, and cannot have any deformation of degree -2 or lower. □

6.2. Finally we consider smoothing components of a special class of Artinian local algebras. These will help us compute T^2 for partition curves. Also they supply us with yet another example where the obstruction map is not surjective, but for a different reason than in the case of scrolls.

Fix a number $r \geqslant 3$, and let $B_r = \mathbb{C}\{x_1, \ldots, x_r\}/\mathbf{m}^2$ be the Artinian local $\mathbb{C}$ algebra of embedding dimension r and minimal multiplicity $r+1$. Denote by Z_r

the corresponding zero dimensional analytic space. Then Z_r is a general hypersurface section of a partition curve of embedding dimension $r+1$.

6.2.1. Pick any one parameter smoothing $f\colon Y \to (\mathbb{C}, 0)$ of a smoothable point scheme Z. We have an exact sequence

$$0 \to \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{\cdot f} \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_Y, \mathcal{O}_Y) \xrightarrow{\beta} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Z, \mathcal{O}_Z)$$

By Wahl's conjecture [Greuel-Looijenga], [Laudal-Pfister, Corollary 3.10] the dimension of the corresponding smoothing component is the length of the cokernel of β. (This can be seen e.g. from the exact sequence in our Main Lemma (1) by tracking back kernels and cokernels to the T^0 level.) As remarked by David Mond and Duco van Straten, this dimension is just the length of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_Z, \mathcal{O}_Z)$ since $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ as it is a rank zero torsion free module over $\mathcal{O}_1$.

The derivations of the Artinian algebra B_r are just the endomorphisms of the maximal ideal $\mathbf{m}$ as a vectorspace, which form a $\mathbb{C}$-vectorspace of r^2 dimensions. Another easy computation using $\mathbf{m}^2 = 0$ yields the dimensions of $T^1_{Z_r}$ and $T^2_{Z_r}$.

PROPOSITION. (1) *The dimension of the smoothing components of Z_r is r^2.*
(2) $\dim_{\mathbb{C}} T^1_{Z_r} = \frac{1}{2}(r-1)r(r+2)$.
(3) $\dim_{\mathbb{C}} T^2_{Z_r} = \frac{1}{6}r(r+1)(2r^2-2r-3)$.

From the Main Lemma we get

COROLLARY. *A partition curve Y of embedding dimension n has* $\dim_{\mathbb{C}} T^2_Y = \frac{1}{2}n(n-1)(n-3)$. □

6.2.2. In [Galligo, §5] the author computed the semi-universal deformation of Z_r for $r=3$. His result is not correct since he ends up with an irreducible versal base space of dimension 10. We would like to thank J. Stevens for repeating this computation and for letting us reproduce his result.

PROPOSITION. *Let for $i = 1, 2, 3$ s_i, t_i, u_i, v_i, w_i be local parameters for the versal base space of Z_3. Let D be the 3×3-matrix*

$$D = \begin{pmatrix} w_1 & v_2 & u_3 \\ u_1 & w_2 & v_3 \\ v_1 & u_2 & w_3 \end{pmatrix}$$

and let $r = (r_1, r_2, r_3)$ and $s = (s_1, s_2, s_3)$. Denote by $\wedge^2 D$ the classical adjoint of the matrix D.

Then the base of the semi-universal base space of Z_3 is given by the 15 *equations*

$$D \cdot r = 0 \quad D^t \cdot s = 0 \quad r \cdot s^t = \overset{2}{\wedge} D$$

In particular the base space is irreducible of dimension 9. □

Notice that the base space is given by 15 equations while T^2 has dimension 18 by Proposition 6.3.1, so the obstruction map is not surjective. Since T^1 is concentrated in degree -1 and T^2 in degree -2, we know a priori that the equations are quadratic so the dual of the obstruction map can easily be computed. To check the result of Jan Stevens we did it using *Macaulay*. This computation has a surprising conclusion; every basis element in T^2 corresponds to an obstruction, i.e. an equation that must vanish on the base space, *but* the equations obtained this way are not a *minimal* generating set. This of course is a completely different situation than for the scrolls, where no element of T^2 gave an obstruction.

Appendix

JAN STEVENS
Mathematisches Seminar der Universität Hamburg

In this appendix we give a formula for the dimension of T^1 for decomposable curves. In order to formulate it we define a subspace of T^1. Let $(X, 0) \in (\mathbb{C}^e, 0)$ be minimally embedded. In the description of T^1_X as $\operatorname{Coker} \Theta_{\mathbb{C}^e,0} \to \operatorname{Hom}(I/I^2, \mathcal{O}_{X,0})$, where I denotes the ideal of $(X, 0)$, the image of $\Theta_{\mathbb{C}^e,0}$ is contained in $\operatorname{Hom}(I/I^2, \mathbf{m}_{X,0})$. So we can make the following definition:

$$\tau_0(X) = \dim_{\mathbb{C}} \operatorname{Coker} \Theta_{\mathbb{C}^e,0} \to \operatorname{Hom}(I/I^2, \mathbf{m}_{X,0}).$$

PROPOSITION. *Let $(X, 0)$ and $(Y, 0)$ be curve singularities. Then:*

$$\begin{aligned}\tau(X \vee Y) &= \tau_0(X \vee Y)\\ &= \tau_0(X) + \tau_0(Y) + \operatorname{edim}(X)(t(Y) + 1) + \operatorname{edim}(Y)(t(X) + 1).\end{aligned}$$

For a curve singularity X and a smooth branch L one has:

$$\tau(X \vee L) = \tau_0(X \vee L) = \tau_0(X) + \operatorname{edim}(X) + t(X).$$

Proof. Let $\operatorname{edim}(X) = m$ and let the ideal I of X in $\mathbb{C}\{x_1, \dots, x_m\}$ be minimally generated by $f_1, \dots, f_s$, with relations $\mathbf{r}_k$: $\sum r_{ki} f_i = 0$. For Y we have $J \subset \mathbb{C}\{y_1, \dots, y_n\}$ generated by $g_1, \dots, g_t$ with relations $\mathbf{s}_k$.

The ideal $\mathscr{I}$ of $X \vee Y$ in $\mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ is generated by the f_i, g_j and $x_i y_j$, $i = 1, \dots, m$, $j = 1, \dots, n$. To write the relations we note that $f_i = \sum f_{ij} x_j$ and $g_i = \sum g_{ij} y_j$ for some f_{ij} and g_{ij}. Then the relations are:

$$(*)\begin{cases} \mathbf{r}_k,\ \mathbf{s}_k \\ x_i(x_j y_k) - x_j(x_i y_k) = 0, \quad y_i(x_k y_j) - y_j(x_k y_i) = 0 \\ y_k f_i - \sum f_{ij}(x_j y_k) = 0,\ x_k g_i - \sum g_{ij}(x_k y_j) = 0. \end{cases}$$

We deform the equations of $X \vee Y$:

$$(**) \quad \begin{cases} f_i(\mathbf{x}) + \varepsilon(f_i^{(1)}(\mathbf{x}) + f_i^{(2)}(\mathbf{y})), & f_i^{(2)} \in \mathbf{m}_n \\ g_j(\mathbf{y}) + \varepsilon(g_j^{(1)}(\mathbf{y}) + g_j^{(2)}(\mathbf{x})), & g_j^{(2)} \in \mathbf{m}_m \\ x_i y_j + \varepsilon(h_{ij}^{(0)} + h_{ij}^{(x)}(\mathbf{x}) + h_{ij}^{(y)}(\mathbf{y})), & h_{ij}^{(x)} \in \mathbf{m}_m,\ h_{ij}^{(y)} \in \mathbf{m}_n. \end{cases}$$

This yields an infinitesimal deformation of $X \vee Y$ if and only if insertion of the equations $(**)$ in the relations $(*)$ gives elements of $\mathscr{I}$. Because $f_{ij} \in \mathbf{m}_m$ and $g_{ij} \in \mathbf{m}_n$, and the relations have also coefficients in $\mathbf{m}$, the condition boils down to:

$$\sum r_{ki} f_i^{(1)} \in I \tag{1}$$

$$\sum s_{kj} g_j^{(1)} \in J \tag{2}$$

$$x_i(h_{jk}^{(0)} + h_{jk}^{(x)}) - x_j(h_{ik}^{(0)} + h_{ik}^{(x)}) \in I \tag{3}$$

$$y_i(h_{kj}^{(0)} + h_{kj}^{(y)}) - y_j(h_{ki}^{(0)} + h_{ki}^{(y)}) \in J \tag{4}$$

$$y_k(f_i^{(1)} + f_i^{(2)}) - \sum f_{ij}(h_{jk}^{(0)} + h_{jk}^{(x)}) \in \mathscr{I} \tag{5}$$

$$x_k(g_i^{(1)} + g_i^{(2)}) - \sum g_{ij}(h_{kj}^{(0)} + h_{kj}^{(y)}) \in \mathscr{I}. \tag{6}$$

From equation (5) it follows, by putting $\mathbf{x} = 0$, that $f_i^{(1)} \in \mathbf{m}_m$. Furthermore, $y_k f_i^{(2)} \in J$ for all k, so $f_i^{(2)} \in J$; we do not change the deformation if we take $f_i^{(2)} = 0$. Likewise $g_j^{(2)} = 0$ and $g_j^{(1)} \in \mathbf{m}_n$. From (3) or (4) we find $h_{ij} = 0$.

The equations (3) give for fixed k that

$$\operatorname{rk}\begin{pmatrix} x_1 & \cdots & x_m \\ h_{1k}^{(x)} & \cdots & h_{mk}^{(x)} \end{pmatrix} \leqslant 1 \text{ in } \mathcal{O}_X.$$

This condition holds also in the total ring of fractions. We conclude the existence of $h_k^{(x)} \in \mathcal{O}_{\tilde{X}}$ such that $h_k^{(x)} x_i = h_{ik}^{(x)}$ in $\mathcal{O}_X$. Likewise there exist $h_k^{(y)} \in \mathcal{O}_{\tilde{Y}}$ with $h_k^{(y)} y_i = h_{ki}^{(y)}$ in $\mathcal{O}_Y$. The conditions (5) and (6) are now also satisfied.

For the case of $X \vee L$ we find infinitesimal deformations

$$\begin{cases} f_i(\mathbf{x}) + \varepsilon f_i^{(1)}(\mathbf{x}), & f_i^{(1)} \in \mathbf{m}_m \\ x_i + \varepsilon(h_{ij}^{(x)}(\mathbf{x}) + h_{ij}^{(y)}(y)y), \end{cases}$$

Here the $f_i^{(1)}$ define a deformation of X, and $h_i^{(x)} = x_i h^{(x)}$. To find T^1 we have to divide out the trivial deformations. We first consider the terms $h^{(x)}$; the ones obtainable by putting $y + \varepsilon\phi(\mathbf{x})$ in the equations are trivial. Therefore the dimension of the subspace of T^1, spanned by the deformations of the form f_i, $x_i y + \varepsilon h_i^{(x)}$, is equal to the dimension of

$$\{h \in \mathcal{O}_{\tilde{X}}/\mathcal{O}_X \mid h\phi \in \mathcal{O}_X \text{ for all } \phi \in \mathbf{m}_X\}$$

and this dimension is $t(X)$, the Cohen-Macaulay type of X. We now use the trivial deformations $x_i + \varepsilon\phi_i(\mathbf{x})$; we find a subspace of dimension $\tau_0(X)$ of deformations of the form $f_i + \varepsilon f_i^{(1)}$, $x_i y$ (here we use the freedom we still have in the choice of $h_i^{(y)}$). The trivial deformation $x_i + \varepsilon\phi_i(y)$ acts on $x_i y$ to give $x_i y + \varepsilon\phi_i(y)y$. The deformations f_i, $x_k y + \varepsilon y$, $x_j y$ for $j \neq k$, are linearly independent of the ones considered previously: if $f_i(\mathbf{x} + \varepsilon\mathbf{c}) \in I$ for all i, then we have a derivation D of $\mathcal{O}_X$ and an $x \in \mathbf{m}_X$ with $Dx = 1$, so by a lemma of Zariski [1] the singularity X is the germ of a product $X_1 \times \mathbb{C}$ for some X_1, contradicting the fact that X is an isolated singularity. So we find that $\tau_0(X \vee L) = \tau_0(X) + t(X) + m$.

Almost the same considerations give the formula for $\tau(X \vee Y)$; we remark that the trivial deformation $x_i + \varepsilon\phi_i$ changes modulo $\mathscr{I}$ only the f_k if $\phi_i \in \mathbf{m}_m$, only the $x_i y_j$ if $\phi_i \in \mathbf{m}_n$, but a constant ϕ_i changes both the f_k and the $x_i y_j$. So $\tau_0(X \vee Y) = \tau_0(X) + \tau_0(Y) + m(t(Y) + 1) + n(t(X) + 1)$. □

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