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Algebraic cycles, Chow varieties, and Lawson homology

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Following the fundamental work of H. Blaine Lawson [19], [20], we introduce new invariants for projective algebraic varieties which we call Lawson homology groups. These groups are a hybrid of algebraic geometry and algebraic topology: the l -adic Lawson homology group $L_r H_{2r+i}(X, \mathbb{Z}_l)$ of a projective variety X for a given prime l invertible in \mathcal{O}_X can be naively viewed as the group of homotopy classes of S^i -parametrized families of r -dimensional algebraic cycles on X . Lawson homology groups are covariantly functorial, as homology groups should be, and admit Galois actions. If $i = 0$, then $L_r H_{2r+i}(X, \mathbb{Z}_l)$ is the group of algebraic equivalence classes of r -cycles; if $r = 0$, then $L_r H_{2r+i}(X, \mathbb{Z}_l)$ is l -adic étale homology. As discussed in [10], operations on Lawson homology relate these special cases, thereby factoring the cycle map from algebraic cycles to homology. For a complex algebraic variety X , we give a parallel development of analytic Lawson homology groups $L_r H_{2r+i}(X^{an})$ which have the property that $L_r H_{2r+i}(X^{an}) \otimes \mathbb{Z}_l$ is naturally isomorphic to $L_r H_{2r+i}(X, \mathbb{Z}_l)$ whenever $i > 0$. As observed by R. Hain [15], this analytic theory admits (colimits of) mixed Hodge structures. A detailed overview of the primary results presented here can be found in [9].

This paper establishes some of the basic properties of Lawson homology groups for a closed algebraic set X over an arbitrary algebraically closed field. Our starting point is a study in section 1 of Chow varieties associated to X provided with a projective embedding; these varieties parametrize effective cycles on X of a given dimension and degree. In order to obtain homotopy-theoretic information from these Chow varieties, we employ the machinery of étale homotopy theory and homotopy-theoretic group completion. Section 2 presents the definitions and requisite formalism, as well as verifies the good functorial behaviour of our Lawson homology groups.

Section 3 is dedicated to providing algebro-geometric analogues of Lawson's geometric arguments. We employ a perhaps unfamiliar technique of constructing 'continuous algebraic maps' which are not quite morphisms of algebraic varieties but which induce maps of topological types. Even though we define

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Lawson homology only for closed algebraic sets over an algebraically closed field, we work as much as possible with closed algebraic sets over an arbitrary field. This has the immediate benefit of providing Galois-equivariant results. Furthermore, we envision Lawson homology over non-algebraically closed fields (whose usefulness does not yet justify the greatly increased complexity arising from the use of techniques of [6]).

Theorem 4.2 proves Lawson's fundamental result in our general algebraic context: 'algebraic suspension induces an isomorphism of Lawson homology groups.' Our proof, an algebraization of Lawson's arguments, gives a purely algebraic proof of Lawson's analytic result. As a corollary, we compute the Lawson homology groups of projective spaces. In Theorem 4.6, we determine the Lawson homology groups associated to codimension 1 cycles on smooth projective varieties, a computation which suggests the possibility that Lawson homology groups might be remarkably well behaved.

We express our great intellectual debt to numerous friends and guides. First and foremost, Blaine Lawson's original work has served us as a source of insights and techniques. Ofer Gabber patiently introduced us to Chow varieties and sketched the outlines of several key steps in this work. Spencer Bloch, Pierre Deligne, William Dwyer, Lawrence Ein, Gerd Faltings, William Fulton, Richard Hain, Nicholas Katz, and Barry Mazur all helped dispel some of our apprehensions and misconceptions. Finally, we warmly thank I.H.E.S. and ETH-Zurich for their hospitality.

1. Chow varieties

In recent years, Chow varieties of projective algebraic varieties have been eclipsed by Hilbert schemes. Nonetheless, they are the appropriate structures to apply Lawson's constructions. For example, Chow components of 0-cycles are symmetric products of a variety, the 'correct' objects to realize homology (see Theorem 4.3 below), whereas punctual Hilbert schemes exhibit more subtle behaviour. In this section, we recall the construction of Chow varieties and prove a few properties useful for our purposes. The reader interested in Chow varieties in characteristic 0 is referred to [1] for an alternate treatment.

We begin with a base field k of characteristic $p \geq 0$ embedded in an algebraically closed field Ω of infinite transcendence degree over k . All fields considered in this section (except Ω itself) will be extensions of k of finite transcendence degree inside Ω . An important example to keep in mind is that in which k is a number field and Ω is the field of complex numbers \mathbb{C} . By a *projective variety* (respectively, *closed algebraic set*), we shall mean a reduced, irreducible, closed subscheme (resp., reduced, closed subscheme) of some projective space \mathbb{P}_F^N over some field F ; more generally, a *variety* (respectively,

algebraic set) is a Zariski open subset of a projective variety (resp., closed algebraic set). If V is an algebraic set defined over F (i.e., provided with a locally closed embedding $j: V \subset \mathbb{P}_F^N$) and if E is a field extension of F , then we denote by V_E the algebraic set over E given by base change from F to E (i.e., given by the same equations). A *cycle* Z on \mathbb{P}_F^N is a formal, finite sum $\sum n_i \cdot V_i$ of closed subvarieties $V_i \subset \mathbb{P}_F^N$ with each n_i a non-zero integer; if $Z = \sum n_i V_i$ with $V_i \neq V_j$ for $i \neq j$, then n_i is called the *multiplicity* of V_i in Z ; the cycle Z is said to be an *r-cycle* if each component has dimension r and is said to be *effective* if each n_i is positive. We recall that the *degree* $\deg(V)$ of a closed subvariety $V \subset \mathbb{P}_F^N$ of dimension r is the number of points of intersection of V_K with a sufficiently general linear subspace $L^{N-r} \subset \mathbb{P}_K^N$, where K is the algebraic closure of F (inside Ω); the degree of the cycle $Z = \sum n_i \cdot V_i$ equals $\sum n_i \cdot \deg(V_i)$. If $X \subset \mathbb{P}_F^N$ is a closed algebraic set and Z is a cycle on \mathbb{P}_K^N , then Z is said to be a cycle on \mathbb{P}_K^N with *support* on X if each component of Z is a subvariety of X_K . For each dimension r , the ‘empty cycle’ will be admitted as the unique ‘effective’ r -cycle of degree 0.

Recall from [28] the existence and basic property of Chow varieties. An elementary example of a Chow variety is given by the special case of hypersurfaces of some degree d in \mathbb{P}^N . In this case, $C_{N-1,d}(\mathbb{P}^N)$ is a projective space of dimension $\binom{N+d}{d} - 1$: after choosing a basis for the monomials of degree d in $N + 1$ variables, one associates to a point in this projective space the hypersurface whose defining equation is given by the coordinates of that point.

1.1 PROPOSITION. *There is a closed algebraic subset $C_{r,d}(\mathbb{P}^N)$ of $\mathbb{P}_k^{N(r,d)}$, $N(r,d) = (r + 1)\binom{N+d}{d} - 1$, such that for any algebraically closed field K morphisms $\text{Spec } K \rightarrow C_{r,d}(\mathbb{P}^N)$ over k are in natural 1–1 correspondence with effective, degree d , r -cycles on \mathbb{P}_K^N . Moreover, if $j: X \rightarrow \mathbb{P}_F^N$ is a closed immersion of a closed algebraic set X , then there is a closed algebraic subset $C_{r,d}(X, j) \subset C_{r,d}(\mathbb{P}^N)_F$ with the property that morphisms $\text{Spec } K \rightarrow C_{r,d}(X, j)$ over F with K an algebraically closed field extension of F are in natural 1–1 correspondence with effective, degree d , r -cycles on \mathbb{P}_K^N with support on $j(X)$. Furthermore, if E/F is a field extension, then $C_{r,d}(X_E, j)$ equals $(C_{r,d}(X, j))_E$.*

Proof. To a closed subvariety V of dimension r and degree d in \mathbb{P}_K^N , one associates the point in $\mathbb{P}_K^{N(r,d)}$ whose coordinates are the coefficients of the irreducible form $F_V(T_0^d, \dots, T_N^d)$ homogeneous of degree d in each of $r + 1$ $(N + 1)$ -tuples such that $F_V(a_0^d, \dots, a_N^d) = 0$ if and only if the $r + 1$ linear forms $\sum a_j^i X_j$ have a common zero in V ; to $Z = \sum n_i V_i$ with $n_i \geq 0$, one associates the point whose coordinates are the coefficients of $F_Z = \prod F_i^{n_i}$. As shown in [28; I.9.5], the points so obtained in $\mathbb{P}_K^{N(r,d)}$ are the K -points of a closed algebraic set of $\mathbb{P}_k^{N(r,d)}$ which we have denoted $C_{r,d}(\mathbb{P}^N)$. The equations defining $C_{r,d}(\mathbb{P}^N)$ are independent of k , so that for any field extension E/k the closed algebraic set $C_{r,d}(\mathbb{P}^N)_E \subset \mathbb{P}_E^{N(r,d)}$ equals the corresponding Chow variety over E . Should the points be constrained to those associated to cycles with support on $j(X)$, then

they too are the K -points of a closed subset $C_{r,d}(X, j) \subset C_{r,d}(\mathbb{P}^N)_F$. Since the equations defining this closed immersion depend only on X and have coefficients in F , we conclude that the closed subset $(C_{r,d}(X, j))_E$ of $C_{r,d}(\mathbb{P}^N)_E$ equals $C_{r,d}(X_E, j)$ for any field extension E/F . \square

Proposition 1.1 implies in particular that if K is an algebraically closed field, then the effective, degree d , r -cycles of \mathbb{P}_K^N are in natural 1–1 correspondence with closed points of $C_{r,d}(\mathbb{P}^N)_K$. We let $\langle Z \rangle \in C_{r,d}(\mathbb{P}^N)_K$ denote the Chow point of the cycle Z . We say that a cycle Z *specializes* to a cycle Z_0 (denoted $Z \searrow Z_0$) if the corresponding Chow points are so related (i.e., if $\langle Z \rangle \searrow \langle Z_0 \rangle$).

Let $F \subset E$ be a field extension. The cycle $Z = \sum n_i V_i$ on \mathbb{P}_E^N is said to be *defined* over F if Z can be written as a sym of cycles associated to closed algebraic subsets of \mathbb{P}_F^N . (The *associated cycle* on \mathbb{P}_E^N of a closed algebraic subset $T \subset \mathbb{P}_F^N$ is the formal sum of the irreducible components of $T_E \subset \mathbb{P}_E^N$.) The cycle $Z = \sum n_i V_i$ is said to be *radicial* over some subfield $F \subset E$ if every automorphism of E over F fixes Z . Equivalently, Z is radicial if it can be written as a sum of cycles defined over some purely inseparable, finite extension of F .

In our next proposition, we identify morphisms from $\text{Spec } F$ into a Chow variety, where F is not assumed to be algebraically closed.

1.2. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$, let E/F be a field extension, and let K be the algebraic closure of E . Then a morphism $z: \text{Spec } E \rightarrow C_{r,d}(X, j)$ is in natural 1–1 correspondence with an effective, degree d , r -cycle Z on \mathbb{P}_K^N with support on $j(X)$ whose Chow point $\langle Z \rangle$ is rational over E . Such an r -cycle Z is always radicial over E and is actually defined over E if it has no multiplicity greater than 1. If $z: \text{Spec } E \rightarrow C_{r,d}(X, j)$ maps $\text{Spec } E$ to a generic point of $C_{r,d}(X, j)$, then the corresponding cycle Z is defined over E .*

Proof. Since the closed point $z(\text{Spec } E) \in C_{r,d}(X, j)_K$ is rational over E , the cycle Z is radicial over E (cf. [28; I.9.4.g]). Moreover, the support of the cycle Z (i.e., the closed algebraic subset of \mathbb{P}_K^N given as the union of the irreducible components of Z) is defined over E [28; I.9.7.g]. If no multiplicity of Z is greater than 1, then Z is the cycle associated to its support and is therefore defined over E .

Finally, if $z: \text{Spec } E \rightarrow C_{r,d}(X, j)$ maps $\text{Spec } E$ to a generic point of $C_{r,d}(X, j)$ we verify that each component of the corresponding cycle Z with multiplicity greater than one must be defined over k and thus Z itself is rational over E . Assume to the contrary, that the component Y_i of Z is a subvariety of \mathbb{P}_K^N not defined over k and that the multiplicity n_i of Y_i of Z is greater than 1. Then $\langle n_i Y_i \rangle$ is a non-equivalent specialization of $\langle (n_i - 1)Y_i' + Y_i'' \rangle$ where $\langle Y_i' \rangle \searrow \langle Y_i \rangle$, $\langle Y_i \rangle \searrow \langle Y_i'' \rangle$ are independent equivalent specializations of Y_i , so that $\langle n_i Y_i \rangle$ is not a constituent of a generic cycle. \square

Let Y be an algebraic set defined over F . We denote by $Y \times \mathbb{P}^N$ the fibre product over $\text{Spec } F$ of Y and \mathbb{P}_F^N . We say that a cycle Z of $Y \times \mathbb{P}^N$ is a Y -relative r -

cycle if $Z = \sum n_i V_i$, where each V_i is an irreducible subvariety of $Y \times \mathbb{P}^N$ whose scheme-theoretic fibre above each point y of Y is a subscheme of $\mathbb{P}_{k(y)}^N$ of pure dimension r , $k(y)$ the residue field of Y at y . If Y is smooth, then cycle-theoretic intersections in $Y \times \mathbb{P}^N$ are well defined in an appropriate Chow group. Since a Y -relative r -cycle Z meets properly each cycle $\{y\} \times \mathbb{P}^N$ for y a closed point of Y , the intersection $Z \cdot (\{y\} \times \mathbb{P}^N)$ is well defined as a cycle. We denote this intersection-theoretic fibre cycle of Z above y by $[Z]_y$,

$$[Z]_y = Z \cdot (\{y\} \times \mathbb{P}^N).$$

This intersection can be computed by using the deformation of the regular embedding $\{y\} \times \mathbb{P}^N \subset Y \times \mathbb{P}^N$ to the embedding of the zero section of its normal bundle [11; 6.1].

1.3. PROPOSITION. *Let Y be a smooth variety defined over F with generic point $\eta = \text{Spec } k(Y)$ and let y be a closed point of Y . For any effective, Y -relative r -cycle Z of $Y \times \mathbb{P}^N$ with generic fibre cycle Z_η , the degree of Z_η equals that of $[Z]_y$. Moreover, the specialization $(\eta, \langle Z_\eta \rangle) \downarrow (y, \langle [Z]_y \rangle)$ in $Y \times C_{r,d}(\mathbb{P}^N)$ is the unique extension of $\eta \downarrow y$ in Y , where d equals the degree of Z_η . Furthermore, if Z is flat over Y , then the r -cycle $[Z]_y$ is the cycle Z_y associated to the scheme-theoretic fibre $Z \times_Y \{y\}$.*

Proof. Since the degree of an r -cycle in \mathbb{P}^N can be computed as the degree of the 0-cycle obtained by intersection with a sufficiently general linear subspace L^{N-r} , the equality of the degrees of Z_η and $[Z]_y$ is a special case of “conservation of number” [11; 10.2]. The fact that $(\eta, \langle Z_\eta \rangle) \downarrow (y, \langle [Z]_y \rangle)$ in $Y \times C_{r,d}(\mathbb{P}^N)$ is a specialization which uniquely extends $\eta \downarrow y$ in Y is demonstrated in [28; II.6.8]. Finally, for any $y \in Y$, let $R_y \subset \text{Spec } k(Y)$ be a discrete valuation ring dominating $\mathcal{O}_{y,Y}$ (associated to the specialization $\eta \downarrow y$ in Y). If Z is flat over Y with pull-back via $\text{Spec } R_y \rightarrow Y$ denoted Z' , then Z' is flat over $\text{Spec } R_y$ with generic fibre Z_η and special fibre $Z_{y'}$ over the closed point $y' \in \text{Spec } R_y$. For such an equidimensional cycle Z' over $\text{Spec } R_y$, the intersection-theoretic fibre $Z' \cdot (\{y'\} \times \mathbb{P}^N)$ equals the scheme-theoretic fibre $Z_{y'}$ [11; 7.1]. On the other hand, $Z' \cdot (\{y'\} \times \mathbb{P}^N)$ is obtained from $[Z]_y = Z \cdot (\{y\} \times \mathbb{P}^N)$ and $Z_{y'}$ from Z_y by base change via $\text{Spec } R_y / \max(R_y) \rightarrow \text{Spec } \mathcal{O}_{y,Y} / \mathfrak{m}_y$. \square

Let $j: X \rightarrow \mathbb{P}_F^N$ be a closed immersion of a closed algebraic set X , E a field extension of F , and Y an algebraic set defined over E . A Y -relative r -cycle Z is said to have support on $j(X)$ if Z_η is a cycle on $\mathbb{P}_{k(\eta)}^N$ with support on $j(X_{k(\eta)})$ for each generic point η of Y . Modulo the awkwardness of possible inseparability of the field of definition of a cycle over the residue field of its Chow point, the following theorem shows that $C_{r,d}(X, j)$ represents the functor which sends a smooth variety Y to the set of effective, Y -relative r -cycles of degree d with support on $j(X)$.

1.4. THEOREM. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$, and let Y be a smooth variety defined over a field extension E of F with generic point $\eta = \text{Spec } k(Y)$. Then effective, Y -relative r -cycles Z on $Y \times \mathbb{P}^N$ whose generic fibres Z_η have degree d and support on $j(X)$ are in natural 1–1 correspondence with morphisms $f: Y \rightarrow C_{r,d}(X, j)$ (over F) such that the cycle with Chow point $f(\eta)$ is defined over $k(Y)$. Moreover, if Z_f is the Y -relative r -cycle corresponding to the map $f: Y \rightarrow C_{r,d}(X, j)$, then $f(\eta) = \langle (Z_f)_\eta \rangle$ and $f(y) = \langle [Z_f]_y \rangle$ for each closed point $y \in Y$.*

Proof. Let $f: Y \rightarrow C_{r,d}(X, j)$ be a morphism over F with the property that the cycle $Z_{f(\eta)}$ with Chow point $f(\eta)$ is defined over $k(Y)$. Write $Z_{f(\eta)}$ as $\sum n_i V_{i,\eta}$, where each $V_{i,\eta}$ is a $k(Y)$ -variety. We define V_i to be the closure of $V_{i,\eta} \subset \mathbb{P}_{k(Y)}^N \subset \mathbb{P}^N \times Y$ in $\mathbb{P}^N \times Y$ and set $Z_f = \sum n_i V_i$. To prove that the fibres of Z_f are purely r -dimensional, it suffices to prove that no component of any fibre has dimension greater than r . For this, it suffices to observe that the morphism from Z to the incidence correspondence $\Lambda \subset X \times C_{r,d}(X, j)$ over $C_{r,d}(X, j)$ [28; I.9.] induces an embedding $V_i \subset \Lambda \times_{C_{r,d}(X, j)} Y$ over Y for each V_i .

Conversely, let Z be an effective, Y -relative r -cycle on $\mathbb{P}^N \times Y$ with generic fibre Z_η of degree d . The associated Chow point $\langle Z_\eta \rangle$ is rational over $k(Y)$ and thereby determines a morphism $\text{Spec } k(Y) \rightarrow C_{r,d}(X, j)$. Equivalently, Z determines a rational map $f: Y \rightarrow C_{r,d}(X, j)$ with $f(\eta) = \langle Z_\eta \rangle$. To prove that f extends to a morphism, it suffices by the normality of Y to observe that the closure of the graph of f projects bijectively onto Y (and thus isomorphically, by Zariski's Main Theorem) in view of the (existence and) uniqueness property of $(\eta, \langle Z_\eta \rangle) \downarrow (y, \langle [Z]_y \rangle)$ given in Proposition 1.3. This also shows that for any morphism $f: Y \rightarrow C_{r,d}(X, j)$ with $f(\eta)$ defined over $k(Y)$ that $f(y) = \langle [Z_f]_y \rangle$.

To verify that these constructions are inverse of each other, we observe that morphisms $Y \rightarrow C_{r,d}(X, j)$ are determined by their restrictions to $\eta \in Y$ (i.e., rationally), whereas Y -relative r -cycles $Z \subset Y \times \mathbb{P}^N$ are the closures of their generic fibres Z_{gen} . \square

We shall frequently use the following special case of Theorem 1.4 in order to identify specializations of cycles.

1.5. COROLLARY. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$, and let C be a smooth quasi-projective curve defined over some field extension E of F with generic point $\eta = \text{Spec } k(C)$. Then an $(r + 1)$ -cycle Z on $C \times \mathbb{P}^N$ is a C -relative r -cycle if and only if Z is flat over C . An effective, C -relative r -cycle Z whose generic fibre Z_η has degree d and support on $j(X)$ determines a morphism $f_Z: C \rightarrow C_{r,d}(X, j)$. Furthermore, if $f: C \rightarrow C_{r,d}(X, j)$ is a morphism with the property that the cycle with Chow point $f(\eta)$ is defined over $k(C)$, then the corresponding effective, C -relative r -cycle Z_f has the property that $f(c) = \langle (Z_f)_c \rangle$ for each closed point $c \in C$.*

Proof. The first assertion follows from the fact that over a smooth curve,

equidimensional is equivalent to flat. The second assertion follows from the equalities $[Z_f]_c = (Z_f)_c$ of (1.3) and $f(y) = \langle [Z_f]_c \rangle$ of (1.4). \square

In order to formulate in what sense the Chow variety of a closed algebraic set is independent of its projective embedding, we introduce the following concept.

1.6. *Definition.* A proper morphism $g: X' \rightarrow X$ of locally noetherian schemes (e.g., a disjoint union of algebraic sets over k) is said to be a bicontinuous algebraic morphism if it is a set-theoretic bijection and if for every $x \in X$ the associated map of residue fields $k(x) \rightarrow k(g^{-1}(x))$ is purely inseparable. A continuous algebraic map $f: X \rightarrow Y$ is a pair $(g: X' \rightarrow X, f': X' \rightarrow Y)$ in which g is a bicontinuous algebraic morphism. Finally, a bicontinuous algebraic map $f: X \rightarrow Y$ is a pair $(g: X' \rightarrow X, f': X' \rightarrow Y)$ in which both f' and g are bicontinuous algebraic morphisms.

The usefulness of the notion of a continuous algebraic map is that it induces a map on etale homotopy types (cf. (2.1.b)) and, in the special case of complex algebraic varieties, a continuous map of underlying analytic spaces (cf. (2.2.b)).

For our purposes, the basic example of a bicontinuous algebraic morphism is a birational morphism $g: X' \rightarrow X$ of projective varieties which induces a set-theoretic bijection on closed geometric points. The fact that g induces a purely inseparable extension $k(x) \rightarrow k(g^{-1}(x))$ for every $x \in X$ is an immediate consequence of the set-theoretic bijection of closed geometric points of the closures of $\{x\}$ and $\{g^{-1}(x)\}$. For example, if X is a plane curve with a single cusp and if $g: X' \rightarrow X$ is a desingularization of X , then g is a bicontinuous algebraic morphism (but not an isomorphism).

The following proposition is a reformulation of a theorem of W. Hoyt [18]. We point out that even in the special case of dimension 0 cycles, the isomorphism type of Chow varieties of a projective variety X can depend on the embedding $j: X \rightarrow \mathbb{P}_F^N$ if F has positive characteristic (cf. [26]).

1.7. **PROPOSITION.** *Let X be a closed algebraic set provided with two embeddings $j: X \rightarrow \mathbb{P}_F^N, j': X \rightarrow \mathbb{P}_F^M$. Then there is a naturally constructed bicontinuous algebraic map over F*

$$\bigcup_{d \geq 0} C_{r,d}(X, j) \rightarrow \bigcup_{d \geq 0} C_{r,d}(X, j').$$

Proof. Let $j'': X \rightarrow \mathbb{P}_F^R$ be the composition of $j \times j': X \rightarrow \mathbb{P}_F^N \times \mathbb{P}_F^M$ and the Segre embedding

$$\mathbb{P}_F^N \times \mathbb{P}_F^M \rightarrow \mathbb{P}_F^R, \quad R = (N + 1)(M + 1) - 1.$$

Then Hoyt in [18] constructs a morphism $\bigcup_{d \geq 0} C_{r,d}(X, j'') \rightarrow \bigcup_{d \geq 0} C_{r,d}(X, j)$ (not, of course, preserving degree) which extends the evident bijection on

geometric points. Namely, a Chow point $\langle Z \rangle$ whose coordinates are the coefficients of the form $G_Z(\{T_{ij}\})$ is sent to the point whose coordinates are the coefficients of the greatest common divisor of the form $(A_1(\{X_i\}), \dots, A_t(\{X_i\}))$, where

$$G_Z(\{X_i, Y_j\}) = \sum_{s=1, \dots, t} A_s(\{X_i\}) B_s(\{Y_j\}).$$

Since the greatest common divisor of $(A_1(\{X_i\}), \dots, A_t(\{X_i\}))$ has coefficients in F whenever each of the A_i does, we conclude that this morphism is defined over F . Since this morphism is proper, bijective on geometric points, and birational, we conclude that it is a bicontinuous algebraic morphism. The asserted bicontinuous algebraic map is then obtained as the composition of the “inverse” of this morphism and the corresponding morphism

$$\bigcup_{d \geq 0} C_{r,d}(X, j'') \rightarrow \bigcup_{d \geq 0} C_{r,d}(X, j'). \quad \square$$

We conclude this section by introducing the “Chow monoid” $\mathcal{C}_r(X, j)$ and investigating its (discrete) monoid $\pi_0(\mathcal{C}_r(X, j))$ of connected components. We formulate our description of $\pi_0(\mathcal{C}_r(X, j))^+$ in terms of algebraic equivalence of cycles on X_K , where K denotes the algebraic closure of F . (For closed algebraic sets defined over an algebraically closed field, algebraic equivalence of cycles has a generally accepted and classical definition [11; 10.3.2]). By Proposition 1.7, $\pi_0(\mathcal{C}_r(X, j))$ is independent of the choice of embedding $j: X \rightarrow \mathbb{P}_F^N$.

1.8. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$, and let K denote the algebraic closure of F . Then addition of cycles determines the structure of an abelian monoid (in the category of disjoint unions of algebraic sets/ F) on*

$$\mathcal{C}_r(X, j) \equiv \bigcup_{d \geq 0} C_{r,d}(X, j).$$

The group completion $\pi_0(\mathcal{C}_r(X_K, j))^+$ of the discrete monoid $\pi_0(\mathcal{C}_r(X_K, j))$ is the group of r -cycles on X_K modulo algebraic equivalence, whereas $\pi_0(\mathcal{C}_r(X, j))^+$ equals the quotient (i.e., coinvariants) of $\pi_0(C_r(X_K, j))^+$ under the action of $\text{Gal}(K/F)$.

Proof. The pairing $\mathbb{P}^{N(r,d)} \times \mathbb{P}^{N(r,e)} \rightarrow \mathbb{P}^{N(r,d+e)}$ sending forms homogeneous of degree d and e in each of $r+1$ ($N+1$)-tuples to their product restricts to addition of cycles $C_{r,d}(\mathbb{P}^N) \times C_{r,e}(\mathbb{P}^N) \rightarrow C_{r,d+e}(\mathbb{P}^N)$. Clearly, this pairing further

restricts to a morphism $C_{r,d}(X, j) \times C_{r,e}(X, j) \rightarrow C_{r,d+e}(X, j)$, thereby determining an abelian monoid structure on $\mathcal{C}_r(X, j)$.

Let $y, y' \in C_{r,d}(X_K, j)$ be two closed points lying in a single irreducible component of $C_{r,d}(X_K, j)$ and let Z, Z' denote the corresponding r -cycles on \mathbb{P}_K^N with support on $j(X)$. Choose a connected curve in $C_{r,d}(X_K, j)$ containing both y, y' (cf. [24; §6]) and let C be the normalization of this curve. Thus, we have a morphism $f: C \rightarrow C_{r,d}(X, j)$ and two points $c, c' \in C$ with $f(c) = y, f(c') = y'$. After replacing C by a finite, flat, radicial extension C' if necessary, we may apply Corollary 1.5 to conclude that f determines an effective, flat family Z_f over C with $Z_{f(c)} = Z$ and $Z_{f(c')} = Z'$. Thus, Z and Z' are (effectively) algebraically equivalent. More generally, assume $y, y' \in C_{r,d}(X, j)$ are two closed points lying in the same connected component of $C_{r,d}(X, j)$. Then we may find a sequence of closed points $y = y_0, y_1, \dots, y_m = y'$ with y_{i-1}, y_i lying in the same irreducible component of $C_{r,d}(X, j)$ for each i , so that the cycles Z, Z' corresponding to y, y' are algebraically equivalent.

Consequently, the group completion $\pi_0(\mathcal{C}_r(X_K, j))^+$ of $\pi_0(\mathcal{C}_r(X_K, j))$ maps onto the group of (not necessarily effective) r -cycles on X_K modulo algebraic equivalence. To prove injectivity, it suffices to verify for any two effective r -cycles Z, Z' of degree d on \mathbb{P}_K^N with support on $j(X)$ which are algebraically equivalent that there exist effective r -cycles T, T' of degree e on \mathbb{P}_K^N with support on $j(X)$ with $\langle T \rangle, \langle T' \rangle$ in the same connected component of $C_{r,e}(X_K, j)$ such that $\langle Z + T \rangle, \langle Z' + T' \rangle$ are in the same connected component of $C_{r,d+e}(X_K, j)$. As argued previously, we may assume that Z, Z' are fibers above points $c, c' \in C$ of a (not necessarily effective) flat family W over a smooth curve C of r -cycles of degree d on \mathbb{P}_K^N with support on $j(X)$. Write $W = \sum n_i W_i$ as $W = W' - W''$, where W' (respectively, W'') is the sum of those $n_i W_i$ for which n_i is positive (resp., negative), and set T (respectively, T') equal to the fiber of W'' above c (resp., c'), so that $Z + T$ (resp. $Z' + T'$) equals the fibre of W' above c (resp., c'). Then $\langle T \rangle$ and $\langle T' \rangle$ (respectively, $\langle Z + T \rangle, \langle Z' + T' \rangle$) lie in the same irreducible component of $C_{r,e}(X_K, j)$ (resp., $C_{r,d+e}(X_K, j)$) by Corollary 1.5.

Finally, the Leray spectral sequence in étale cohomology [23; III.1.15] for a proper morphism $Y \rightarrow \text{Spec } F$ and a constant torsion sheaf \mathbb{Z}/l has the form

$$E_2^{s,t} = H^s(\text{Gal}(K/F), H^t(Y_K, \mathbb{Z}/l)) \Rightarrow H^{s+t}(Y, \mathbb{Z}/l)$$

In particular,

$$\text{Hom}_{(\text{sets})}(\pi_0(Y), \mathbb{Z}/l) \approx H^0(Y, \mathbb{Z}/l) \approx H^0(\text{Gal}(K/F), H^0(Y_K, \mathbb{Z}/l))$$

so that $\pi_0(Y) \approx H_0(\text{Gal}(K/F), \pi_0(Y_K))$. Since the group completion of the abelian monoid $\pi_0(\mathcal{C}_r(X_K, j))$ is obtained as the colimit of self-maps of the monoid given

by multiplication by elements and since $H_0(\text{Gal}(K/F), -)$ commutes with filtered colimits, we conclude that

$$\begin{aligned} \pi_0(\mathcal{C}_r(X, j))^+ &\approx (H_0(\text{Gal}(K/F), \pi_0(\mathcal{C}_r(X_K, j))))^+ \\ &\approx H_0(\text{Gal}(K/F), \pi_0(\mathcal{C}_r(X_K, j))^+). \end{aligned}$$

2. Lawson homology

In this section, we introduce the l -adic Lawson homology groups of a closed algebraic set X provided with an embedding $j: X \rightarrow \mathbb{P}_K^N$ for some algebraically closed field K . These groups, denoted $L_r H_{2r+i}(X, \mathbb{Z}_l)$, depend upon a choice of prime $l \neq p$ fixed throughout, a nonnegative integer r reflecting the consideration of r -cycles on X , and another nonnegative integer i reflecting a choice of dimension of families of r -cycles. In the special case in which $K = \mathbb{C}$, we provide a parallel development of analytic Lawson homology groups $L_r H_{2r+i}(X^{an})$ which satisfy $L_r H_{2r+i}(X^{an}) \otimes \mathbb{Z}_l \cong L_r H_{2r+i}(X, \mathbb{Z}_l)$. Throughout this section, F and K will denote subfields of our universal field Ω of finite transcendence degree over our base field k of characteristic $p \geq 0$; moreover, K will be always taken to be algebraically closed.

We require two basic constructions to define Lawson homology groups. The first is a “topological realization” functor which transforms (closed) algebraic sets to topological spaces, whereas the second is a “homotopy-theoretic group completion” functor applicable to spaces with a multiplication. Lawson homology groups are then defined as the homotopy groups of the homotopy-theoretic group completion of the topological realization of the Chow monoid $\mathcal{C}_r(X, j)$.

In the special case in which $K = \mathbb{C}$, the functor $(-)^{an}$ which associates to a complex algebraic set its underlying topological space with the analytic topology is a suitable “topological realization functor” and provides us with the analytic Lawson homology groups $L_r H_{2r+i}(X^{an})$. We recall that complex algebraic sets with the analytic topology can be triangulated; moreover, such a triangulation can be achieved in such a manner that any system of sub-algebraic sets is a system of subcomplexes (cf. [17]). Consequently, the spaces we construct for $K = \mathbb{C}$ will all have the homotopy type of C.W. complexes.

In the general algebraic (as opposed to analytic) context, we require the more elaborate machinery of étale homotopy theory [8] and functors of Bousfield and Kan [3]. We shall employ a composition of the étale topological type functor $(-)_{et}$ from simplicial schemes to pro-simplicial sets, the Bousfield-Kan (\mathbb{Z}/l) -completion functor $(\mathbb{Z}/l)_\infty(-)$ from simplicial sets to simplicial sets, the Bousfield-Kan homotopy inverse limit functor $\text{holim}(-)$ from indexed families

of simplicial sets to simplicial sets, and the geometric realization functor $\text{Re}(-)$ from simplicial sets to topological spaces.

We begin by introducing these ‘topological realization’ functors and isolating those properties which we shall require. For readers uncomfortable with hypercoverings occurring in the definition of the etale topological type functor $(-)\text{et}$, [8; 8.2] permits the replacement of $(-)\text{et}$ in (2.1) by the simpler ‘Cech topological type’ functor (denoted $(-)\text{ret}$ in [8]) which involves only Cech nerves of a coverings.

2.1. PROPOSITION. *Let $|(-)\text{et}|: (\text{algebraic sets}) \rightarrow (\text{topological spaces})$ denote the composition $\text{Re}(-) \circ \text{holim}(-) \circ (\mathbb{Z}/l)_\infty(-) \circ (-)\text{et}$, where the maps in (algebraic sets) are all scheme-theoretic maps (not necessarily over k).*

- (a) *The set of connected components of the algebraic set X , $\pi_0(X)$, is in natural 1-1 correspondence with $\pi_0(|X\text{et}|)$.*
- (b) *A continuous algebraic map $f: X \rightarrow Y$ induces a continuous map $f: |X\text{et}| \rightarrow |Y\text{et}|$.*
- (c) *If X, Y are algebraic sets over K (assumed algebraically closed), then the canonical map $|(X \times Y)\text{et}| \rightarrow |X\text{et}| \times |Y\text{et}|$ is a homotopy equivalence.*

Proof. By construction, $\pi_0(X) \approx \pi_0(X\text{et})$, so that (a) follows from [8; 6.10]. To prove (b), we recall from [13; IX.4.10] that if $g: X' \rightarrow X$ is a finite, surjective, and radicial morphism of locally noetherian schemes, then pull-back via g (i.e., $(-)\times_X X'$) induces an equivalence between the categories of etale opens of X and X' . We readily observe that a morphism g of locally noetherian schemes is finite, surjective, and radicial if and only if it is bicontinuous algebraic morphism: finite is equivalent to finite-to-one and proper, whereas radicial (i.e., universally injective) is equivalent to the condition that the induced maps on residue fields are purely inseparable [14; I.3.7.1]. Consequently, $g: X'_{\text{et}} \rightarrow X_{\text{et}}$ is an isomorphism whenever g is a bicontinuous algebraic morphism, so that $g: |X'_{\text{et}}| \rightarrow |X_{\text{et}}|$ is a homeomorphism. Thus, a continuous algebraic map $f: X \rightarrow Y$ induces a continuous map $f: |X_{\text{et}}| \rightarrow |Y_{\text{et}}|$.

Finally, to prove (c), we use the Kunnet theorem in etale cohomology [23; 8.13] which implies that $(X \times Y)\text{et} \rightarrow X_{\text{et}} \times Y_{\text{et}}$ induces an isomorphism in \mathbb{Z}/l -cohomology. This readily implies that the induced map on Artin-Mazur l -adic completions

$$((X \times Y)\text{et})_{l^\wedge} \rightarrow (X_{\text{et}})_{l^\wedge} \times (Y_{\text{et}})_{l^\wedge}$$

is a weak equivalence (cf. [2; 4.3]). Because X_{et} and Y_{et} are weakly equivalent to pro-simplicial sets in which each simplicial set is finite in each dimension [8; 7.2], we conclude using [8; 6.10] that the Sullivan homotopy inverse limit of this weak equivalence of Artin-Mazur l -adic completions is equivalent to

$|(X \times Y)_{\text{et}}| \rightarrow |X_{\text{et}}| \times |Y_{\text{et}}|$, thereby verifying that the latter map is a homotopy equivalence. \square

The analytic analogue of Proposition 2.1 is undoubtedly more familiar.

2.2. PROPOSITION. *Let $(-)^{\text{an}}$: (algebraic sets/ \mathbb{C}) \rightarrow (topological spaces) be defined by setting X^{an} equal to the space of (closed) points of the complex algebraic set X provided with the analytic topology.*

- (a) *The set of connected components of the algebraic set X , $\pi_0(X)$, is in natural 1–1 correspondence with $\pi_0(X^{\text{an}})$.*
- (b) *A continuous algebraic map $f: X \rightarrow Y$ induces a continuous map $f: X^{\text{an}} \rightarrow Y^{\text{an}}$.*
- (c) *$(X \times Y)^{\text{an}}$ is homeomorphic to $X^{\text{an}} \times Y^{\text{an}}$.*

Proof. Assertion (a) is an immediate consequence of [25; 2.23]. To prove (b), observe that a bicontinuous algebraic map $g: X' \rightarrow X$ induces a proper, bijective, continuous map $g: (X')^{\text{an}} \rightarrow X^{\text{an}}$ of Hausdorff spaces which is necessarily a homeomorphism. Finally, (c) is a standard property of the analytic topology. \square

The functoriality of $|(-)_{\text{et}}|$ and $(-)^{\text{an}}$ leads to construction of the simplicial spaces $\mathcal{B}_r(X_{\text{et}})$ and $\mathcal{B}_r(X^{\text{an}})$ associated to the Chow monoid $\mathcal{C}_r(X, j)$. In Definition 2.3, we implicitly assert that the homotopy types of $\mathcal{B}_r(X_{\text{et}})$ and $\mathcal{B}_r(X^{\text{an}})$ are independent (up to natural homotopy equivalence) of the embedding $j: X \rightarrow \mathbb{P}_K^N$, a fact which is an immediate consequence of Propositions 1.7, 2.1, and 2.2. We restrict our attention to closed algebraic sets over the algebraically closed field K in anticipation of Proposition 2.4.

We frequently employ the geometric realization $\text{Re}(T)$ of a simplicial space T . Recall that $\text{Re}(T)$ is the topological space defined as the quotient of the disjoint union of the product spaces $T_n \times \Delta[n]$ by the equivalence relation generated by $(t, \mu(x)) \sim (\mu(t), x)$, where $\mu: \{0, 1, \dots, k\} \rightarrow \{0, 1, \dots, n\}$ is any non-decreasing map, $t \in T_n$, $x \in \Delta[n]$. If each of the spaces T_n has the homotopy type of a C.W. complex, so does $\text{Re}(T)$ (cf. [29]). For typographic simplicity, we shall display the simplicial space T as

$$T_0 \cdots T_1 \cdots T_2 \cdots T_3 \cdots$$

omitting any attempt to indicate the $n + 1$ face maps and n degeneracy maps relating T_n and T_{n-1} .

2.3. DEFINITION. *Let X be a closed algebraic set provided with some embedding $j: X \rightarrow \mathbb{P}_K^N$ (K assumed algebraically closed) and let r be a non-negative integer less than or equal to $\dim(X)$. We denote by $\mathcal{C}_r(X_{\text{et}})$ (alternatively, $\mathcal{C}_r(X^{\text{an}})$ if $K = \mathbb{C}$) the space $\bigcup_{\alpha \in A} |C_{\alpha}|$, where*

$$A = \pi_0(\mathcal{C}_r(X, j)) = \bigcup_{d \geq 0} \pi_0(C_{r,d}(X, j)),$$

C_α denotes the component of $\mathcal{C}_r(X, j)$ indexed by $\alpha \in A$, and $|-|$ denotes $|(-)_{\text{et}}|$ (alternatively, $(-)^{\text{an}}$). We denote by $\mathcal{B}_r(X_{\text{et}})$ (alt., $\mathcal{B}_r(X^{\text{an}})$) the geometric realization of the simplicial space

$$* \cdots \bigcup_{\alpha \in A} |C_\alpha| \cdots \bigcup_{\alpha, \beta \in A} |C_\alpha \times C_\beta| \cdots \bigcup_{\alpha, \beta, \gamma \in A} |C_\alpha \times C_\beta \times C_\gamma| \cdots$$

associated to the bar construction applied to the monoid $\mathcal{C}_r(X, j)$). Let $\mathcal{C}_r(X_{\text{et}})^+$ denote $\Omega\mathcal{B}_r(X_{\text{et}})$, the loop space of the topological space $\mathcal{B}_r(X_{\text{et}})$, and similarly let $\mathcal{C}_r(X^{\text{an}})^+$ denote $\Omega\mathcal{B}_r(X^{\text{an}})$. Then for any $i \geq 0$, we define the l -adic Lawson homology groups by

$$L_r H_{2r+i}(X, \mathbb{Z}_l) = \pi_i(\mathcal{C}_r(X_{\text{et}})^+)$$

and the analytic Lawson homology groups by

$$L_r H_{2r+i}(X^{\text{an}}) = \pi_i(\mathcal{C}_r(X^{\text{an}})^+), \quad K = \mathbb{C}.$$

In the following proposition, we exploit the fact that the functors $|(-)_{\text{et}}|$ and $(-)^{\text{an}}$ commute up to homotopy with products. Of course, $(-)^{\text{an}}$ satisfies the stronger property of commuting with products.

2.4. PROPOSITION. *Let X be a closed algebraic set provided with some embedding $j: X \rightarrow \mathbb{P}_K^N$, and let $r \leq \dim(X)$. Let $|-|$ denote $|(-)_{\text{et}}|$ (alternatively, $(-)^{\text{an}}$ if $F = \mathbb{C}$) and let $\mathcal{C}_r(|X|)$, $\mathcal{C}_r(|X|)^+$ denote the spaces $\mathcal{C}_r(X_{\text{et}})$, $\mathcal{C}_r(X_{\text{et}})^+$ (alt., $\mathcal{C}_r(X^{\text{an}})$, $\mathcal{C}_r(X^{\text{an}})^+$) of (2.3). Then $\mathcal{C}_r(|X|)^+$ is an infinite loop space. Moreover, there is a natural map of H -spaces*

$$i: \mathcal{C}_r(|X|) = \bigcup_{\alpha \in A} |C_\alpha| \rightarrow \mathcal{C}_r(|X|)^+$$

which is a homotopy-theoretic group completion (i.e., has the effect of localizing $H_*(\mathcal{C}_r(|X|))$ with respect to the action of $A = \pi_0(\mathcal{C}_r(|X|))$). Consequently, the Lawson homology groups $L_r H_{2r}(X, \mathbb{Z}_l)$ and $L_r H_{2r}(X^{\text{an}})$ are isomorphic to the group of algebraic r -cycles on X modulo algebraic equivalence.

Proof. Let \mathcal{F} denote the full subcategory of the category of pointed sets whose objects are the finite pointed sets $\mathbf{n} = \{0, 1, \dots, n\}$ pointed by 0. (The category \mathcal{F} is the opposite category of the category Γ of [30]). The abelian monoid structure on $\mathcal{C}_r(X, j)$ naturally determines a functor

$$\mathcal{C}_r(X): \mathcal{F} \rightarrow (\text{disjoint unions of closed algebraic sets}/F),$$

where

$$\mathcal{C}_r(X)(\mathbf{n}) = \bigcup_{\alpha_1, \alpha_2, \dots, \alpha_n} C_{\alpha_1} \times C_{\alpha_2} \times \cdots \times C_{\alpha_n}$$

for $n > 0$ and $\mathcal{C}_r(X)(\mathbf{0}) = \text{Spec } F$. Namely, if $s: \mathbf{n} \rightarrow \mathbf{m}$ is a pointed map, then $\mathcal{C}_r(X)(\mathbf{n}) \rightarrow \mathcal{C}_r(X)(\mathbf{m})$ maps $C_{\alpha_1} \times C_{\alpha_2} \times \cdots \times C_{\alpha_n}$ via addition of cycles to $C_{\beta_1} \times C_{\beta_2} \times \cdots \times C_{\beta_m}$, where β_j is the sum of α_i with $s(i) = j$. Applying the functor $|-|$, we obtain a ‘Segal Γ -space’ (cf. [30])

$$|\mathcal{C}_r(X)|: \mathcal{F} \rightarrow (\text{pointed spaces}).$$

As shown in [7; 1.4], the fact that $|-|$ commutes up to homotopy with products implies that $\mathcal{B}_r(|X|) = |\mathcal{C}_r(X)|(\Sigma^1)$ is an infinite loop space, where Σ^1 is the minimal simplicial model for the circle and the \mathcal{F} -space $|\mathcal{C}_r(X)|$ is extended in the evident manner to the finite simplicial set Σ^1 .

The space $\mathcal{C}_r(|X|) = |\mathcal{C}_r(X)|(\mathbf{1})$ admits a product structure given by composing the homotopy inverse of $|\mathcal{C}_r(X)|(\mathbf{2}) \rightarrow |\mathcal{C}_r(X)|(\mathbf{1}) \times |\mathcal{C}_r(X)|(\mathbf{1})$ with the sum $|\mathcal{C}_r(X)|(\mathbf{2}) \rightarrow |\mathcal{C}_r(X)|(\mathbf{1})$. As discussed in [21] (see also [30]), the natural map

$$i: |\mathcal{C}_r(X)|(\mathbf{1}) \rightarrow \Omega |\mathcal{C}_r(X)|(\Sigma^1) = \mathcal{C}_r(|X|)^+$$

is a homotopy-theoretic group completion.

By Propositions 2.1 and 2.2, $\pi_0(\mathcal{C}_r(|X|)^+)$ is the group completion of $\pi_0(\mathcal{C}_r(X, j))$, which by Proposition 1.8 is the group of algebraic r -cycles on X modulo algebraic equivalence. \square

Using the group completion property of $\mathcal{C}_r(|X|)^+$, we next conclude that this space has the homology of the ‘space of stable r -cycles.’

2.5. PROPOSITION. *Let X be a closed algebraic set provided with some embedding $j: X \rightarrow \mathbb{P}_K^N$ and let $r \leq \dim(X)$. For each $\alpha \in A = \pi_0(\mathcal{C}_r(X, j))$, choose some $Z_\alpha \in C_\alpha$. Let $\beta_1, \beta_2, \dots, \beta_n, \dots$ be a sequence of elements of A such that each element of \bar{A} occurs infinitely often among the β_n 's. Then the map of H -spaces of Proposition 2.4*

$$i: \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|)^+$$

factors up to homotopy through a homology equivalence

$$\tilde{r}: \text{Tel}(\mathcal{C}_r(|X|), \zeta_n) \rightarrow \mathcal{C}_r(|X|)^+$$

where $\text{Tel}(\mathcal{C}_r(|X|), \zeta_n)$ is the mapping telescope of the sequence of maps

$$\zeta_n = Z_{\beta_n} + (-): \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|).$$

Moreover, the maps on homotopy and homology groups induced by \tilde{i} are uniquely determined as the direct limit of the corresponding maps induced by i .

Proof. We remind the reader that the mapping telescope $\text{Tel}(T_n, f_n)$ of a sequence of continuous maps, $\{f_n: T_n \rightarrow T_{n+1}, n \geq 0\}$, is the topological space defined as the disjoint union of the product spaces $T_n \times \Delta[1]$ modulo the equivalence relation generated by $(t, 1) \sim (f_n(t), 0)$ for any $t \in T_n$ (where 0, 1 are the endpoints of the interval $\Delta[1]$). This space has the property that a sequence of maps $g_n: T_n \rightarrow S$ with the property that $g_{n+1} \circ f_n$ is homotopic to g_n for all n determines a map $g: \text{Tel}(T_n, f_n) \rightarrow S$ with the property that the composition of g with the natural inclusion $T_n \rightarrow \text{Tel}(T_n, f_n)$ is homotopic to g_n . We caution the reader that the homotopy type of the map g is determined by the data of the homotopy type of each g_n and the homotopy type of each homotopy relating $g_{n+1} \circ f_n$ to g_n . On the other hand, the maps on homotopy and homology groups induced by g are simply the direct limits of the corresponding maps induced by the family $\{g_n\}$. Of course, if T_0 consists of a single point, this point serves as a distinguished base point for $\text{Tel}(T_n, f_n)$.

Since $i: \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|)^+$ is a map of H -spaces, ζ_n extends to a self-map of $\mathcal{C}_r(|X|)^+$. We construct a homotopy inverse, $s_n: \mathcal{C}_r(|X|)^+ \rightarrow \mathcal{C}_r(|X|)^+$, to this extension by defining s_n to be given as pairing via the H -space structure of $\mathcal{C}_r(|X|)^+$ with any point of the component of $\mathcal{C}_r(|X|)^+$ inverse in A^+ to β_n . Since the composition $s_n \circ i \circ \zeta_n: \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|)^+$ is homotopic to i , the system of maps

$$s_1 \circ \cdots \circ s_n \circ \zeta_n \circ \cdots \circ \zeta_1: \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|)^+$$

determines an extension of i to $\tilde{i}: \text{Tel}(\mathcal{C}_r(|X|), \zeta_n) \rightarrow \mathcal{C}_r(|X|)^+$.

Since homology commutes with direct limits, the inclusion $j: \mathcal{C}_r(|X|) \rightarrow \text{Tel}(\mathcal{C}_r(|X|), \zeta_n)$ induces $j_*: H_*(\mathcal{C}_r(|X|)) \rightarrow \text{colim}\{H_*(\mathcal{C}_r(|X|)), \zeta_n\}$, which is precisely the localization of $H_*(\mathcal{C}_r(|X|))$ with respect to the action of A . Since $i: \mathcal{C}_r(|X|) \rightarrow \mathcal{C}_r(|X|)^+$ is a homotopy-theoretic group completion by Proposition 2.4, $i_*: H_*(\text{Tel}(\mathcal{C}_r(|X|), \zeta_n)) \rightarrow H_*(\mathcal{C}_r(|X|)^+)$ is an isomorphism between localizations of $H_*(\mathcal{C}_r(|X|))$. \square

In order to compute the Lawson homology groups $\pi_*(\mathcal{C}_r(|X|)^+)$, it is very useful to know that \tilde{i} of Proposition 2.5 is a homotopy equivalence. Because $\mathcal{C}_r(X^{\text{an}})$ is an abelian monoid when $K = \mathbb{C}$ (whereas $\mathcal{C}_r(X_{\text{et}})$ is only a homotopy commutative H -space), we can demonstrate this in the analytic context. This is done in the corollary below, whose key step was shown to us by W. Dwyer. Theorems 4.3 and 4.6 provide special cases in which \tilde{i} is a homotopy equivalence in the algebraic (i.e., etale homotopy-theoretic) content.

2.6. **COROLLARY.** *Let X be a closed algebraic set over \mathbb{C} , provided with an embedding $j: X \rightarrow \mathbb{P}_{\mathbb{C}}^N$, and let $r \leq \dim_{\mathbb{C}}(X)$. Then the map of (2.5)*

$$\tilde{i}: \text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n) \rightarrow \mathcal{C}_r(X^{\text{an}})^+$$

is a homotopy equivalence.

Proof. Let T denote $\text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n)$ and let $j_n: \mathcal{C}_r(X^{\text{an}}) \rightarrow T$ denote the inclusion of the n -th stage of the telescope, so that j_{n-1} is homotopic to $j_n \circ \zeta_n$. Consider elements $t \in T, g \in \pi_1(T, t)$, and $h \in \pi_m(T, t)$. Choose $n \geq 0$ and $\alpha \in A$ such that $j_n((C_{\alpha})^{\text{an}})$ lies in the same component of T as t and such that g, h lie in the image of $(j_n)_{\#}: \pi_{*}((C_{\alpha})^{\text{an}}, Z_{\alpha}) \rightarrow \pi_{*}(T, t)$ (where we have replaced t by $j_n(Z_{\alpha})$). Choose $s > n$ such that $\beta_s = \alpha$. Then the pointed map $j_n: (C_{\alpha})^{\text{an}} \rightarrow T$ is homotopic to

$$j_s \circ \zeta_{n+1} \circ \cdots \circ \zeta_s: (C_{\alpha})^{\text{an}} \rightarrow (C_{2\alpha})^{\text{an}} \rightarrow \cdots \rightarrow T$$

which equals each of the following two compositions

$$\begin{aligned} j_s \lim \zeta_{n+1} \circ \cdots \circ \zeta_{s-1} \circ \mu \circ (\text{id} \times Z_{\alpha}): (C_{\alpha})^{\text{an}} &\rightarrow (C_{\alpha} \times C_{\alpha})^{\text{an}} \\ &\rightarrow (C_{2\alpha})^{\text{an}} \rightarrow \cdots \rightarrow T \end{aligned}$$

$$\begin{aligned} j_s \circ \zeta_{n+1} \circ \cdots \circ \zeta_{s-1} \circ \mu \circ (Z_{\alpha} \times \text{id}): (C_{\alpha})^{\text{an}} &\rightarrow (C_{\alpha} \times C_{\alpha})^{\text{an}} \\ &\rightarrow (C_{2\alpha})^{\text{an}} \rightarrow \cdots \rightarrow T \end{aligned}$$

where Z_{α} is the point map and μ the multiplication map.

Since $(\text{id} \times Z_{\alpha})_{\#}(\pi_1((C_{\alpha})^{\text{an}}, Z_{\alpha}))$ acts trivially on $(Z_{\alpha} \times \text{id})_{\#}(\pi_m((C_{\alpha})^{\text{an}}, Z_{\alpha}))$, we conclude that $g \in \pi_1(T, t)$ acts trivially on $h \in \pi_m(T, t)$. In other words, T is a simple space. Because a loop space is always a simple space, Proposition 2.5 implies that $\tilde{i}: \text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n) \rightarrow \mathcal{C}_r(X^{\text{an}})^+$ is a homology equivalence between simple spaces each of which has the homotopy type of a C.W. complex (cf. [17]) and thus a homotopy equivalence. \square

The comparison theorems in étale homotopy in conjunction with Corollary 2.6 permit us to relate l -adic and analytic Lawson homology of complex algebraic varieties.

2.7. **PROPOSITION.** *Let X be a closed complex algebraic set provided with some embedding $j: X \rightarrow \mathbb{P}_{\mathbb{C}}^N$, and let $r \leq \dim(X)$. Then for any $i > 0$, there is a natural identification*

$$L_r H_{2r+i}(X, \mathbb{Z}_l) \approx L_r H_{2r+i}(X^{\text{an}}) \otimes \mathbb{Z}_l.$$

Proof. Write $\mathcal{C}_r(X) = \bigcup_{\alpha \in A} C_\alpha$, with each C_α connected. By [8; 8.5], for each $\alpha \in A$ there is a natural chain of homotopy equivalences

$$(\mathbb{Z}/l)_\infty \circ (C_\alpha)^{\text{an}} \leftarrow |(\mathbb{Z}/l)_\infty(C_\alpha)_{s,\text{et}}| \rightarrow |(C_\alpha)_{\text{et}}|$$

where $(\mathbb{Z}/l)_\infty \circ (-)^{\text{an}}$ denotes the composition $\text{Re}(-) \circ (\mathbb{Z}/l)_\infty \circ \text{Sing}(-) \circ (-)^{\text{an}}$ and where $|(\mathbb{Z}/l)_\infty(-)_{s,\text{et}}|$ denotes the composition $\text{holim}(-) \circ (\mathbb{Z}/l)_\infty \circ \text{diag}(-) \circ \text{Sing}(-)$ applied to the category of Čech nerves of rigid étale coverings (cf. (8; 8.4]). Let

$$\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}}) \leftarrow \mathcal{C}_r((\mathbb{Z}/l)_\infty X_{s,\text{et}}) \rightarrow \mathcal{C}_r(X_{\text{et}}) \quad (2.7.1)$$

denote the disjoint union (indexed by $\alpha \in A$) of these equivalences.

The equivalences of (2.7.1) determine a commutative diagram of homotopy groups whose vertical maps are isomorphisms

$$\begin{array}{ccc} \pi_\star(\text{Tel}(\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}})^+) \\ \uparrow & & \uparrow \\ \pi_\star(\text{Tel}(\mathcal{C}_r(((\mathbb{Z}/l)_\infty X_{s,\text{et}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r((\mathbb{Z}/l)_\infty X_{s,\text{et}})^+) \\ \downarrow & & \downarrow \\ \pi_\star(\text{Tel}(\mathcal{C}_r(X_{\text{et}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r(X_{\text{et}})^+). \end{array} \quad (2.7.2)$$

On the other hand, we have a commutative diagram of homotopy groups

$$\begin{array}{ccc} \pi_\star(\text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r(X^{\text{an}})^+) \\ \uparrow & & \uparrow \\ \pi_\star(\text{Tel}(\mathcal{C}_r(\text{Sing}.X^{\text{an}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r(\text{Sing}.X^{\text{an}})^+) \\ \downarrow & & \downarrow \\ \pi_\star(\text{Tel}(\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}}), \zeta_n)) & \rightarrow & \pi_\star(\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}})^+) \end{array} \quad (2.7.3)$$

where $\mathcal{C}_r(\text{Sing}.X^{\text{an}})$ is the disjoint union of the spaces $\text{Re}(\text{Sing}(C_\alpha))$. Because $(\mathbb{Z}/l)_\infty(-)$ commutes up to homotopy with products, the proof of Corollary 2.6 applies to prove that the lower (as well as the middle and upper) horizontal map of (2.7.3) is an isomorphism. Using (2.7.2), we conclude that the proposition will follow once we prove that the lower left vertical map of (2.7.3) is given by tensoring with \mathbb{Z}_l .

As in Proposition 2.5, let $\beta_1, \dots, \beta_n, \dots$ be a sequence of elements of A such that each $\alpha \in A$ occurs infinitely often among the β_n 's. Then the distinguished component of $\text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n)$ is given by the telescope of the sequence

$$(C_{\beta_1})^{\text{an}} \rightarrow (C_{\beta_1 + \beta_2})^{\text{an}} \rightarrow \cdots \rightarrow (C_{\beta_1 + \cdots + \beta_n})^{\text{an}} \rightarrow \cdots$$

For notational convenience, let C_n denote $(C_{\beta_n})^{\text{an}}$.

As discussed previously, the triangulability of complex algebraic sets permits us to assume that the inductive system $\{C_n\}$ satisfies the condition that each $C_n \rightarrow C_{n+1}$ is a simplicial inclusion. For any C.W. complex D , let $sk_n D$ denote the subcomplex consisting of the union of cells of D of dimension less than or equal to n .

Let C denote the union of the C_n 's. Since $C \approx \text{colim}(C_n)$ is homotopy equivalent to $C^\sim \times K(\pi_1(C), 1)$ (where C^\sim denotes the universal cover of C), the canonical map $C_n \rightarrow C$ factors (up to homotopy) through

$$C_n \rightarrow C'_n \equiv C''_n \times sk_{d(n)}K(\pi_1(C_n)^\sim, 1)$$

where C''_n is a simply connected, finite C.W. complex obtained from C_n by adding 2-cells in C but not in C_n to kill the fundamental group of C_n , where $\pi_1(C_n) \equiv \text{im}\{\pi_1(C_n) \rightarrow \pi_1(C)\}$ is a finitely generated abelian group, and where $d(n) = \dim(C_n)$. Observe that C'_n is a finite complex which is $d(n)$ -simple. The finiteness of C_n implies that there exists some $f(n) > n$ such that $C_n \rightarrow C_{f(n)}$ factors (up to homotopy) through $C_n \rightarrow C'_n$. We give $\{C'_n\}$ the structure of an inductive system with maps $C'_n \rightarrow C'_{f(n)}$ defined to be the composition

$C'_n \rightarrow C_{f(n)} \rightarrow C'_{f(n)}$. Thus,

$$\text{colim } \pi_i(C_n) \otimes \mathbb{Z}_l \approx \text{colim } \pi_i(C'_n) \otimes \mathbb{Z}_l,$$

$$\text{colim } \pi_i((\mathbb{Z}/l)_\infty C_n) \approx \text{colim } \pi_i((\mathbb{Z}/l)_\infty C'_n)$$

(where the ambiguous notation $\text{colim } \pi_i(C_n) \otimes \mathbb{Z}_l$ is acceptable in view of the fact that $\text{colim}(-)$ and $(-)\otimes \mathbb{Z}_l$ commute).

Recall that if T is a simple, finite, connected C.W. complex, then by [8; 6.6, 6.10]

$$\pi_i((\mathbb{Z}/l)_\infty(T)) \approx \pi_i(T) \otimes \mathbb{Z}_l, \quad i \geq 0.$$

Since the homotopy type $(\mathbb{Z}/l)_\infty(C'_n)$ in dimensions $\leq d(n)$ depends only upon the homotopy type of C'_n in dimensions $\leq d(n)$ by [3; IV.5.1], we conclude that

$$\pi_i((\mathbb{Z}/l)_\infty(C'_n)) \approx \pi_i(C'_n) \otimes \mathbb{Z}_l, \quad i < d(n).$$

Consequently, the left vertical map of (2.7.3),

$$\pi_i(\text{Tel}(\mathcal{C}_r(X^{\text{an}}), \zeta_n)) \rightarrow \pi_i(\text{Tel}(\mathcal{C}_r((\mathbb{Z}/l)_\infty X^{\text{an}}), \zeta_n))$$

can be identified with the composition

$$\begin{aligned} \operatorname{colim} \pi_i(C_n) &\rightarrow \operatorname{colim} \pi_i((\mathbb{Z}/l)_\infty(C_n)) \approx \operatorname{colim} \pi_i((\mathbb{Z}/l)_\infty(C'_n)) \\ &\approx \operatorname{colim} \pi_i(C'_n) \otimes \mathbb{Z}_l \approx \operatorname{colim} \pi_i(C_n) \otimes \mathbb{Z}_l \end{aligned}$$

which is readily seen to be the map ‘tensor with \mathbb{Z}_l .’ □

We now proceed to show that the usual functoriality of algebraic equivalence classes of cycles extends to all Lawson homology groups. The following lemma formalizes our frequently used method of constructing continuous algebraic maps.

2.8. LEMMA. *Let X, Y be algebraic sets defined over a field F and assume that Y is closed. Consider a rational map $f: X \rightarrow Y$ over F (i.e., a morphism over F defined on some open, dense subset of X) and let Γ_f denote the graph of f (closed in $X \times Y$). Then $\operatorname{pr}_1: \Gamma_f \rightarrow X$ is a bicontinuous algebraic morphism (and hence f is a continuous algebraic map) if and only for every generic point η of X and every specialization $\eta \searrow x$ in X_K , there exists at most one point y of Y_K such that $(\eta, f(\eta)) \searrow (x, y)$ is a specialization in $X_K \times Y_K$ extending $\eta \searrow x$, where K is some algebraically closed extension of F .*

Proof. Since Y is closed, $\operatorname{pr}_1: \Gamma_f \rightarrow X$ is proper; since each generic point η of X lies in the image of pr_1 , we conclude that pr_1 is surjective. By construction, pr_1 is birational. Hence, pr_1 is a bicontinuous algebraic morphism if and only if it is injective on geometric points with values in K .

Since $\Gamma_f \subset X \times Y$ is closed, $(\Gamma_f)_K$ contains for each generic point $\eta \in X$ every specialization of $(\eta, f(\eta))$ in $X_K \times Y_K$. Since Γ_f is the closure of the graph of f , every geometric point with values in K of Γ_f is a specialization in $X_K \times Y_K$ of $(\eta, f(\eta))$ for some generic point η of X . Therefore, pr_1 is injective if and only if for each specialization $\eta \searrow x$ in X_K there is a unique y in Y_K such that $(\eta, f(\eta)) \searrow (x, y)$ is a specialization in $X_K \times Y_K$ extending $\eta \searrow x$. □

If $f: X \rightarrow Y$ is a morphism of closed algebraic sets and V a subvariety of X , we recall that $\deg(V/f(V))$ is defined to be 0 if $\dim(V) > \dim(f(V))$ and $[k(V): k(f(V))]$ if $\dim(V) = \dim(f(V))$, where $k(V), k(f(V))$ are the function fields of $V, f(V)$. If $g: W \rightarrow X$ is a flat morphism of some fixed relative dimension and if V is a subvariety of X , we recall that $g^{-1}(V)$ is defined to be the cycle associated to the subscheme $V \times_X W \subset W$. (The cycle associated to a subscheme $T \subset \mathbb{P}_F^R$ is the formal sum $\sum n_i T_i$, where η_1, \dots, η_m are the generic points of the irreducible components of T_1, \dots, T_m of T and n_i is the length of the local ring of T at η_i .) We refer to the reader to the first chapter of [11] for further details.

2.9. PROPOSITION. *Let X, Y, W be closed algebraic sets, provided with embeddings $j: X \rightarrow \mathbb{P}_F^N, j': Y \rightarrow \mathbb{P}_F^M, j'': W \rightarrow \mathbb{P}_F^R$. For any morphism $f: X \rightarrow Y$ over F*

and any nonnegative integer $r \leq \dim(X)$, there exists a continuous algebraic map over F as indicated

$$f_{\star}: \mathcal{C}_r(X, j) \rightarrow \mathcal{C}_r(Y, j'), \quad f_{\star} \left(\sum n_i V_i \right) = \sum n_i \deg(V_i/f(V_i)) \cdot f(V_i).$$

For any flat morphism $g: W \rightarrow X$ over F of relative dimension $s \geq 0$ and any nonnegative integer $r \leq \dim(X)$, there exists a continuous algebraic map over F as indicated

$$g^*: \mathcal{C}_r(X, j) \rightarrow \mathcal{C}_{r+s}(W, j''), \quad g^* \left(\sum n_i V_i \right) = \sum n_i g^{-1}(V_i).$$

These continuous algebraic maps determine maps on l -adic Lawson homology groups for any $i \geq 0$

$$\begin{aligned} f_{\#}: L_r H_{2r+i}(X_K, \mathbb{Z}_l) &\rightarrow L_r H_{2r+i}(Y_K, \mathbb{Z}_l) \\ g^{\#}: L_r H_{2r+i}(X_K, \mathbb{Z}_l) &\rightarrow L_{r+s} H_{2r+2s+i}(W_K, \mathbb{Z}_l) \end{aligned}$$

where K denotes the algebraic closure of F . Moreover, in the special case $F = \mathbb{C}$, f_{\star} and g^* determine maps on analytic Lawson homology groups for any $i \geq 0$

$$\begin{aligned} f_{\#}: L_r H_{2r+i}(X^{\text{an}}) &\rightarrow L_r H_{2r+i}(Y^{\text{an}}), \\ g^{\#}: L_r H_{2r+i}(X^{\text{an}}) &\rightarrow L_{r+s} H_{2r+2s+i}(W^{\text{an}}). \end{aligned}$$

Proof. To define $f_{\star}: \mathcal{C}_r(X, j) \rightarrow \mathcal{C}_r(Y, j')$ rationally, let $\eta = \text{Spec } E \in C_{r,d}(X, j)$ be a generic point. By Proposition 1.2, η corresponds to an r -cycle Z_{η} defined over X_E and therefore determines an r -cycle $f_{\star}(Z_{\eta})$ on Y_E of some degree $d(\eta)$ corresponding to a morphism $\text{Spec } E \rightarrow C_{r,d(\eta)}(Y, j')$. These morphisms determine a rational map $C_{r,d}(X, j) \rightarrow \mathcal{C}_r(Y, j')$ over F . To prove that this rational map is a continuous algebraic map, by Lemma 2.8 we must show that the uniqueness of specializations of $(\eta, f(\eta))$ in $C_{r,d}(X, j)_K \times C_{r,d(\eta)}(Y, j')_K$ extending a given specialization $\eta \downarrow x$ in $C_{r,d}(X, j)_K$, where $f(\eta) = \langle f_{\star}(Z_{\eta}) \rangle$ and K is an algebraic closure of F .

Let $(\eta, f(\eta)) \downarrow (x, y)$ be a specialization in $C_{r,d}(X, j)_K \times C_{r,d(\eta)}(Y, j')_K$ and let

$$h: C \rightarrow C_{r,d}(X, j)_K \times C_{r,d(\eta)}(Y, j')_K$$

be a morphism from a smooth curve sending the generic point ξ of C to $(\eta, f(\eta))$ and a special point $c \in C$ to (x, y) . Then $\text{pr}_1 \circ h$ corresponds to a C -relative r -cycle Z with support on $j(X)$ with generic fibre above ξ equal to Z_{η} and scheme-theoretic fibre above c equal to Z_x , the cycle with $\langle Z_x \rangle = x$.

For any irreducible component V of the C -relative r -cycle Z , flatness of $(1 \times f)(V) \subset C \times Y$ over C is implied by flatness of $V \subset C \times X$ over C . This follows from the observation that because C is a smooth curve, it suffices to check that $(1 \times f)(V)$ dominates C (cf. [16; III.9.7]). Hence we conclude that $(1 \times f)_*(Z)$ is a C -relative r -cycle with support on $j'(Y)$ with $((1 \times f)_*(Z))_\xi = f_*(Z_\eta)$ and $((1 \times f)_*(Z))_c = f_*(Z_x)$. We conclude using Proposition 1.3 that y must equal $\langle f_*(Z_k) \rangle$. In other words, $(\eta, f(\eta)) \downarrow (x, \langle f_*(Z_x) \rangle)$ is the unique specialization of $(\eta, f(\eta))$ extending $\eta \downarrow x$.

We construct $g^*: \mathcal{C}_r(X, j) \rightarrow \mathcal{C}_{r+s}(W, j'')$ similarly. Namely, g^* is first defined rationally by sending a generic point $\eta = \text{Spec } E \in C_{r,d}(X, j)$ corresponding to an r -cycle Z_η with support on $j(X)$ to the Chow point $g^*(\eta)$ of the $(r+s)$ -cycle $g^*(Z_\eta)$. As for f_* , we verify the uniqueness of specializations of $(\eta, g^*(\eta))$ in $C_{r,d}(X, j)_K \times C_{r+s,d(\eta)}(Y, j')_K$ extending a given specialization $\eta \downarrow x$ in $C_{r,d}(X, j)_K$. The proof is similar to that for f_* : if Z is a C -relative r -cycle with support on $j(X)$ for some smooth curve C with generic fibre above ξ equals to Z_η and scheme-theoretic fibre above $c \in C$ equal to Z_x , then $(1 \times g)^*(Z)$ is a C -relative $(r+s)$ -cycle with support on $j''(W)$ with generic fibre above ξ equal to $g^*(Z_\eta)$ and scheme-theoretic fibre above $c \in C$ equal to $g^*(Z_x)$.

The previous construction applies essentially verbatim to provide continuous algebraic maps f_*, g^* from $\bigcup_{\alpha_1 \dots \alpha_n} C_{\alpha_1} \times \dots \times C_{\alpha_n}$ for any $n \geq 1$ and thus continuous maps from $\bigcup_{\alpha_1 \dots \alpha_n} |C_{\alpha_1} \times \dots \times C_{\alpha_n}|$. The naturality of f_*, g^* implies that these maps after base change from F to K determine maps of simplicial spaces exhibited in Definition 2.3 and thus maps

$$f_*: \mathcal{C}_r(|X_K|)^+ \rightarrow \mathcal{C}_r(|Y_K|)^+, \quad g^*: \mathcal{C}_r(|X_K|)^+ \rightarrow \mathcal{C}_{r+s}(|W_K|)^+.$$

The induced maps on homotopy groups are the asserted maps $f_\#, g^\#$. \square

We conclude this section with the following assertion concerning the existence of Galois actions on $L_r H_{2r+i}(X, \mathbb{Z}_l)$. We caution the reader that the analytic Lawson homology groups $L_r H_{2r+i}(X^{\text{an}})$ for complex varieties do not admit actions associated to discontinuous Galois automorphisms of \mathbb{C} .

2.10. PROPOSITION. *Let X be a closed algebraic set, provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ and let K denote the algebraic closure of F . Then the Galois group $\text{Gal}(K/F)$ naturally acts on $L_r H_{2r+i}(X_K, \mathbb{Z}_l)$, any $0 \leq r \leq \dim(X)$, $0 \leq i$. Moreover, if $f: X \rightarrow Y$ and $g: W \rightarrow X$ are morphisms over F as in (2.9), then the maps of Proposition 2.9*

$$f_\#: L_r H_{2r+i}(X_K, \mathbb{Z}_l) \rightarrow L_r H_{2r+i}(Y_K, \mathbb{Z}_l), \quad \text{any } i \geq 0.$$

$$g^\#: L_r H_{2r+i}(X_K, \mathbb{Z}_l) \rightarrow L_{r+s} H_{2r+2s+i}(W_K, \mathbb{Z}_l)$$

are $\text{Gal}(K/F)$ -equivariant.

Proof. Since $C_{r,d}(X, j)_K = C_{r,d}(X_K, j)$ by Proposition 1.1, $\text{Gal}(K, F)$ naturally acts on $C_{r,d}(X_K, j)$. Clearly this action commutes with the monoid structure on $\mathcal{C}_r(X_K, j)$. The functoriality of $|(-)_{\text{ét}}|$ for all morphisms of schemes (not merely K -maps) thus implies that $\text{Gal}(K/F)$ acts on $\mathcal{B}_r((X_K)_{\text{ét}})$ and therefore on l -adic Lawson homology groups $L_r H_{2r+i}(X_K, \mathbb{Z}_l)$. Since the constructions in (2.9) involved in defining $f_{\#}$, g^* are continuous algebraic maps over F whenever $f: X \rightarrow Y$, $g: W \rightarrow X$ are morphisms over F , functoriality of $|(-)_{\text{ét}}|$ implies that these maps are $\text{Gal}(K/F)$ -equivariant. \square

3. Constructions for algebraic suspensions

Lawson's main theorem is the assertion that 'algebraic suspension determines an isomorphism in Lawson homology.' In this section, we lay the foundations for an algebraic proof of this theorem in section 4. We re-work Lawson's original constructions [19], [20] in order that they apply in our present algebraic context. Geometric arguments analyzing spaces of currents are reformulated, so that continuous maps in the complex analytic context are replaced by continuous algebraic maps of algebraic sets. Our strategy is to first show that Lawson's constructions determine rational maps $f: X \rightarrow Y$ of algebraic sets and then verify that the condition in Lemma 2.8 is fulfilled.

As in previous sections, F is a field extension of finite transcendence degree over our base field k and contained in our universal field Ω , and l is a prime not equal to $p = \text{char}(k)$. As the reader will see, our arguments are achieved without assumption on the field F , except that F is required to be infinite in Proposition 3.5.

We begin with an analysis of the (algebraic) *suspension* construction, which (following Lawson) we denote by $\Sigma(-)$. If $j: X \rightarrow \mathbb{P}_F^N$ is an embedding of a closed algebraic set, then $\Sigma j: \Sigma X \rightarrow \mathbb{P}_F^{N+1}$ is the embedding given by the same homogeneous equations (i.e., not involving the new homogeneous coordinate of \mathbb{P}_F^{N+1}), so that X is the intersection of ΣX with the 'standard embedding' $\mathbb{P}_F^N \subset \mathbb{P}_F^{N+1}$. If $Z = \sum n_i Z_i$ is any r -cycle on X , define ΣZ to be $\sum n_i (\Sigma Z_i)$.

If V and W are closed algebraic subsets of \mathbb{P}_F^N , we define the (projective) *join* of V and W , denoted $V \# W$, to be the closed algebraic subset of \mathbb{P}_F^N consisting of the union of all lines in \mathbb{P}_F^N which meet both V and W . (In the grassmannian \mathcal{L} of lines in \mathbb{P}_F^N , those lines L which meet both V and W form a closed algebraic set \mathcal{S} ; $V \# W \subset \mathbb{P}_F^N$ is the projection of the incidence correspondence in $\mathbb{P}_F^N \times \mathcal{L}$ restricted to $\mathbb{P}_F^N \times \mathcal{S}$). We shall often view \mathbb{P}_F^{M+N+1} as the 'external join' of disjoint linear subspaces $L \approx \mathbb{P}_F^M$ and $L' \approx \mathbb{P}_F^N$. If X, Y are closed algebraic sets provided with embeddings $j_X: X \rightarrow \mathbb{P}_F^M$, $j_Y: Y \rightarrow \mathbb{P}_F^N$, then the *external join* $j_{X \# Y}: X \# Y \rightarrow \mathbb{P}_F^{M+N+1}$ of X and Y is the join of

$$j_X(X) \subset \mathbb{P}_F^M \approx L \subset \mathbb{P}_F^{M+N+1}, \quad j_Y(Y) \subset \mathbb{P}_F^N \approx L' \subset \mathbb{P}_F^{M+N+1}.$$

If V is a subvariety of \mathbb{P}_F^N , then $\Sigma V = x_\infty \# V$ where $x_\infty = \langle 0, \dots, 0, 1 \rangle \in \mathbb{P}_F^{N+1}$.

We recall that two cycles $Z = \sum n_i Z_i$, $Z' = \sum m_j Z'_j$ in an ambient algebraic variety V are said to *intersect properly* (in V) if the codimension in V of each irreducible component of $Z_i \cap Z'_j$ is the sum of the codimensions of Z_i and Z'_j .

3.1. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$. Then sending an effective, degree d , r -cycle Z with support on $j(X)$ to ΣZ is given by a continuous algebraic map over F*

$$\Sigma(-): C_{r,d}(X, j) \rightarrow C_{r+1,d}(\Sigma X, \Sigma j).$$

Those effective $(r + 1)$ -cycles Z of degree d on \mathbb{P}_F^{N+1} with support on ΣX which meet \mathbb{P}_F^N properly (in \mathbb{P}_F^{N+1}) form an open subset of $C_{r+1,d}(\Sigma X, \Sigma j)$ which we denote by $T_{r+1,d}(\Sigma X, \Sigma j)$. Moreover, $T_{r+1,d}(\Sigma X, \Sigma j)$ contains the image of $\Sigma(-)$.

Proof. To define $\Sigma(-): C_{r,d}(X, j) \rightarrow C_{r+1,d}(\Sigma X, \Sigma j)$ rationally, let $\eta = \text{Spec } L_\eta$ be a generic point $C_{r,d}(X, j)$, with corresponding effective r -cycle Z_η defined over L_η by Proposition 1.2. Then $\Sigma(Z_\eta)$ determines a morphism $\text{Spec } L_\eta \rightarrow C_{r+1,d}(\Sigma X, \Sigma j)$ over F . These morphisms for all generic η determine a rational map $C_{r,d}(X, j) \rightarrow C_{r+1,d}(\Sigma X, \Sigma j)$ over F , sending η to $\Sigma(\eta) = \langle \Sigma(Z_\eta) \rangle$.

To prove that this rational map is a continuous algebraic map, we proceed as in the proof of Proposition 2.9. Let $(\eta, \Sigma(\eta)) \downarrow (x, y)$ be a specialization in $C_{r,d}(X, j)_K \times C_{r+1,d}(\Sigma X, \Sigma j)_K$ extending a given specialization $\eta \downarrow x$ in $C_{r,d}(X, j)_K$, where K is an algebraically closed extension of F . Let

$$h: C \rightarrow C_{r,d}(X, j)_K \times C_{r+1,d}(\Sigma X, \Sigma j)_K$$

be a map from a smooth curve C sending the generic point ξ of C to $(\eta, \Sigma(\eta))$ and a special point $c \in C$ to (x, y) . Then $\text{pr}_1 \circ h$ corresponds to a C -relative r -cycle $Z = \sum n_i V_i$ on $C \times \mathbb{P}_K^{N+1}$ with support on $j(X)$ with generic fibre above ξ equal to Z_η and scheme-theoretic fibre above c equal to Z_x , the cycle with $\langle Z_x \rangle = x$.

Define $\Sigma_C(Z) = \sum n_i \Sigma_C(V_i)$ on $C \times \mathbb{P}_K^{N+1}$ with support on $\Sigma j(\Sigma X)$ by setting $\Sigma_C(V_i)$ equal to that subvariety of $C \times \mathbb{P}_K^{N+1}$ given by the same equations (not involving the extra homogeneous coordinate of \mathbb{P}_K^{N+1}) as $V_i \subset C \times \mathbb{P}_K^N$. Since $\Sigma_C(V_i)$ dominates C if V_i does, we conclude that $\Sigma_C(Z)$ is a C -relative $(r + 1)$ -cycle with support on $\Sigma j(\Sigma X)$. Moreover, the generic fibre above ξ equals ΣZ_η and scheme-theoretic fibre above c equals ΣZ_x . We conclude that y must equal $\langle \Sigma Z_x \rangle$ if $(\eta, \Sigma(\eta)) \downarrow (x, y)$ extends $\eta \downarrow x$. By Lemma 2.8, $\Sigma(-)$ is a continuous algebraic map.

By [28; II.6.7], if $Z \downarrow Z_0$ is a specialization in $C_{r,d}(X, j)$ and if Z_0 meets \mathbb{P}_F^N

properly (in \mathbb{P}_F^{N+1}), then Z also meets \mathbb{P}_F^N properly. Thus, $T_{r+1,d}(\Sigma X, \Sigma j)$ is an open subset of $C_{r+1,d}(\Sigma X, \Sigma j)$. Finally, the (set-theoretic) intersection of the $(r+1)$ -cycle ΣZ with \mathbb{P}_F^N (in \mathbb{P}_F^{N+1}) is the r -cycle Z , so that ΣZ meets \mathbb{P}_F^N properly; in other words, $\Sigma(-)$ has image contained in $T_{r+1,d}(\Sigma X, \Sigma j)$. \square

The following proposition is our algebraic version of the first of Lawson’s two geometric arguments, the argument Lawson calls “holomorphic taffy” (cf. [20; §4]).

3.2. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ and let \mathbb{A}_F^1 denote the affine line over F . For any $d > 0$, $r \leq \dim(X)$, there exists a continuous algebraic map over F*

$$\Phi: T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 \rightarrow T_{r+1,d}(\Sigma X, \Sigma j)$$

whose restrictions to $T_{r+1,d}(\Sigma X, \Sigma j) \times \{1\}$ and to $\Sigma(C_{r,d}(X, j)) \times \{s\}$ for any $s \in \mathbb{A}_F^1$ are the identity and whose restriction to $T_{r+1,d}(\Sigma X, \Sigma j) \times \{0\}$ is a continuous algebraic map

$$\phi: T_{r+1,d}(\Sigma X, \Sigma j) \rightarrow \Sigma C_{r,d}(X, j).$$

(In other words, Φ is a strong deformation retraction of $\Sigma(C_{r,d}(X, j)) \subset T_{r+1,d}(\Sigma X, \Sigma j)$).

Proof. We define

$$\Lambda \subset \mathbb{P}_F^{N+1} \times \mathbb{P}_F^1 \times \mathbb{P}_F^{N+1}$$

to be the graph of the rational map $\mathbb{P}_F^{N+1} \times \mathbb{P}_F^1 \rightarrow \mathbb{P}_F^{N+1}$ whose restriction to $\mathbb{P}_F^{N+1} \times s$ for $s \in \mathbb{A}_F^1 - \{0\}$ is the linear automorphism $\Theta_s: \mathbb{P}_F^{N+1} \rightarrow \mathbb{P}_F^{N+1}$ sending $\langle x_0, \dots, x_N, x_{N+1} \rangle$ to $\langle x_0, \dots, x_N, x_{N+1}/s \rangle$. More explicitly, Λ is the closed subvariety given by the homogeneous equations

$$\{X_i Y_j - X_j Y_i, T X_{N+1} Y_j - S X_j Y_{N+1}; 0 \leq i, j \leq N\},$$

where $\{X_i\}$ (respectively, $\{S, T\}$; resp., $\{Y_j\}$) are “the standard linear forms” on \mathbb{P}_F^{N+1} (resp., \mathbb{P}_F^1 ; resp., \mathbb{P}_F^{N+1}).

We define Φ rationally by sending a generic point

$$\text{Spec } L = (\eta, \xi) \in T_{r+1}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1$$

to the point

$$\langle \text{pr}_3((Z_\eta \times \xi \times \mathbb{P}_F^{N+1}) \cdot \Lambda) \rangle \in C_{r+1,d}(\Sigma X, \Sigma j).$$

Observe that for any $s \in \mathbb{A}_F^1 - \{0\}$ and any r -cycle Z_x with $\langle Z_x \rangle = x \in C_{r,d}(X, j)$, the cycle $\text{pr}_3((Z_x \times s \times \mathbb{P}_F^{N+1}) \cdot \Lambda)$ can be computed as the projection of the scheme-theoretic intersection of $Z_x \times s \times \mathbb{P}_F^{N+1}$ with Λ , and thus equals $\Theta_s(Z_x)$. Namely, for any irreducible component V of Z_x , the subvarieties $V \times s \times \mathbb{P}_F^{N+1}$ and Λ meet transversally at their intersection $V \times s \times \Theta_s(V)$ so that the multiplicity of $V \times s \times \Theta_s(V)$ in $(Z_x \times s \times \mathbb{P}_F^{N+1}) \cdot \Lambda$ is 1; furthermore, the projection $\text{pr}_3: \Lambda \rightarrow \mathbb{P}_F^{N+1}$ restricts to an isomorphism on $(Z_x \times s \times \mathbb{P}_F^{N+1}) \cdot \Lambda$ viewed as a cycle on Λ . Since the cycle $(Z_x \times s \times \mathbb{P}_F^{N+1}) \cdot \Lambda$ is defined over L [28; II.6.6] (and implicit in [11]), this construction determines a morphism $\text{Spec } L \rightarrow C_{r+1,d}(\Sigma X, \Sigma j)$ and thus a rational map over F

$$\Phi: T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 \rightarrow C_{r+1,d}(\Sigma X, \Sigma j).$$

We employ Lemma 2.8. to show that Φ extends to a continuous algebraic map

$$\begin{aligned} \Phi: T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 &\rightarrow C_{r+1,d}(\Sigma X, \Sigma j); \\ \Phi(Z_x \times s) &= \text{pr}_3((Z_x \times s \times \mathbb{P}_F^{N+1}) \cdot \Lambda). \end{aligned}$$

Let K be an algebraically closed extension of F . Recall that if

$$(\eta, \xi, \langle (Z_\eta \times \xi \times \mathbb{P}_K^{N+1}) \cdot \Lambda_K \rangle) \downarrow (x, s, y) \text{ in } C_{r+1,d}(X, j)_K \times \mathbb{P}_K^1 \times C_{r+1,d}(X, J)_K$$

extends a given specialization $(\eta, \xi) \downarrow (x, s)$ and if $Z_x \times s \times \mathbb{P}_K^{N+1}$ intersects Λ_K properly in $\mathbb{P}_K^{N+1} \times \mathbb{P}_K^1 \times \mathbb{P}_K^{N+1}$, then y must equal $\text{pr}_3((Z_x \times s \times \mathbb{P}_K^{N+1}) \cdot \Lambda_K)$ [28; I.9.7.d; II.6.7]. Thus, to prove that Φ extends to a continuous algebraic map, it suffices to verify that $Z_x \times s \times \mathbb{P}_K^{N+1}$ intersects Λ_K properly for any $(x, s) \in T_{r+1,d}(\Sigma X, Xj)_K \times \mathbb{A}_K^1$. For $s \neq 0$, this follows from the identification of $\text{pr}_3((Z_x \times s \times \mathbb{P}_K^{N+1}) \cdot \Lambda_K)$ with $\Theta_s(Z_x)$. If W is a subvariety of \mathbb{P}_K^{N+1} , then the intersection of $W \times 0 \times \mathbb{P}_K^{N+1}$ and Λ_K in $\mathbb{P}_K^{N+1} \times \mathbb{P}_K^1 \times \mathbb{P}_K^{N+1}$ consists of those triples $(x, 0, y)$ such that either

- i. $x \in W \cap \mathbb{P}_K^N$ and $y \in x_\infty \neq x$ (the projective line through x_∞ and x), or
- ii. $x \in W$ and $y = x_\infty$.

Hence, $Z_x \times 0 \times \mathbb{P}_K^{N+1}$ intersects Λ_K properly for any $x \in T_{r+1,d}(\Sigma X, \Sigma j)_K$.

As seen above, Φ restricted to

$$T_{r+1,d}(\Sigma X, \Sigma j) \times s \subset T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 - \{0\}$$

is induced by Θ_s . In particular, Φ restricted to $T_{r+1,d}(\Sigma X, \Sigma j) \times 1$ is the identity.

Moreover, the preceding explicit description of the intersection of $W \times 0 \times \mathbb{P}_K^{N+1}$ and Λ_K shows that

$$\text{pr}_3((W \times 0 \times \mathbb{P}_F^{N+1}) \cap \Lambda) \text{ equals } \Sigma(W_x \cap \mathbb{P}_F^N).$$

Thus, Φ restricted to $T_{r+1,d}(\Sigma X, \Sigma j) \times \{0\}$ has image contained in $\Sigma(C_{r,d}(X, j))$. Finally, if $W = \Sigma V$ for some subvariety V of \mathbb{P}_F^N , then our explicit description shows that the intersection of $\Sigma V \times 0 \times \mathbb{P}_F^{N+1}$ and Λ projects isomorphically onto ΣV . Since $\Sigma V \times 0 \times \mathbb{P}_F^{N+1}$ and Λ intersect transversally, Φ restricted to

$$\Sigma(C_{r,d}(X, j)) \times 0 \subset T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1$$

is also the identity.

In particular the graph of Φ , Γ_Φ , satisfies $\text{pr}_3(\Gamma_\Phi) \subset T_{r+1,d}(\Sigma X, \Sigma j)$ so that Φ is a continuous algebraic map $T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 \rightarrow T_{r+1,d}(\Sigma X, \Sigma j)$. Moreover,

$$\text{pr}_1: \Gamma_\Phi \rightarrow T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1$$

restricts to an isomorphism above any regular point of $T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1$ by Zariski's Main Theorem. In particular, the restriction of this projection above $T_{r+1,d}(\Sigma X, \Sigma j) \times 0$ is birational, as well as proper and bijective, thereby determining a continuous algebraic map with domain $T_{r+1,d}(\Sigma X, \Sigma j) \times 0$. As seen above, the range of this map is $\Sigma C_{r,d}(X, j)$, so that this continuous algebraic map is of the form

$$\phi: T_{r+1,d}(\Sigma X, \Sigma j) \rightarrow \Sigma C_{r,d}(X, j). \quad \square$$

3.3. LEMMA *Let X, Y be algebraic sets over F with Y closed, $U \subset X$ a Zariski open subset, $W \subset U \times Y$ a closed subset, and $W^\sim \subset X \times Y$ the Zariski closure of W in $X \times Y$. Then any $w \in W^\sim$ with $\text{pr}_1(w) \in U$ is necessarily in W . Consequently, if $f: X \rightarrow Y$ is a rational map over F , then there exists a maximal open subset $D_f \subset X$ (the “domain of continuity” of f) such that $f|_{D_f}: D_f \rightarrow Y$ is a continuous algebraic map.*

Proof. Since Y is a closed algebraic set, $\text{pr}_1: U \times Y \rightarrow U$ is proper and so therefore is its restriction $\rho: W \rightarrow U$. Consider some point $w \in W^\sim$ with $\text{pr}_1(w) \in U$. Choose $w' \in W$ with $w' \searrow w$. Let R be a valuation ring with residue field E and function field L together with a map $\omega: \text{Spec } R \rightarrow W^\sim$ mapping $\text{Spec } E$ to w and mapping $\text{Spec } L$ to w' . Since $\rho: W \rightarrow U$ is proper, the pair $(\rho \circ \omega: \text{Spec } R \rightarrow U, \rho|: \text{Spec } L \rightarrow W)$ extends to a map $\omega^\sim: \text{Spec } R \rightarrow W$ by the valuative criterion for properness. Since $W^\sim \rightarrow X$ is separated (because Y is a separated over $\text{Spec } k$) ω^\sim must equal ω , thereby implying that $w \in W$.

Let $D_f = \{x \in X; \#((\Gamma_f \cap (x \times Y))_K) = 1\}$. Since f is a rational map, D_f is a

dense Zariski open subset of X . By applying the first part of the lemma with W equal to the graph of $f|_U$ and \tilde{W} equal to the graph of f , we conclude that $D_f \supset U$ whenever $f|_U$ is a continuous algebraic map. \square

We now algebraicize Lawson's second geometric construction, which he calls 'magic fans' (cf. [20; §5]).

3.4. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$. For any $d, e > 0, r \leq \dim(X)$, there exists a rational map over F*

$$\Psi_e: C_{r+1,d}(\Sigma X, \Sigma j) \times C_{N+1,e}(\mathbb{P}_F^{N+2}) \rightarrow C_{r+1,de}(\Sigma X, \Sigma j)$$

such that if $x_1 \# Z$ meets D properly (in \mathbb{P}_F^{N+2}) and if x_2 does not lie in any component of $(x_1 \# Z) \cap D$, then (Z, D) is in the domain of continuity of Ψ_e and

$$\Psi_e(Z, D) = p_2((x_1 \# Z) \cdot D),$$

where $x_1 = \langle 0, \dots, 0, 0, 1 \rangle$ and $x_2 = \langle 0, \dots, 0, 1, 1 \rangle$ are points of \mathbb{P}_F^{N+2} , and $p_2: \mathbb{P}_F^{N+2} \rightarrow \mathbb{P}_F^{N+1}$ is the projection off x_2 . Moreover, if $Z = \Sigma W$ for some r -cycle W on X with $x_1 \# Z$ meeting D properly and x_2 not lying on $(x_1 \# Z) \cap D$ or if $D = e \cdot \mathbb{P}_F^{N+1}$, then $\Psi_e(Z, D)$ (where defined) equals $e \cdot Z$.

Proof. As seen in Proposition 3.1,

$$x_1 \# (-): C_{r+1,d}(\Sigma X, \Sigma j) \rightarrow C_{r+2,d}(x_1 \# \Sigma X, j')$$

is a continuous algebraic map over F , where $j' = x_1 \# \Sigma j: x_1 \# \Sigma X \rightarrow \mathbb{P}_F^{N+2}$. Moreover, intersection determines a rational map over F

$$C_{r+2,d}(x_1 \# \Sigma X, j') \times C_{N+1,e}(\mathbb{P}_F^{N+2}) \rightarrow C_{r+1,de}(x_1 \# \Sigma X, j').$$

Namely, if $(\eta, \delta) = \text{Spec } L$ is a generic point of $C_{r+2,d}(x_1 \# \Sigma X, j') \times C_{N+1,e}(\mathbb{P}_F^{N+2})$ with corresponding pair of cycles (Z_η, D_δ) , then both Z_η and D_δ are defined over L by Proposition 1.2 and hence $Z_\eta \cdot D_\delta$ is also defined over L (cf. [28; II.6.6] or implicit in [11]); hence (η, δ) determines a morphism $\text{Spec } L \rightarrow C_{r+1,de}(x_1 \# \Sigma X, j')$ over F . We verify that a pair (Z, D) lies in the domain of continuity of this rational map whenever Z and D intersect properly by applying the 'Theorem of Specialization' [28; II.6.7] (cf. also [11; 10.1]).

Furthermore, p_2 determines a rational map over F

$$C_{r+1,de}(x_1 \# \Sigma X, j') \rightarrow C_{r,de}(\Sigma X, \Sigma j)$$

($x_1 \# \Sigma X$ is a double suspension and projecting off x_2 is projecting along a suspension coordinate). Namely, if $\xi = \text{Spec } E$ is a generic point of

$C_{r+1,de}(x_1 \# \Sigma X, j')$ with corresponding $(r+1)$ -cycle Z_ξ , then Z_ξ has no component containing x_2 so that $p_2(Z_\xi)$ is a well defined r -cycle supported on ΣX and defined over E [28; I.9.7.d]; hence, ξ determines a morphism $\text{Spec } E \rightarrow C_{r,de}(\Sigma X, \Sigma j)$. Applying [28; I.9.7.d] once again, we conclude that Z is in the domain of continuity of this map whenever no component of Z contains x_2 .

We define Ψ_e to be the composition of these rational maps. The assertion concerning the domain of continuity of Ψ_e is immediate from the above discussion. If $D = e \cdot \mathbb{P}_F^{N+1}$ as a (degenerate) degree e hypersurface in \mathbb{P}_F^{N+2} , then

$$p_2((x_1 \# Z) \cdot D) = e \cdot p_2((x_1 \# Z) \cdot \mathbb{P}_F^{N+1}) = e \cdot Z,$$

since $(x_1 \# Z)$ intersects \mathbb{P}_F^{N+1} transversally.

Finally, if $Z = \Sigma W = x_\infty \# W$ is a suspended cycle, then $p_2((x_1 \# Z) \cdot D)$ equals $p_2((x_2 \# Z) \cdot D)$. Since D has degree e , we conclude that D meets each line $x_2 \# y \subset \mathbb{P}_F^{N+2}$ exactly e times (counting multiplicities) for any $y \in \mathbb{P}_F^{N+1}$. Thus, $(x_2 \# Z) \cdot D$ is an e -sheeted covering of Z . Hence, $p_2((x_2 \# \Sigma W) \cdot D) = e \cdot \Sigma W$. \square

Investigating the rational map Ψ_e of Proposition 3.4 more closely, we obtain a 'continuous algebraic homotopy' between multiplication by e on $C_{r+1,d}(\Sigma X, \Sigma j)$ and a map which factors through $T_{r+1,de}(\Sigma X, \Sigma j)$. The determination of codimensions in the proof of Proposition 3.5 closely follows an argument of Lawson (cf. [20; §5]).

3.5. PROPOSITION. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ for some infinite field F . For any $d > 0$, $r \leq \dim(X)$, there exists a positive integer $E(d, r)$ such that for any $e > E(d, r)$ the rational map Ψ_e of (3.4) has the following property: there exists an F -rational projective line $L_e \subset C_{N+1,de}(\mathbb{P}_F^{N+2})$ with $e \cdot \mathbb{P}_F^{N+1} \in L_e$ such that Ψ_e restricts to a continuous algebraic map over F*

$$\Psi_e: C_{r+1,d}(\Sigma X, \Sigma j) \times L_e \rightarrow C_{r+1,de}(\Sigma X, \Sigma j)$$

with the property that the further restrictions of Ψ_e yield continuous algebraic maps over F

$$\begin{aligned} \psi_D: C_{r+1,d}(\Sigma X, \Sigma j) &\rightarrow T_{r+1,de}(\Sigma X, \Sigma j), & D \in L_e - e \cdot \mathbb{P}_F^{N+1} \\ \psi_{D_0} = e \cdot (-): C_{r+1,d}(\Sigma X, \Sigma j) &\rightarrow C_{r+1,de}(\Sigma X, \Sigma j), & D_0 = e \cdot \mathbb{P}_F^{N+1}. \end{aligned}$$

Proof. We consider the closed subset $\Delta \subset C_{r+1,d}(\Sigma X, \Sigma j) \times C_{N+1,e}(\mathbb{P}_F^{N+2})$ defined by

$$\Delta = \{(Z, D); x_2 \# \mathbb{P}_F^N \text{ meets } (x_1 \# Z) \cap D \text{ improperly}\}.$$

For any $r + 1$ -cycle Z with support on $\Sigma_j(\Sigma X)$, we let $\Delta(Z)$ denote the fibre of Δ above Z . We proceed to verify that

$$\min\{\text{codim}(\Delta(Z) \subset C_{N+1,e}(\mathbb{P}_F^{N+2})); Z \in C_{r+1,d'}(\Sigma X, \Sigma j), d' \leq d\} \rightarrow \infty \quad (3.5.1)$$

as $e \rightarrow \infty$. This will clearly imply the existence of some $E(d, r)$ such that

$$\text{codim}\{\text{pr}_2(\Delta) \subset C_{N+1,e}(\mathbb{P}_F^{N+2})\} \geq 2 \quad \text{for any } e > E(d, r). \quad (3.5.2)$$

Since $\Delta(Z) = \cup \Delta(V_i)$ whenever $Z = \sum n_i V_i$, it suffices to restrict attention to irreducible cycles $Z = V$ in (3.5.1). Let V' denote $x_1 \# V \subset \mathbb{P}_F^{N+2}$ and H denote $x_2 \# \mathbb{P}_F^N \subset \mathbb{P}_F^{N+2}$. Observe that H meets $V' \cap D$ improperly if and only if H contains some component of $V' \cap D$ if and only if the image of G in $\Gamma(V', \mathcal{O}(e))$ lies in the image of

$$H + (-): \Gamma(V', \mathcal{O}_{V'}(e - 1)) \rightarrow \Gamma(V', \mathcal{O}_{V'}(e))$$

where G is a form of degree e with associated divisor D . Thus,

$$\begin{aligned} &\text{codim}(\Delta(Z) \subset C_{N+1,e}(\mathbb{P}_F^{N+2})) \\ &\geq \dim(\Gamma(V', \mathcal{O}_{V'}(e)) - \dim(\Gamma(V', \mathcal{O}_{V'}(e - 1))) - 1. \end{aligned}$$

As a function of e , the right hand side of the above inequality grows as a polynomial of degree $r - 1$ with leading coefficient $\deg(V')/(r + 1)!$. This clearly implies inequality (3.5.1).

Consider the projective variety L of all lines in the projective space $C_{N+1,e}(\mathbb{P}_F^{N+2})$ passing through the F -rational point $e \cdot \mathbb{P}_F^N \in C_{N+1,e}(\mathbb{P}_F^{N+2})$. Assume $e > E(d, r)$, so that a generic $L_\eta \in L$ intersects $\text{pr}_2(\Delta)$ only at $e \cdot \mathbb{P}_F^N$ by (3.5.2). Observe that neither x_1 nor x_2 lies on such a generic L_η . Since F is infinite, the F -rational points of L are dense, so that there exists some F -rational line $L_e \in L$ intersecting $\text{pr}_2(\Delta)$ only at $e \cdot \mathbb{P}_F^N$ and such that neither x_1 nor x_2 belong to any $D \in L_e - e \cdot \mathbb{P}_F^N$.

Since D meets $x_1 \# Z$ properly for any $Z \in C_{r+1,d}(\Sigma X, \Sigma j)$ whenever D does not contain x_1 , Proposition 3.4 implies that $C_{r+1,d}(\Sigma X, \Sigma j) \times L_e$ is contained in the domain of continuity of Ψ_e . As argued at the end of the proof of Proposition 3.2, Ψ_e therefore restricts to a continuous algebraic map

$$\Psi_e: C_{r+1,d}(\Sigma X, \Sigma j) \times L_e \rightarrow C_{r+1,de}(\Sigma X, \Sigma j).$$

Moreover, for $D_0 = e \cdot \mathbb{P}_F^N \in C_{N+1,e}(\mathbb{P}_F^{N+2})$, Proposition 3.4 implies that Ψ_e restricted to $C_{r+1,d}(\Sigma X, \Sigma j) \times \{D_0\}$ is multiplication by e . For $e \cdot \mathbb{P}_F^N \neq D \in L_e$, D does not lie in $\text{pr}_2(\Delta)$; in other words, $x_2 \# \mathbb{P}_F^N$ meets $(x_1 \# Z) \cap D$ properly so

that \mathbb{P}_F^N meets $\Psi_e(Z, D)$ properly. We conclude that Ψ_e restricted to $C_{r+1,d}(\Sigma X, \Sigma j) \times \{D\}$ has image in $T_{r+1,de}(\Sigma X, \Sigma j)$. \square

4. Theorems for Lawson homology

In this section, we reap the benefits of the formal developments of the preceding sections. Theorem 4.2 is our algebraic/analytic generalization of Lawson's suspension theorem applicable to an arbitrary algebraic set over an algebraically closed field K . Following Lawson, we derive as a corollary (our Corollary 4.4) the computation of l -adic Lawson homology for projective spaces over algebraically closed fields. However, to derive this corollary, we require an algebraic version of the Dold-Thom theorem (our Theorem 4.3), a result of independent interest because it provides a new definition of etale homology. Finally, in Theorem 4.6, we determine the Lawson homology in codimension 1 of a smooth projective variety over K .

As in section 2, F and K will denote subfields of Ω of finite transcendence degree over k of characteristic $p \geq 0$ and a prime $l \neq p$. We shall only use K to denote algebraically closed fields.

4.1. LEMMA. *Let V be an algebraic set defined over a field F and let $\Theta: V \times \mathbb{A}_F^1 \rightarrow Y$ be a continuous algebraic map over F . Then for any pair of F -rational points $a, b \in \mathbb{A}_F^1$, the restrictions Θ_a, Θ_b of Θ to $V \times a, V \times b$ determine homotopic maps $\Theta_a, \Theta_b: |V_{\text{et}}| \rightarrow |Y_{\text{et}}|$. Moreover, if $F = \mathbb{C}$, then $\Theta_a, \Theta_b: V^{\text{an}} \rightarrow Y^{\text{an}}$ are also homotopic.*

Proof. We recall that $\text{pr}_1: V \times \mathbb{A}_F^1 \rightarrow V$ induces an isomorphism in \mathbb{Z}/l -cohomology [23; VI.4.12, VI.4.15]. As argued in the proof of Proposition 2.1, this implies that $\text{pr}_1: (V \times \mathbb{A}_F^1)_{\text{et}} \rightarrow |V_{\text{et}}|$ is a homotopy equivalence. Since $\text{pr}_1 \circ i_c: V \rightarrow V \times \mathbb{A}_F^1 \rightarrow V$ equals the identity of V for any F -rational point $c \in \mathbb{A}_F^1$, we conclude that the homotopy type of $\Theta \circ i_c = \Theta_c: |V_{\text{et}}| \rightarrow |Y_{\text{et}}|$ is independent of c . The analytic assertion is immediate from the contractibility of $(\mathbb{A}_{\mathbb{C}}^1)^{\text{an}}$. \square

Having rephrased Lawson's analytic arguments into our algebro-geometric context in section 3, we can now prove the following algebraic/analytic generalization of Lawson's suspension theorem, the main theorem of [20].

4.2. THEOREM. *Let X be a closed algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ and let K be the algebraic closure of F . For any $r \leq \dim(X)$, algebraic suspension determines a homotopy equivalence*

$$\Sigma: \mathcal{C}_r((X_K)_{\text{et}})^+ \rightarrow \mathcal{C}_{r+1}((\Sigma X)_{\text{et}})^+$$

which is $\text{Gal}(K/F)$ -equivariant. This homotopy equivalence induces $\text{Gal}(K/F)$ -equivariant isomorphisms of l -adic Lawson homology groups

$$\Sigma_{\#}: L_r H_{2r+i}(X_K, \mathbb{Z}/l) \approx L_{r+1} H_{2r+2+i}(\Sigma X_K, \mathbb{Z}/l), \quad i \geq 0.$$

Furthermore, if $F = \mathbb{C}$, then algebraic suspension determines a homotopy equivalence

$$\Sigma: \mathcal{C}_r(X^{\text{an}})^+ \rightarrow \mathcal{C}_r(\Sigma X^{\text{an}})^+$$

and thus isomorphisms of analytic Lawson homology groups

$$\Sigma_{\#}: L_r H_{2r+i}(X^{\text{an}}) \approx L_{r+1} H_{2r+2+i}(\Sigma X^{\text{an}}), \quad i \geq 0.$$

Proof. As in section 2, we let $|-|$ denote either $|(-)_{\text{et}}|$ or $(-)^{\text{an}}$ in order to consider both $\mathcal{C}_r((X_K)_{\text{et}})^+$ and $\mathcal{C}_r(X^{\text{an}})^+$ simultaneously. As argued in the proof of Proposition 2.10, the fact that $\Sigma(-)$ in Proposition 3.1 is defined over F implies the Galois equivariance of $\Sigma: \mathcal{C}_r((X_K)_{\text{et}})^+ \rightarrow \mathcal{C}_{r+1}((\Sigma X_K)_{\text{et}})^+$. For notational convenience, we now assume $F = K$ and proceed to verify the asserted homotopy equivalences. Observe that

$$\mathcal{T}_{r+1}(\Sigma X, \Sigma j) \equiv \bigcup_{d \geq 0} T_{r,d}(\Sigma X_K, \Sigma j) = \bigcup_{\alpha \in A} T_{\alpha}$$

is a submonoid of $\mathcal{C}_{r+1}(\Sigma X, \Sigma j) = \bigcup_{d \geq 0} C_{r,d}(\Sigma X_K, \Sigma j)$, with $\pi_0(\mathcal{T}_{r+1}(\Sigma X, \Sigma j))$ equal to $A = \pi_0(\mathcal{C}_r(X, j))$ by Proposition 3.2. As in Definition 2.3 for $\mathcal{B}_r(|X|)$, denote by $\mathcal{D}_{r+1}(|\Sigma X|)$ the geometric realization of the simplicial space

$$* \cdots \bigcup_{\alpha \in A} |T_{\alpha}| \cdots \bigcup_{\alpha, \beta \in A} |T_{\alpha} \times T_{\beta}| \cdots \bigcup_{\alpha, \beta, \gamma \in A} |T_{\alpha} \times T_{\beta} \times T_{\gamma}| \cdots$$

As in Proposition 2.5, we conclude that

$$i: \mathcal{T}_{r+1}(|\Sigma X|) = \bigcup_{\alpha \in A} |T_{\alpha}| \rightarrow \Omega \mathcal{D}_{r+1}(|\Sigma X|) \equiv \mathcal{T}_{r+1}(|\Sigma X|)^+$$

is a map of H -spaces factoring through a homology equivalence

$$\tilde{i}: \text{Tel}(\mathcal{T}_{r+1}(|\Sigma X|), \zeta_n) \rightarrow \mathcal{T}_{r+1}(|\Sigma X|)^+,$$

where $\zeta_n = Z_{\beta_n} + (-): \mathcal{T}_{r+1}(|\Sigma X|) \rightarrow \mathcal{T}_{r+1}(|\Sigma X|)$ as in (2.5) with the added condition that $Z_{\beta_n} \in T_{\beta_n}$.

By Lemma 4.1, the existence of the continuous algebraic map

$$\Phi: T_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}^1 \rightarrow T_{r+1,d}(\Sigma X, \Sigma j)$$

of Proposition 3.2 implies that $\Sigma \circ \phi$ and the identity of $|T_\alpha|$ are homotopic (via a $\text{Gal}(K/F)$ -equivariant homotopy), whereas $\phi \circ \Sigma: |C_\alpha| \rightarrow |T_\alpha| \rightarrow C_\alpha$ equals the identity for each $\alpha \in A$. Consequently,

$$\Sigma: \text{Tel}(\mathcal{C}_r(|X|), \zeta_n) \rightarrow \text{Tel}(\mathcal{T}_{r+1}(|\Sigma X|), \zeta_n)$$

is a homology equivalence whose induced map on homotopy groups fits in a commutative square

$$\begin{array}{ccc} H_*(\text{Tel}(\mathcal{C}_r(|X|), \zeta_n)) & \rightarrow & H_*(\text{Tel}(\mathcal{T}_{r+1}(|\Sigma X|), \zeta_n)) \\ \downarrow & & \downarrow \\ H_*(\mathcal{C}_r(|X|)^+) & \rightarrow & H_*(\mathcal{T}_{r+1}(|\Sigma X|)^+) \end{array}$$

whose vertical maps are isomorphisms. Thus, the map of H -spaces

$$\mathcal{C}_r(|X|)^+ \rightarrow \mathcal{T}_{r+1}(|\Sigma X|)^+ \tag{4.2.1}$$

induces an isomorphism on homology groups and is consequently a homotopy equivalence.

In a similar fashion, we apply Lemma 4.1 to the restriction

$$\Psi_e: C_{r+1,d}(\Sigma X, \Sigma j) \times \mathbb{A}_F^1 \rightarrow C_{r+1,d}(\Sigma X, \Sigma j)$$

of the continuous algebraic map of Proposition 3.6, where \mathbb{A}_K^1 denotes the complement in L_e of some point different from $e \cdot \mathbb{P}_K^{N+1}$. Choose some $D \neq e \cdot \mathbb{P}_K^{N+1} \in L_e$. Let $A' = \pi_0(\mathcal{C}_{r+1}(\Sigma X, \Sigma j))$, let $\Sigma: A \rightarrow A'$ be the map induced by $\Sigma: \mathcal{C}_r(X, j) \rightarrow \mathcal{C}_{r+1}(\Sigma X, \Sigma j)$, and let $\psi: A' \rightarrow A$ be the map induced by

$$\psi_D: \mathcal{C}_{r+1}(\Sigma X, \Sigma j) \rightarrow \mathcal{T}_{r+1}(\Sigma X, \Sigma j)$$

of Proposition 3.5. By Proposition 3.5, for any $\beta \in \pi_0(C_{r+1,d}(\Sigma X, \Sigma j)) \subset A'$ the maps

$$e(-), \iota \circ \psi_D: |C_\beta| \rightarrow |T_{\psi(\beta)}| \rightarrow |C_{e\beta}|$$

are homotopy equivalent, as are the maps $\alpha \in \pi_0(T_{r+1,d}(\Sigma X, \Sigma j)) \subset A$

$$e(-), \psi_D \circ \iota: |T_\alpha| \rightarrow |C_{\Sigma\alpha}| \rightarrow |T_{e\alpha}|$$

where $\iota: \mathcal{F}_{r+1}(\Sigma X, \Sigma j) \rightarrow \mathcal{C}_{r+1}(\Sigma X, \Sigma j)$ is the natural inclusion. Consequently,

$$j: \text{Tel}(\mathcal{F}_{r+1}(|\Sigma X|), \zeta_n) \rightarrow \text{Tel}(\mathcal{C}_{r+1}(|\Sigma X|), \zeta_n)$$

is a homotopy equivalence, implying as above that

$$\mathcal{F}_{r+1}(|\Sigma X|)^+ \rightarrow \mathcal{C}_{r+1}(|\Sigma X|)^+ \tag{4.2.2}$$

is also a homotopy equivalence.

The asserted homotopy equivalences are the compositions of (4.2.1) and (4.2.2). The asserted isomorphisms of Lawson homology groups are the maps on homotopy groups induced by these equivalences. \square

The Dold-Thom theorem [5] asserts that if T is a pointed, connected C.W. complex and if $\text{SP}^\infty(T) \equiv \text{colim}_d \text{SP}^d(T)$ denotes the infinite symmetric product of T , then the map on homotopy groups induced by the inclusion $T \rightarrow \text{SP}^\infty(T)$ can be identified with the Hurewicz homomorphism $\pi_*(T) \rightarrow H_*(T)$. In view of the triangulability of complex algebraic sets [17], this immediately implies the natural isomorphisms

$$L_0 H_i(X) \cong H_i(X)$$

for any complex projective algebraic variety X .

Using P. Deligne’s analysis of the étale cohomology of symmetric products of an algebraic variety [4], we prove the analogous statement in the context of étale homotopy and étale homology of algebraic varieties.

4.3. THEOREM. *Let X be a closed, geometrically connected, algebraic set provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$, let K be the algebraic closure of F , and let $x_0 \in X$ be an F -rational point. Then the homology equivalence of (2.5)*

$$\tilde{\tau}: \text{Tel}(\mathcal{C}_0((X_K)_{\text{ét}}), \zeta_n) \rightarrow \mathcal{C}_0((X_K)_{\text{ét}})^+$$

is a homotopy equivalence. Moreover, the identity component of $\text{Tel}(\mathcal{C}_0((X_K)_{\text{ét}}), \zeta_n)$ can be identified in a $\text{Gal}(K/F)$ -equivariant manner with $\text{colim}_d |\text{SP}^d(X_K)_{\text{ét}}|$, where $\text{SP}^d(X_K)$ is the d -fold symmetric product of X_K with itself and where $\text{SP}^d(X_K) \rightarrow \text{SP}^{d+1}(X_K)$ is given by addition with x_0 . Finally, there are $\text{Gal}(K/F)$ -natural identifications

$$L_0 H_i(X_K, \mathbb{Z}_l) = \begin{cases} \mathbb{Z} & i = 0 \\ \lim H_i((X_K)_{\text{ét}}, \mathbb{Z}/l^n) & i > 0 \end{cases}$$

where the (inverse) limit is indexed by pairs arising from the indexing category of the pro-simplicial set $(X_K)_{\text{ét}}$ and the natural numbers $n > 0$.

Proof. As discussed in [26; §6], there is a natural bicontinuous algebraic morphism $\mathrm{SP}^d(X) \rightarrow C_{0,d}(X, j)$: functions on $C_{0,d}(X, j)$ are those symmetric functions expressible in terms of ‘fundamental symmetric forms.’ Moreover, $\mathrm{SP}^d(X_K)$ is connected for any $d > 0$, so that we may choose ζ_n to be addition by $\deg(\zeta_n) \cdot x_0$. Hence, these natural morphisms $\mathrm{SP}^d(X) \rightarrow C_{0,d}(X, j)$ determine a $\mathrm{Gal}(K/F)$ -equivariant homotopy equivalence

$$\mathbb{Z} \times \mathrm{colim}_d |\mathrm{SP}^d(X_K)_{\mathrm{et}}| \rightarrow \mathrm{Tel}(\mathcal{C}_0((X_K)_{\mathrm{et}}), \zeta_n) \quad (4.3.1)$$

By [4; 5.5.21],

$$H^*(\mathrm{SP}^d(X_K)_{\mathrm{et}}, \mathbb{Z}/l) = \Gamma_d(H^*((X_K)_{\mathrm{et}}, \mathbb{Z}/l))$$

where $\Gamma_d(-)$ is the construction of mod- l cohomology which satisfies $\Gamma_d(H^*(T, \mathbb{Z}/l)) = H^*(\mathrm{SP}^d T, \mathbb{Z}/l)$ for any finite connected simplicial set T . We recall that $\Gamma_d(-)$ applied to a finite \mathbb{Z}/l vector space W in a given cohomology degree t consists of ‘constructions with length satisfying constraints depending upon d ’ among those ‘constructions’ which give $H^*(K(W, t), \mathbb{Z}/l)$ where $K(W, t)$ is the Eilenberg-MacLane space with $\pi_t(K(W, t)) = W$ (cf. [22]). Let $\{T_\alpha\}$ be an inverse system in the homotopy category of connected simplicial sets finite in each dimension weakly equivalent to $(X_K)_{\mathrm{et}}$ (cf. [8; 7.2]). Then there are natural isomorphisms

$$\begin{aligned} H^*(\mathrm{SP}^d(X_K)_{\mathrm{et}}, \mathbb{Z}/l) &\approx \mathrm{colim}_\alpha H^*(\mathrm{SP}^d(T_\alpha), \mathbb{Z}/l) \\ &\approx H^*(\mathrm{SP}^d((X_K)_{\mathrm{et}}), \mathbb{Z}/l) \end{aligned} \quad (4.3.2)$$

Observe that if $U \rightarrow X_K$ is étale and surjective, then $\mathrm{SP}^d(U) \rightarrow \mathrm{SP}^d(X_K)$ is also étale and surjective as can be seen by identifying the geometric fibres of this map. Hence, the natural maps

$$\pi_0 \mathrm{Nerve}(\mathrm{SP}^d(U)/\mathrm{SP}^d(X_K)) \rightarrow \mathrm{SP}^d(\pi_0 \mathrm{Nerve}(U/X_K))$$

determine a natural (in particular, $\mathrm{Gal}(K/F)$ -equivariant) map $\mathrm{SP}^d(X_K)_{\mathrm{et}} \rightarrow \mathrm{SP}^d((X_K)_{\mathrm{et}})$. As argued in the proof of Proposition 2.1, (4.3.2) implies that this map induces a homotopy equivalence

$$|\mathrm{SP}^d(X_K)_{\mathrm{et}}| \approx \mathrm{holim}(\mathbb{Z}/l)_\infty \mathrm{SP}^d((X_K)_{\mathrm{et}}) \equiv |\mathrm{SP}^d((X_K)_{\mathrm{et}})|.$$

Since the homological connectivity of $\mathrm{SP}^d(T_\alpha) \rightarrow \mathrm{SP}^\infty(T_\alpha)$ goes to infinity at least as fast as a function $c(d)$ (approximately equal to $2d$) which satisfies $\mathrm{colim}_d c(d) = \infty$ [22] and since $\mathrm{SP}^\infty(T_\alpha)$ is a generalized Eilenberg-MacLane space for any α , we conclude that the action of $\pi_1(|\mathrm{SP}^d((X_K)_{\mathrm{et}})|)$ on

$\pi_i(|\mathrm{SP}^d((X_K)_{\mathrm{et}})|)$ is trivial for all $i < c(d)$. Consequently, $\mathrm{colim}_d |\mathrm{SP}^d((X_K)_{\mathrm{et}})|$ is a simple space, so that (4.3.1) implies that $\mathrm{Tel}(\mathcal{C}_0((X_K)_{\mathrm{et}}), \zeta_n)$ is also simple. Thus, the homology equivalence

$$\tilde{r}: \mathrm{Tel}(\mathcal{C}_0((X_K)_{\mathrm{et}}), \zeta_n) \rightarrow \mathcal{C}_0((X_K)_{\mathrm{et}})^+$$

is a homotopy equivalence.

By [8; 6.10],

$$\pi_i(|\mathrm{SP}^d((X_K)_{\mathrm{et}})|) \approx \lim(\pi_i(\mathrm{SP}^d(T_\alpha)_l)^\wedge),$$

where $(-)_l^\wedge$ denotes the l -adic completion functor on abelian groups. Hence, for $0 < i < c(d)$, the Dold-Thom theorem applied to each T_α determines $\mathrm{Gal}(K/F)$ -equivariant isomorphisms

$$\begin{aligned} \pi_i(\mathcal{C}_0((X_K)_{\mathrm{et}})^+) &\approx \pi_i(|\mathrm{SP}^d((X_K)_{\mathrm{et}})|) \\ &\approx \lim H_i(T_\alpha)_l^\wedge = \lim_\alpha \lim_n H_i(T_\alpha, \mathbb{Z}/l^n) \\ &\approx \lim H_i((X_K)_{\mathrm{et}}, \mathbb{Z}/l^n). \end{aligned} \quad \square$$

Using the fact that \mathbb{P}_K^N is the r -fold algebraic suspension of \mathbb{P}_K^{N-r} for any $r \leq N$, we easily conclude the following corollary of Theorems 4.2 and 4.3.

4.4. COROLLARY. *Let K be an algebraically closed field. Then*

$$L_r H_{2r+i}(\mathbb{P}_K^N, \mathbb{Z}_l) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_l & i = 2, 4, \dots, 2N - 2r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorems 4.2, $L_r H_{2r+i}(\mathbb{P}_K^N, \mathbb{Z}_l) \approx L_0 H_i(\mathbb{P}_K^{N-r}, \mathbb{Z}_l)$. By Theorem 4.3, the latter is isomorphic to \mathbb{Z} for $i = 0$ and is isomorphic to $\lim_n H_i((\mathbb{P}_K^{N-r})_{\mathrm{et}}, \mathbb{Z}/l^n)$ for $i > 0$. These groups are well known to be as asserted in the statement of the corollary (cf. [8; 8.7]). \square

Let $\mathrm{Pic}(T)$ denote the group of isomorphism classes of invertible sheaves on a scheme T . If X is a smooth, projective variety of dimension m over a field F with an F -rational point, then the functor sending an F -scheme S to $\mathrm{Pic}(X \times S)/\mathrm{Pic}(S)$ is representable by a complete scheme Pic_X provided with a universal invertible sheaf \mathcal{L} on $\mathrm{Pic}_X \times X$. Moreover, the reduced scheme $(\mathrm{Pic}_X)_{\mathrm{red}}$ has connected component $(\mathrm{Pic}_X^0)_{\mathrm{red}}$ which is an abelian variety (cf. [12]). The group $\pi_0(\mathrm{Pic}_X)$ is the group completion A^+ of the abelian monoid $A = \pi_0(\mathcal{C}_{m-1}(X, j))$, where the natural map $\pi_0(\mathcal{C}_{m-1}(X, j)) \rightarrow \pi_0(\mathrm{Pic}_X)$ sends the algebraic equivalence class of an effective divisor D to the algebraic equivalence class of the associated line bundle $L(D)$.

4.5. PROPOSITION. *Let X be a projective, smooth variety of dimension m provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ and an F -rational point. Write $\mathcal{C}_{m-1}(X, j) = \bigcup_{\alpha \in A} C_\alpha$, and for each $\alpha \in A$ let $\text{Pic}_{\alpha, X}$ denote the corresponding component of $(\text{Pic}_X)_{\text{red}}$ with universal invertible sheaf \mathcal{L}_α on $\text{Pic}_{\alpha, X} \times X$. Let $\pi_\alpha: \text{Pic}_{\alpha, X} \times X \rightarrow \text{Pic}_{\alpha, X}$ and $\tau_\alpha: \mathbb{P}(\alpha) = \text{Proj}(\pi_{\alpha*} \mathcal{L}_\alpha) \rightarrow \text{Pic}_{\alpha, X}$ denote the projections, and let $\eta \in A$ denote the component of a hyperplane section. Then for each α there exists an integer n_α such that if $n > n_\alpha$ and $\beta = \alpha + n\eta$*

- (a) $\tau_\beta: \mathbb{P}(\beta) \rightarrow \text{Pic}_{\beta, X}$ is a projective bundle with fibres $\mathbb{P}^{N(\beta)}$, $N(\beta) = \dim \Gamma(X_{k(z)}, L_z) - 1$, where L_z represents some $z \in \text{Pic}_{\beta, X}$.
- (b) there exists a natural bicontinuous algebraic morphism $\phi_\beta: \mathbb{P}(\beta) \rightarrow C_\beta$.

Proof. For each $y \in \text{Pic}_{\alpha, X}$ there exists an $n_y > 0$ such that $H^i(X_{k(y)}, L_y(n)) = 0$ for all $i > 0$ whenever $n > n(y)$, where L_y represents y and $L_y(n) = L_y \otimes \mathcal{O}_X(n)$. Observe that $L_y(n)$ is the restriction of $\mathcal{L}_\alpha \otimes \text{pr}_1^*(\mathcal{O}_X(n))$ on $X \times \text{Pic}_{\alpha, X}$. By the upper semi-continuity of (Zariski) sheaf cohomology for the projective morphism $\text{pr}_2: X \times \text{Pic}_{\alpha, X} \rightarrow \text{Pic}_{\alpha, X}$ and the coherent sheaf $\mathcal{L}_\alpha \otimes \text{pr}_1^*(\mathcal{O}_X(n))$ flat over $\text{Pic}_{\alpha, X}$ [16; III.12.8], for each $i > 0$ there is some neighborhood of $y \in \text{Pic}_{\alpha, X}$ such that $H^i(X_{k(y')}, L_{y'}(n)) = 0$ for all y' in this neighborhood. Using the finite dimensionality and compactness of X , we conclude the existence of some $n(\alpha)$ such that $H^i(X_{k(y)}, L_y(n)) = 0$ for all $i > 0$ and all $y \in \text{Pic}_{\alpha, X}$ whenever $n > n(\alpha)$. Because the Hilbert polynomial of L_y is independent of $y \in \text{Pic}_{\alpha, X}$ [16; III.9.9], we conclude that $\dim(H^0(X_{k(y)}, L_y(n)))$ is independent of $y \in \text{Pic}_{\alpha, X}$ whenever $n > n(\alpha)$. Since the pull-back of \mathcal{L}_β on $\text{Pic}_{\beta, X} \times X$ via the translation isomorphism $\text{Pic}_{\alpha, X} \rightarrow \text{Pic}_{\beta, X}$ is $\mathcal{L}_\alpha \otimes \text{pr}_1^*(\mathcal{O}_X(n))$, [24; §5, Cor. 2] implies that $\pi_* \mathcal{L}_\beta$ is locally free on $\text{Pic}_{\beta, X}$ with fibre above $z \in \text{Pic}_{\beta, X}$ equal to $\Gamma(X_{k(z)}, L_z)$. This implies (a).

To construct $\phi_\beta: \mathbb{P}(\beta) \rightarrow C_\beta$, it suffices by Theorem 1.4 to exhibit an effective $\mathbb{P}(\beta)$ -relative $(m - 1)$ -cycle on $\mathbb{P}(\beta) \times \mathbb{P}^N$ with support on $j(X)$. For each open subset $U \subset \text{Pic}_{\beta, X}$ restricted to which $\pi_* \mathcal{L}_\beta$ is trivial, the invertible sheaf $(\tau_\beta \times 1)^* \mathcal{L}_\beta$ on $\mathbb{P}(\beta) \times X$ has the property that the restriction of $(\tau_\beta \times 1)^* \mathcal{L}_\beta$ to $(\mathbb{P}(\beta) \times X) \times_{\text{Pic}_{\beta, X}} U$ has a section canonically determined up to scalar multiple. The zero loci of these local sections of $(\tau_\beta \times 1)^* \mathcal{L}_\beta$ patch together to determine the natural $\mathbb{P}(\beta)$ -relative $(m - 1)$ -cycle $D(\beta)$ on $\mathbb{P}(\beta) \times \mathbb{P}^N$ with support on $j(X)$.

Since both $\mathbb{P}(\beta)$ and C_β are projective, ϕ_β is proper. If $f: C \rightarrow \mathbb{P}(\beta)_K$ is a map with domain a smooth curve over K , then the pull-back $f^{-1}(D(\beta))$ via f is the zero locus of the distinguished local sections of the pull-back sheaf $(f \circ \tau_\beta \times 1)^* \mathcal{L}_\beta$ on $C \times X$. Thus, $D(\beta)_t = [D(\beta)]_t$ for any geometric point t of $\mathbb{P}(\beta)_K$. Consequently, we can verify by inspection that ϕ_β is bijective on geometric points: for any point $t \in \mathbb{P}(\beta)_K$ mapping to $y \in \text{Pic}_{\beta, X_K}$, $D(\beta)_t$ is the zero locus of $s_t \in \Gamma(X, L_y)$ (where s_t is determined up to scalar multiple by $t \in \tau_\beta^{-1}(y) \in \text{Proj}(\Gamma(X, L_y))$). Finally, ϕ_β is birational: by Proposition 1.2, the

generic divisor $D_\beta \in C_\beta$ (whose field of definition is the field of fractions of $\mathbb{P}(\beta)$) is defined over the residue field of its Chow point, the field of fractions of C_β . \square

Using Proposition 4.5, we identify the Lawson homology in codimension 1 of a smooth projective variety. We recall that the Neron-Severi group $\text{NS}(X)$ of a projective variety X is the group of divisors modulo algebraic equivalence.

4.6. THEOREM. *Let X be a smooth, projective algebraic variety of dimension m provided with an embedding $j: X \rightarrow \mathbb{P}_F^N$ and let K be the algebraic closure of F . Then the homology equivalence of (2.5)*

$$\tilde{r}: \text{Tel}(\mathcal{C}_{m-1}((X_K)_{\text{et}}), \zeta_n) \rightarrow \mathcal{C}_{m-1}((X_K)_{\text{et}})^+$$

is a homotopy equivalence. Moreover, there are $\text{Gal}(K/F)$ -natural identifications

$$L_{m-1}H_{2m-2+i}(X_K, \mathbb{Z}_l) = \begin{cases} \text{NS}(X_K) & i = 0 \\ \lim_{\mathbb{Z}_l} H_1((\text{Pic}_{X_K}^0)_{\text{et}}, \mathbb{Z}/l^n) & i = 1 \\ \mathbb{Z}_l & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $F = \mathbb{C}$, then

$$L_{m-1}H_{2m-2+i}(X^{\text{an}}) = \begin{cases} \text{NS}(X) & i = 0 \\ H_1((\text{Pic}_X^0)^{\text{an}}) & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The determination of $L_{m-1}H_{2m-2}(X_K, \mathbb{Z}_l)$ and $L_{m-1}H_{2m-2}(X^{\text{an}})$ is achieved in Proposition 2.4.

Let $\beta = \alpha + n\eta$ be as in the conclusion of (4.5). For any geometric point $\text{Spec } K \rightarrow \text{Pic}_{\beta, X}$, the natural map $(\mathbb{P}^{N(\beta)})_{\text{et}} \rightarrow \text{fib}((\tau_\beta)_{\text{et}})$ induces an isomorphism in \mathbb{Z}/l cohomology [8; 10.8]. This implies that the induced map of Artin-Mazur l -adic completions

$$(\mathbb{P}_K^{N(\beta)})_{\text{et}}^\wedge \rightarrow \text{fib}((\tau_\beta)_{\text{et}})^\wedge$$

is a weak homotopy equivalence [2; 4.3]. Since τ_β is a projective bundle, the monodromy action of $\pi_1(\text{Pic}_{X_K}^0)$ on $H^*((\mathbb{P}_K^{N(\beta)})_{\text{et}}, \mathbb{Z}/l^n)$ is trivial. This implies that the natural map

$$\text{fib}((\tau_\beta)_{\text{et}})^\wedge \rightarrow \text{fib}((\tau_\beta)_{\text{et}})$$

is also a weak homotopy equivalence [2; 5.9; 4.11]. Applying [8; 6.10] to $(\tau_\beta)_{\text{et}}^\wedge$

and replacing $|\mathbb{P}(\beta)_{\text{et}}|$ by $|(C_\beta)_{\text{et}}|$ in view of the bicontinuous algebraic morphism $\phi_\beta: \mathbb{P}(\beta) \rightarrow C_\beta$ of Proposition 4.5, we conclude that the triple

$$|(\mathbb{P}_K^{N(\beta)})_{\text{et}}| \rightarrow |(C_\beta)_{\text{et}}| \rightarrow |(\text{Pic}_{\beta, X_K})_{\text{et}}| \tag{4.6.1}$$

is a fibration sequence.

Since translation determines isomorphisms $\text{Pic}_{\beta, X_K} \approx (\text{Pic}_{X_K}^0)_{\text{red}}$ and since $(\text{Pic}_{X_K}^0)_{\text{red}}$ is an abelian variety, $|(\text{Pic}_{\beta, X_K})_{\text{et}}|$ is a simple space. By the comparison theorems in étale cohomology (cf. [8; 8.7] which provides an explicit comparison on the level of homotopy types), $|(\mathbb{P}_K^{N(\beta)})_{\text{et}}|$ has one non-vanishing homotopy group in degrees $\leq 2N(\beta)$ which is $\pi_2(|\mathbb{P}_K^{N(\beta)})_{\text{et}}| = \mathbb{Z}_l$. Hence, the triviality of the monodromy action of $\pi_1(\text{Pic}_{\beta, X_K})$ on $H^2(|(\mathbb{P}_K^{N(\beta)})_{\text{et}}|, \mathbb{Z}/l^n)$ implies that $|(C_\beta)_{\text{et}}|$ is $2N(\beta)$ -simple. We conclude that each component of $\text{Tel}(\mathcal{C}_{m-1}(X_K)_{\text{et}}, \zeta_n)$ is homotopy equivalent to the simple space obtained as an infinite telescope of such $|(C_\beta)_{\text{et}}|$, so that i^\sim is a homotopy equivalence.

The liftability to characteristic 0 of abelian varieties in positive characteristics [27] implies that $\text{Pic}_{X_K}^0$ is liftable to an abelian variety over a field of characteristic 0 whose analytic type is that of a complex torus. Hence, the comparison theorem [8; 8.7] implies that $|(\text{Pic}_{\beta, X_K})_{\text{et}}|$ has one non-vanishing homotopy group which is $\pi_1(|\text{Pic}_{X_K}^0)_{\text{et}}| = \lim H_1((\text{Pic}_{X_K}^0)_{\text{et}}, \mathbb{Z}/l^n)$. If $\alpha' > \alpha$, $n > n_{\alpha'}$, $n > n_\alpha$, $\beta' = \alpha' + n\eta$, and $\beta = \alpha + n\eta$, then $\langle Z_{\alpha'-\alpha} \rangle + (-): \mathbb{P}(\beta) \rightarrow \mathbb{P}(\beta')$ covers the translation isomorphism $\text{Pic}_{\beta, X_K} \rightarrow \text{Pic}_{\beta', X_K}$ and restricts to a linear embedding $\mathbb{P}_K^{N(\beta)} \rightarrow \mathbb{P}_K^{N(\beta')}$ on fibres. We now apply the long exact sequence in homotopy for the colimit of the fibration sequences in (4.6.1) together with the fact that $(\text{Pic}_{X_K}^0)_{\text{red}} \rightarrow \text{Pic}_{X_K}^0$ induces an equivalence of étale topologies [23; I.3.23] to conclude the asserted computation for $L_{m-1}H_{2m-2+i}(X_K, \mathbb{Z}_l)$, in view of the identifications

$$\text{colim}_\beta \pi_2(|(\mathbb{P}_K^{N(\beta)})_{\text{et}}|) = \mathbb{Z}_l, \quad \text{colim}_\beta \pi_i(|(\mathbb{P}_K^{N(\beta)})_{\text{et}}|) = 0 \quad \text{for } i \neq 2.$$

Finally, if $K = \mathbb{C}$, Pic_X is reduced. Proposition 4.5 implies that

$$(\mathbb{P}_K^{N(\beta)})^{\text{an}} \rightarrow (C_\beta)^{\text{an}} \rightarrow (\text{Pic}_{\beta, X})^{\text{an}} \tag{4.6.2}$$

is a fibration sequence. Recall that i^\sim is a homotopy equivalence in the analytic context by Corollary 2.6. Hence, upon replacing (4.6.1) by (4.6.2), the above discussion applies to prove the asserted computation for $L_{m-1}H_{2m-2+i}(X^{\text{an}})$. □

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