Compositio Mathematica

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Errata to the papers : “Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ” and “Class numbers of appropriate imaginary quadratic fields, parts I and II”

Compositio Mathematica, tome 77, nº 1 (1991), p. 119-125
<http://www.numdam.org/item?id=CM_1991__77_1_119_0>

Compositio Mathematica **77**: 119–125, 1991.

Errata to the papers: Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, Parts I and II
(published in Compositio Mathematica **75**: *247-270 and 271-285, 1990)*

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Due to a printer's error, 74 pairs of parentheses in Kronecker symbols were omitted, which completely changed the sense of the theorems of both papers. Here are their correct versions.

Part I:

THEOREM 1. *Let for* $k = 0, 1, 2$ *and* 3

$$s_k = \sum_{l \in [kD/8, (k+1)D/8)} \left(\frac{D}{l}\right) l.$$

Then for $D \neq 5$:

(i) $k_2(D) = \frac{16}{45}\left(2\left(\frac{D}{2}\right) - 7\right)(s_0 + s_1) - \frac{2}{45}\left(2\left(\frac{D}{2}\right) - 7\right) Dh(-4D)$,

(ii) $$k_2(D) = -\frac{32}{75}\left(\left(\frac{D}{2}\right) + 4\right)(s_0 + s_2)$$
$$+ \frac{2}{75}\left(\left(\frac{D}{2}\right) + 4\right) D\left(-\left(\left(\frac{D}{2}\right) + 2\right) h(-4D) + 2h(-8D)\right),$$

(iii) $k_2(8D) = -32(s_1 + s_2) - 2D\left(2\left(\frac{D}{2}\right) h(-4D) - h(-8D)\right)$,

(iv) $k_2(8D) + \left(\left(\frac{D}{2}\right) - 34\right) k_2(D) = 64 s_0 - 2D\left(\left(\frac{D}{2}\right) h(-4D) + h(-8D)\right)$,

$$k_2(8D) + 3\left(3\left(\frac{D}{2}\right) - 2\right) k_2(D)$$

$$= -64s_1 - 2D\left(\left(\left(\frac{D}{2}\right) - 4\right)h(-4D) + h(-8D)\right),$$

$$k_2(8D) - 3\left(3\left(\frac{D}{2}\right) - 2\right)k_2(D)$$

$$= -64s_2 - 2D\left(\left(3\left(\frac{D}{2}\right) + 4\right)h(-4D) - 3h(-8D)\right),$$

$$k_2(8D) + 15\left(\left(\frac{D}{2}\right) - 2\right)k_2(D) = 64s_3 - 6D\left(\left(\frac{D}{2}\right)h(-4D) - h(-8D)\right).$$

COROLLARY 1. *Let φ denote Euler's totient function.*

(i) $k_2(D) \equiv 2h(-4D) + 2\varphi(D) + \varepsilon \pmod{32}$,

where $\varepsilon = 0$ unless $D = p \equiv -3 \pmod 8$ a prime or $D = pq$, where $p \equiv q \not\equiv 1 \pmod 8$ or $p \equiv q + 4 \equiv 3 \pmod 8$, p, q-primes. In these cases $\varepsilon = 16$ if $p \equiv q \equiv -3 \pmod 8$, $\varepsilon = -8$ if $p \equiv q \equiv -1 \pmod 8$ and $\varepsilon = 8$ otherwise.

(ii) $k_2(D) \equiv 6h(-4D) - 4\left(2 - \left(\frac{D}{2}\right)\right)h(-8D) \pmod{32}$,

(iii) $k_2(D) \equiv -2\left(2 - \left(\frac{D}{2}\right)\right)\left(2h(-4D) - \left(\frac{D}{2}\right)h(-8D)\right) \pmod{32}$,

(iv) $k_2(8D) + \left(\left(\frac{D}{2}\right) - 34\right)k_2(D)$

$$\equiv -2\left(2\left(\frac{D}{2}\right) - 1\right)\left(\left(\frac{D}{2}\right)h(-4D) + h(-8D)\right) \pmod{64},$$

$$k_2(8D) + 3\left(3\left(\frac{D}{2}\right) - 2\right)k_2(D)$$

$$\equiv -2\left(2\left(\frac{D}{2}\right) - 1\right)\left(\left(\left(\frac{D}{2}\right) - 4\right)h(-4D) + h(-8D)\right) \pmod{64},$$

$$k_2(8D) - 3\left(3\left(\frac{D}{2}\right) - 2\right)k_2(D)$$

$$\equiv -2\left(2\left(\frac{D}{2}\right) - 1\right)\left(\left(3\left(\frac{D}{2}\right) + 4\right)h(-4D) - 3h(-8D)\right) \pmod{64},$$

$$k_2(8D) + 15\left(\left(\frac{D}{2}\right) - 2\right)k_2(D)$$

$$\equiv -6\left(2\left(\frac{D}{2}\right) - 1\right)\left(\left(\frac{D}{2}\right)h(-4D) - h(-8D)\right) \pmod{64}.$$

(v) *If $D = p = 8t + 1$ or $8t - 3$ a prime then*:

$k_2(D) \equiv 2h(-4D) + 16t \pmod{32}$,

$$k_2(D) \equiv 32\alpha + 2\beta\left(-\left(2 + \left(\frac{D}{2}\right)\right)h(-4D) + 2h(-8D)\right)(\text{mod } 64),$$

where $\alpha = 1$ *if* $p \equiv -3(\text{mod } 16)$ *and* $\alpha = 0$ *otherwise, and* $\beta = -1, -3$, *resp.* 5 *if* $p \equiv 1(\text{mod } 8)$, $p \equiv 5(\text{mod } 16)$, *resp.* $p \equiv -3(\text{mod } 16)$,

$$k_2(8D) \equiv 32\alpha + 2\beta\left(2\left(\frac{D}{2}\right)h(-4D) - h(-8D)\right)(\text{mod } 64),$$

where $\alpha = 0$ *if* $p \equiv 1(\text{mod } 16)$ *and* $\alpha = 1$ *otherwise, and* $\beta = -1, -3$, *resp.* 5 *if* $p \equiv 1(\text{mod } 8)$, $p \equiv -3(\text{mod } 16)$, *resp.* $p \equiv 5(\text{mod } 16)$.

THEOREM 2. *Let for* $k = 0, 1, 2$ *and* 3

$$s_k = \sum_{l \in [k\Delta/8, (k+1)\Delta/8)} \left(\frac{-\Delta}{l}\right) l.$$

Then for $\Delta \neq 3$:

(i) $k_2(4\Delta) = 16(s_0 + s_1) - 2\Delta\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta)$ (*see* [5], *too*),

(ii) $$k_2(4\Delta) = 32\left(\frac{-\Delta}{2}\right)(s_0 + s_3) + 2\Delta\left(\frac{-\Delta}{2}\right)\left(7\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + 2h(-8\Delta)\right),$$

(iii) $$k_2(8\Delta) = 32(s_0 - s_3) - 2\Delta\left(6\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + h(-8\Delta)\right),$$

(iv) $$k_2(8\Delta) + \left(\frac{-\Delta}{2}\right)k_2(4\Delta) = 64s_0 + 2\Delta\left(\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + h(-8\Delta)\right),$$

$$k_2(8\Delta) + \left(\left(\frac{-\Delta}{2}\right) - 4\right)k_2(4\Delta) = -64s_1 + 2\Delta\left(5\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + h(-8\Delta)\right),$$

$$k_2(8\Delta) - \left(\left(\frac{-\Delta}{2}\right) + 4\right)k_2(4\Delta) = 64s_2 + 2\Delta\left(7\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) - 3h(-8\Delta)\right),$$

$$k_2(8\Delta) - \left(\frac{-\Delta}{2}\right)k_2(4\Delta) = -64s_3 - 2\Delta\left(13\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + 3h(-8\Delta)\right).$$

COROLLARY 1.

(i) $k_2(4\Delta) \equiv -6h(-\Delta)\left(\left(\frac{-\Delta}{2}\right) - 1\right) + 2\varphi(\Delta) + \varepsilon (\bmod 32),$

where $\varepsilon = 0$ *unless* $\Delta = p \equiv 3 (\bmod 4)$ *a prime or* $\Delta = pq$, *where* $p \equiv q + 2 \equiv -1 (\bmod 8)$, *p, q-primes, or* $\Delta = pqr$, *where* $p \equiv q \equiv r \equiv -1, 3 (\bmod 8)$, *or* $p \equiv q \equiv -1$, *resp.* $3 (\bmod 8)$ *and* $r \equiv 3$, *resp.* $-1 (\bmod 8)$, *p, q, r-primes.*

In these cases $\varepsilon = 4$ *if* $\Delta = p \equiv -1 (\bmod 8)$, $\varepsilon = -4$ *if* $\Delta = p \equiv 3 (\bmod 8)$ *and* $\varepsilon = 16$ *otherwise.*

(ii) $k_2(4\Delta) \equiv 6\left(\frac{-\Delta}{2}\right)\left(7\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + 2h(-8\Delta)\right)(\bmod 32),$

$k_2(4\Delta) \equiv -4h(-8\Delta))(\bmod 32),$ *if* $\Delta \equiv -1 (\bmod 8)$, *in particular,*

(iii) $k_2(8\Delta) \equiv 2\left(1 - \left(\frac{-\Delta}{2}\right)\right)\left(6\left(1 - \left(\frac{-\Delta}{2}\right)\right)h(-\Delta) - h(-8\Delta)\right)(\bmod 32),$

$k_2(8\Delta) \equiv 2h(-8\Delta))(\bmod 32),$ *if* $\Delta \equiv -1 (\bmod 8)$, *in particular,*

(iv) $k_2(8\Delta) + \left(\frac{-\Delta}{2}\right)k_2(4\Delta)$

$$\equiv -2\left(2\left(\frac{-\Delta}{2}\right) - 1\right)\left(\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + h(-8\Delta)\right)(\bmod 64),$$

$k_2(8\Delta) + \left(\left(\frac{-\Delta}{2}\right) - 4\right)k_2(4\Delta)$

$$\equiv -2\left(2\left(\frac{-\Delta}{2}\right) - 1\right)\left(5\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + h(-8\Delta)\right)(\bmod 64),$$

$k_2(8\Delta) - \left(\left(\frac{-\Delta}{2}\right) + 4\right)k_2(4\Delta)$

$$\equiv -2\left(2\left(\frac{-\Delta}{2}\right) - 1\right)\left(7\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) - 3h(-8\Delta)\right)(\bmod 64),$$

$k_2(8\Delta) - \left(\frac{-\Delta}{2}\right)k_2(4\Delta)$

$$\equiv 2\left(2\left(\frac{-\Delta}{2}\right) - 1\right)\left(13\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + 3h(-8\Delta)\right)(\bmod 64).$$

(v) *If* $\Delta = p = 8t - 1$ *or* $8t + 3$ *a prime then*:

$$k_2(4\Delta) \equiv -6h(-\Delta)\left(\left(\frac{-\Delta}{2}\right) - 1\right) + 16t \pmod{32},$$

$$k_2(4\Delta) \equiv 32\alpha + 2\beta\left(\frac{-\Delta}{2}\right)\left(7\left(\left(\frac{-\Delta}{2}\right) - 1\right)h(-\Delta) + 2h(-8\Delta)\right) \pmod{64},$$

$$k_2(8\Delta) \equiv 32\alpha + 2\beta\left(13\left(1 - \left(\frac{-\Delta}{2}\right)\right)h(-\Delta) + h(-8D)\right) \pmod{64},$$

where $\alpha = 1$ *if* $p \equiv 7 \pmod{16}$ *and* $\alpha = 0$ *otherwise, and* $\beta = -1, 3$, *resp.* 11 *if* $p \equiv -1 \pmod 8$, $p \equiv 3 \pmod{16}$, *resp.* $p \equiv 11 \pmod{16}$.

Part II:

LEMMA 1 ([5], [1]). *We have*:

$$T_1 = \begin{cases} \frac{1}{4}\left(\frac{e}{2}\right)h(-4e) + \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{1}{4}\left(5 - \left(\frac{e}{2}\right)\right)h(e) - \frac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_2 = \begin{cases} \frac{1}{4}\left(2 - \left(\frac{e}{2}\right)\right)h(-4e) - \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{1}{4}\left(-1 + \left(\frac{e}{2}\right)\right)h(e) + \frac{1}{4}h(8e) + \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_3 = \begin{cases} \frac{1}{4}\left(-2 - \left(\frac{e}{2}\right)\right)h(-4e) + \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{3}{4}\left(1 - \left(\frac{e}{2}\right)\right)h(e) + \frac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_4 = \begin{cases} \frac{1}{4}\left(\frac{e}{2}\right)h(-4e) - \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{3}{4}\left(1 - \left(\frac{e}{2}\right)\right)h(e) - \frac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

where $\lambda(e) = 1$, *if* $e = -3$, *and* $\lambda(e) = 0$, *otherwise.*

Moreover we have for $k = 5, 6, 7, 8$

$$T_k = \left(\frac{e}{-1}\right)T_{9-k}.$$

LEMMA 2 ([8]). *We have*:

$$S_1 = \begin{cases} \frac{1}{64}k_2(8e) - \frac{1}{64}\left(34 - \left(\frac{e}{2}\right)\right)k_2(e) \\ \quad + \frac{1}{32}e\left(\left(\frac{e}{2}\right)h(-4e) + h(-8e)\right) + 7\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) + \frac{1}{64}\left(\frac{e}{2}\right)k_2(-4e) \\ \quad - \frac{1}{32}e\left(\left(1 - \left(\frac{e}{2}\right)\right)h(e) - h(8e)\right) - v(e), & \text{if } e < 0, \end{cases}$$

$$S_2 = \begin{cases} -\frac{1}{64}k_2(8e) + \frac{3}{64}\left(2 - 3\left(\frac{e}{2}\right)\right)k_2(e) \\ \quad + \frac{1}{32}e\left(\left(4 - \left(\frac{e}{2}\right)\right)h(-4e) - h(-8e)\right) - 3\omega(e), & \text{if } e > 0, \\ -\frac{1}{64}k_2(-8e) + \frac{1}{64}\left(4 - \left(\frac{e}{2}\right)\right)k_2(-4e) \\ \quad - \frac{1}{32}e\left(5\left(-1 + \left(\frac{e}{2}\right)\right)h(e) + h(8e)\right) + 5v(e), & \text{if } e < 0, \end{cases}$$

$$S_3 = \begin{cases} -\frac{1}{64}k_2(8e) - \frac{3}{64}\left(2 - 3\left(\frac{e}{2}\right)\right)k_2(e) \\ \quad + \frac{1}{32}e\left(\left(-4 - \left(\frac{e}{2}\right)\right)h(-4e) + 3h(-8e)\right) + 3\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) - \frac{1}{64}\left(4 + \left(\frac{e}{2}\right)\right)k_2(-4e) \\ \quad - \frac{1}{32}e\left(7\left(1 - \left(\frac{e}{2}\right)\right)h(e) + 3h(8e)\right) - 7v(e), & \text{if } e < 0, \end{cases}$$

$$S_4 = \begin{cases} \frac{1}{64}k_2(8e) - \frac{15}{64}\left(2 - \left(\frac{e}{2}\right)\right)k_2(e) \\ \quad + \frac{1}{32}e\left(3\left(\frac{e}{2}\right)h(-4e) - 3h(-8e)\right) + 9\omega(e), & \text{if } e > 0, \\ -\frac{1}{64}k_2(-8e) + \frac{1}{64}\left(\frac{e}{2}\right)k_2(-4e) \\ \quad - \frac{1}{32}e\left(13\left(1 - \left(\frac{e}{2}\right)\right)h(e) - 3h(8e)\right) - 13v(e), & \text{if } e < 0, \end{cases}$$

where $\omega(e) = \frac{1}{4}$, *if* $e = 5$, $\omega(e) = 0$, *otherwise, and* $\nu(e) = \frac{1}{8}$, *if* $e = -3$, $\nu(e) = 0$, *otherwise.*

Moreover we have for $k = 5, 6, 7, 8$

$$S_k = eT_{9-k} - \left(\frac{e}{-1}\right) S_{9-k}.$$