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C. J. ATKIN

J. GRABOWSKI

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Homomorphisms of the Lie algebras associated with a symplectic manifold

C. J. ATKIN and J. GRABOWSKI
[1]*Department of Mathematics, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand,* [2]*J. Grabowski, Instytut Matematyki U.W., P.K. i N. IXp. 00-901 Warszawa, Poland*

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1. Introduction

We have each given (in [1] and [7]) proofs of an algebraic result – restated for our present purposes as (6.2) below – which in effect (see (6.5)) constructs a one-one correspondence between the points of a symplectic manifold and certain subalgebras of its Lie algebra of Poisson brackets. Our aim here is, firstly, to extend this result to the Lie algebras of locally, globally, and conformally Hamiltonian vector fields determined by the symplectic structure; and then to utilise it to prove that each of these algebras determines the manifold, as far as that is possible. In fact, we approach these 'uniqueness theorems' by studying certain types of Lie homomorphism (which we classify in Section 7) between such Lie algebras, rather than by reconstructing the manifold from the algebra; this method (modelled on that in [6]) both gives and requires less structural information, but yields more facts about homomorphisms. Indeed, we have taken no pains to delve more deeply into the structure of our algebras than our techniques demand, and those techniques are perhaps more interesting than the results which motivated them.

In Section 2 we present some definitions and facts not related to symplectic structures; Section 3 introduces the notion of n-ample algebras of vector fields. In Sections 4 and 5 we review some definitions and notations, and give proofs of some properties which will be needed subsequently (and one or two which will not, such as (5.8)). Here we mostly follow, and often refer to, the well-known paper [2]. Although we allow both the real-analytic and the holomorphic (Stein) differentiability classes, the results from [2] which we quote are merely local, and as such hold in these cases without any modification. Then Section 6 gives the algebraic characterisations of the points of a symplectic manifold, Section 7 classifies suitable homomorphisms, and Section 8 considers the application to epimorphisms and isomorphisms.

Special cases of the 'uniqueness theorem' have been proved before (though not, we believe, published). Our method, however, seems to be the first which applies simultaneously to so many cases.

2. Preliminaries on manifolds and on Lie algebras

(2.1) In speaking of an n-dimensional manifold M of class $\mathscr{C}$ over the field F, we shall mean one of three things:

(a) a real paracompact manifold of differentiability class $\mathbf{C}^\infty$ and of real dimension n, when $\mathscr{C}$ denotes $\mathbf{C}^\infty$ and F denotes the real field $\mathbb{R}$;
(b) a real paracompact manifold of differentiability class $\mathbf{C}^\omega$ and of real dimension n (when $\mathscr{C}$ denotes $\mathbf{C}^\omega$ and F denotes $\mathbb{R}$);
(c) a complex manifold of complex dimension n for which each connected component is Stein (when $\mathscr{C}$ denotes the holomorphic differentiability class $\mathscr{H}$ and F denotes the complex field $\mathbb{C}$).

We shall not consider complex manifolds whose components are not Stein, and shall often omit explicit mention of $\mathscr{C}$, F, or n.

(2.2) For each of (a), (b), (c), one has an embedding theorem (due to Whitney [14] in case (a), to Remmert and to Narasimhan [12] in case (c), and to Grauert [8] in case (b)): a connected manifold of class $\mathscr{C}$ and dimension n is $\mathscr{C}$-diffeomorphic to a closed $\mathscr{C}$-submanifold of F^{2n+1}. (Note that by a 'closed submanifold' we understand a 'properly and regularly embedded submanifold'.)

(2.3) For a manifold M of class $\mathscr{C}$, we denote by TM the bundle of tangent vectors (meaning, in case (c), the tangent vectors of type $(1, 0)$), and by T^*M the corresponding cotangent bundle. The vector space over F of exterior forms of class $\mathscr{C}$ on M (again, in case (c), these forms are to be of type $(k, 0)$ and holomorphic) will be called $\Omega^k(M)$. The exterior derivative $\mathrm{d}: \Omega^k(M) \to \Omega^{k+1}(M)$ is defined as usual; we write its kernel as $Z^k(M)$ and its image as $B^{k+1}(M)$, with the convention that $\Omega^k(M) = 0$ when $k < 0$. The chain complex $(\Omega^k(M), \mathrm{d})$ is the de Rham complex of M.

(2.4) LEMMA. *Let M be a manifold of class $\mathscr{C}$; let $p, q \in M$, $p \neq q$, and let k be a nonnegative integer. Given any k-jets (of F-valued functions) at p and q, there exists $f \in \Omega^0(M)$ which has those k-jets.*

Proof. When $M = F^n$, this is trivial. (2.2) then gives it in general.

(2.5) For each of the cases of (2.1), there is a de Rham isomorphism

$$Z^k(M)/B^k(M) \cong H^k(M; F),$$

where the cohomology may conveniently be assumed singular. For case (a), this is de Rham's theorem. For case (c), it is a well-known consequence of Cartan's Theorem B; see, for instance, [4], exposé XX, or p. 80 of [9]. The same argument may be applied in case (b), where Tognoli [13] has pointed out the validity of Theorems A and B in a real-analytic version.

(2.6) PROPOSITION. *Let M be a manifold of class $\mathscr{C}$ and dimension n; take $m = 2n + 1$. Then there exist F-valued functions $x_1, \ldots, x_m$ of class $\mathscr{C}$ on M, such that, for any integer $k \geqq 0$ and any form $\psi \in \Omega^k(M)$, there exist F-valued functions $f_{i_1, i_2, \ldots, i_k}$ of class $\mathscr{C}$ on M (for all $i_1, i_2, \ldots, i_k$ with $1 \leqq i_1 < i_2 < \cdots < i_k \leqq m$), for which*

$$\psi = \sum_{i_1 < i_2 < \cdots < i_k} f_{i_1, i_2, \ldots, i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

Proof. It will clearly suffice to prove the result for each component of M individually. So we may suppose (see (2.2)) that M is $\mathscr{C}$-embedded in F^m, with embedding $j: M \to F^m$. The natural monomorphism $TM \to j^*TF^m$ defined by Tj dualises to epimorphisms π_ℓ: $\Lambda^\ell T^*F^m | j(M) \to \Lambda^\ell T^*M$ for any ℓ. Let $\mathbf{\Omega}^k(M)$, $\mathbf{\Omega}^k(F^m)$ denote the sheaves of germs of k-forms of class $\mathscr{C}$ on M, F^m respectively; then π_k induces a sheaf epimorphism (over M)

$$J: j^*\mathbf{\Omega}^k(F^m) \to \mathbf{\Omega}^k(M).$$

Let $\mathbf{Q} = \ker J$, so that there is an exact sequence

$$0 \to \mathbf{Q} \to j^*\mathbf{\Omega}^k(F^m) \xrightarrow{J} \mathbf{\Omega}^k(M) \to 0. \tag{1}$$

Let $\mathscr{O}_M$ be the structure sheaf over M of germs of C^∞ functions in case (a), C^ω functions in case (b), and holomorphic functions in case (c). Then J is a homomorphism of $\mathscr{O}_M$-modules, so that $\mathbf{Q}$ is also an $\mathscr{O}_M$-module.

Let $y_1, \ldots, y_m$ be the coordinate functions on F^m. Then $\mathbf{\Omega}^k(F^m)$ is free over the appropriate structure sheaf $\mathscr{O}_{F^m}$; indeed, it has free generators given by the sections $\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k}$ for $1 \leqq i_1 < \cdots < i_k \leqq m$. Hence, as $j^*\mathscr{O}_{F^m} = \mathscr{O}_M$ trivially, $j^*\mathbf{\Omega}^k(F^m)$ is free over $\mathscr{O}_M$, and it has free generators $(\mathrm{d}y_i \wedge \cdots \wedge \mathrm{d}y_{i_k})^\sim$ induced from the sections

$$\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k} \quad \text{of } \mathbf{\Omega}^k(F^m). \tag{2}$$

In the $\mathbf{C}^\omega$ and holomorphic cases $\mathbf{\Omega}^k(M)$ is coherent over $\mathscr{O}_M$, and so of course is the free $\mathscr{O}_M$-module $j^*\mathbf{\Omega}^k(F^m)$. By Serre's 3-lemma, then, $\mathbf{Q}$ is also coherent over $\mathscr{O}_M$. From Theorem B (see (2.5)) we know that $H^k(\mathbf{Q}) = 0$ for $k > 1$.

In the $\mathbf{C}^\infty$ case, $\mathscr{O}_M$ is soft (see [3] or [5]) so that $\mathbf{Q}$ is also soft and $H^1(\mathbf{Q}) = 0$.

In all three cases, the cohomology exact sequence of (1)

$$0 \to H^0(\mathbf{Q}) \to H^0(j^*\mathbf{\Omega}^k(F^m)) \xrightarrow{J_*} H^0(\mathbf{\Omega}^k(M)) \to H^1(\mathbf{Q}) \to \cdots$$

leads to the result that J_* is onto. The given form $\psi \in \Omega^k(M)$ is consequently the image J_* of a section ϕ of $j^*\mathbf{\Omega}^k(F^m)$; but, as remarked at (2), $j^*\mathbf{\Omega}^k(F^m)$ is free over $\mathcal{O}_M$, so that there are sections $f_{i_1\ldots i_k}$ of $\mathcal{O}_M$ for which

$$\phi = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\ldots i_k}(\mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_k})^{\sim}.$$

Applying J_* (that is, compounding with J), we obtain after some bookkeeping

$$\psi = \sum_{i_1<i_2<\cdots<i_k} f_{i_1\ldots i_k}\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

where, for each i, $x_i = y_i \circ j$; this is clearly the result.

(2.7) Again let M be a manifold of class $\mathscr{C}$. Then $\Gamma(M)$ will denote the Lie algebra of sections of TM of class $\mathscr{C}$. If $\mathscr{F}$ is a foliation of M of class $\mathscr{C}$, let $\Gamma(\mathscr{F})$ be the Lie subalgebra of $\Gamma(M)$ consisting of vector fields everywhere tangent to $\mathscr{F}$. In general, if K is a vector subspace of $\Gamma(M)$ and $p \in M$, set

$$K(p) = \{X(p)\colon x \in K\} \subseteq T_p(M), \quad \text{and}$$
$$K_p = \{X \in K\colon X(p) = 0\}.$$

(2.8) If $\alpha \in \Omega^k(M)$, let $\Gamma_0(\alpha)$ denote the class of vector fields of class $\mathscr{C}$ which leave α invariant, and let $\Gamma(\alpha)$ be the class of vector fields which operate on α as multiplication by a locally constant function. In other words, if $\mathscr{L}$ denotes the Lie derivative,

$$\Gamma_0(\alpha) = \{X \in \Gamma(M)\colon \mathscr{L}_X\alpha = 0\}, \quad \text{and}$$
$$\Gamma(\alpha) = \{X \in \Gamma(M)\colon (\exists f \in Z^0(M))\mathscr{L}_X\alpha = f\alpha\}.$$

As $\mathscr{L}_{[X,Y]} = [\mathscr{L}_X, \mathscr{L}_Y]$, it follows that

$$[\Gamma(\alpha), \Gamma(\alpha)] \subseteq \Gamma_0(\alpha) \subseteq \Gamma(\alpha),$$

so that $\Gamma(\alpha)$ is a Lie subalgebra of $\Gamma(M)$ and $\Gamma_0(\alpha)$ a Lie ideal in $\Gamma(\alpha)$ (including its commutator).

(2.9) For any Lie algebra L over F, let $\Sigma(L)$ denote the class of all self-normalising maximal proper finite-codimensional Lie subalgebras of L. (Notice that a maximal subalgebra is self-normalising if and only if it is not an ideal, and that it can be an ideal if and only if it includes the commutator.)

We may call $\Sigma(L)$ the 'spectrum' of L.

Let $L^{(n)}$ denote the nth. derived ideal of L, for $n = 0, 1, 2, \ldots$; thus $L^{(0)} = L$ and, for each n, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. It will be convenient for technical reasons to define the 'n-spectrum' $\Sigma^n(L)$ (for $n = 1, 2, 3, \ldots$) as the class of subalgebras Q of L such that $Q \in \Sigma(L)$ and also

$$Q \not\supseteq L^{(n)}. \tag{1}$$

We have already observed that

$$\Sigma(L) = \Sigma^1(L). \tag{2}$$

Since $L^{(n)}$ is an ideal in L, $Q + L^{(n)}$ is a subalgebra; thus for $Q \in \Sigma(L)$, (1) is equivalent to

$$Q + L^{(n)} = L. \tag{3}$$

(2.10) LEMMA. *Let $\Phi: L_1 \to L_2$ be a surjective homomorphism of Lie algebras over F. Then, for any positive integer n,*

(a) *for any $Q \in \Sigma^n(L_2)$, $\Phi^{-1}(Q) \in \Sigma^n(L_1)$;*
(b) *for any $Q' \in \Sigma^n(L_1)$, either $\Phi(Q') = L_2$ or $\Phi(Q') \in \Sigma^n(L_2)$.*

Proof. Certainly $\Phi(Q')$, $\Phi^{-1}(Q)$ are finite-codimensional Lie subalgebras of L_2, L_1 respectively. Let R be a Lie subalgebra of L_2 such that $R \supseteq \Phi(Q')$. Then $\Phi^{-1}(R) \supseteq Q'$; consequently, either $\Phi^{-1}(R) = Q'$ or $\Phi^{-1}(R) = L_1$, and, as Φ is surjective, $R = \Phi(\Phi^{-1}(R))$ is either $\Phi(Q')$ or $\Phi(L_1) = L_2$. Hence $\Phi(Q')$ is either L_2 or a maximal proper Lie subalgebra. Similarly, let S be a Lie subalgebra of L_1 such that $S \supseteq \Phi^{-1}(Q)$; then as $S \supseteq \Phi^{-1}(0)$, $S = \Phi^{-1}(\Phi(S))$. But $\Phi(S) \supseteq Q$. Therefore $\Phi(S) = Q$ or $\Phi(S) = L_2$, and either $S = \Phi^{-1}(Q)$ or $S = L_1$; hence $\Phi^{-1}(Q)$ is a maximal proper Lie subalgebra of L_1.

Now, if $\Phi^{-1}(Q)$ were a proper Lie ideal of L_1, $Q = \Phi(\Phi^{-1}(Q))$ would be a proper Lie ideal of L_2, since Φ is surjective; if $\Phi(Q')$ were a proper Lie ideal of L_2, $\Phi^{-1}(\Phi(Q''))$ would be a proper Lie ideal in L_1 including Q', and therefore would be equal to Q'. But neither Q nor Q' is a Lie ideal (in L_2, L_1 respectively); so $\Phi^{-1}(Q)$ and $\Phi(Q')$ are not Lie ideals. This proves (a) and (b) when $n = 1$.

Finally, suppose $n > 1$. Then, as Φ is epimorphic,

$$\begin{aligned}\Phi(Q') + L_2^{(n)} &= \Phi(Q') + \Phi(L_1^{(n)}) = \Phi(Q' + L_1^{(n)}) \\ &= \Phi(L_1) = L_2;\end{aligned}$$

whilst, if $\Phi^{-1}(Q) \supseteq L_1^{(n)}$, then $Q = \Phi(\Phi^{-1}(Q)) \supseteq \Phi(L_1^{(n)}) = L_2^{(n)}$. The results now follow, by (2.9)(3) and (2.9)(1) respectively.

(2.11) LEMMA. *Let L be a Lie algebra over F, and let K be a Lie ideal of L including $L^{(1)}$.*

Then

(a) *if* $(L, K] = K^{(1)}$ *and* $Q \in \Sigma(L)$, *there exists* $N \in \Sigma(K)$ *such that* $Q \cap K \subseteq N$;
(b) *if, for some positive integer* n, $Q \in \Sigma^{n+1}(L)$, *there exists* $N \in \Sigma^n(K)$ *such that* $Q \cap K \subseteq N$.

Proof. Consider case (a). As K is an ideal, $K + Q$ is a Lie subalgebra of L. However, $K \nsubseteq Q$; for otherwise, as $K \supseteq L^{(1)}$, we should have $Q \supseteq L^{(1)}$ and Q would be an ideal. Hence $K + Q \neq Q$ and, by maximality, $K + Q = L$.

By hypothesis, $[Q, K] \subseteq [L, K] = K^{(1)}$. Thus, if $K^{(1)} \subseteq Q$, K must normalise Q and (as $K + Q = L$) Q must be an ideal in L. This is false, as $Q \in \Sigma(L)$; we deduce that

$$K^{(1)} \nsubseteq Q. \tag{0}$$

In case (b), $K^{(n)} \supseteq L^{(n+1)}$ and, as $Q \in \Sigma^{n+1}(L)$, it follows immediately that

$$K^{(n)} \nsubseteq Q. \tag{1}$$

In either case, $K^{(1)} + Q$ is a subalgebra (as $K^{(1)}$ is an ideal) which is not equal to Q; thus $K^{(1)} + Q = L$, and as a consequence

$$K^{(1)} + (Q \cap K) = K. \tag{2}$$

As $K \nsubseteq Q$, $K \cap Q$ is of finite positive codimension in K. Let N be a maximal proper subalgebra of K including $Q \cap K$ (which we may construct by finite induction). Then, by (2), $K^{(1)} + N = K$; in view of the maximality of N, this implies that $N \nsupseteq K^{(1)}$ and N is not a Lie ideal in K (see (2.9)). This proves (a). For (b), observe that (1) gives

$$K^{(n)} + (Q \cap K) = K,$$

exactly as (0) led to (2). Ergo, $K^{(n)} + N = K$, which shows $N \in \Sigma^n(K)$, by (2.9)(3).

3. Lie algebras of vector fields

(3.1) Once more, let M be a manifold of class $\mathscr{C}$ and let L be a Lie subalgebra (over F) of $\Gamma(M)$ (see (2.7)). We shall say that L is n-ample, where n is a positive integer — more precisely, L is an n-ample subalgebra of $\Gamma(M)$ – if, for each $p \in M$, $L_p \in \Sigma^n(L)$ (see (2.9)).

(3.2) LEMMA. (a) *Let* L *be an* n*-ample subalgebra of* $\Gamma(M)$. *Then*

$$(\forall p \in M) L^{(n)}(p) = L(p). \tag{1}$$

(b) *Suppose L is a 1-ample subalgebra of* $\Gamma(M)$ *and satisfies* (1). *Then L is n-ample.*

Proof. (a) By hypothesis, $L_p + L^{(n)} = L$ (see (2.9)(3)). The result follows, as $L_p(p) = 0$ by definition.

(b) If $L^{(n)}(p) = L(p)$, then evidently $L_p + L^{(n)} = L$. Apply (2.9)(3).

(3.3) LEMMA. *Suppose L_1, L_2 are Lie subalgebras of* $\Gamma(M)$ *such that* $L_1 \subseteq L_2$ *and, for any* $p \in M$, $L_1(p) = L_2(p)$. *Then, if L_1 is n-ample, so is L_2.*

Proof. Take $p \in M$, and let R be a Lie subalgebra of L_2 which includes $(L_2)_p$. Then $R \cap L_1 \supseteq (L_1)_p$, and, as L_1 is 1-ample, either $R \cap L_1 = L_1$ or $R \cap L_1 = (L_1)_p$. As $L_1(p) = L_2(p)$, certainly

$$L_2 = L_1 + (L_2)_p. \tag{1}$$

Consequently

$$R = (R \cap L_1) + (L_2)_p \text{ (as } R \supseteq (L_2)_p). \tag{2}$$

If $R \cap L_1 = L_1$, (2) and (1) show that $R = L_2$; whilst, if $R \cap L_1 = (L_1)_p$, (2) shows that $R = (L_2)_p$. This establishes the maximality of the subalgebra $(L_2)_p$ of L_2, and it is evidently of finite codimension therein. If it were an ideal in L_2, $(L_1)_p = L_1 \cap (L_2)_p$ would be an ideal in L_1, which it is not. This shows that, if L_1 is 1-ample, so is L_2. If L_1 is n-ample, by (3.2)(a)

$$L_1(p) = L_2^{(n)}(p) \subseteq L_2^{(n)}(p) \subseteq L_2(p),$$

so that L_2 is n-ample by (3.2)(b).

4. Symplectic structures and the associated Lie algebras

(4.1) A symplectic manifold (M, ω) of class $\mathscr{C}$ and dimension $2n$ is a manifold M of class $\mathscr{C}$ and dimension $2n$, furnished with an everywhere non-degenerate closed 2-form $\omega \in Z^2(M)$. Following [2], p. 2, we have then a bundle isomorphism of class $\mathscr{C}$,

$$\mu_\omega = \mu \colon TM \to T^*M \colon X \to -i(X)\omega$$

(where i denotes the internal product), which induces isomorphisms, also denoted by μ_ω, of the tensor bundles and their spaces of sections.

(4.2) We have also $\Lambda = \Lambda_\omega = i(\mu_\omega^{-1}(\omega)) \colon \Omega^{r+2}(M) \to \Omega^r(M)$ (ibid., p. 3).

(4.3) Let $L^*(\omega) = \mu_\omega^{-1}(B^1(M))$ and $L(\omega) = \mu_\omega^{-1}(Z^1(M))$ denote respectively the spaces of globally and of locally Hamiltonian vector fields on M. Certainly

$$[L(\omega), L(\omega)] \subseteq L^*(\omega) \subseteq L(\omega), \qquad ([2], \text{p. } 7)$$

so that $L(\omega)$ and $L^*(\omega)$ are Lie subalgebras of the algebra of vector fields, and (recall (2.8)) $[\Gamma(\omega), \Gamma(\omega)] \subseteq \Gamma_0(\omega) = L(\omega)$ ((3.1) on p. 6 of [2]). Following ([2], p. 11), we describe fields in $\Gamma(\omega)$ as 'conformally Hamiltonian'.

(4.4) A foliation $\mathscr{F}$ (of class $\mathscr{C}$) of the symplectic manifold (M, ω) of class $\mathscr{C}$ will itself be described as 'symplectic' if ω restricts to an everywhere non-degenerate form on each leaf of $\mathscr{F}$. Thus the leaves also become symplectic manifolds. Similarly, a subbundle S (of class $\mathscr{C}$) of TM is 'symplectic' if ω restricts to a non-degenerate form on each fibre of S. Clearly there is the usual correspondence between symplectic foliations and integrable symplectic subbundles.

(4.5) Given $f, g \in \Omega^0(M)$, one defines the Poisson bracket (relative to ω) by

$$[f, g]_\omega = \Lambda_\omega(\mathrm{d}f \wedge \mathrm{d}g) \tag{1}$$

$$= \omega(\mu_\omega^{-1}(\mathrm{d}f), \mu_\omega^{-1}(\mathrm{d}g)) = (\mu_\omega^{-1}(\mathrm{d}f))g, \tag{2}$$

by an easy computation. This makes $\Omega^0(M)$ into a Lie algebra, which we denote by $A(M)$. Using the non-degeneracy of ω and (2.4), one sees that the centre $C(M)$ of $A(M)$ consists of the locally constant functions of class $\mathscr{C}$ on M; that is,

$$C(M) = Z^0(M). \tag{3}$$

Thus d induces a linear isomorphism $A(M)/C(M) \to B^1(M)$, which determines a Lie algebra structure on $B^1(M)$; then $\mu_\omega : L^*(\omega) \to B^1(M)$ is a Lie algebra isomorphism. There is a Lie algebra exact sequence

$$0 \to C(M) \to A(M) \xrightarrow[\mu_\omega^{-1}\mathrm{d}]{} L^*(\omega) \to 0. \tag{4}$$

(4.6) For $X, Y \in L(\omega)$, one has ([2], (3.3), p. 7)

$$[X, Y] = \mu_\omega^{-1}\, \mathrm{d}\Lambda(\mu_\omega(X) \wedge \mu_\omega(Y)) = \mu_\omega^{-1}\, \mathrm{d}\{\omega(X, Y)\}.$$

(4.7) The Lie algebra $A(M)$ is also a commutative associative algebra under pointwise multiplication of functions, which is related to the Lie algebra structure by the structural equation derived from (4.5)(1)

$$(\forall f, g, h \in A(M))\ [fg, h]_\omega = f[g, h]_\omega + [f, h]_\omega g.$$

(4.8) PROPOSITION. *For every $n \geqq 1$ and every $p \in M$,*

$$(L^*(\omega))^{(n)}(p) = T_pM = (L^*(\omega))(p).$$

Proof. To avoid messy calculations, let us use the theorem of Darboux to construct coordinates $(x_1, \ldots x_{2n})$ of class $\mathscr{C}$ on a neighbourhood of p such that ω is represented on that neighbourhood as $\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}x_{n+i}$. In these coordinates, μ_ω^{-1} has the form

$$\mu_\omega^{-1}(\mathrm{d}x_i) = \frac{\partial}{\partial x_{n+i}}, \quad \mu_\omega^{-1}(\mathrm{d}x_{n+1}) = -\frac{\partial}{\partial x_i} \quad \text{for } 1 \leqq i \leqq n;$$

thus, in terms of principal parts, μ_ω^{-1} is represented by a constant linear isomorphism

$$J\colon (F^{2n})^* \to F^{2n}.$$

For a scalar-valued function f, and x in the chart in question, $\mathrm{d}f(x)$ is represented by the derivative $Df(x) \in (F^{2n})^*$; thus $J(Df(x))$ represents $\mu_\omega^{-1}(\mathrm{d}f)(x)$, and the kth. derivative in these coordinates of $X_f = \mu_\omega^{-1}(\mathrm{d}f)$ at p is obtained by identifying $D^{k+1}f(p) \in (\otimes^{k+1}F^{2n})^*$ with a linear mapping $\otimes^k F^{2n} \to (F^{2n})^*$ and compounding with J.

(A) We now claim inductively that, given integers $k \geqq 0$, $0 \leqq \ell \leqq k$, and nonzero vectors $\xi \in \otimes^\ell F^{2n}$, $\eta \in F^{2n}$, there exists $X \in (L^*(\omega))^{(n)}$ such that $D^rX(p) = 0$ when $r \leqq k$ and $r \neq \ell$, whilst $D^\ell X(p).\,\xi = \eta$. For $n = 0$, take $X = X_f$, where $D^r f(p) = 0$ for $r < k+1$ and $r \neq \ell + 1$, and $D^{\ell+1}f(p)$ is a symmetric element of $(\otimes^{\ell+1}F^{2n})^*$ such that $D^{\ell+1}f(p)\cdot(\xi \otimes \tau) = J^{-1}(\eta)\cdot\tau$ for each element τ of a basis of F^{2n}. The existence of such a symmetric multilinear map is an elementary exercise using multinomials; the existence of a suitable $f \in A(M)$ follows from (2.4).

Suppose the claim established for arbitrary k, ℓ, ξ, η and given $n \geqq 0$, and take $X \in (L^*(\omega))^{(n)}$ so that $X(p) \neq 0$, $DX(p) = 0$, $D^2X(p) = 0, \ldots, D^{k+1}X(p) = 0$. Then for any $Y \in \Gamma(M)$, the local coordinate representations of $[X, Y]$ are

$$\begin{aligned}
[X, Y](p) &= DY(p)\cdot X(p) - DX(p)\cdot Y(p), \qquad (1)\\
D[X, Y](p) &= D^2Y(p)\cdot(\ , X(p)) + DY(p)\circ DX(p)\\
&\quad - D^2X(p)\cdot(\ , Y(p)) - DX(p)\circ DY(p),
\end{aligned}$$

and so on to $D^k[X, Y](p) = D^{k+1}Y(p)\cdot(\ , X(p)) +$ terms combining lower derivatives of Y and higher derivatives (to order $k+1$) of X; by choice of X, then,

$$D^j[X, Y](p) = D^{j+1}Y(p)\cdot(\ , X(p)) \quad \text{for } 0 \leqq j \leqq k.$$

If Y is chosen in $(L^*(\omega))^{(n)}$ so that $D^1Y(p) = 0, \ldots, D^\ell Y(p) = 0, D^{\ell+2}Y(p) = 0, \ldots$ $D^{k+1}Y(p) = 0$, and $D^{\ell+1}Y(p)\cdot(\xi \otimes X(p)) = \eta$, it follows that $[X, Y]$ satisfies the required conditions, and of course it is an element of $(L^*(\omega))^{(n+1)}$. This proves the claim.

Now take $k = \ell = 0$ and the Proposition follows.

Note. The result when M is real follows instantly from (5.5). In the complex case, however, the above argument seems the simplest (though not the most conclusive) available, and is largely needed anyway for the next result.

(4.9) LEMMA. *Let $X \in \Gamma(M)$, $p \in M$, and $X(p) \neq 0$; suppose B is a subspace of $\Gamma(M)$ which includes $L^*(\omega)$. Then the subspace of B*

$$\{Z \in B\colon (\forall n \geqq 1)\mathscr{L}_X^n Z \in B_p\} \quad \text{(see (2.7))}$$

is of infinite codimension in B.

Proof. Use the coordinate system introduced in the proof of (4.8). Applying (4.8)(1) inductively, we find that in terms of these coordinates

$$\mathscr{L}_X^n Y(p) = D^{n+1}Y(p)\cdot(X(p), X(p), \ldots, X(p)) + \\ + \text{terms in lower-order derivatives of } Y \text{ at } p;$$

but, by the assertion (4.8)(A), there exists for each $n \geqq 1$ a field $Y \in L^*(\omega)$ such that $Y(p) = 0, DY(p) = 0, \ldots, D^n Y(p) = 0, D^{n+1}Y(p)\cdot(X(p), X(p), \ldots, X(p)) \neq 0$. Thus $\mathscr{L}_X^j Y(p) = 0$ for $0 \leqq j < n$, whilst $\mathscr{L}_X^n Y(p) \neq 0$. This evidently proves the result.

5. Commutators in the Lie algebras

(5.1) In this section, (M, ω) is a fixed symplectic manifold of class $\mathscr{C}$ and dimension $2n$, and we abbreviate the previous notations (from §§2, 4) $A(M)$, $\Omega^r(M)$, $Z^r(M)$, $B^r(M)$, μ_ω, Λ_ω, $[\ ,\ ]_\omega$, to A, Ω^r, Z^r, B^r, μ, Λ, $[\ ,\ ]$ respectively. As in [2], p. 2, set $\eta = \omega^n/n! \in \Omega^{2n}$, the symplectic volume form. We have

$$(\forall X \in L(\omega))\mathscr{L}_X \eta = 0. \tag{1}$$

This follows instantly from $L(\omega) = \Gamma_0(\omega)$ (see (4.3)).

The symplectic adjunction operator $*$, defined on p. 2 of [2], satisfies

$$*^2 = I \text{ (the identity)}, \quad *\eta = 1. \tag{2}$$

For the symplectic coderivative $\delta: \Omega^p \to \Omega^{p-1}: \alpha \to (-1)^p * \mathrm{d} * \alpha$, one has

$$\mathrm{d}\Lambda - \Lambda\mathrm{d} = -\delta, \tag{3}$$

as is proved in [10] and quoted as (1.8)(a) on p. 3 of [2].

(5.2) THEOREM.

$$A^{(1)} = \{f \in \Omega^0 : f\eta \in B^{2n}\}.$$

Moreover, there exist elements $x_1, x_2, \ldots, x_m$ *of* A (*where* $k = 4n + 1$) *such that the mapping*

$$A^m \to A^{(1)}: (f_1, f_2, \ldots, f_m) \to \sum_{i=1}^{m} [f_i, x_i]$$

is surjective.

Proof. Suppose first that $g, h \in A$. Then, by (4.5)(2) and (5.1)(1),

$$[g, h]\eta = \{\mu^{-1}(\mathrm{d}g)h\}\eta = \mathscr{L}_{\mu^{-1}(\mathrm{d}g)}(h\eta) = d\{i(\mu^{-1}(\mathrm{d}g))(h\eta)\}$$

(i denotes the interior product, and we have used the fact that $\Omega^{2n} = Z^{2n}$). Hence $[g, h]\eta \in B^{2n}$.

For the converse, suppose $f \in A$ and $f\eta \in B^{2n}$. So there exists $\beta \in \Omega^{2n-1}$ such that $f\eta = \mathrm{d}\beta$, and consequently

$$f = *(f\eta) = *(\mathrm{d}\beta) = -\delta * \beta.$$

Let $x_1, x_2, \ldots, x_m$ be as in (2.6) (with the difference that the dimension of M is now $2n$). Hence, as $*\beta \in \Omega^1$, there exist functions $f_1, f_2, \ldots, f_m \in A$ such that

$$*\beta = \sum_{i=1}^{m} f_i \mathrm{d}x_i.$$

It follows that

$$\delta * \beta = \Lambda \mathrm{d} * \beta$$

by (5.1)(3), since Λ has degree -2

$$= \sum_{i=1}^{m} \Lambda(\mathrm{d}f_i \wedge \mathrm{d}x_i) = \sum_{i=1}^{m} [f_i, x_i]$$

by (4.5)(1). This shows that $f \in A^{(1)}$, and completes the proof.

(5.3) COROLLARY. *There is a canonical isomorphism* $A/A^{(1)} \cong H^{2n}(M; F)$.

Proof. The map $A \to \Omega^{2n}$: $f \to f\eta$ is an isomorphism; by (5.2), it carries $A^{(1)}$ on to B^{2n}. As $\Omega^{2n} = Z^{2n}$, the result follows.

(5.4) NOTES. Both A and $H^{2n}(M; F)$ are direct products of the corresponding functors of the individual components of M. When M is real (whether $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$), η will define the fundamental class of each compact component; for each noncompact component, the top cohomology vanishes. Thus the isomorphism $A \to \Omega^{2n}$: $f \to f\eta$ of (5.3) induces, via inclusion $C_0 \to A$ and projection $\Omega^{2n} \to H^{2n}(M; F)$, an isomorphism of the space C_0 of locally constant F-valued functions ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$ as appropriate) of compact support on M with $H^{2n}(M; F)$. Since C_0 is clearly an abelian ideal of A, (5.3) now yields a Lie direct sum decomposition $A = C_0 \oplus A^{(1)}$. As C_0 is central, one deduces in turn that $A^{(1)} = A^{(2)}$.

These arguments do not hold in the complex case (2.1)(c). In that case, the top dimension for F-cohomology, namely $2n$, is only the middle topological dimension, and, even for connected M, the dimension of $H_{2n}(M; F)$ may be any finite integer or countably infinite. Indeed, let E be any discrete subset of $\mathbb{C}$, and endow $M = (\mathbb{C}\backslash E) \times (\mathbb{C}\backslash\{0\})^{2n-1}$ with the trivial symplectic structure (as a subset of $\mathbb{C}^{2n}$). The $2n$th Betti number of M is the number of points of E, and M is clearly Stein.

Surprisingly few examples of compact real symplectic manifolds are known. For a fairly recent, though inconclusive, survey, see [11]. There has been some progress since (by Gromov and McDuff in particular).

(5.5) LEMMA. *Let M be a real symplectic manifold, of class $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$. Then*

$$[L^*(\omega), L^*(\omega)] = [L(\omega), L(\omega)] = L^*(\omega).$$

Proof. By (4.5)(4), $L^*(\omega) \cong A/C(M)$; by (5.4), $A = C_0 \oplus A^{(1)}$, where $C_0 \subseteq C(M)$. Hence $[L^*(\omega), L^*(\omega)] = L^*(\omega)$. This suffices (see (4.3)).

(5.6) LEMMA. *Let M be a real symplectic manifold of class $\mathbf{C}^\infty$ or $\mathbf{C}^\omega$, and suppose the symplectic form ω is exact. Then*

$$[\Gamma(\omega), \Gamma(\omega)] = [\Gamma(\omega), L(\omega)] = L(\omega).$$

Furthermore, $\Gamma(\omega)$ is the semi-direct product of its derived ideal $L(\omega)$ with an abelian subalgebra isomorphic to $C(M)$.

Proof. See [2], p. 12. (We write Γ instead of L^c, ω in place of F; (5.5) must be invoked in the $\mathbf{C}^\omega$ case, and our formulation allows M not to be connected).

(5.7) LEMMA. *Let M be a symplectic manifold of class $\mathscr{C}$. Then*

$$[L(\omega), L^*(\omega)] = [L^*(\omega), L^*(\omega)].$$

(*Compare* (5.5)).

Proof. In view of (4.5)(4), $[L^*(\omega), L^*(\omega)]$ consists of the images under $\mu^{-1}\mathrm{d}$ of elements of $A^{(1)}$, which are characterized by (5.2). Suppose $X \in L(\omega)$ and $Y \in L^*(\omega)$, so that $Y = \mu^{-1}(\mathrm{d}f)$ for some $f \in A$ and $\mathscr{L}_X\omega = \mathscr{L}_Y\omega = 0$ (see (4.3)). Hence, by (4.6),

$$[X, Y] = \mu^{-1}\mathrm{d}\Lambda(\mu(X) \wedge \mathrm{d}f),$$

and it is enough, by (5.2), to show that $\Lambda(\mu(X) \wedge \mathrm{d}f)\eta \in B^{2n}$.

Now

$$\begin{aligned}\Lambda(\mu(X) \wedge \mathrm{d}f) &= i(\mu^{-1}(\omega))(\mu(X) \wedge \mathrm{d}f) && \text{by definition, (4.2)}\\ &= \omega(X, \mu^{-1}(\mathrm{d}f)) = Xf && \text{by definition, (4.1).}\end{aligned}$$

But

$$\begin{aligned}(Xf)\eta &= \mathscr{L}_X(f\eta) - f(\mathscr{L}_X\eta) = \mathscr{L}_X(f\eta) && \text{by (5.1)(1)}\\ &= \mathrm{d}\{i(X)(f\eta)\}, && \text{as } f\eta \in Z^{2n}.\end{aligned}$$

This completes the proof.

(5.8) LEMMA. *Let M be a connected symplectic manifold of class $\mathscr{C}$, and suppose the symplectic form ω is not exact. Then $\Gamma(\omega) = L(\omega) = \Gamma_0(\omega)$.*

Proof. See p. 11 of [2] (after (5.4)); no change is needed.

Note that when M is compact, ω cannot be exact (as η is not).

6. Spectra

(6.1) By a Poisson algebra over the field F, we mean a commutative associative algebra A furnished with an F-bilinear operation

$$(f, g) \to [f, g]: A \times A \to A$$

with respect to which it is also a Lie algebra over F, and satisfies the structural relation

$$[f, gh] = [f, g]h + g[f, h]$$

for all f, g, $h \in A$. Thus, in particular, $A(M)$ is a Poisson algebra over F when (M, ω) is a symplectic manifold of class $\mathscr{C}$ (see (4.7)).

Given a Poisson algebra A over F, let $\mathfrak{M}(A)$ denote the set of all maximal finite-codimensional proper associative ideals J in A, and let $\Sigma(A)$ be as in (2.9). The theorem which follows is proved (in superficially different formulations) in [7] and in [1].

(6.2) THEOREM. *Suppose the Poisson algebra A satisfies*:

(a) $A^2 = A$,

(b) *for any $J \in \mathfrak{M}(A)$, the Lie normaliser*

$$\mathfrak{N}_A(J) = \{f \in A\colon [f, J] \subseteq J\}$$

is a proper finite-codimensional linear subspace of A.

Then the mapping $J \mapsto \mathfrak{N}_A(J)$ establishes a one-one correspondence between $\mathfrak{M}(A)$ and $\Sigma(A)$.

(6.3) Now let (M, ω) be a symplectic manifold of class $\mathscr{C}$ and positive dimension. Given $p \in M$, define

$$p^* = \{f \in A(M)\colon f(p) = 0\},$$
$$N(M, p) = \{f \in A(M)\colon \mathrm{d}f(p) = 0\}.$$

On p. 17 of [6] it is proved that the map $p \mapsto p^*$ furnishes a bijection between M and $\mathfrak{M}(A(M))$. In addition,

(6.4) LEMMA.

$$\mathfrak{N}_{A(M)}(p^*) = N(M, p).$$

Proof. Suppose $g \in A(M)$ and $\mathrm{d}g(p) = 0$; then, by (4.5)(1), $[f, g]_\omega \in p^*$. This proves that $N(M, p) \subseteq \mathfrak{N}_{A(M)}(p^*)$. Conversely, suppose $g \in \mathfrak{N}_{A(M)}(p^*)$ but $\mathrm{d}g(p) = 0$. As ω is non-degenerate at p, there exists $\chi \in T_p^*M$ for which

$$\omega(\mu_\omega^{-1}(\chi), \mu_\omega^{-1}(\mathrm{d}g(p))) \neq 0.$$

By (2.4), there exists $f \in p^*$ with $\mathrm{d}f(p) = \chi$. Thus $[f, g]_\omega(p) \neq 0$ and $g \notin \mathfrak{N}_{A(M)}(p^*)$. This completes the proof.

(6.5) THEOREM. *The map $p \mapsto N(M, p)$ constitutes a bijection of M with $\Sigma(A(M))$.*

Proof. As remarked in (6.3), $p \mapsto p^*\colon M \to \Sigma(A(M))$ is bijective; hypothesis (6.2)(a) is trivial, and (6.2)(b) follows from (6.4). Thus the result follows from (6.2) and (6.4).

(6.6) THEOREM. *The map* $p \mapsto N^*(M, p) = (L^*(\omega))_p$ *(see* (2.7)) *constitutes a bijection of* M *with* $\Sigma(L^*(\omega))$.

Proof. Any proper self-normalising Lie subalgebra of $A(M)$ necessarily includes the Lie centre $C(M)$, so its image under $\mu_\omega^{-1}\mathrm{d}$ must be proper (see (4.5)(4)); thus, by (2.10), $\mu_\omega^{-1}\mathrm{d}$ induces a bijection $\Sigma(A(M)) \to \Sigma(L^*(\omega))$. The result follows from (6.5) (and the definition (4.1)).

(6.7) LEMMA. *The subalgebras* $L^*(\omega)$, $L(\omega)$, $\Gamma(\omega)$, $\Gamma(M)$ *of* $\Gamma(M)$ *are n-ample for each* $n \geqq 1$.

Proof. By (4.8), $(L^*(\omega))^{(n)}(p) = T_pM = (L^*(\omega))(p)$ for each $p \in M$ and each n. By (6.6), $L^*(\omega)$ is 1-ample. By (3.2)(b), $L^*(\omega)$ is n-ample. By (3.3), the same holds for $L(\omega)$, $\Gamma(\omega)$, and $\Gamma(M)$.

(6.8) THEOREM. *The map* $p \to \hat{N}(M, p) = (L(\omega))_p$ *(see* (2.7)) *constitutes a bijection of* M *with* $\Sigma(L(\omega))$.

Proof. By (6.7), $\hat{N}(M, p) \in \Sigma(L(\omega))$ for each $p \in M$. Secondly, for each p

$$(L^*(\omega))_p = L^*(\omega) \cap (L(\omega))_p \quad \text{obviously,}$$

so that (by (6.6))$\hat{N}(M, \ )$ must be one-one. It remains to show that $\hat{N}(M, \ )$ is surjective. So let $Q \in \Sigma(L(\omega))$. In (2.11)(a), take $L = L(\omega)$ and $K = L^*(\omega)$; the hypotheses are satisfied, by (4.3) and (5.7). Thus there exists $N \in \Sigma(L^*(\omega))$ with $Q \cap L^*(\omega) \subseteq N$. By (6.6) again, there exists $p \in M$ such that $N = (L^*(\omega))_p$. This does not immediately show $Q \subseteq (L(\omega))_p$.

Now $[Q, Q] \subseteq [L(\omega), L(\omega)] \subseteq L^*(\omega)$, and consequently $Q^{(1)} \subseteq Q \cap L^*(\omega)$. If there exists $X \in Q$ for which $X(p) \neq 0$, it follows that

$$(\forall Y \in Q)\mathscr{L}_X Y = [X, Y] \in Q \cap L^*(\omega),$$

and, by induction, $\mathscr{L}_X^n Y \in Q \cap L^*(\omega) \subseteq N \subseteq (L(\omega))_p$ for all $n \geqq 1$; but, as $X(p) \neq 0$, the subspace

$$\{Z \in L(\omega)\colon (\forall n \geqq 1)\mathscr{L}_X^n Z \in (L(\omega))_p\}$$

is of infinite codimension in $L(\omega)$, by (4.9). This contradicts the definition of Q, and we conclude that each $X \in Q$ vanishes at p; that is, $Q \subseteq (L(\omega))_p$. But, as Q is maximal, it follows that $Q = (L(\omega))_p$, and this completes the proof.

(6.9) LEMMA. *For each* $n \geqq 1$, $\Sigma(A(M)) = \Sigma^n(A(M))$, $\Sigma(L^*(\omega)) = \Sigma^n(L^*(\omega))$, *and* $\Sigma(L(\omega)) = \Sigma^n(L(\omega))$. *(See* (2.9)).

Proof. By (3.2)(b), (4.8), both $L^*(\omega)$ and $L(\omega)$ are n-ample; by (6.7), (6.8), this means every element of their spectra is also in the corresponding n-spectrum. The converse inclusion is trivial (see (2.9)). As in the proof of (6.6), (2.10) gives a one-one correspondence $\Sigma^n(A(M)) \leftrightarrow \Sigma^n(L^*(\omega))$ induced by $\mu_\omega^{-1}\mathrm{d}$; thus the first assertion follows.

(6.10) THEOREM. *For any $n \geqq 2$, the map $p \mapsto \tilde{N}(M, p) = (\Gamma(\omega))_p$ (see (2.7)) constitutes a bijection of M with $\Sigma^n(\Gamma(\omega))$. In particular, $\Sigma^n(\Gamma(\omega))$ does not depend on n when $n \geqq 2$.*

Proof. Repeat the proof of (6.8), reading $L^*(\omega)$ for $L(\omega)$, $\Gamma(\omega)$ for $L(\omega)$, (6.8) for (6.6), $\tilde{N}(M, p)$ for $\hat{N}(M, p)$, Σ^n for Σ, (2.11)(b) for (2.11)(a); and suppressing mention of (5.7).

(6.11) REMARKS. It is definitely untrue in general that every element of $\Sigma(\Gamma(\omega))$ is of the form $(\Gamma(\omega))_p$ for some $p \in M$. Suppose that M is real, with first Betti number 1, and that ω is exact. (An example would be $M = S^1 \times \mathbb{R}$, as quotient of $\mathbb{R}^2$ with the 'constant' symplectic structure $dx_1 \wedge dx_2$). Then (5.5), (5.6), (4.3) show that

$$\Gamma(\omega)^{(1)} = L(\omega), \quad L(\omega)^{(1)} = L^*(\omega), \quad L(\omega)/L^*(\omega) \cong \mathbb{R} \cong H^1(M; \mathbb{R}),$$

$$\Gamma(\omega)/L(\omega) \cong H^0(M; \mathbb{R}); \ L(\omega) \text{ has a complementary subalgebra } B \text{ in } \Gamma(\omega).$$

Thus $L^*(\omega) + B = \Gamma(\omega)^{(2)} + B$ is a subalgebra of $\Gamma(\omega)$ which is proper and maximal (being of codimension 1), and does not include $\Gamma(\omega)^{(1)}$; so it belongs to $\Sigma(\Gamma(\omega))$. However, it cannot be of the form $(\Gamma(\omega))_p$, since it includes $L^*(\omega)$. This example suggests how (6.10) might be proved otherwise, at least in the real case; and clarifies why it is precisely Σ^2 that is crucial.

It is worth noting that (6.8) and (6.10) in effect prove that, in their specific circumstances, the subalgebras N of (2.11)(a), (b) respectively are unique and equal to $Q \cap K$.

7. Homomorphisms

(7.1) LEMMA. *Let $M_i (i = 1, 2)$ be a manifold of class $\mathscr{C}$, and let ω be a closed 2-form of class $\mathscr{C}$ on M_2 which is non-zero on a dense subset of M_2. Assume M_1 has dimension greater than 1, and write A_i for $A(M_i)$.*

Suppose $\hat{\varphi}$: $B^1(M_1) \to \Omega^1(M_2)$ is an F-linear map such that, for any $f \in A_1$, there exists $c(f) \in C(M_2)$ for which

$$d(\hat{\varphi}(df)) = c(f)\omega, \tag{1}$$

and ψ: $M_2 \to M_1$ is any map which is not constant on any component of M_2. If, for any $p \in M_2$ and $f \in A_1$,

$$[\hat{\varphi}(df)](p) = 0 \Leftrightarrow df(\psi(p)) = 0, \tag{2}$$

then ψ is a submersion of class $\mathscr{C}$, and there exists $c_0 \in C(M_2)$ such that, for all $f \in A_1$, $\hat{\varphi}(df) = c_0\psi^*(df)$.

Proof. For $f, g \in A_1$ and $p \in M_2$,

$$\{d(fg) - f(\psi(p))dg - g(\psi(p))df\}(\psi(p)) = 0.$$

Thus (2) gives at each point of M_2

$$\hat{\varphi}(d(fg)) = (f \circ \psi)\hat{\varphi}(dg) + (g \circ \psi)\hat{\varphi}(df). \tag{3}$$

Applying (3) with $g = f$ and with $g = f^2$ in turn, we find

$$\hat{\varphi}(d(f^2)) = 2(f \circ \psi)\hat{\varphi}(df) \tag{4}$$

and

$$\hat{\varphi}(d(f^3)) = 3(f^2 \circ \psi)\hat{\varphi}(df). \tag{5}$$

Choose $g \in A_1$ so that $dg(\psi(p)) \neq 0$ (see (2.4)). Then, by (2), $[\hat{\varphi}(dg)](p) \neq 0$, and there is a neighbourhood U of p on which $\hat{\varphi}(dg)$ does not vanish. Now $\hat{\varphi}(d(g^2))$ and $\hat{\varphi}(dg)$ are forms of class $\mathscr{C}$; as $\hat{\varphi}(dg)$ does not vanish on U, (4) shows that $g \circ \psi$ is of class $\mathscr{C}$ on U.

Retaining the same g, consider (3) with arbitrary $f \in A_1$. Since $\hat{\varphi}(d(fg))$, $\hat{\varphi}(dg)$, $\hat{\varphi}(df)$ are forms of class $\mathscr{C}$, $\hat{\varphi}(dg)$ does not vanish on U, and $g \circ \psi$ is a function of class $\mathscr{C}$ on U, we deduce that $f \circ \psi$ is of class $\mathscr{C}$ on U. However, as p was arbitrary, it follows that $f \circ \psi \in A_2$. Hence – for instance by the embedding theorems (2.2) – we find that $\psi: M_2 \to M_1$ is of class $\mathscr{C}$.

Again, take arbitrary $f \in A_1$. Differentiate (4), and apply (1):

$$c(f^2)\omega = 2d(f \circ \psi) \wedge \hat{\varphi}(df) + 2(f \circ \psi)c(f)\omega. \tag{6}$$

(Notice we are using the fact, just proved, that $f \circ \psi \in A_2$.) Similarly, from (5),

$$c(f^3)\omega = 6(f \circ \psi)d(f \circ \psi) \wedge \hat{\varphi}(df) + 3(f^2 \circ \varphi)c(f)\omega. \tag{7}$$

Substituting from (6) in (7), and recalling that ω is nonzero on a dense subset, we have

$$c(f^3) = 3(f \circ \psi)c(f^2) - 3(f^2 \circ \psi)c(f).$$

In this equality, $c(f^3)$, $c(f^2)$, and $c(f)$ are locally constant; so differentiate:

$$c(f^2)d(f \circ \psi) = c(f) \cdot 2(f \circ \psi)d(f \circ \psi). \tag{8}$$

Suppose now that $\psi^*\mathrm{d}(f^2) = 2(f \circ \psi)\mathrm{d}(f \circ \psi)$ does not vanish at p. Thus it does not vanish on an open neighbourhood U of p; and, on U, (8) reduces to

$$c(f^2) = c(f) \cdot 2(f \circ \psi).$$

Differentiating, $0 = c(f) \cdot \mathrm{d}(f \circ \psi)$ on U; but as $\mathrm{d}(f \circ \psi)$ is there nonzero, $c(f) = 0$ on U. In effect, then, $c(f) = 0$ at any point where $\psi^*(\mathrm{d}(f^2))$ does not vanish. It is easy to see from (2.4) that, for any $q \in M_2$, and any $f \in A_1$, there exist g_q, $h_q \in A_1$ such that $f = g_q + h_q$ and $\psi^*\mathrm{d}(g_q^2)(q) \neq 0$, $\psi^*\mathrm{d}(h_q^2)(q) \neq 0$. Thus

$$c(f)(q) = c(g_q)(q) + c(h_q)(q) = 0,$$

and $c(f) = 0$ at all points of M_2; so that (2) becomes

$$(\forall f \in A_1)\mathrm{d}(\hat{\varphi}(\mathrm{d}f)) = 0. \tag{9}$$

Differentiate (3), and use (9). We find that, for any f, $g \in A_1$,

$$\mathrm{d}(f \circ \psi) \wedge \hat{\varphi}(\mathrm{d}g) + \mathrm{d}(g \circ \psi) \wedge \hat{\varphi}(\mathrm{d}f) = 0. \tag{10}$$

Taking $f = g$, we have in particular

$$\psi^*(\mathrm{d}f) \wedge \hat{\varphi}(\mathrm{d}f) = 0. \tag{11}$$

Define

$$\begin{aligned} S(f) &= \{x \in M_2 \colon \mathrm{d}f(\psi(x)) \neq 0\} \\ &= \{x \in M_2 \colon [\hat{\varphi}(\mathrm{d}f)](x) \neq 0\}, \quad \text{by (2).} \end{aligned}$$

In (11), both factors are 1-forms. It follows that, at each point $p \in S(f)$, there is a scalar $e_f(p)$ such that

$$\psi^*(\mathrm{d}f)(p) = e_f(p)[\varphi(\mathrm{d}f)](p). \tag{12}$$

Again by (2) – and by (2.4) – $\hat{\varphi}$ induces a linear monomorphism

$$\hat{\varphi}_p \colon T^*_{\psi(p)}M_1 \to T^*_p M_2 \tag{13}$$

for each $p \in M_2$. As M_1 is of dimension greater than 1, so is the image of $\hat{\varphi}_p$.

Given $g, h \in A_1$ and $p \in S(g) \cap S(h)$, one may therefore choose $k \in A_1$, by (2.4), so that

$$[\hat{\varphi}(\mathrm{d}k)](p) \wedge [\hat{\varphi}(\mathrm{d}g)](p) \neq 0 \neq [\hat{\varphi}(\mathrm{d}k)](p) \wedge [\hat{\varphi}(\mathrm{d}h)](p). \tag{14}$$

Taking h in place of f and k in place of g in (10), and substituting from (12),

$$\begin{aligned} 0 = e_g(p)&[\hat{\varphi}(\mathrm{d}g)](p) \wedge [\hat{\varphi}(\mathrm{d}k)](p) \\ &+ e_k(p)[\hat{\varphi}(\mathrm{d}k)](p) \wedge [\hat{\varphi}(\mathrm{d}g)](p), \end{aligned}$$

so that $e_g(p) = e_k(p)$ by (14); and similarly $e_k(p) = e_h(p)$.

Once more, (2.4) shows that the sets $S(f)$ cover M_2 as f varies over A_1. So we have shown that there is a well-defined function e on M_2 such that, for any $f \in A_1$ and any $p \in S(f)$,

$$\psi^*(\mathrm{d}f)(p) = e(p)[\hat{\varphi}(\mathrm{d}f)](p). \tag{15}$$

But, in view of (2), this equality holds automatically when $p \notin S(f)$. It also shows that, on $S(f)$ (and hence everywhere), e is of class $\mathscr{C}$.

Differentiate (15), recalling (9). Thus

$$\mathrm{d}e \wedge \hat{\varphi}(\mathrm{d}f) = 0$$

or, for each $p \in M_2$,

$$\mathrm{d}e(p) \wedge \hat{\varphi}_p(\mathrm{d}f(\psi(p))) = 0. \tag{16}$$

As remarked at (13), $\hat{\varphi}_p$ has image of dimension not less than 2, and (by (2.4)) all its elements are of the form $\hat{\varphi}_p(\mathrm{d}f(\psi(p)))$ for some $f \in A_1$. Thus (16) can only hold for all $f \in A_1$ if $\mathrm{d}e(p) = 0$; and p was arbitrary, so e is locally constant. Now, if e vanishes at any point, it vanishes on a whole component of M_2, and (12) implies that $\psi^*(\mathrm{d}f)$ vanishes at all points of this component for any $f \in A_1$. Ergo, the tangent map of ψ vanishes at each point of the component (using (2.4)), and ψ is constant thereon. This is contrary to hypothesis; so e does not vanish.

Finally, as $\hat{\varphi}_p$ is monomorphic (see (13)), (12) shows that

$$\psi_p^*\colon T^*_{\psi(p)}M_1 \to T^*_pM_2$$

is also monomorphic, for each $p \in M_2$. Thus $\psi\colon M_2 \to M_1$ is a submersion. (Of course c_0 is just the reciprocal of the non-vanishing function e.)

(7.2) THEOREM. *Let* (M_i, ω_i) *be a symplectic manifold of positive dimension and of class* $\mathscr{C}$, *for* $i = 1, 2$. *Write* μ_i *for* μ_{ω_i}. *Then*

$$\varphi: L^*(\omega_1) \to \Gamma(\omega_2)$$

is a Lie algebra homomorphism over F whose image is a 1-*ample subalgebra of* $\Gamma(M_2)$ (*see* (3.1)) *if and only if there are a submersion of class* $\mathscr{C}$

$$\psi: M_2 \to M_1,$$

and a nowhere vanishing function $c_0 \in C(M_2)$, *with the properties that*

(a) $D = \mu_2^{-1}\psi^*(T^*M_1)$
$= \{\mu_2^{-1}\psi_p^*\xi: p \in M_2,\ \xi \in T^*_{\psi(p)}M_1\}$

is a symplectic subbundle of TM_2 *of class* $\mathscr{C}$;

(b) ω_2 agrees on D *with* $c_0\psi^*(\omega_1)$;

(c) $\varphi = c_0\mu_2^{-1}\psi^*\mu_1$.

In these circumstances, the following additional properties must also hold:

(d) *D is an integrable subbundle of* TM_2, *with corresponding symplectic foliation* $\mathscr{F}$ *of* (M_2, ω_2) (*see* (4.4));

(e) *on each leaf of* $\mathscr{F}$, ψ *is a local diffeomorphism* (*both a submersion and an immersion*);

(f) *the formula* (c) *extends* φ *to a Lie algebra homomorphism*

$$\bar{\varphi}: \Gamma(M_1) \to \Gamma(M_2)$$

such that $\bar{\varphi}(\Gamma(M_1)) \subseteq \Gamma(\mathscr{F})$ (*see* (2.7)) *and* $\bar{\varphi}(L(\omega_1)) \subseteq L(\omega_2)$, $\bar{\varphi}(L^*(\omega_1)) \subseteq L^*(\omega_2)$;

(g) $(\forall p \in M_2)(\forall X \in \Gamma(M_1))\ \psi_{*p}(\bar{\varphi}(X))(p) = X(\psi(p))$ (*that is, X and* $\bar{\varphi}(X)$ *are* ψ-*related*);

(h) $\tilde{\varphi} = c_0\psi^*: A(M_1) \to A(M_2)$ *is also a Lie algebra homomorphism, which quotients* (*see* (4.5)(4)) *to* $\bar{\varphi}: L^*(\omega_1) \to L^*(\omega_2)$; *that is,*

$$\bar{\varphi}\mu_1^{-1}\mathrm{d} = \mu_2^{-1}\mathrm{d}\tilde{\varphi};$$

(i) $\bar{\varphi}(L^*(\omega_1))$, $\bar{\varphi}(L(\omega_1))$, $\bar{\varphi}(\Gamma(\omega_1))$, *and* $\bar{\varphi}(\Gamma(M_1))$ *are n-ample subalgebras of* $\Gamma(M_2)$, *for any* $n \geqslant 1$;

(j) ψ *and* c_0 *are uniquely determined by* φ.

(7.3) NOTES. For convenience, we shall usually write A_i for $A(M_i)$, L_i^* *for* $L^*(\omega_i)$, L_i for $L(\omega_i)$, Γ_i for $\Gamma(\omega_i)$. However, $\Gamma(M_i)$ will not be abbreviated. With these conventions, (6.2)(f) and (h) may be described by the commutative diagram of Lie algebra homomorphisms

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\mu_1^{-1}\mathrm{d}} & L_1^* \subseteq & L_1 & \subseteq \Gamma_1 \subseteq & \Gamma(M_1) \\ \Big\downarrow{\scriptstyle \tilde{\varphi} = c_0\psi^*} & & \Big\downarrow{\scriptstyle \varphi} & \Big\downarrow{\scriptstyle \bar{\varphi}|L_1} & & \Big\downarrow{\scriptstyle \bar{\varphi}} \\ A_2 & \xrightarrow[\mu_2^{-1}\mathrm{d}]{} & L_2^* \subseteq & L_2 & \subseteq \Gamma_2 \subseteq & \Gamma(M_2). \end{array}$$

It is not in general true that $\bar{\varphi}(\Gamma_1) \subseteq \Gamma_2$. The reason is that the requirements of the theorem only specify the restriction of ω_2 to D, whilst Γ_2 is defined in terms of ω_2 on TM_2. (Counterexamples are easily constructed in which $M_1 = F^{2m}$, with trivial symplectic structure ω_1, and $M_2 = F^{2n}$, with ψ as a linear epimorphism but ω_2 non-constant.) We do not know of any conditions appropriate to the present context that would make $\bar{\varphi}(\Gamma_1) \subseteq \Gamma_2$.

It is worth noting that (7.2)(e) expresses the fact that $\mathscr{F}$ is transverse to the foliation of M_2 by the fibres of ψ. Thus M_2 decomposes locally as a product, in a fashion which (unlike the local representation of a submersion) is completely determined by the data.

(7.4) PROOF OF (7.2). Suppose in the first place that $\varphi: L_1^* \to \Gamma_2$ is a Lie algebra homomorphism whose image $K^* = \varphi(L_1^*)$ is 1-ample. By (2.10),

$$(\forall p \in M_2)\varphi^{-1}(K_p^*) \in \Sigma^1(L_1^*) = \Sigma(L_1^*).$$

Write $N_1^*(q)$ for $N^*(M_1, q) = (L^*(\omega_1))_q$; then use (6.6) to define

$$(\forall p \in M_2)\psi(p) = (N_1^*)^{-1}\{\varphi^{-1}(K_p^*)\} \in M_1. \tag{1}$$

It follows immediately that, for any $X \in L_1^*$ and $p \in M_2$,

$$\varphi(X)(p) = 0 \Leftrightarrow X(\psi(p)) = 0. \tag{2}$$

Set

$$\hat{\varphi} = \mu_2 \varphi \mu_1^{-1}. \tag{3}$$

Then $\hat{\varphi}: B^1(M_1) \to \Omega^1(M_2)$ is certainly linear over F, and, whenever $f \in A_1$, $p \in M_2$, (2) gives

$$[\hat{\varphi}(\mathrm{d}f)](p) = 0 \Leftrightarrow \mathrm{d}f(\psi(p)) = 0. \tag{4}$$

Suppose there were a component Q of M_2 on which ψ were constant with value $a \in M_1$. Then, if $X \in L_1^*$ were such that $\varphi(X)$ vanish at $q \in Q$, (2) would give $X(a) = 0$, and, in turn, $\varphi(X)$ would therefore vanish at all points of Q. Hence, for any $q \in Q$, the Lie subalgebra K_q^* of $\varphi(L_1^*)$ would actually be a Lie ideal; this contradicts the hypothesis that $\varphi(L_1^*)$ is 1-ample.

Now apply (7.1), whose hypotheses have all been verified. Thus ψ is a submersion of class $\mathscr{C}$, and there is an everywhere non-zero function $c_0 \in C(M_2)$ such that

$$\hat{\varphi} = c_0 \psi^*. \tag{5}$$

However, φ is a Lie algebra homomorphism. Thus, for any $X, Y \in L_1^*$, we have from (4.6)

$$\varphi(\mu_1^{-1}\mathrm{d}\{\omega_1(X, Y)\}) = \mu_2^{-1}\mathrm{d}\{\omega_2(\varphi(X), \varphi(Y))\}.$$

Subsituting (5) and (3),

$$c_0\psi^*(\mathrm{d}\{\omega_1(X, Y)\}) = \mathrm{d}\{\omega_2(\varphi(X), \varphi(Y))\};$$

as c_0 is locally constant, this means

$$\mathrm{d}(c_0\{\omega_1(X, Y) \circ \psi\} - \omega_2(\varphi(X), \varphi(Y))) = 0.$$

Hence the function on M_2

$$F(X, Y) = c_0\{\omega_1(X, Y) \circ \psi\} - \omega_2(\varphi(X), \varphi(Y)) \tag{6}$$

is locally constant. Once more, let Q be a component of M_2, and take $q \in Q$; suppose first that $\varphi(X)$ is not identically zero on Q.

Certainly $X = \mu_1^{-1}(\mathrm{d}f)$ for some $f \in A_1$; clearly one may adjust f by a constant to make $f \circ \psi(q) = 0$. By (2), $\mathrm{d}f$ cannot vanish at all points of $\psi(Q)$ (since $\varphi(X)$ is not identically zero on Q), and so there exists $p \in Q$ with $f \circ \psi(p) \neq 0$. Now set

$$X_1 = (2f(\psi(p)))^{-1}\mu_1^{-1}\mathrm{d}(f^2),$$
$$X_2 = -(2f(\psi(p)))^{-1}\mu_1^{-1}\mathrm{d}\{(f - f(\psi(p)))^2\}.$$

Then $X = X_1 + X_2$ and $X_1(\psi(q)) = 0$, $X_2(\psi(p)) = 0$. By (2),

$$\varphi(X_1)(q) = 0 \quad \text{and} \quad \varphi(X_2)(p) = 0.$$

Applying (6), $F(X_1, Y)(q) = F(X_2, Y)(p) = 0$. Since $F(X_1, Y)$ and $F(X_2, Y)$ are constant on Q (see (6)), it follows that

$$F(X, Y) = F(X_1, Y) + F(X_2, Y)$$

vanishes at all points of Q. The same conclusion is trivial when $\varphi(X)$ is identically zero on Q. We deduce that

$$(\forall X, Y \in L_1^*) \quad \omega_2(\varphi(X), \varphi(Y)) = c_0\{\omega_1(X, Y) \circ \psi\}. \tag{7}$$

Use (4.1) to rewrite this (in a self-explanatory notation) as

$$(\forall p \in M_2) \quad \langle \varphi(X), \mu_2 \varphi(Y) \rangle_p = c_0(p) \langle X, \mu_1(Y) \rangle_{\psi(p)}.$$

From (5) and (3), this can be expressed as

$$(\forall p \in M_2) \quad \langle \varphi(X), c_0(p) \psi^* \mu_1(Y) \rangle_p = c_0(p) \langle X, \mu_1(Y) \rangle_{\psi(p)}.$$

As c_0 is everywhere non-zero, it follows that, for all $X, Y \in L_1^*$, $p \in M_2$,

$$\langle \psi_{*p}(\varphi(X)(p)), \mu_1(Y)(\psi(p)) \rangle_{\psi(p)} = \langle X(\psi(p)), \mu_1(Y)(\psi(p)) \rangle_{\psi(p)}.$$

In view of (2.4), $\mu_1(Y)$ may take arbitrary values at $\psi(p)$. Ergo,

$$(\forall X \in L_1^*)(\forall p \in M_2) \psi_{*p}(\varphi(X)(p)) = X(\psi(p)). \tag{8}$$

(That is, X and $\varphi(X)$ are ψ-related when $X \in L_1^*$.)

Again, applying (5) and (3),

$$K^* = \varphi(L_1^*) = c_0 \mu_2^{-1} \psi^* \mu_1(L_1^*) = c_0 \mu_2^{-1} \psi^*(B^1(M_1)),$$

and, because of (2.4), we deduce

$$(\forall p \in M_2) \quad K^*(p) = \mu_2^{-1} \psi^*(T^*_{\psi(p)} M_1). \tag{9}$$

As ψ is a submersion, D, as defined in the statement of (7.2)(a), is a subbundle of TM_2; (9) states in effect that

$$(\forall p \in M_2) \quad K^*(p) = D_p, \text{ the fibre of } D \text{ over } p. \tag{10}$$

It follows that D is an integrable subbundle of TM_2, since, for each $p \in M_2$, it has a base of sections in a neighbourhood of p furnished by vector fields in the Lie subalgebra K^* of $\Gamma(M_2)$, and the bracket of two such fields also takes values in D.

Now, if $\xi \in D_p$, and $(\forall \eta \in D_p)\, \omega_2(\xi, \eta) = 0$, we may use (10) to write $\xi = \varphi(X)(p)$ for some $X \in L_1^*$. Then, again by (10),

$$(\forall Y \in L_1^*) \quad \omega_2(\varphi(X)(p), \varphi(Y)(p)) = 0.$$

Recalling (7), we see that this entails

$$(\forall Y \in L_1^*) \quad \omega_1(X(\psi(p)), Y(\psi(p))) = 0,$$

so that, by (2.4) and the non-degeneracy of ω_1 at $\psi(p)$,

$$X(\psi(p)) = 0$$

and, by (2), $\varphi(X)(p) = \xi = 0$. This proves that ω_2 is non-degenerate on D. By (4.4), D generates a symplectic foliation $\mathscr{F}$ of M_2, and of course $K^* \subseteq \Gamma(\mathscr{F})$ by (10) (see (2.7)).

Substitute (8) in (7); we find that, for $p \in M_2$ and X, $Y \in L_1^*$,

$$\begin{aligned} c_0(p)\omega_1(X(\psi(p)), Y(\psi(p))) &= c_0(p)\{\psi^*\omega_1\}(\varphi(X)(p), \varphi(Y)(p)) \\ &= \omega_2(\varphi(X)(p), \varphi(Y)(p)), \end{aligned}$$

which, with (10), shows that ω_2 and $c_0\psi^*(\omega_1)$ agree on D. We have now proved that the conditions (a), (b), (c) are necessary.

Suppose in turn that ψ and c_0 are given and (a), (b) are satisfied. Let $p \in M_2$, and let $\xi \in T^*_{\psi(p)}M_1$ be such that

$$(\forall Y_p \in D_p)\langle \psi_{*p} Y_p, \xi \rangle_{\psi(p)} = 0. \tag{11}$$

Then

$$(\forall Y_p \in D_p)\ \langle Y_p, \psi_p^* \xi \rangle_p = 0$$

or (see (4.1))

$$\omega_2(Y_p, \mu_2^{-1}\psi_p^*\xi) = 0.$$

By (a), $\mu_2^{-1}\psi_p^*\xi \in D_p$. Since ω_2 is non-degenerate on D_p, this shows $\mu_2^{-1}\psi_p^*\xi = 0$. However, ψ is a submersion, and consequently

$$\psi_p^*\colon T^*_{\psi(p)}M_1 \to T_p^*M_2 \tag{12}$$

is injective; therefore $\xi = 0$. Referring back to our hypothesis (11), we have shown that

$$\psi_{*p}(D_p) = T_{\psi(p)}M_1. \tag{13}$$

Take X_p, $Z_p \in D_p$. By (b),

$$\begin{aligned}\omega_2(X_p, Z_p) &= c_0(p)\{\psi^*\omega_1\}(X_p, Z_p) = c_0(p)\omega_1(\psi_{*p}X_p, \psi_{*p}Z_p)\\ &= \langle \psi_{*p}X_p, c_0(p)\mu_1\psi_{*p}Z_p\rangle_{\psi(p)}\\ &= \langle X_p, c_0(p)\psi_p^*\mu_1\psi_{*p}Z_p\rangle_p \quad \text{(see (12))}\\ &= \omega_2(X_p, c_0(p)\mu_2^{-1}\psi_p^*\mu_1\psi_{*p}Z_p) \quad \text{by (4.1).}\end{aligned}$$

Since ω_2 is non-degenerate on D_p, by (a), this implies that

$$(\forall Z_p \in D_p) \quad Z_p = c_0(p)\mu_2^{-1}\psi_p^*\mu_1\psi_{*p}Z_p. \tag{14}$$

Observe that (13) establishes the surjectivity of $\psi_{*p}|D_p$, whilst (14) gives its injectivity; this proves (7.2)(e). Also, (13) and (14) together lead to

$$(\forall W \in T_{\psi(p)}M_1) \quad W = c_0(p)\psi_{*p}\mu_2^{-1}\psi_p^*\mu_1 W. \tag{15}$$

Define

$$\bar{\varphi}: \Gamma(M_1) \to \Gamma(M_2): X \mapsto c_0\mu_2^{-1}\psi^*\mu_1(X). \tag{16}$$

Then, by (15), we have instantly

$$(\forall X \in \Gamma(M_1))(\forall p \in M_2) \quad \psi_{*p}(\bar{\varphi}(X)(p)) = X(\psi(p)), \tag{17}$$

as asserted by (7.2)(g). That $\bar{\varphi}(\Gamma(M_1)) \subseteq \Gamma(\mathscr{F})$, as stated in (7.2)(f), is immediate from the definitions of $\bar{\varphi}$ and D.

By definition (see (4.3)), $X \in L_1 \Leftrightarrow \mathrm{d}(\mu_1(X)) = 0$. In that case, $\mathrm{d}(\mu_2\bar{\varphi}(X)) = \mathrm{d}(c_0\psi^*\mu_1 X) = c_0\psi^*(\mathrm{d}(\mu_1 X)) = 0$, so that $\bar{\varphi}(L_1) \subseteq L_2$. Likewise, if $X = \mu_1^{-1}(\mathrm{d}f) \in L_1^*$, where $f \in A_1$, then

$$\bar{\varphi}(X) = c_0\mu_2^1\psi^*(\mathrm{d}f) = \mu_2^{-1}\mathrm{d}\{c_0(f \circ \psi)\} \in L_2^*. \tag{18}$$

We write φ for $\bar{\varphi}|L_1^*: L_1^* \to \Gamma_2$. Notice that (f) has been proved.

Set $\tilde{\varphi} = c_0\psi^*: A_1 \to A_2$. Then, firstly,

$$\mu_2^{-1}\mathrm{d}\tilde{\varphi} = c_0\mu_2^{-1}\,\mathrm{d}\psi^* = c_0\mu_2^{-1}\psi^*\mathrm{d} = \varphi\mu_1^{-1}\mathrm{d}, \tag{19}$$

so that φ is the quotient linear map of $\tilde{\varphi}$ (see (4.5)(4)). Secondly, given $f, g \in A_1$ and $p \in M_2$, apply (4.5)(2) and (19).

$$\begin{aligned}
[\tilde{\varphi}(f), \tilde{\varphi}(g)]_2(p) &= \omega_2(\mu_2^{-1}\,\mathrm{d}(\tilde{\varphi}(f)), \mu_2^{-1}\,\mathrm{d}(\tilde{\varphi}(g)))(p) \\
&= \omega_2(c_0(p)\mu_2^{-1}\psi_p^*\{\mathrm{d}f(\psi(p))\}, c_0(p)\mu_2^{-1}\psi_p^*\{\mathrm{d}g(\psi(p))\}) \\
&= c_0(p)\{\psi_p^*(\omega_1)\}(c_0(p)\mu_2^{-1}\psi_p^*\{\mathrm{d}f(\psi(p))\}, c_0(p)\mu_2^{-1}\psi_p^*\{\mathrm{d}g(\psi(p))\})
\end{aligned}$$

(by the hypotheses (a) and (b))

$$\begin{aligned}
&= c_0(p)\omega_1(c_0(p)\psi_{*p}\mu_2^{-1}\psi_p^*\{\mathrm{d}f(\psi(p))\}, c_0(p)\psi_{*p}\mu_2^{-1}\psi_p^*\{\mathrm{d}g(\psi(p))\}) \\
&= c_0(p)\omega_1(\mu_1^{-1}\,\mathrm{d}f(\psi(p)), \mu_1^{-1}\,\mathrm{d}g(\psi(p))) \quad \text{by (15)} \\
&= c_0(p)[f, g]_1(\psi(p)) = (\tilde{\varphi}[f, g]_1)(p).
\end{aligned}$$

Consequently $\tilde{\varphi}\colon A_1 \to A_2$ is a Lie algebra homomorphism (and so must be its quotient map $\varphi\colon L_1^* \to L_2^*$). This proves (h).

Again, take $p \in M_2$. Repeating an earlier argument (see (10) and its sequel), use (2.4) to take functions $f_1, f_2, \ldots, f_m \in A_1$ such that the fields $\mu_1^{-1}\,\mathrm{d}f_i$, for $1 \leqslant i \leqslant m$, form a local basis for TM_1 over a neighbourhood of $\psi(p)$. The definitions of φ and of D (see (16), (7.2)(a)) show – since ψ is a submersion, so that φ is injective in each fibre – that φ transforms these fields into a local basis for D over a neighbourhood of p. Because φ is a Lie homomorphism, it follows that D is an integrable subbundle, as stated in (7.2)(d).

Let $X, Y \in \Gamma(M_1)$. Then, as X and $\bar{\varphi}(X)$, Y and $\bar{\varphi}(Y)$, are ψ-related (see (17)), so are $[X, Y]$ and $[\bar{\varphi}(X), \bar{\varphi}(Y)]$; that is,

$$\begin{aligned}
(\forall p \in M_2)\psi_{*p}([\bar{\varphi}(X), \bar{\varphi}(Y)](p)) &= [X, Y](\psi(p)) \\
&= \psi_{*p}(\bar{\varphi}([X, Y])(p)), \quad \text{again by (17).}
\end{aligned} \tag{20}$$

However, $\bar{\varphi}(X)$ and $\bar{\varphi}(Y)$ are in $\Gamma(\mathscr{F})$; therefore so is $[\bar{\varphi}(X), \bar{\varphi}(Y)]$, and $[\bar{\varphi}(X), \bar{\varphi}(Y)](p) \in D_p$. But, as already remarked, $\psi_{*p} \mid D_p$ is injective (see (14)), and (20) must imply that

$$[\bar{\varphi}(X), \bar{\varphi}(Y)] = \bar{\varphi}([X, Y]) \tag{21}$$

(at the arbitrary point $p \in M_2$). This completes the proof of (f); (k) is obvious.

The only assertion which remains to be proved is (7.2)(j). Now, from the definition (16), and by the injectivity of ψ_p^*,

$$\varphi(X)(p) = 0 \Leftrightarrow X(\psi(p)) = 0$$

for any $p \in M_2$ and $X \in \Gamma(M_1)$. As an immediate consequence,

$$(\bar{\varphi}(L_1^*))_p = \bar{\varphi}\{(L_1^*)_{\psi(p)}\}, (\bar{\varphi}(L_1))_p = \bar{\varphi}\{(L_1)_{\psi(p)}\},$$
$$(\bar{\varphi}(\Gamma(M_1)))_p = \bar{\varphi}\{(\Gamma(M_1))_{\psi(p)}\}, (\bar{\varphi}(\Gamma_1))_p = \bar{\varphi}\{(\Gamma_1)_{\psi(p)}\}. \tag{22}$$

On the other hand, $\bar{\varphi}(L_1^*)(p) = \bar{\varphi}(L_1)(p) = \bar{\varphi}(\Gamma_1)(p) = \bar{\varphi}(\Gamma(M_1))(p) = D_p$ by the definitions (7.2)(a) and (16), and by (2.4). Therefore (22) entails that

$$\bar{\varphi}\{(L_1^*)_{\psi(p)}\} \neq \bar{\varphi}(L_1^*), \bar{\varphi}\{(L_1)_{\psi(p)}\} \neq \bar{\varphi}(L_1),$$
$$\bar{\varphi}\{\Gamma(M_1))_{\psi(p)}\} \neq \bar{\varphi}(\Gamma(M_1)), \bar{\varphi}\{(\Gamma_1)_{\psi(p)}\} \neq \bar{\varphi}(\Gamma_1). \tag{23}$$

As $\bar{\varphi}$ is a Lie algebra homomorphism (see (21) above), we may apply (2.10)(b). Since $(L_1^*)_{\psi(p)} \in \Sigma^n(L_1^*)$, $(L_1)_{\psi(p)} \in \Sigma^n(L_1)$, $(\Gamma(M_1))_{\psi(p)} \in \Sigma^n(\Gamma(M_1))$, $(\Gamma_1)_{\psi(p)} \in \Sigma^n(\Gamma_1)$ for all $n \geqslant 1$, by (6.7), we deduce from (23) and (2.10)(b) that $\bar{\varphi}\{(L_1^*)_{\psi(p)}\} \in \Sigma^n(\bar{\varphi}(L_1^*))$ and so on. In turn, (22) now proves that $(\bar{\varphi}(L_1^*))_p \in \Sigma^n(\bar{\varphi}(L_1^*))$, and similarly in the other cases. This completes the proof of (7.2)(j).

REMARK. In this theorem, $L^*(\omega_1)$ occupies a special position because of the use of $B^1(M_1)$ in (7.1). To extend the result to $L(\omega_1)$ and $\Gamma(\omega_1)$, we require a technical lemma.

(7.5) LEMMA. *Let R be a Lie algebra over F, and S an ideal of R. Suppose that σ_1, σ_2 are Lie algebra homomorphisms $R \to \Gamma(M_2)$, and*

(a) $\sigma_1 | S = \sigma_2 | S = \sigma$,
(b) *$\sigma(S)$ is a 1-ample subalgebra of $\Gamma(M_2)$* (see (3.1)),
(c) *for every $p \in M_2$, $\sigma(S)(p) = \sigma_1(R)(p) = \sigma_2(R)(p)$.*

Then $\sigma_1 = \sigma_2$.

(Note that (b), (c) imply that $\sigma_1(R)$, $\sigma_2(R)$ are 1-ample, by (3.3)).

Proof. Take any $p \in M_2$ and $X \in R$. By (c),

$$\sigma_1(X)(p) - \sigma_2(X)(p) \in \sigma(S)(p),$$

and so there exists $Y \in S$ such that

$$\sigma(Y) - \sigma_1(X) + \sigma_2(X) \in (\Gamma(M_2))_p. \tag{1}$$

Now, for any $Z \in S$, apply (a):

$$[\sigma(Z), \sigma_1(X) - \sigma_2(X)] = \sigma_1[Z, X] - \sigma_2[Z, X] = 0$$

(as S is an ideal and $\sigma_1|S = \sigma_2|S$). Hence

$$[\sigma(Z), \sigma(Y)] = [\sigma(Z), \sigma(Y) - \sigma_1(X) + \sigma_2(X)]. \tag{2}$$

Suppose $\sigma(Z) \in (\sigma(S))_p$. Then $[\sigma(Z), \sigma(Y)] = \sigma[Z, Y] \in \sigma(S)$, as S is an ideal of R; and (2) expresses $[\sigma(Z), \sigma(Y)]$ as the bracket of two elements of $(\Gamma(M_2))_p$ – see (1). Consequently,

$$[(\sigma(S))_p, \sigma(Y)] \subseteq (\sigma(S))_p. \tag{3}$$

It follows that $(\sigma(S))_p + F\sigma(Y)$ is a subalgebra of $\sigma(S)$. If $\sigma(Y) \notin (\sigma(S))_p$, then (b) implies that $(\sigma(S))_p + F\sigma(Y) = \sigma(S)$; in turn, (3) now tells us that $(\sigma(S))_p$ is an ideal in $\sigma(S)$, which contradicts (b). So $\sigma(Y) \in (\sigma(S))_p$, and, by (1), this means that $\sigma_1(X)(p) = \sigma_2(X)(p)$. The result follows.

(7.6) THEOREM. *Let (M_i, ω_i) be a symplectic manifold of positive dimension and of class $\mathscr{C}$, for $i = 1, 2$. Then*

$$\varphi: L(\omega_1) \to \Gamma(\omega_2)$$

is a Lie algebra homomorphism over F whose image is a 1-ample subalgebra of $\Gamma(M_2)$ (see (3.1)*) if and only if there exist ψ and c_0, as in* (7.2)*, such that* (7.2)(a)–(c) *hold. In this case* (7.2)(d)–(k) *also hold.*

Proof. Let $\varphi(L_1) = K$ and $\varphi(L_1^*) = K^*$. Given $p \in M_2$, $K_p \in \Sigma(K)$ by hypothesis; so $\varphi^{-1}(K_p) \in \Sigma(L_1)$, by (2.10)(a), and there exists $q \in M_1$ such that $\varphi^{-1}(K_p) = (L_1)_q$ by (6.8). But now

$$(L_1^*)_q = L_1^* \cap (L_1)_q$$

and

$$K_p^* = (\varphi|L_1^*)(\varphi^{-1}(K_p) \cap L_1^*) = \varphi((L_1^*)_q). \tag{1}$$

However, $(L_1^*)_q \in \Sigma(L_1^*)$ by (6.6), so that, by (2.10)(b) applied to (1), either $K_p^* \in \Sigma(K^*)$ or $K_p^* = K^*$. Suppose, if possible, that $K_p^* = K^*$. Then $K_p \supseteq K^*$ and $\varphi^{-1}(K_p) = (L_1)_q \supseteq L_1^*$, which is impossible. The contradiction establishes that $K_p^* \in \Sigma(K^*)$. Thus K^* is 1-ample, and (7.2) may be applied to $\varphi|L_1^*$. In particular, $\varphi|L_1^*$ extends to a Lie algebra homomorphism $(\varphi|L_1^*)^-: \Gamma(M_1) \to \Gamma(M_2)$ such that $(\varphi|L_1^*)^-(L_1) \subseteq L_2$, which is given by the formula (7.2)(c).

To complete the proof that (7.2)(a) – (c) are necessary, it is therefore only necessary to demonstrate that

$$(\varphi|L_1^*)^-|L_1 = \varphi. \tag{2}$$

Take in (7.5) $R = L_1$, $S = L_1^*$ (and recall that $L_1^* \supseteq [L_1, L_1]$, by (4.3)); $\sigma_1 = \varphi$, $\sigma_2 = (\varphi|L_1^*)^-|L_1$. The hypothesis (7.5)(a) is automatic, whilst (7.5)(b) has just been proved (K^* is 1-ample). As for (7.5)(c), we have for each $p \in M_2$

$$\varphi(L_1)(p) \supseteq \varphi(L_1^*)(p) \supseteq \varphi(L_1)^{(1)}(p) = \varphi(L_1)(p),$$

by (3.2)(a), since $\varphi(L_1)$ is 1-ample; and, since (7.2)(j) assures us that $(\varphi|L_1^*)^-(L_1)$ is also 1-ample, the same argument applies to it. This proves (7.5)(c). The required equality (2) now follows. (The converse implication, that (7.2)(a)–(c) are sufficient, is already contained in (7.2)(f), (j).)

(7.7) THEOREM. *Let (M_i, ω_i) be a symplectic manifold of positive dimension and of class $\mathscr{C}$, for $i = 1, 2$. Then, if*

$$\varphi: \Gamma(\omega_1) \to \Gamma(\omega_2)$$

is a Lie algebra homomorphism over F whose image is a 2-ample subalgebra of $\Gamma(M_2)$ (see (3.1)), there exist ψ and c_0, as in (7.2), such that (7.2)(a)–(k) hold.

Proof. Repeat the proof of (7.6), reading Σ^2 in place of Σ, L_1 instead of L_1^*, Γ_1 instead of L_1, and (7.6) in place of (7.2). (Note also that 2-ample implies 1-ample.)

(7.8) NOTES. As remarked in (7.3), the formula (7.2)(c) defines a homomorphism $\Gamma(M_1) \to \Gamma(M_2)$ which need not carry Γ_1 into Γ_2. Thus there can be no converse implication in (7.7). Nor would it be sufficient to require only that $\varphi(\Gamma_1)$ be 1-ample. Take $M_1 = M_2 = S^1 \times \mathbb{R}$, and let x denote the standard coordinate in $\mathbb{R}$, θ the standard local coordinate in S^1 (defined modulo 2π). The symplectic form is to be $d\theta \wedge dx = d(-x\, d\theta)$, as in (6.11). Now – again compare (6.11) –

$$L_1 = L_1^* \oplus \mathbb{R}X, \quad \text{where } X = \frac{\partial}{\partial x} \text{ for instance,}$$

$$\Gamma_1 = L_1 \oplus \mathbb{R}Y, \text{ where } Y = x\frac{\partial}{\partial x} \text{ for instance.}$$

(Direct computation shows that these choices for X and Y are possible.) Observe that $[X, Y] = X$, so that X and Y span a subalgebra Q of Γ_1 which is complementary to the ideal L_1^*. Define the quotient-inclusion homomorphism $\varphi: \Gamma_1 \to \Gamma_2$ by: $\varphi|L_1^* = 0$, $\varphi|Q$ is the identity map of Q. Then $\varphi(\Gamma_1) = Q$ is 1-ample (but not 2-ample, since $Q^{(2)} = 0$). Indeed,

$$Q = \{(\alpha + \beta x)X: \alpha, \beta \in \mathbb{R}\}, \qquad Q^{(1)} = \{\alpha X: \alpha \in \mathbb{R}\},$$

and so $Q_p = \{(\alpha + \beta x)X: \alpha + \beta x(p) = 0\}$ does not include $Q^{(1)}$ and is of codimension 1, which shows it is in $\Sigma(Q)$. However, φ cannot be derived from ψ and c_0 as in (7.2); if it were, its image would have to be infinite-dimensional.

Although it is convenient to consider 1-ample or 2-ample images in order to deduce (7.6) and (7.7) from (7.2), these are not conditions of a very explicit kind, and from the algebraic point of view they are quite unsatisfactory.

8. Epimorphisms

We retain the conventions of (7.3).

(8.1) THEOREM. *Let (M_i, ω_i) be a symplectic manifold of class $\mathscr{C}$ and of positive dimension, for $i = 1, 2$. Then*

(a) *if $\varphi: L_1^* \to \Gamma_2$ is a Lie algebra epimorphism, then $\Gamma_2 = L_2^*$;*
(b) *if $\varphi: L_1 \to \Gamma_2$ is a Lie algebra epimorphism, then $\Gamma_2 = L_2$;*
(c) *if $\varphi: L_1^* \to L_2$ is a Lie algebra epimorphism, then $L_2 = L_2^*$;*
(d) *if $\varphi: \Gamma_1 \to L_2^*$ is a Lie algebra epimorphism, then $L_2^* = \Gamma_2$;*
(e) *if $\varphi: \Gamma_1 \to L_2$ is a Lie algebra epimorphism, then $L_2 = \Gamma_2$;*
(f) *if $\varphi: L_1 \to L_2^*$ is a Lie algebra epimorphism, then $L_2^* = L_2$.*

(See Section 5 for the significance in certain situations of the equalities asserted.)

The proof of this result, with that of (8.2), will be given at (8.3).

(8.2) THEOREM. *Given (M_i, ω_i) as in* (8.1), *suppose that either* (a) $\varphi: L_1^* \to L_2^*$, *or* (b) $\varphi: L_1 \to L_2$, *or* (c) $\varphi: \Gamma_1 \to \Gamma_2$, *is a Lie algebra epimorphism. In each case, there is a diffeomorphism ψ of M_2 with an open and closed subset of M_1, and a function $c_0 \in C(M_2)$ which is everywhere non-zero, such that $\omega_2 = c_0\psi^*(\omega_1)$ and the map $\bar{\varphi}: \Gamma(M_1) \to \Gamma(M_2)$ defined by*

$$(\forall X \in \Gamma(M_1)) \quad \bar{\varphi}(X) = c_0\mu_2^{-1}\psi^*\mu_1(X)$$

is a Lie algebra homomorphism satisfying the equality

$$(\forall X \in \Gamma(M_1)) \quad \bar{\varphi}(X) = \psi_*^{-1}(X \mid \psi(C)) \tag{1}$$

and agreeing with φ on the domain of φ. Both ψ and c_0 are uniquely determined by φ.

(8.3) *Proof of* (8.1) *and* (8.2). Let us write V for the domain, and W for the range, of φ, in each of the nine cases. By (6.7), W is 2-ample. Thus, in every case, either (7.2) or (7.6) or (7.7) applies, and the assertions (7.2)(a)–(k) hold.

By (7.2)(f), $\bar{\varphi}(\Gamma(M_1)) \subseteq \Gamma(\mathscr{F})$. As φ is an epimorphism and $W \supseteq L_2^*$, it follows that $L_2^* \subseteq \varphi(V) \subseteq \Gamma(\mathscr{F})$. Use (4.8): the subbundle D tangent to $\mathscr{F}$ must be the whole of TM_2. Hence, by (7.2)(e), ψ is a local diffeomorphism of M_2 with an open subset of M_1.

Suppose $p, q \in M_2$ and $p \neq q$. By (2.4), there exists $X \in L_2^*$ with $X(p) = 0$ but $X(q) \neq 0$. But there exists $Y \in V$ such that $X = \varphi(Y)$; by (7.2)(c), then, $Y(\psi(p)) = 0$ and $Y(\psi(q)) \neq 0$, and so $\psi(p) \neq \psi(q)$. This proves that ψ is one-one, and therefore maps each individual component of M_2 diffeomorphically on to an open set in M_1 (although we do not yet know that it is a homeomorphism of M_2 with $\psi(M_2)$). Ergo, we may define $\psi_*^{-1}(X \mid \psi(M_2)) \in \Gamma(M_2)$, for given $X \in \Gamma(M_1)$, by treating each component of M_2 separately.

Take $p \in M_2$ and $Y \in T_pM_2$. Then

$$\begin{aligned}
&\omega_1(\psi_* c_0(p)\mu_2^{-1}\psi^*\mu_1(X),\ \psi_*(Y))(\psi(p))\\
&\quad = \{(\psi^*\omega_1)(c_0\mu_2^{-1}\psi^*\mu_1 X,\ Y)\}(p)\\
&\quad = \omega_2(\mu_2^{-1}\psi^*\mu_1 X,\ Y)(p) \quad \text{as } c_0\psi^*\omega_1 = \omega_2\\
&\quad = \langle Y,\ -\psi^*\mu_1(X)\rangle_p \quad \text{by definition (4.1)}\\
&\quad = \langle \psi_* Y,\ -\mu_1(X)\rangle_{\psi(p)}\\
&\quad = \omega_1(X,\ \psi_* Y)(\psi(p)), \quad \text{by (4.1).}
\end{aligned}$$

As p is an arbitrary point of M_2 and $\psi_* Y$ is an arbitrary element of $T_{\psi(p)}M_1$, we deduce

$$c_0\mu_2^{-1}\psi^*\mu_1(X) = \psi_*^{-1}(X \mid \psi(M_2)),$$

thus proving (8.2)(1).

Let C be any component of M_2, and C_1 the component of M_1 which includes $\psi(C)$. Suppose x is in the closure of $\psi(C)$ (and therefore in C_1). By (4.8), there exists $X \in L_1^*$ such that $X(x) \neq 0$. Define $Y \in \Gamma(M_2)$ to agree with $\psi_*^{-1}(X)$ on C and to vanish elsewhere. Then $Y \in L_2^*$, since (1) applies on C (compare (7.4)(18)), and, off C, it is obvious.

By construction, $\psi_* Y \mid \psi(C) = X \mid \psi(C)$. Hence, if there exists $Z \in V$ such that $Y = \varphi(Z) = \psi_*^{-1}(Z \mid \psi(M_2))$, necessarily $Z \mid \psi(C) = X \mid \psi(C)$ and, by continuity, $Z(x) = X(x) \neq 0$. It follows that $Y \notin \varphi(V_x)$. Since $V_x \in \Sigma^2(V)$ and $\varphi(V_x) \neq W$, (2.10)(b) yields that $\varphi(V_x) \in \Sigma^2(W)$. From (6.6)–(6.10), we know that then

$$\varphi(V_x) = W_y \text{ for some } y \in M_2. \tag{2}$$

If possible, suppose $\psi(y) \neq x$. Then, by (2.4), there exists $U \in L_1^*$ such that $U(x) = 0$ and $U(\psi(y)) \neq 0$. The formula (1) shows that

$$\varphi(U) = \psi_*^{-1}(U \mid \psi(M_2)),$$

and therefore $\varphi(U) \notin W_y$ and $U \in V_x$, contradicting (2). Hence $\psi(y) = x$. Furthermore, $y \in C$; for, as C is closed in M_2 and ψ is both open and one-one, the assumption that $y \notin C$ is incompatible with our hypothesis that $x = \psi(y)$ is in the closure of $\psi(C)$. In fact, then $x \in \psi(C)$, and $\psi(C)$ must be closed. Since it is also open and connected, $\psi(C) = C_1$. This evidently proves that ψ is a diffeomorphism of M_2 with the union of certain components of M_1; and (8.2) is therefore proved in full. However, (8.1) is now almost obvious: φ may be factorised as $\psi_*^{-1}j$, where j is the map which transforms a vector field on M_1 to its restriction over $\psi(M_2)$, and both ψ_*^{-1} and j clearly carry the fields of a given kind (globally, locally, or conformally Hamiltonian) onto all fields of the same kind on M_2 or $\psi(M_2)$ respectively; the assertions (8.1)(a)–(f) follow.

(8.4) COROLLARY. *In* (8.1)(a)–(f), *each of the epimorphisms must split in the category of Lie homomorphisms.*

(8.5) REMARKS.

(a) The conclusion of (8.1) holds under weaker hypotheses. The proof (8.3) requires only that the image of φ be 1-ample when the domain is L_1 or L_1^*, 2-ample when the domain is Γ_1; and that it satisfy certain 'separation hypotheses' which were ensured in (8.3) by its including L_2^*.
(b) We have of course tacitly (though largely unnecessarily) assumed our manifolds have empty boundary. If they were allowed to have boundaries, we could not prove as in (8.3) that $\psi(C) = C_1$ or that the image of ψ is open. However, our method may be somewhat tediously modified to prove that, in this case also, ψ is a diffeomorphism with its image.

(8.6) THEOREM. *Suppose that, in any of the cases* (8.1)(a)–(f), (8.2)(a)–(c), *the Lie homomorphism φ is an isomorphism. Then there exist a diffeomorphism ψ of M_2 with M_1 and an everywhere non-zero function $c_0 \in C(M_2)$ such that $\omega_2 = c_0\psi^*(\omega_1)$ and*

$$\varphi = \psi_*^{-1};$$

in particular, the domain and range of φ must consist of vector fields of the same kind.

(8.7) THEOREM. *Suppose that, for $i = 1, 2$, (M_i, ω_i) is a symplectic manifold of class $\mathscr{C}$ and of positive dimension; and let $\varphi: A(M_1) \to A(M_2)$ be an epimorphism of Lie algebras. Then there are a diffeomorphism ψ of M_2 with an open and closed subset of M_1 and an everywhere non-zero function $c_0 \in C(M_2)$, and an F-linear map $\Phi: A_1 \to C(M_2)$ vanishing on $A_1^{(1)}$, such that $\omega_2 = c_0\psi^*(\omega_1)$ and*

$$\varphi = c_0\psi^* + \Phi. \tag{1}$$

Each of ψ, c_0 and Φ is uniquely determined by φ.

Proof. Use the notations of (7.3). By (4.5)(4), $\mu_i^{-1}\mathrm{d}$ is a Lie epimorphism with kernel $C(M_i)$. As φ is epimorphic,

$$\varphi(C(M_1)) \subseteq C(M_2);$$

consequently φ induces a Lie epimorphism $\kappa\colon L_1^* \to L_2^*$ such that $\mu_2^{-1}\mathrm{d}\varphi = \kappa\mu_1^{-1}\mathrm{d}$. By (8.2), there is a diffeomorphism ψ of M_2 with an open and closed subset of M_1, and there is a non-vanishing $c_0 \in C(M_2)$, such that $\omega_2 = c_0\psi^*(\omega_1)$ and $\kappa = c_0\mu_2^{-1}\psi^*\mu_1$. Hence $\mu_2^{-1}\mathrm{d}\varphi = c_0\mu_2^{-1}\mathrm{d}\psi^*$ and $\mathrm{d}(\varphi - c_0\psi^*) = 0$. It follows that

$$\Phi = \varphi - c_0\psi^* \text{ maps } A_1 \text{ into } C(M_2). \tag{2}$$

However, it is easily checked, for instance from (4.5)(2), that $c_0\psi^*$ is a Lie algebra epimorphism. Thus, for any $f, g \in A_1$,

$$\Phi[f, g] = [\varphi f, \varphi g - c_0\psi^* g] + [\varphi f - c_0\psi^* f, c_0\psi^* g] = 0,$$

from (2). This completes the proof of the Theorem.

(8.8) ADDENDA. It is trivial that, for ψ, c_0, and Φ as in the theorem, $\Phi + c_0\psi^*$ is a Lie homomorphism $A_1 \to A_2$. In general, it is not onto (for instance, ψ might be the identity of a compact real symplectic manifold M, $A_1 = A_2 = A(M)$, and c_0 might be identically equal to unity. Then $c_0\psi^*$ is the identity of $A(M)$, but it is clear from (5.4) that – in the notation used there – if

$$\Phi\colon C_0 \oplus A^{(1)} \to C(M)\colon (f, g) \mapsto -f,$$

then $\Phi + c_0\psi^*$ is not epimorphic, being the projection on $A^{(1)}$). Thus the homomorphisms to which Theorem (8.7) applies are, more generally, those which differ from epimorphisms by linear maps whose kernel includes $A_1^{(1)}$ and whose image is included in $C(M_2)$. When $H^{2n}(M_1, F) = 0$, as for the real case when M_1 has no compact components, then (5.3) shows that all such linear maps vanish, so that all epimorphisms $A(M_1) \to A(M_2)$ are of the form $c_0\psi^*$.

(8.9) NOTES

(a) As in (8.5)(a), one may prove (8.6) under weaker hypotheses (those which ensure that (8.3) remains valid). An alternative approach would be to construct ψ directly from (6.5) – but in the present context it would be uneconomical to do so.

(b) Suppose that M_1 is real and compact, with components $Q_1, \ldots, Q_a$; and M_2 has components $R_1, \ldots, R_b$. Whether in (8.2) or (8.6), M_2 must also be compact and $b \leqslant a$. We may suppose $\psi(R_j) = Q_j$ for $1 \leqslant j \leqslant b$, and identify $C(M_1)$ with $\mathbb{R}^a$, $C(M_2)$ with $\mathbb{R}^b$, and $\psi^*\colon C(M_1) \to C(M_2)$ with the projection on the first a coordinates $\mathbb{R}^b \to \mathbb{R}^a$, in the obvious way, with isomorphisms

$$\tau_1\colon \mathbb{R}^a \to C(M_1), \quad \tau_2\colon \mathbb{R}^b \to C(M_2).$$

The isomorphisms $A_i \cong C(M_i) \oplus A_i^{(1)}$ of (5.4) give rise to projections $\pi_i\colon A_i \to C(M_i)$ which, in view of (5.3), may be expressed by

$$\pi_1(f) = \tau_1(\alpha_1, \ldots, \alpha_a), \quad \text{where } \alpha_j = \left(\int_{Q_j} f\eta\right) \Big/ \left(\int_{Q_j} \eta\right) \tag{1}$$

and similarly for π_2. Hence the Φ of (8.6) takes the form $\Phi = \tau_2 \Delta \tau_1^{-1} \pi_1$, where $\Delta\colon \mathbb{R}^a \to \mathbb{R}^b$ is a linear map, and π_1 is given by (1). However, $c_0\psi^*$ carries $C(M_1)$ on to $C(M_2)$:

$$c_0\psi^*\tau_1(\alpha_1, \ldots, \alpha_a) = \tau_2(\lambda_1\alpha_1, \ldots, \lambda_b\alpha_b) \tag{2}$$

where $\lambda_1, \ldots, \lambda_b$ are the values of c_0 on $R_1, \ldots, R_b$ respectively; and $c_0\psi^*$ also carries $A_1^{(1)}$ on to $A_2^{(1)}$. It follows that $\varphi = c_0\psi^* + \Phi$ will be surjective if and only if $T + \Delta\colon \mathbb{R}^a \to \mathbb{R}^b$ is surjective, where

$$T(\alpha_1, \ldots, \alpha_a) = (\lambda_1\alpha_1, \ldots, \lambda_a\alpha_a).$$

(Any noncompact components of M_1 or M_2 may be ignored for the purposes of the question of surjectivity; on the corresponding factors of A_1, Φ vanishes and $c_0\psi^*$ is onto.)

(8.10) THEOREM. *Let (M_i, ω_i) be a symplectic manifold of positive dimension and of class $\mathscr{C}$, for $i = 1, 2$. If $\varphi\colon A(M_1) \to A(M_2)$ is a Lie algebra isomorphism, then there is a diffeomorphism ψ of M_2 with M_1, and there is a nonvanishing function $c_0 \in C(M_2)$, such that $\omega_2 = C_0\psi^*(\omega_1)$ and $\varphi - c_0\psi^*$ vanishes on the commutator of $A(M_1)$ and takes values in the centre of $A(M_2)$.*

(8.11) REMARK. A symplectic manifold (M, ω) of class $\mathscr{C}$ has dimension zero if and only if $\Gamma(\omega) = 0$; or if and only if $L(\omega) = 0$; or if and only if $L^*(\omega) = 0$; or if and only if $A(M)$ is abelian. It is therefore trivial to describe what happens to the preceding results if one omits the requirement that the manifolds be of positive dimension.

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