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Complex analytic compactifications of $\mathbb{C}^3$

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Introduction

The purpose of this paper is to give a proof of the announcement [7].

Let X be an n-dimensional compact connected complex manifold and Y an analytic subset of X. We call the pair (X, Y) a complex analytic compactification of $\mathbb{C}^n$ if $X - Y$ is biholomorphic to $\mathbb{C}^n$. By a theorem of Hartogs, Y is a divisor on X.

In this paper, we will consider only the case of $n = 3$. Let (X, Y) be a complex analytic compactification of $\mathbb{C}^3$. Assume that Y has at most isolated singularities. Then Y is normal. Thus, by Peternell-Schneider [18] (cf. Brenton [2]), X is projective. In particular, X is a Fano 3-fold of index $r(1 \leqslant r \leqslant 4)$ with $b_2(X) = 1$. In the case of $r \geqslant 2$, such a (X, Y) is completely determined (cf. [3], [5], [6], [18]). In the case of $r = 1$, by a detailed analysis of the singularities of the boundary Y, we can prove that such a compactification (X, Y) does not exist. Thus, we have:

THEOREM. *Let (X, Y) be a complex analytic compactification of $\mathbb{C}^3$. Assume that Y has at most isolated singularities. Then X is a Fano 3-fold of index $r(2 \leqslant r \leqslant 4)$ with $b_2(X) = 1$, and*

(1) $r = 4 \Rightarrow$ $(X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$,

(2) $r = 3 \Rightarrow$ $(X, Y) \cong (\mathbb{Q}^3, \mathbb{Q}_0^2)$, *where $\mathbb{Q}^3$ is a smooth quadric hypersurface in $\mathbb{P}^4$ and $\mathbb{Q}_0^2$ is a quadric cone in $\mathbb{P}^3$,*

(3) $r = 2 \Rightarrow$ $(X, Y) \cong (V_5, H_5)$, *where V_5 is a complete intersection of three hyperplanes in the Grassmannian $G(2, 5) \hookrightarrow \mathbb{P}^9$, and H_5 is a normal hyperplane section of V_5 with exactly one rational double point of A_4-type.*

This paper consists of five sections. In Section 1, we will prove that $(X, Y) \cong (V_{22}, H_{22})$ if such a (X, Y) exists in the case of $r = 1$, where $V_{22} \hookrightarrow \mathbb{P}H^0(V_{22}, \mathcal{O}(-K_{V_{22}})) \cong \mathbb{P}^{13}$ is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ (index 1, genus 12) and H_{22} is a normal hyperplane section which is rational (Proposition 1.13). In Section 2, we will determine the singularities of $Y = H_{22}$ (Proposition 2.5). In Sections 3 and 4, we will prove that such a $(X, Y) = (V_{22}, H_{22})$ does not exist. In Section 5, we will refer to a recent work of Peternell-Schneider [18] (c.f. Peternell [19]) on a projective compactification (X, Y) of $\mathbb{C}^3$ with $b_2(X) = 1$ (especially, the case where the boundary Y is non-normal), and prove that there is a compactification (X, Y) of $\mathbb{C}^3$ with a non-normal boundary Y in the case of the index $r = 1$.

Notations

- K_M: a canonical divisor on a projective manifold M.
- $b_i(M)$: the ith Betti number of M.
- $N_{C|M}$: the normal bundle of C in M.
- $c_1(\mathscr{F}_M)$: the first Chern class of a locally free sheaf $\mathscr{F}_M$ on M.
- $m(\mathcal{O}_{Y,x})$: the multiplicity of the local ring $\mathcal{O}_{Y,x}$ at x.

1. The structure in the case of $r = 1$

1. Let (X, Y) be an analytic compactification of $\mathbb{C}^3$ such that Y has at most isolated singularities. Assume that the index $r = 1$. Then X is a Fano 3-fold of index 1 with $\operatorname{Pic} X \cong \mathbb{Z} c_1 \mathcal{O}_X(Y)$ ([3], [9], [18]). Then, by Proposition 1, Proposition 2 and Proposition 3 in [3], we have:

LEMMA 1.1. (1) $K_Y = 0$,
(2) $H^1(Y, \mathcal{O}_Y) = 0, H^2(Y, \mathcal{O}_Y) \cong \mathbb{C}$,
(3) $H^1(Y; \mathbb{Z}) = 0, H^2(Y; \mathbb{Z}) \cong \mathbb{Z} c_1(N_{Y|X})$.

Let Sing Y be the singular locus of Y and put $S := \{ y \in \text{Sing } Y; y \text{ is not a rational singularity}\}$. Let $\pi\colon \tilde{Y} \to Y$ be the minimal resolution of singularities of Y and Z be the fundamental cycle of S associated with the resolution $(\tilde{Y}, \pi)$. We put $E := \pi^{-1}(\text{Sing } Y), C := \pi^{-1}(S) = \bigcup_{i=1}^{s} C_i$ (C_i's are irreducible).

LEMMA 1.2. $S \neq \phi$

Proof. Let us consider the following exact sequence (see [2]):

$$\to H^1(Y; \mathbb{R}) \to H^1(\tilde{Y}; \mathbb{R}) \to H^1(E; \mathbb{R}) \to H^2(Y; \mathbb{R}) \to$$
$$\to H^2(\tilde{Y}; \mathbb{R}) \to H^2(E; \mathbb{R}) \to 0. \tag{1.1}$$

Assume that $S = \phi$. Then we have $K_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}$ and $H^1(E; \mathbb{R}) = 0$. By Lemma 1.1(3), $H^1(\tilde{Y}; \mathbb{R}) = 0$. Thus $\tilde{Y}$ is a K-3 surface. Since $b_2(Y) = 1$ and Y is projective, we have $b^+(Y) = 1$. On the other hand, by Brenton [2], $b^+(Y) = b^+(\tilde{Y})$. Thus we have $b^+(\tilde{Y}) = 1$. This is a contradiction. Therefore $S \neq \phi$. Q.E.D.

Thus, by Umezu [22], we have:

LEMMA *U*.

(1) $K_{\tilde{Y}} = -\Sigma_{i=1}^{s} n_i C_i (n_i > 0,\ n_i \in \mathbb{Z})$, *and thus* $\tilde{Y}$ *is a ruled surface over a non-singular compact algebraic curve R of genus* $q = \dim H^1(\tilde{Y}; \mathcal{O}_{\tilde{Y}})$ *(namely,* $\tilde{Y}$ *is birationally equivalent to a* $\mathbb{P}^1$*-bundle over R),*

(2) *if* $q \neq 1$, *then S consists of one point with* $p_g := \dim(R^1\pi_* \mathcal{O}_{\tilde{Y}})_s = q + 1$,

(3) *if* $q = 1$, *then S consists of either one point with* $p_g = 2$ *or two points with* $p_g = 1$. *Moreover, in the second case of* (3), *both of the two points are simply elliptic.*

LEMMA 1.3. *S consists of one point with* $p_g = q + 1$ *and* $b_2(\tilde{Y}) = b_2(E) + 1$.

Proof. Assume that *S* consists of two points. By Lemma *U*(3), these two points are simply elliptic, and $C = \pi^{-1}(S) = C_1 \cup C_2$, where C_1, C_2 are distinct sections of $\tilde{Y}$. Since $b_2(Y) = 1$, by (1.1), we have $b_1(\tilde{Y}) = b_1(E)$. Since

$$2 = b_1(\tilde{Y}) = b_1(E) \geqslant b_1(C) = b_1(C_1) + b_1(C_2) = 4,$$

we have a contradiction. Therefore *S* consists of one point with $p_g = q + 1$. Since $b_1(\tilde{Y}) = b_1(E)$ and $b_2(Y) = 1$, we have $b_2(\tilde{Y}) = b_2(Y) + b_2(E) = 1 + b_2(E)$.

Q.E.D.

2. Let *U* be a strongly pseudoconvex neighborhood of *C* in $\tilde{Y}$. A cycle *D* on *U* is an integral combination of the C_i, $D = \Sigma d_i C_i$ $(1 \leqslant i \leqslant s)$, with $d_i \in \mathbb{Z}$. We denote the support of *D* by $|D| = \bigcup C_i, d_i \neq 0$. We put $\mathcal{O}_D = \mathcal{O}_U / \mathcal{O}_U(-D)$. Let K_U be a canonical divisor on *U*. We put $\chi(D) := \dim H^0(U, \mathcal{O}_D) - \dim H^1(U, \mathcal{O}_D)$. Then, by the Riemann-Roch theorem [21],

$$\chi(D) = -\tfrac{1}{2}\{(D \cdot D) + (D \cdot K_U)\}. \tag{1.2}$$

For two cycles *A*, *B*, we have, by (1.2),

$$\chi(A + B) = \chi(A) + \chi(B) - (A \cdot B). \tag{1.3}$$

LEMMA 1.4.

(1) $q = 0 \Rightarrow \tilde{Y}$ *is a rational surface, and* $K_{\tilde{Y}} = K_U = -Z$.

(2) $q \neq 0 \Rightarrow$ *there is an irreducible component* C_{i_1} *of C such that* C_{i_1} *is a section of* $\tilde{Y}$ *and the rest* $\overline{C - C_{i_1}} = \bigcup_{i \neq i_1} C_i (\neq \phi)$ *is contained in the singular fibers of* $\tilde{Y}$, *and* $-K_{\tilde{Y}} = Z + C_{i_1}$.

Proof. (1) Since $q = 0$, we have $p_g = 1$. Thus *S* consists of a minimally elliptic singularity. By Laufer [11], we have $K_{\tilde{Y}} = K_U = -Z$.

(2) Since $K_{\tilde{Y}} = -\Sigma_{i=1}^{s} n_i C_i$ ($n_i > 0, n_i \in \mathbb{Z}$), for a general fiber f of $\tilde{Y}$, we have

$$2 = (-K_{\tilde{Y}} \cdot f) = \sum_{i=1}^{s} n_i (C_i \cdot f).$$

Thus we have the following:

(i) there is an irreducible component C_{i_1} of C such that $n_{i_1} = 2$, $(C_{i_1} \cdot f) = 2$ and $(C_i \cdot f) = 0$ $(i \neq i_1)$,

(ii) there are two irreducible components C_1, C_2 of C such that $n_1 = n_2 = 1$, $(C_i \cdot f) = 1$ $(i = 1, 2)$, $(C_i \cdot f) = 0$ $(i \geqslant 3)$, and

(iii) there is an irreducible component C_1 of C such that $n_1 = 1$, $(C_1 \cdot f) = 2$, $(C_i \cdot f) = 0$ $(i \neq 1)$.

Claim 1. The case (ii) can not occur.

Indeed, by the adjunction formula, the curve C_i $(i = 1, 2)$ is a non-singular elliptic curve with $(C_1 \cdot C_2) = 0$ and there is no other irreducible component of C which intersects C_i $(i = 1, 2)$. Thus $C = C_1 \cup C_2$ $(C_1 \cap C_2 = \phi)$, namely, S consists of two points. This contradicts Corollary 1.4.

Claim 2. The case (iii) can not occur.

Indeed, by the adjunction formula, C_1 is a non-singular elliptic curve and there is no other irreducible component of C which intersects C_1. By Corollary 1.4, we have $C = C_1$, hence, $K_{\tilde{Y}} = -C_1$. This contradicts Lemma 1 and Lemma 2 in Umezu [22].

Thus we have the case (i). In particular, C_{i_1} is a section of the ruled surface $\tilde{Y}$ and C_i's $(i \neq i_1)$ are all contained singular fibers of $\tilde{Y}$. We also have $C - C_{i_1} \neq \phi$ by the same reason as above. Since $n_{i_1} = 2$, we have $-K_{\tilde{Y}} = 2C_{i_1} + \Sigma_{i \neq i_1} n_i C_i$ $(n_i > 0)$. We remark that the genus of C_{i_1} is equal to $q = h^1(\mathcal{O}_{\tilde{Y}}) \geqslant 1$.

Claim 3. $-K_{\tilde{Y}} = C_{i_1} + Z$.

Indeed, since $(-K_{\tilde{Y}} - C_{i_1}) \cdot C_i \leqslant 0$ $(1 \leqslant i \leqslant s)$, by definition of the fundamental cycle, $-K_{\tilde{Y}} - C_{i_1} \geqslant Z$. Now, assume that $-K_{\tilde{Y}} = C_{i_1} + Z + D$, where $D > 0$. For a general fiber f of $\tilde{Y}$, we have $2 = (-K_{\tilde{Y}} \cdot f) = (C_{i_1} \cdot f) + (Z \cdot f) + (D \cdot f)$. Since $C_{i_1} \subset |Z|$, we have $(C_{i_1} \cdot f) = (Z \cdot f) = 1$, and $(D \cdot f) = 0$. This means that the support $|D|$ is contained in the singular fibers of $\tilde{Y}$. Since

$$H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-Z)) \cong H^0(\tilde{Y}; \mathcal{O}_{\tilde{Y}}(-C_{i_1} - D)) = 0 \quad \text{and } H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0,$$

by the Riemann-Roch theorem, we have

$$0 \geqslant -\dim H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-Z)) = \tfrac{1}{2}(Z \cdot Z + Z \cdot K_{\tilde{Y}}) + 1 - q.$$

Since $K_M = K_U$, by (1.2), we have $\chi(Z) \geqslant 1 - q$. Since $H^0(U; \mathcal{O}_Z) \cong \mathbb{C}$ (cf. [11,

p. 1260]), $\chi(Z) = 1 - \dim H^1(U, \mathcal{O}_Z) \leqslant 1$. Since S does not consists of a rational singularity, $\chi(Z) \neq 1$ by Artin [1]. Thus we have

$$1 - q \leqslant \chi(Z) \leqslant 0. \tag{1.4}$$

Since $1 - q = \chi(C_{i_1}) = \chi(-K_U - C_{i_1}) = \chi(Z + D) = \chi(Z) + \chi(D) - (D \cdot Z)$, we have

$$\chi(Z) = -\chi(D) + 1 - q + (D \cdot Z). \tag{1.5}$$

Since $(D \cdot Z) \leqslant 0$, by (1.4), (1.5), we have $\chi(D) \leqslant 0$. On the other hand, the support $|D|$ is contained in the singular fibers of $\tilde{Y}$. Thus, the contraction of $|D|$ in $\tilde{Y}$ yields rational singularities. Hence $\chi(D) \geqslant 1$. This is a contradiction. Therefore $D = 0$, namely, $-K_{\tilde{Y}} = Z + C_{i_1}$. Q.E.D.

COROLLARY 1.5. *Assume that* $q \neq 0$. *Then*
(1) $(C_{i_1} \cdot Z) = 2 - 2q$.
(2) $(Z \cdot Z) \leqslant (C_{i_1} \cdot C_{i_1})$.

Proof. Since $K_{\tilde{Y}} = -Z - C_{i_1}$, by the adjunction formula, we have (1) and (2). Q.E.D.

LEMMA 1.6.
(1) $q \neq 0 \Rightarrow b_2(\tilde{Y}) \leqslant 9 - 4q + \sqrt{9 + 8q}$.
(2) $q = 0 \Rightarrow 11 \leqslant b_2(\tilde{Y}) \leqslant 13$.

Proof. (1) By the Noether formula, we have

$$10 - 8q = (K_{\tilde{Y}} \cdot K_{\tilde{Y}}) + b_2(\tilde{Y}). \tag{1.6}$$

Since $K_{\tilde{Y}} = -Z - C_{i_1}$, we have

$$(K_{\tilde{Y}} \cdot K_{\tilde{Y}}) = (Z \cdot Z) + 2(Z \cdot C_{i_1}) + (C_{i_1} \cdot C_{i_1}). \tag{1.7}$$

By (1.6), (1.7) and Corollary 1.5, we have

$$b_2(\tilde{Y}) = 6 - 4q - (Z \cdot Z) - (C_{i_1} \cdot C_{i_1}). \tag{1.8}$$

$$\leqslant 6 - 4q - 2(Z \cdot Z). \tag{1.9}$$

Since $S = \{\text{one point}\}$ is a hypersurface singularity, we have

$$-(Z \cdot Z) \leqslant n := m(\mathcal{O}_{Y,S}) \quad \text{(Wagreich [23])}. \tag{1.10}$$

$$p_g \geqslant \tfrac{1}{2}(n-1)(n-2) \quad \text{(Yau [24])}. \tag{1.11}$$

Since $p_g = q + 1$, by (1.10), (1.11), we have

$$-(Z \cdot Z) \leqslant \tfrac{1}{2}(3 + \sqrt{9 + 8q}). \tag{1.12}$$

By (1.9), (1.12), we have the claim.

(2) By the Noether formula, we have

$$b_2(\tilde{Y}) = 10 - (K_{\tilde{Y}} \cdot K_{\tilde{Y}}). \tag{1.13}$$

Since $p_g = 1$ and S is a hypersurface singularity, by Laufer [11], we have $-3 \leqslant (Z \cdot Z) \leqslant -1$. By Lemma 1.4(1) and (1.13), we have the claim. Q.E.D.

COROLLARY 1.7. $0 \leqslant q \leqslant 3$.

Proof. Assume that $q \neq 0$. By Lemma 1.6(1), we have

$2 \leqslant b_2(\tilde{Y}) \leqslant 9 - 4q + \sqrt{9 + 8q}$. This implies $q \leqslant 3$. Q.E.D.

3. By the classification of Fano 3-Folds with the second Betti numbers one due to Iskovskih [9] (see also Mukai [15], [16]), we have:

(Table 1)

g	2	3	4	5	6	7	8	9	10	12
$\frac{1}{2}b_3(X)$	52	30	20	14	10	7	5	3	2	0

,

where $g := \frac{1}{2}(K_X^3) + 1 = \frac{1}{2}(Y^3) + 1$.

Since $2q = b_1(\tilde{Y}) = b_3(\tilde{Y}) = b_3(Y) = b_3(X)$ (cf. [2]), by Corollary 1.7, $0 \leqslant \frac{1}{2}b_3(X) \leqslant 3$. Thus, by the Table 1 above, we have $(g, q) = (9, 3), (10, 2)$ or $(12, 0)$.

LEMMA 1.8. $q \neq 3$.

Proof. Assume that $q = 3$. By Lemma 1.6(1), we have $2 \leqslant b_2(\tilde{Y}) \leqslant -3 + \sqrt{33} < 3$, namely, $b_2(\tilde{Y}) = 2$. Hence, $\tilde{Y}$ is a $\mathbb{P}^1$-bundle over a smooth compact algebraic curve R of genus 3. Therefore, Y is a cone over R. This is a contradiction, by Table (I) in [3]. Q.E.D.

LEMMA 1.9. *Assume that $q \neq 0$. Then there is exactly one exceptional curve of the first kind in every singular fiber of the ruled surface $\tilde{Y}$, and the other irreducible components of the singular fiber are all contained in $E := \pi^{-1}(\operatorname{Sing} Y)$.*

Proof. Since $q \neq 0$, the rest $\overline{E - C_{i_1}}$ must be contained in the singular fibers of $\tilde{Y}$. Let $F_1, \ldots, F_r$ be the singular fibers of $\tilde{Y}$, $1 + \alpha_i$ ($\alpha_i > 0$) the "number" of the irreducible components of F_i and δ_i the "number" of the irreducible components of F_i which are not contained in E. Then we have

$$\begin{cases} 1 + t = b_2(\tilde{Y}) = 2 + \sum_{i=1}^{r} \alpha_i. \\ \sum_{i=1}^{r} (1 + \alpha_i - \delta_i) + 1 = t. \end{cases}$$

Thus we have $\Sigma_{i=1}^{r}(1 - \delta_i) = 0$. Since each singular fiber F_i contains at least an exceptional curve of the first kind, we have $\delta_i \geqslant 1$ for $1 \leqslant i \leqslant r$, hence, $\delta_i = 1$ for $1 \leqslant i \leqslant r$. Q.E.D.

LEMMA 1.10. *Assume that $q = 2$. Then the dual graphs of all the exceptional curves in $\tilde{Y}$ look like the Figure* 1 *below.*

Proof. By Lemma 1.6(1), we have $2 \leqslant b_2(\tilde{Y}) \leqslant 6$. Since Y is not a cone (see Table 1 in [3], we have $b_2(\tilde{Y}) \neq 2$. If $b_2(\tilde{Y}) = 3$, then $\tilde{Y}$ contains two exceptional curves of the first kind in a singular fiber. This contradicts Lemma 1.9. Hence we have $4 \leqslant b_2(\tilde{Y}) \leqslant 6$. Thus, by (1.9), (1.12), we have $-4 \leqslant (Z \cdot Z) \leqslant -3$. We put $n := (C_{i_1} \cdot C_{i_1}) < 0$. Then, by Lemma 1.3 and (1.8), we have

(i) $b_2(\tilde{Y}) = 6 \Rightarrow (n, t) = (-4, 5)$ and $(Z \cdot Z) = -4$.
(ii) $b_2(\tilde{Y}) = 5 \Rightarrow (n, t) = (-3, 4)$ and $(Z \cdot Z) = -4$.
(iii) $b_2(\tilde{Y}) = 4 \Rightarrow (n, t) = (-3, 3)$ and $(Z \cdot Z) = -3$,
$= (-2, 3)$ and $(Z \cdot Z) = -4$.

Thus, we have

(a) $(Z \cdot Z) = -4 \Rightarrow (n, t) = (-2, 3), (-3, 4), (-4, 5)$.
(b) $(Z \cdot Z) = -3 \Rightarrow (n, t) = (-3, 3)$.

Since Sing $Y - S$ consists of rational double points, by Lemma 1.9 and (a), (b), the configuration of the exceptional curves of $\tilde{Y}$ can be easily described. Thus we have the lemma. Q.E.D.

Notation. The vertex $\boxed{k}$ represents a non-singular compact algebraic curve of genus 2 with the self-intersection number $-k$, (which is corresponding to the section C_{i_1} of $\tilde{Y}$), ⓚ a non-singular rational curve with the self-intersection number $-k$. We denote ② simply by ○. Adjacent to the graph, we write a basis $\{e_i\}$ $(0 \leqslant i \leqslant t)$ of $H^2(\tilde{Y}: \mathbb{Z})$, where $t = \dim H^2(E; \mathbb{R})$.

LEMMA 1.11. *Assume that $q = 2$. Then there is a canonical curve D of genus* 10 *and* $\deg D = 18$ *such that*

(i) Sing $Y \cap D = \phi$,
(ii) $\mathcal{O}_Y(Y) = \mathcal{O}_Y(D)$.

Proof. Since $q = 2$, by Table 1, X is a Fano 3-fold of degree 18 in $\mathbb{P}^{11}$ and Y is a hyperplane section of X (see [9]). For a sufficiently general hyperplane section H, we put $D = H \cdot Y$, which is desired. Q.E.D.

4. We put $\tilde{D} := \pi^{-1}(D) \hookrightarrow \tilde{Y}$. Since $D \cap \operatorname{Sing} Y = \phi$ by Lemma 1.11, we have $\tilde{D} \cong D$ (isomorphism), $(\tilde{D} \cdot \tilde{D}) = 18$ and $(\tilde{D} \cdot E_j) = 0$ for each irreducible component E_j of $E = \pi^{-1}(\operatorname{Sing} Y)$.

Let $\{e_i\}$ $(0 \leqslant i \leqslant t)$ be a basis of $H^2(\tilde{Y}; \mathbb{Z}) \cong \mathbb{Z}^{t+1}$ (see Fig. 1). Then, we have

$$c_1(\mathcal{O}(\tilde{D})) = \sum_{i=0}^{t} \alpha_i e_i \quad (\alpha_i \in \mathbb{Z}), \tag{*}$$

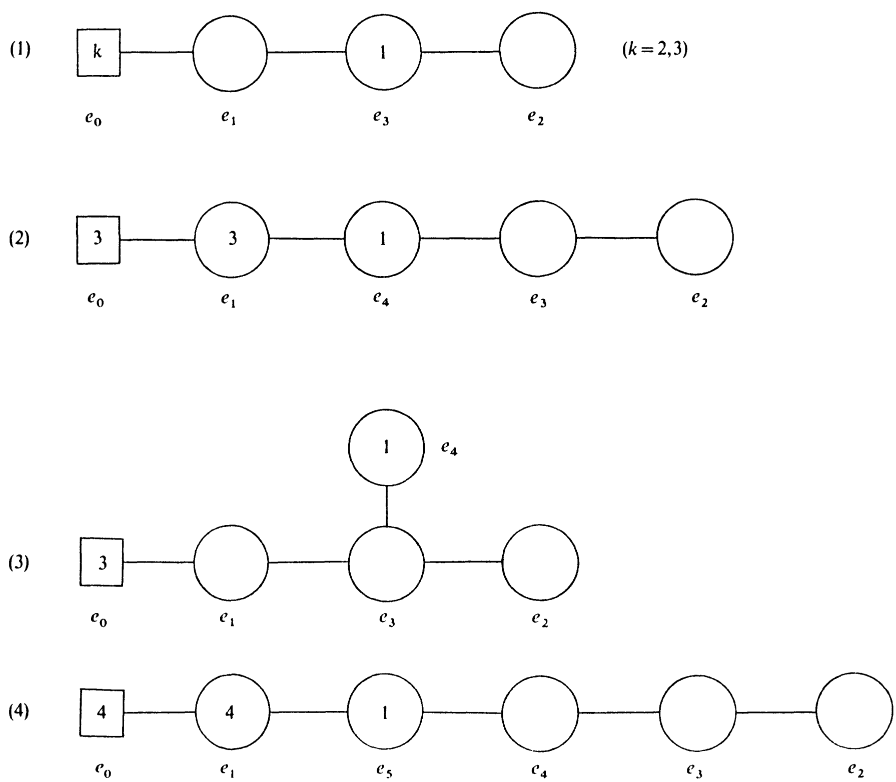
where $c_1(\mathcal{O}(\tilde{D})) \in H^2(\tilde{Y} : \mathbb{Z})$ is the first Chern class of $\mathcal{O}(\tilde{D})$, and

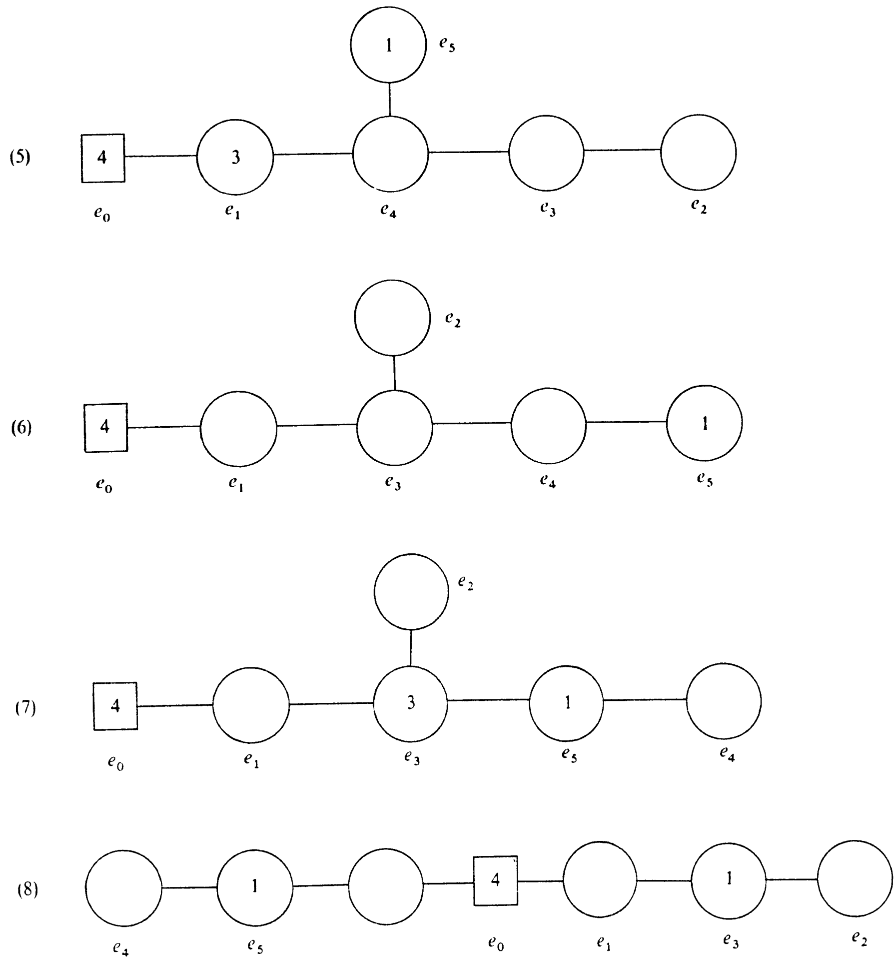
(i) the intersection number $e_i \cdot e_j$ is determined by the graph in Fig. 1,

(ii) $c_1(\mathcal{O}(\tilde{D})) \cdot e_i = 0$ $(0 \leqslant i \leqslant t)$,

(iii) $c_1(\mathcal{O}(\tilde{D})) \cdot c_1(\mathcal{O}(\tilde{D})) = 18$,

(iv) $d_{i_0} := c_1(\mathcal{O}(\tilde{D})) \cdot e_{i_0} \neq 0$, where e_{i_0} is a class corresponding to the exceptional curve of the first kind.





(Fig. 1).

By (*) and (i)–(iv) above, for each graph in the Fig. 1, we have the equations of α_i $(0 \leqslant i \leqslant t)$ and d_{i_0} over $\mathbb{Z}$ below:

Case (1)

$$\begin{aligned} \alpha_1 - k\alpha_0 &= 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_2 - \alpha_3 + \alpha_1 &= d_3 \\ \alpha_3 \cdot d_3 &= 18 \end{aligned} \qquad \text{(C-1)}$$

$\therefore \alpha_0^2 = 36/2k - 1$ $(k = 2, 3)$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (2).

$$\begin{aligned} \alpha_1 - 3\alpha_0 &= 0 \\ \alpha_4 - 3\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\ \alpha_1 - \alpha_4 + \alpha_3 &= d_4 \\ \alpha_4 \cdot d_5 &= 18 \end{aligned} \tag{C-2}$$

$\therefore \alpha_0^2 = \frac{27}{4}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (3).

$$\begin{aligned} \alpha_1 - 3\alpha_0 &= 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_0 &= 0 \\ \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 &= 0 \\ \alpha_3 - \alpha_4 &= d_4 \\ \alpha_5 \cdot d_5 &= 18 \end{aligned} \tag{C-3}$$

$\therefore \alpha_0^2 = 8$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (4).

$$\begin{aligned} \alpha_1 - 4\alpha_0 &= 0 \\ \alpha_5 - 4\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\ \alpha_5 - 2\alpha_4 + \alpha_3 &= 0 \\ \alpha_1 - \alpha_5 + \alpha_4 &= d_5 \\ \alpha_5 \cdot d_5 &= 18 \end{aligned} \tag{C-4}$$

$\therefore \alpha_0^2 = \frac{24}{5}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (5).

$$\begin{aligned} \alpha_1 - 4\alpha_0 &= 0 \\ \alpha_4 - 3\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\ \alpha_5 + \alpha_1 - 2\alpha_4 + \alpha_3 &= 0 \\ \alpha_4 - \alpha_5 &= d_5 \\ \alpha_5 \cdot d_5 &= 18 \end{aligned} \tag{C-5}$$

$\therefore \alpha_0^2 = \frac{81}{16}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (6).

$$\begin{aligned} \alpha_1 - 4\alpha_0 &= 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 &= 0 \\ \alpha_5 - 2\alpha_4 + \alpha_3 &= 0 \\ \alpha_4 - \alpha_5 &= d_5 \\ a_5 \cdot d_5 &= 18 \end{aligned} \tag{C-6}$$

$\therefore \alpha_0^2 = 6$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (7).

$$\begin{aligned} \alpha_1 - 4\alpha_0 &= 0 \\ \alpha_0 - 2\alpha_1 + \alpha_3 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_5 + \alpha_1 - 3\alpha_3 + \alpha_2 &= 0 \\ \alpha_5 - 2\alpha_4 &= 0 \\ \alpha_4 - \alpha_5 + \alpha_3 &= d_5 \\ \alpha_5 \cdot d_5 &= 18 \end{aligned} \tag{C-7}$$

$\therefore \alpha_0^2 = \frac{16}{3}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (8).

$$\begin{aligned} \alpha_1 - 4\alpha_0 &= 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\ \alpha_3 - 2\alpha_2 &= 0 \\ \alpha_1 - \alpha_3 + \alpha_2 &= d_3 \\ \alpha_5 - 2\alpha_4 &= 0 \\ \alpha_4 - \alpha_5 &= d_5 \\ \alpha_0 + \alpha_5 &= 0 \\ \alpha_5 d_5 + \alpha_3 d_3 &= 18 \end{aligned} \tag{C-8}$$

$\therefore \alpha_0^2 = 6$. Hence $\alpha_0 \notin \mathbb{Z}$.

By the computations (C-1)–(C-8), we find that these equations have no integral solutions. Thus, we have:

LEMMA 1.12. $q \neq 2$.

By Lemma 1.8, Lemma 1.12, and Table 1, we have the following

PROPOSITION 1.13. (cf. [18], [19]). *Assume that the index* $r = 1$. *Then,*

$(X, Y) \cong (V_{22}, H_{22})$, *where* V_{22} *is a Fano 3-fold of degree* 22 *in* $\mathbb{P}^{13}$(*index* 1, *genus* 12) *and* H_{22} *is a hyperplane section of* V_{22} *which is rational.*

REMARK 1.14. Among Fano 3-folds of degree 22 in $\mathbb{P}^{13}$(index 1, genus 12), there is a special one, $V'_{22} \hookrightarrow \mathbb{P}^{12}$, which has been overlooked by Iskovskih [8] (see Mukai-Umemura [14]).

Recently, Mukai has succeeded in classifying Fano 3-folds of index 1 with $b_2(X) = 1$, applying the theory of vector bundles on K-3 surfaces (see [15], [16]).

2. Determination of the boundary

1. Let $(X, Y) = (V_{22}, H_{22})$ be as in Proposition 1.13. Since $q = 0$, by Lemma 1.3, S consists of one point x with $p_g = 1$, namely, x is a minimally elliptic singularity. We put Sing $Y - \{x\} =: \{y_1, \ldots, y_k\}$ $(k \geqslant 0)$, and $B := \pi^{-1}(\{y_1, \ldots, y_k\})$. Then y_j's are all rational double points.

By Lemma 1.6(2), we have:

$$b_2(B) + b_2(C) = 10 \quad \text{if} \quad (Z \cdot Z) = -1, \tag{2.1}$$

$$b_2(B) + b_2(C) = 11 \quad \text{if} \quad (Z \cdot Z) = -2, \tag{2.2}$$

$$b_2(B) + b_2(C) = 12 \quad \text{if} \quad (Z \cdot Z) = -3. \tag{2.3}$$

2. Let T_0 (resp. T_i) be a contractible neighborhood of x (resp. y_i) in Y. We may assume that T_0, T_i $(1 \leqslant i \leqslant k)$ are disjoint. We put $T := \bigcup_{i=0}^{k} T_i$ and $\partial T := \bigcup_{i=0}^{k} \partial T_i$, where ∂T_i is the boundary of T_i. We put $T^* := T -$ Sing Y and $Y^* := Y -$ Sing Y. Since $T^* \approx \partial T$ (deformation retract), by the Mayer-Vietoris exact sequence, we have

$$\begin{aligned} &\to H_i(\partial T; \mathbb{Z}) \to H_i(Y^*; \mathbb{Z}) \oplus H_i(T; \mathbb{Z}) \to \\ &\to H_i(Y; \mathbb{Z}) \to H_{i-1}(\partial T; \mathbb{Z}) \to . \end{aligned} \tag{2.4}$$

Since Sing Y is isolated in Y, we have $H_2(Y^*; \mathbb{Z}) \cong H^2(\tilde{Y}, E; \mathbb{Z}) \cong H^2(Y, \text{Sing}\, Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z}) \cong \mathbb{Z}$. On the other hand, since $X = V_{22}$ is a Fano 3-fold of index 1 and the genus $g = 12$, we have $H^3(X; \mathbb{Z}) = 0$ (cf. [8], [15], [16]). Thus we have $H_3(Y; \mathbb{Z}) \cong H_3(X; \mathbb{Z}) \cong H^3(X; \mathbb{Z}) \cong 0$, and $H_1(Y^*; \mathbb{Z}) \cong H^3(Y; \mathbb{Z}) \cong H^3(X; \mathbb{Z}) \cong 0$ (cf. [2], [3]). Therefore we have finally the Poincaré's exact sequence:

$$0 \to H^2(Y; \mathbb{Z}) \xrightarrow{P_2} H_2(Y; \mathbb{Z}) \to H_1(\partial T; \mathbb{Z}) \to 0 \quad \text{(cf. [18])}. \tag{2.5}$$

By Lemma 2.5 in [18], we have

LEMMA 2.1 (cf. Peternell-Schneider [18]). $H_1(\partial T;\mathbb{Z}) = H_1(\partial T_0;\mathbb{Z}) \oplus \mathscr{H} \cong \mathbb{Z}_{22}$, *where* $\mathscr{H} = \bigoplus_{i=1}^k H_1(\partial T_i;\mathbb{Z})$, *namely, we have*:

(*Table 2*)

$H_1(\partial T_0;\mathbb{Z})$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{11}$	$\mathbb{Z}_2$	0
$\mathscr{H}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{11}$	$\mathbb{Z}_{22}$

LEMMA 2.2. *For the rational double point* $y_j \in \operatorname{Sing} Y - \{x\}$ $(1 \leqslant j \leqslant k)$, *we have*:

(*Table 3*)

Type of y_i	A_n	D_{2n}(resp. D_{2n+1})	E_6	E_7	E_8
$H_1(\partial T_i;\mathbb{Z})$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$(resp. $\mathbb{Z}_4$)	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0

Proof. Apply Lemma M below.

LEMMA M (Mumford [17]).

Let S be a smooth complex surface and consider a divisor $C = \bigcup_{i=1}^n C_i$ (C_i: *a smooth rational curve*) *with normal crossings. Let* ∂T *be the boundary of a tubular neighborhood T of C in S. Then,* $H_1(\partial T;\mathbb{Z})$ *is generated by* $\gamma_1, \dots, \gamma_n$ *with the fundamental relations*:

$$\sum_{j=1}^{n} (C_i \cdot C_j) \cdot \gamma_j \quad (j = 1, 2, \dots, n), \tag{\#}$$

where γ_j *is a loop in* ∂T *which goes around* C_j *with positive orientation.*

REMARK. By Lemma M, one can easily compute the homology group $H_1(\partial T_0:\mathbb{Z})$ for each exceptional divisor C in Table L-1–Table L-9 below.

By Lemma 2.1, Lemma 2.2, and (2.1), (2.2), we have easily the following

LEMMA 2.3.
(1) $(Z \cdot Z) = -1 \Rightarrow H_1(\partial T_0:\mathbb{Z}) \not\cong 0, \mathbb{Z}_2$
(2) $(Z \cdot Z) = -2 \Rightarrow H_1(\partial T_0;\mathbb{Z}) \not\cong 0.$

LEMMA 2.4.
(1) *The case of* $(Z \cdot Z) = -1$. *We have:*
  (i) $H_1(\partial T_0:\mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 1, 3, 9$ (*Table* L-1)
  (ii) $H_1(\partial T_0:\mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 2, 10$ (*Table* L-2)
(2) *The case of* $(Z \cdot Z) = -2$. *We have:*
  (i) $H_1(\partial T_0:\mathbb{Z}) \cong \mathbb{Z}_2 \Rightarrow b_2(C) = 1$ (*Table* L-3)
  (ii) $H_1(\partial T_0;\mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 2, 4, 10$ (*Table* L-4)
  (iii) $H_1(\partial T_0;\mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 3, 11$ (*Table* L-5)

(3) *The case of* $(Z\cdot Z) = -3$. *We have:*

(i) $H_1(\partial T_0: \mathbb{Z}) \cong 0 \Rightarrow b_2(C) = 1$ (*Table* L-6)

(ii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_2 \Rightarrow b_2(C) = 2$ (*Table* L-7)

(iii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 3, 5, 11$ (*Table* L-8)

(iv) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 4, 12$ (*Table* L-9)

Proof. We will prove for the case (3)(iii). The proof for the other cases are similar. Since $(Z\cdot Z) = -3$ and $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11}$, we have $b_2(B) + b_2(C) = 12$ by (2.3), and $\mathscr{H} := \oplus_{i=1}^{r} H_1(\partial T_i; \mathbb{Z}) \cong \mathbb{Z}_2$. By the Table 3 Sing $Y - \{x\} = \{A_1\text{-type}\}$, $\{E_7\text{-type}\}$ or $\{A_1\text{-type} + E_8\text{-type}\}$, hence, $b_2(B) = 1, 7, 9$, respectively. By (2.3), we have $b_2(C) = 3, 5, 11$. Pick out the possible types of the dual graphs with $b_2(C) = 3, 5, 11$, from the Table 3 in Laufer [11], we have finally the Table L-8. We remark that there is no dual graph with $b_2(C) = 4$. Q.E.D.

From Lemma 2.4 and the Table L-1–Table L-9, we have directly the following

(*Table L-1*)

	Dual graph	$A_*\cdot A_*$	$H_1(\partial T_0:\mathbb{Z})$
1	*Cu*	-1	0
2	*Tr*	$-2, -2, -3$	$\mathbb{Z}_3$
3	$A_{5,****}$	$-2, -2, -2, -3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
4	$E_{8,*}$	-3	0

(*Table L-2*)

	Type of x	$A_*\cdot A_*$	$H_1(\partial T_0;\mathbb{Z})$
5	*Ta*	$-2, -3$	$\mathbb{Z}_2$
6	$A_{6,****}$	$-2, -2, -2, -3$	$\mathbb{Z}_4$

(*Table L-3*)

	Type of x	$A_*\cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
7	*Cu*	-2	"$\mathbb{Z}_2$"

(*Table L-4*)

	Dual graph	$A_*\cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
8	*Ta*	$-2, -4$	$\mathbb{Z}_2^{\oplus 2}$
9	*Ta*	$-3, -3$	$\mathbb{Z}_5$

10	$A_{6,****}$	$-2, -2, -2, -4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$
11	$A_{6,****}$	$-2, -2, -3, -3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
12	$A_{6,****}$	$-2, -3, -3, -2$	$\mathbb{Z}_{17}$
13	$A_{*,o} + A_{*,o} + A_{*,o}$ $+ A_{4,**o}$	$-2, -2, -2$ $-2, -2$	$\mathbb{Z}_2^{\oplus 4}$
14	$A_{*,o} + A_{*,o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_2^{\oplus 2}$
15	$A_{*,o} + A_{n,**o}$ $+ A_{m,**o}$ $(m + n = 4)$	$-2, -2, -2,$ $-2, -2$	$\mathbb{Z}_4^{\oplus 2}$ if $(m, n) = (1, 3)$ $\mathbb{Z}_2^{\oplus 4}$ if $(m, n) = (2, 2)$
16	$A_{*,o} + D_{7,*,o}$	$-2, -2$	$\mathbb{Z}_2^{\oplus 4}$
17	$A'_{7,**,o}$	$-2, -2$	$\mathbb{Z}_5$
18	$A_{2,**,o} + A'_{3,**o}$	$-2, -2, -2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
19	$A_{1,**,o} + D_{5,*,o}$	$-2, -2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$
20	$A_{0,**o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

(*Table L-5*)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
21	*Tr*	$-2, -2, -4$	$\mathbb{Z}_6$
22	*Tr*	$-2, -3, -3$	$\mathbb{Z}_8$
23	$A_{7,****}$	$-2, -2, -2, -4$	$\mathbb{Z}_2^{\oplus 3}$
24	$A_{7,****}$	$-2, -2, -3, -3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_6$
25	$A_{7,****}$	$-2, -3, -3, -2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_9$
26	$A_{*,o} + A_{*,o} +$ $+ A_{*,o} + A_{5,**,o}$	$-2, -2$ $-2, -2, -2$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_4$
27	$A_{*,o} + A_{n,**,o} +$ $+ A_{m,**,o}$ $(m + n = 5)$	$-2, -2, -2$ $-2, -2$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_4$
28	$A_{3,**,o} + A'_{3,**,o}$	$-2, -2, -2, -2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_6$
29	$A_{2,**,o} + D_{5,*,o}$	$-2, -2, -2$	$\mathbb{Z}_2^{\oplus 3}$
30	$A_{1,**,o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_4$
31	$D_{9,*,o}$	-2	$\mathbb{Z}_2$

(*Table L-6*)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
32	*Cu*	-3	$\mathbb{Z}_3$

(*Table L-7*)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
33	*Ta*	$-2, -5$	$\mathbb{Z}_6$
34	*Ta*	$-3, -4$	$\mathbb{Z}_8$

(*Table L-8*)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
35	*Tr*	$-2, -2, -5$	$\mathbb{Z}_3^{\oplus 2}$
36	*Tr*	$-2, -3, -4$	$\mathbb{Z}_{13}$
37	*Tr*	$-3, -3, -3$	$\mathbb{Z}_4^{\oplus 2}$
38	$A_{1,****}$	$-2, -2, -2, -5$	$\mathbb{Z}_6^{\oplus 2}$
39	$A_{1,****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{10}^{\oplus 2}$
40	$A_{1,****}$	$-2, -3, -3, -3$	$\mathbb{Z}_9^{\oplus 2}$
41	$A_{7,****}$	$-2, -2, -2, -5$	$\mathbb{Z}_6^{\oplus 2}$
42	$A_{7,****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{10}^{\oplus 2}$
43	$A_{7,****}$	$-2, -3, -4, -2$	$\mathbb{Z}_{16}^{\oplus 2}$
44	$A_{7,****}$	$-2, -3, -3, -3$	$\mathbb{Z}_{15}^{\oplus 2}$
45	$A_{*,o} + A_{*,o} + A_{*,o} +$ $+ A_{5,**,o}$	$-3, -2, -2,$ $-2, -2$	$\mathbb{Z}_8^{\oplus 2}$
46	$A_{*,o} + A_{*,o} + A_{*,o} +$ $+ A_{5,**,o}$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{26}^{\oplus 2}$
47	$A_{*,o} + A_{n,**o} +$ $+ A_{m,**,o} (m + n = 5)$	$-3, -2, -2$ $-2, -2,$	$\mathbb{Z}_8^{\oplus 2}$
48	$A_{*,o} + A_{n,**,o} +$ $+ A_{m,**,o} \ (m + n = 5)$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{18}^{\oplus 2}$ if $(m, n) = (1, 4)$ $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ if $(m, n) = (2, 3)$ $\mathbb{Z}_{22}^{\oplus 2}$ if $(m, n) = (3, 2)$ $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$ if $(m, n) = (4, 1)$
49	$D_{9,*,o}$	-3	$\mathbb{Z}_5$
50	$A_{3,**,o} + A'_{3,**,o}$	$-3, -2, -2, -2$	$\mathbb{Z}_{33}$

51	$A_{3,**,o} + A'_{3,**,o}$	$-2, -2, -2, -3$	$\mathbb{Z}_{14}^{\oplus 2}$
52	$A_{2,**,o} + D_{5,*,o}$	$-3, -2, -2$	$\mathbb{Z}_{10}^{\oplus 2}$
53	$A_{2,**,o} + D_{5,*,o}$	$-2, -2, -3$	$\mathbb{Z}_{30}^{\oplus 2}$
54	$A_{1,**,o} + E_{7,o}$	$-3, -2$	$\mathbb{Z}_9$
55	$A_{1,*,o} + A_{1,*,o} +$ $+ E_{6,o}$	$-2, -2$	$\mathbb{Z}_3^{\oplus 2}$
56	$A_{1,*,o} + A_{7,*,o}$	$-2, -2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_9$
57	$A_{4,*,o} + A_{4,*,o}$	$-2, -2$	$\mathbb{Z}_6^{\oplus 2}$

(*Table L-9*)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
58	$A_{8,****}$	$-2, -2, -2, -5$	$\mathbb{Z}_{12}$
59	$A_{8,****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{20}$
60	$A_{8,****}$	$-2, -3, -4, -2$	$\mathbb{Z}_{34}$
61	$A_{8,****}$	$-2, -3, -3, -3$	$\mathbb{Z}_{48}$
62	$A_{*,o} + A_{*,o} + A_{*,o} +$ $+ A_{6,**,o}$	$-3, -2, -2$ $-2, -2$	$\mathbb{Z}_8^{\oplus 3}$
63	$A_{*,o} + A_{*,o} + A_{*,o} +$ $+ A_{6,**,o}$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{14}^{\oplus 3}$
64	$A_{*,o} + A_{n,**o} +$ $+ A_{m,**,o} (m + n = 6)$	$-3, -2, -2$ $-2, -2$	$\mathbb{Z}_8^{\oplus 3}$
65	$A_{*,o} + A_{n,**,o} +$ $+ A_{m,**,o} (m + n = 6)$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{36}$ if $(m, n) = (1, 5)$ $\mathbb{Z}_{10}^{\oplus 3}$ if $(m, n) = (2, 4)$ $\mathbb{Z}_{44}$ if $(m, n) = (3, 3)$ $\mathbb{Z}_4 \oplus \mathbb{Z}^{\oplus 2}$ if $(m, n) = (4, 2)$ $\mathbb{Z}_{13} \oplus \mathbb{Z}_{52}$ if $(m, n) = (5, 1)$
66	$A_{4,**,o} + A'_{3,**,o}$	$-3, -2, -2, -2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_{12}$
67	$A_{4,**,o} + A'_{3,**,o}$	$-2, -2, -2, -3$	$\mathbb{Z}_{28}$
68	$A_{3,**,o} + D_{5,*,o}$	$-3, -2, -2$	"$\mathbb{Z}_{22}$"
69	$A_{3,**,o} + D_{5,*,o}$	$-2, -2, -3$	$\mathbb{Z}_{20}$
70	$A_{2,**,o} + E_{7,o}$	$-3, -2$	$\mathbb{Z}_{10}$
71	$A_{10,**,o}$	-2	$\mathbb{Z}_4$
72	$A_{4,*,o} + E_{6,o}$	-2	$\mathbb{Z}_6$

PROPOSITION 2.5. *Let* $(X, Y) = (V_{22}, H_{22})$ *be a compactification of* $\mathbb{C}^3$ *as in Proposition* 1.13. *Then,*

(a) Sing $Y = \{x\}$, *where x is a minimally elliptic singularity of* $A_{3,**o} + D_{5,*,o}$ – *Type* (*Table* L-9, (68)), *or*

(b) Sing $Y = \{x, y\}$, *where x is a minimally elliptic singularity of* Cu-*type* (*Table* L-3, (7)) *and y is a rational double point of* A_{10}-*type.*

In the Table L-1–Table L-9, we use the same terminology as that of the Table 1–Table 3 in Laufer [11, p. 1287–1294].

3. Non-existence of the case (a)

Assume that there is a compactification $(X, Y) = (V_{22}, H_{22})$ of the case (a) in Proposition 2.5. Let $\pi: \tilde{Y} \to Y$ be the minimal resolution of the singularity $x := \mathrm{Sing}\, Y$ and Z the fundamental cycle of x associated with the resolution $(\tilde{Y}, \pi)$. By assumption, we have $K_{\tilde{Y}} = -Z$ and $(Z \cdot Z) = -3$. The dual graph of $\pi^{-1}(x)$ looks like the Fig. 2, where we denote by ○ (resp. ⊖3) a smooth rational curve with the self-intersection number -2 (resp. -3). We can represent $\tilde{Y}$ as a ruled surface $v: \tilde{Y} \to \mathbb{P}^1$ over $\mathbb{P}^1$ (see Fig. 3), where

$$\begin{cases} v^{-1}(0) =: \tilde{C} \cup f_1 \cup f_2 \cup f_3 \cup f_4 \cup l_1 \text{ and } v^{-1}(\infty) =: \bigcup_{i=5}^{10} f_i \cup \tilde{B} \text{ are} \\ \text{singular fibers.} \end{cases} \tag{3.1}$$

$$l_2 \text{ is a section} \tag{3.2}$$

$$\begin{cases} (l_1 \cdot l_1)_{\tilde{Y}} = (l_2 \cdot l_2)_{\tilde{Y}} = -3, \\ (f_i \cdot f_i)_{\tilde{Y}} = -2 (1 \leqslant i \leqslant 10), \\ (\tilde{C} \cdot \tilde{C})_{\tilde{Y}} = (\tilde{B} \cdot \tilde{B})_{\tilde{Y}} = -1. \end{cases} \tag{3.3}$$

$$\begin{cases} (\tilde{D} \cdot \tilde{C})_{\tilde{Y}} = 2, (\tilde{D} \cdot \tilde{B})_{\tilde{Y}} = 3, \text{ where } D = \pi(\tilde{D}) \text{ is a canonical} \\ \text{hyperplane section such that } \mathrm{Pic}\, Y \cong \mathbb{Z} \cdot \mathcal{O}_Y(D), \text{ in particular,} \\ \deg D = (D \cdot D)_Y = 22. \end{cases} \tag{3.4}$$

$$\pi^{-1}(x) = \bigcup_{i=1}^{10} f_i \cup l_1 \cup l_2, \tag{3.5}$$

$$Z = f_4 + 2f_3 + 2f_2 + 2f_1 + l_1 + 2l_2 + \\ + 3f_5 + 4f_6 + 2f_7 + 3f_8 + 2f_9 + f_{10}. \tag{3.6}$$

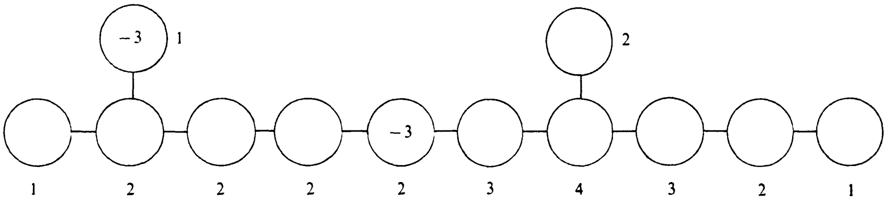


Fig. 2.

LEMMA 3.1.

(1) *there is no line in* X *through the point* $x = \operatorname{Sing} Y \in X$.

(2) $C_0 := \pi(C) \subset Y \subset X$ *is a unique conic on* X *through the point* x.

Proof. Since the multiplicity $m(\mathcal{O}_{Y,x})$ is equal to 3 by Laufer [11] and $\operatorname{Pic} X \cong \mathbb{Z} \cdot \mathcal{O}_X(Y)$, any line or any conic through the point x must be contained in Y. Now, since $(\tilde{C} \cdot \tilde{D})_Y = (C_0 \cdot D)_Y = 2$ and D is a hyperplane section, C_0 is a conic on X. Let F be a line or a conic on X through the point x. Then, we have $F \subset Y$.

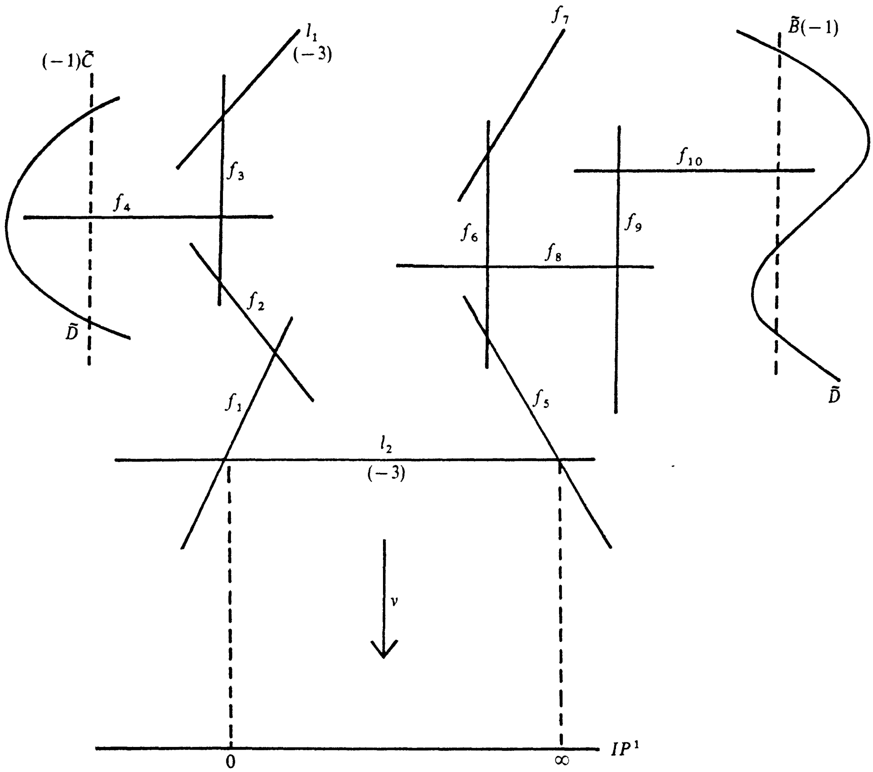


Fig. 3.

Let $\tilde{F}$ be the proper transform of F in $\tilde{Y}$. Since $\tilde{D}$ can be written as follows:

$$\begin{aligned}\tilde{D} = 2\tilde{C} + 4f_4 + 6f_3 + 2l_1 + 6f_2 + 6f_1 + \\ + 6l_2 + 12f_5 + 18f_6 + 9f_7 + 15f_8 + 12f_9 + 9f_{10} + 6\tilde{B},\end{aligned} \tag{3.7}$$

we have $(\tilde{D}\cdot\tilde{F}) \neq 1$, and also have $(\tilde{D}\cdot\tilde{F}) = 2$ if and only if $\tilde{F} = \tilde{C}$. This proves (1) and (2). Q.E.D.

2. Let us consider the triple projection of $X = V_{22}$ from the singularity $x = \mathrm{Sing}\, Y \in X$. For this purpose, we will consider the linear system $|H - 3x| := |\mathcal{O}_X(H) \oplus \mathfrak{m}_{x,X}^3|$, where H is a hyperplane section of X and $\mathfrak{m}_{x,X}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Since the multiplicity $m(O_{Y,x})$ is equal to 3, we have $Y \in |H - 3x|$ (c.f. [16a]).

Let $\sigma_1 \colon X_1 \to X$ be the blowing up of X at the point x, and put $E_1 := \sigma^{-1}(x) \cong \mathbb{P}^2$. Let Y_1 be the proper transform of Y in X_1. Since $-K_X = H$ and $Y \in |H - 3x|$, we have

$$-K_{X_1} = \sigma_1^* H - 2E_1 \tag{3.8}$$

$$Y_1 = \sigma_1^* H - 3E_1. \tag{3.9}$$

By the adjunction formula, we have

$$-K_{Y_1} = E_1|_{Y_1}. \tag{3.10}$$

LEMMA 3.2. *$H^0(X_1, \mathcal{O}_{X_1}(\sigma_1^* H - 3E_1)) \cong \mathbb{C}^4$, and $H^i(X_1, \mathcal{O}_{X_1}(\sigma_1^* H - 3E_1)) = 0$ for $i > 0$.*

Proof. Let us consider the exact sequence

$$0 \to \mathcal{O}_{X_1} \to \mathcal{O}_{X_1}(Y_1) \to \mathcal{O}_{Y_1}(Y_1) \to 0. \tag{3.11}$$

Since $Y_1 = \sigma_1^* H - 3E_1$ and $H^i(X_1, \mathcal{O}_{X_1}) = 0$ for $i > 0$, we have only to prove $H^i(Y_1, \mathcal{O}_{Y_1}(Y_1)) = 0$ for $i > 0$ and $H^0(Y_1, \mathcal{O}_{Y_1}(Y_1)) \cong \mathbb{C}^3$.

By (3.10), we have

$$\begin{aligned}\mathcal{O}_{Y_1}(Y_1) &= \mathcal{O}_{Y_1}(\sigma_1^* H - 3E_1) \\ &= \mathcal{O}_{Y_1}(D_1 + 3K_{Y_1}),\end{aligned} \tag{3.12}$$

where $D_1 := \sigma_1^* H|_{Y_1}$ is linearly equivalent to the proper transform of D in Y_1.

Claim. $\mathcal{O}_{Y_1}(D + 2K_{Y_1})$ is nef and big on Y_1. Indeed, there exists a birational morphism $\mu_1\colon \tilde{Y} \to Y_1$ such that $\pi = (\sigma_1|_{Y_1}) \circ \mu_1$. Then, we have $\mu_1^*(D_1 + 2K_{Y_1}) = \tilde{D} - 2Z$. It is easy to see that $\tilde{D} - 2Z$ is nef and big on $\tilde{Y}$ (see (3.6), (3.7)). Thus $\mathcal{O}_{Y_1}(D_1 + 2K_{Y_1})$ is nef and big on Y_1.

By the Kawamata–Vieweg vanishing theorem, we have $H^i(Y_1, \mathcal{O}_{Y_1}(D_1 + 3K_{Y_1})) = 0$ for $i > 0$, namely, $H^i(Y_1, \mathcal{O}_{Y_1}) = 0$ for $i > 0$. On the other hand, since $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D - 3Z)) = 0$ for $i > 0$, by the Riemann–Roch theorem, we have $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D - 3Z)) \cong \mathbb{C}^3$. Q.E.D.

By Lemma 3.2, the linear system $|H - 3x|$ defines a rational map $\Phi := \Phi_{|H-3x|}\colon X \dashrightarrow \mathbb{P}^3$, called a triple projection.

Let $\{g_1, g_2, g_3\}$ be a basis of $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D} - 3Z))$ such that

$$\begin{cases} (g_1) = 11\tilde{C} + 10f_4 + 9f_3 + 2l_1 + 6f_2 + 3f_1 \\ (g_2) = 5\tilde{C} + 4f_4 + 3f_3 + 2f_2 + f_1 + 2f_5 + \\ \qquad + 4f_6 + 2f_7 + 4f_8 + 4f_9 + 4f_{10} + 4\tilde{B} \\ (g_3) = 8\tilde{C} + 7f_4 + 6f_3 + l_1 + 4f_2 + 2f_1 + \\ \qquad + f_5 + 2f_6 + f_7 + 2f_8 + 2f_9 + 2f_{10} + 2\tilde{B} \end{cases} \tag{3.13}$$

Since $2(g_3) = (g_1) + (g_2)$, $g := (g_1 : g_2 : g_3)$ defines a rational map $g\colon \tilde{Y} \dashrightarrow Q$ of $\tilde{Y}$ onto a conic $Q := \{w_2^2 = w_0 w_1\} \hookrightarrow \mathbb{P}^2(w_0 : w_1 : w_2)$. This implies that $\Phi(Y) = Q \cong \mathbb{P}^1$ and $W = \Phi(X)$ is a quadratic hypersurface in $\mathbb{P}^3$. Thus we have the following

LEMMA 3.3. *Let* $\Phi := \Phi_{|H-3x|}\colon X \dashrightarrow \mathbb{P}^3$ *be the triple projection from the point* x. *Then the image* $W := \Phi(X)$ *is an irreducible quadric hypersurface in* $\mathbb{P}^3$, *and* $Q = \Phi(Y)$ *is a smooth hyperplane section, namely, a conic in* $\mathbb{P}^2$.

3. Next, we will study the resolution of the indeterminancy of the rational map $\Phi\colon X \dashrightarrow \mathbb{P}^3$.

Let $\Phi^{(1)}_{|\sigma_1^* H - 3E_1|}\colon X_1 \dashrightarrow \mathbb{P}^3$ be a rational map defined by the linear system $|\sigma_1^* H - 3E_1|$. Then we have the diagram:

$$\begin{array}{ccc} X_1 & & \\ {\scriptstyle \sigma_1}\downarrow & \overset{\Phi^{(1)}}{\searrow} & \\ X & \underset{\Phi}{\dashrightarrow} & \mathbb{P}^3. \end{array}$$

Let $\Delta \subset X$ be a small neighborhood of x in X with a coordinate system

(Z_1, Z_2, Z_3). By Laufer [11], we may assume that

$$\begin{cases} \Delta \cap Y = \{Z_2 \cdot Z_3^2 = Z_1^3 \cdot Z_2 + Z_1 Z_3^3 + Z_1 Z_2^4\}, \\ x = (0,0,0) \in \Delta. \end{cases} \tag{3.14}$$

By an easy computation, we find that Y_1 has two rational double points q_1 of A_4-type and q_0 of D_6-type (cf. [11, Theorem 3.15]). Let $\mu_1: \tilde{Y} \to Y_1$ be the birational map as above. Then we have $\mu_1^{-1}(q_1) = f_1 \cup f_2 \cup f_3 \cup f_4$, $\mu_1^{-1}(q_0) = \bigcup_{j=5}^{10} f_j$.

We put $l_i^{(1)} := \mu_1(l_i)$, and $C_1 := \mu_1(\tilde{C}) \hookrightarrow Y_1 \subset X_1$. Then C_1 is the proper transform of C_0 in X_1, in particular, C_1 is a smooth rational curve in $Y_1 \subset X_1$ with $q_1 \in C_1, q_0 \notin C_1$. Moreover, we have $Y_1 \cdot E_1 = l_1^{(1)} + 2l_2^{(1)}$, where $l_1^{(1)}, l_2^{(1)}$ are two distinct lines in $E_1 \cong \mathbb{P}^2$, and $l_1^{(1)} \cap l_2^{(1)} = q_1 \in X_1$.

By (3.13), the base locus $Bs|\mathcal{O}_{Y_1}(Y_1)| = C_1 \ni q_1$. Since $H^1(X_1, \mathcal{O}_{X_1}) = 0$ by (3.12), we have the base locus $Bs|\sigma_1^* H - 3E_1| \ni C_1 \ni q_1$.

Since $\operatorname{Pic} X \cong \mathbb{Z}\mathcal{O}_X(Y)$, the linear system $|H - 3x|$ has no fixed component. Thus, we have the following

LEMMA 3.4. *The linear system $|\sigma_1^* H - 3E_1|$ on X_1 has no fixed component, but has the base locus $Bs|\sigma_1^* H - 3E_1| = C_1 \ni q_1$.*

We need the following

LEMMA Mo (Morrison [13]). *Let S be a surface with only one singularity x of A_n-type in a smooth projective 3-fold X. Let $E \subset S \subset X$ be a smooth rational curve in X. Let $\mu: \tilde{S} \to S$ be the minimal resolution of the singularity of S and put*

$$\mu^{-1}(x) = \bigcup_{j=1}^{n} C_j,$$

where C_j's $(1 \leqslant j \leqslant n)$ are smooth rational curves with

$$\begin{aligned} &(C_j \cdot C_j)_S = -2 && (1 \leqslant j \leqslant n), \\ &(C_j \cdot C_{j+1})_S = 1 && (1 \leqslant j \leqslant n-1), \\ &(C_i \cdot C_j)_S = 0 && \text{if } |i-j| \geqslant 2. \end{aligned}$$

Let $\tilde{E}$ be the proper transform of E in $\tilde{S}$. Assume that

(i) $N_{\tilde{E}|S} \cong \mathcal{O}_{\tilde{E}}(-1)$,

(ii) $\deg N_{E|X} = -2$,

where $N_{\tilde{E}|\tilde{S}}$ (resp. $N_{E|X}$) *is the normal bundle of* $\tilde{E}$ (resp. E) *in* $\tilde{S}$ (resp. X). *Then we have*

(1) $N_{E|X} \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2)$ *if* $x \in E$ *and* $(C_j \cdot \tilde{E}) = 1$ *for* $j = 1$ *or* n,

(2) $N_{E|X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$ *if* $x \notin E$.

Proof. In the proof of Theorem 3.2 in Morrison [13], we have only to replace the conormal bundle $N^*_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(2)$ by $N_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(1)$. Q.E.D.

The indeterminancy of the rational map $\Phi^{(1)}: X_1 \dashrightarrow \mathbb{P}^3$ can be resolved by the following way:

Let us consider the following sequence of blowing ups:

$$X_6 \xrightarrow{\sigma_6} X_5 \xrightarrow{\sigma_5} X_4 \xrightarrow{\sigma_4} X_3 \xrightarrow{\sigma_3} X_2 \xrightarrow{\sigma_2} X_1,$$

where

(i) $\sigma_{j+1}: X_{j+1} \to X_j$ is the blowing up of X_j along $C_j \cong \mathbb{P}^1$ $(1 \leqslant j \leqslant 5)$,

(ii) C_{j+1} is the negative section of the $\mathbb{P}^1$-bundle $C'_j = \mathbb{P}(N^*_{C_j|X_j}) \cong \mathbb{F}_2 (1 \leqslant j \leqslant 4)$,

(iii) C_6 is a section of $C'_5 \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $(C_6 \cdot C_6) = 0$.

Then we have the morphism $\bar{\Phi}: X_6 \to \mathbb{P}^3$ and a diagram:

$$\begin{array}{ccc} X_1 & \xleftarrow{\sigma} & X_6 \\ {\scriptstyle\sigma_1}\downarrow & \overset{\Phi^{(1)}}{\searrow} & \downarrow{\scriptstyle\bar{\Phi}} \\ X & \underset{\Phi}{\dashrightarrow} & \mathbb{P}^3, \end{array} \qquad \text{(D-1)}$$

where $\sigma := \sigma_2 \circ \sigma_3 \circ \sigma_4 \circ \sigma_5 \circ \sigma_6$.

This is a desired resolution of the indeterminancy of the rational map

$\Phi^{(1)}: X_1 \dashrightarrow \mathbb{P}^3$ (or $\Phi: X \dashrightarrow \mathbb{P}^3$).

4. We will prove the facts above.

Notations:

- Y_{j+1}: the proper transform of Y_j in X_{j+1}.
- E_{j+1}: the proper transform of E_j in X_{j+1}.
- C_{j+1}: a section of $C'_j = \mathbb{P}(N_{C_j|X_j}{}^*)$.
- q_j: the singularity of Y_j of A_{5-j}-type (A_0-type means the smoothness).
- $\tilde{Y}_0$: the contraction of the exceptional set $\bigcup_{i=5}^{10} f_i$ in $\tilde{Y}$.
- $f^{(j+1)}$: a fiber of the $\mathbb{P}^1$-bundle $C'_j \hookrightarrow X_{j+1}$.
- $l_i^{(j+1)}$: the proper transform of $l_i^{(1)}$ in X_{j+1} $(i = 1, 2)$.

- μ_j: $\tilde{Y}_0 \to Y_j$: a birational morphism with μ_j

$$\tilde{Y}_0 - \bigcup_{i=1}^4 f_i \xrightarrow{\sim} Y_j - \mu_j(\bigcup_{i=1}^4 f_i).$$

Step 1. Let $\sigma_2: X_2 \to X_1$ be the blowing up of X_1 along $C_1 \cong \mathbb{P}^1$. Since $(K_{X_1} \cdot C_1) = (\sigma_1^* H - 2E_1 \cdot C_1) = 0$, we have $\deg N_{C_1|X_1} = -2$. Since $q_1 \in C_1 \subset Y_1$ is the singularity of Y_1 of A_4-type and $(\tilde{C} \cdot f_1)_{\tilde{Y}} = 1$, by Lemma Mo, we have

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-2). \tag{3.15}$$

Thus we have $C_1' \cong \mathbb{F}_2$. It is easy to see that Y_2 has two rational double points q_2 of A_3-type and q_0 of D_6-type with $q_2 \in C_2 \subset Y_2, q_0 \notin C_2$. Since $(K_{X_2} \cdot C_2) = 0$, by Lemma Mo, we have

$$N_{C_2|X_1} \cong \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(-2). \tag{3.16}$$

In particular, we have

$$\begin{aligned} &\mu_2^{-1}(q_2) = f_2 \cup f_3 \cup f_4, \\ &\tilde{Y}_0 - (f_2 \cup f_3 \cup f_4) \cong Y_2 - \{q_2\}, \\ &\mu_2(\tilde{C}) = C_2, \\ &\mu_2(f_1) = f^{(2)}, \\ &\mu_2(l_1) = l_i^{(2)} \quad (i = 1, 2). \end{aligned} \tag{3.17}$$

(*Step* $k, 2 \leqslant k \leqslant 5$). Let σ_k: $X_k \to X_{k-1}$ be the blowing up of X_{k-1} along $C_{k-1} \cong \mathbb{P}^1$. Then Y_k has two rational double points q_k of A_{5-k}-type and q_0 of D_6-type with $q_k \in C_k \subset Y_k, q_0 \notin C_k$ $(k \leqslant 5)$. Since $(K_{X_k} \cdot C_k) = 0$, we have $\deg N_{C_k|X_k} = -2$. By Lemma Mo, we have

$$\begin{aligned} &N_{C_k|X_k} \cong \mathcal{O}_{C_k} \oplus \mathcal{O}_{C_k}(-2) \quad (2 \leqslant k \leqslant 4) \\ &N_{C_5|X_5} \cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1) \quad (k = 5). \end{aligned} \tag{3.18}$$

In particular,

$$\begin{aligned} &\mu_k^{-1}(q_k) = f_k \cup \cdots \cup f_4 \\ &\tilde{Y}_0 - (f_k \cup \cdots \cup f_4) \cong Y_k - \{q_k\} \\ &\mu_k(\tilde{C}) = C_k, \\ &\mu_k(f_{k-1}) = f^{(k)}. \end{aligned} \tag{3.19}$$

Step 6. Let $\sigma_6: X_6 \to X_5$ be the blowing up of X_5 along $C_5 \cong \mathbb{P}^1$. By (3.18), we have $C'_5 = \sigma_5^{-1}(C_5) \cong \mathbb{P}^1 \times P^1$. Then we have an isomorphism $\mu_6: \tilde{Y}_0 \cong Y_6$. We identify $\tilde{Y}_0$ with Y_6 (see Fig. 6). Thus we put

$$\mu_6(\tilde{C}) =: \tilde{C}, \quad \mu_6(\tilde{D}) =: \tilde{D}$$
$$\mu_6(f_i) =: f_i, \quad \mu_6(l_i) =: l_i. \tag{3.20}$$

Then $\tilde{C} = Y_6 \cdot C'_5$ gives another ruling on C'_5. Let $\bar{C}'_j$ $(1 \leqslant j \leqslant 4)$ be the proper transform of C'_j in X_6. Then we have Figure 4 (see also Pagoda (5.8) in Reid [20]).

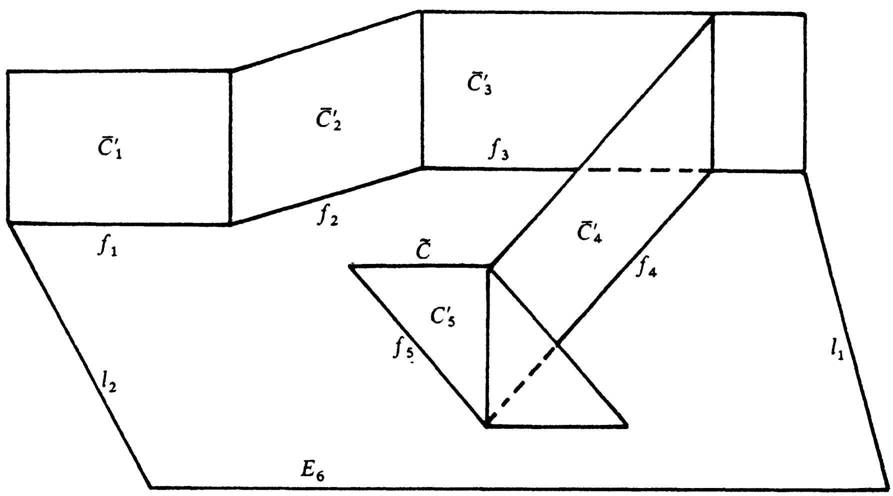


Figure 4

Now, since

$$Y_6 = \sigma_6^*\sigma_5^*\sigma_4^*\sigma_3^*\sigma_2^*\sigma_1^* H - 3\sigma_6^*\sigma_5^*\sigma_4^*\sigma_3^*\sigma_2^* E_1$$
$$-5C'_5 - 4\bar{C}'_4 - 3\bar{C}'_3 - 2\bar{C}'_2 - \bar{C}'_1, \tag{3.21}$$

we have

$$\mathcal{O}_{Y_6}(Y_6) = \mathcal{O}_{Y_6}(\tilde{D} + 3K_{Y_6} - 5\tilde{C} - 4f_4 - 3f_3 - 2f_2 - f_1)$$
$$\cong \mathcal{O}_{\tilde{Y}_0}(\tilde{D} - 3Z - 5\tilde{C} - 4f_4 - 3f_3 - 2f_2 - f_1)$$
$$\cong \mathcal{O}_{\tilde{Y}_0}(2f), \tag{3.22}$$

where f is a general fiber of $\nu: \tilde{Y} \to \mathbb{P}^1$(see Fig. 4). This shows that the linear

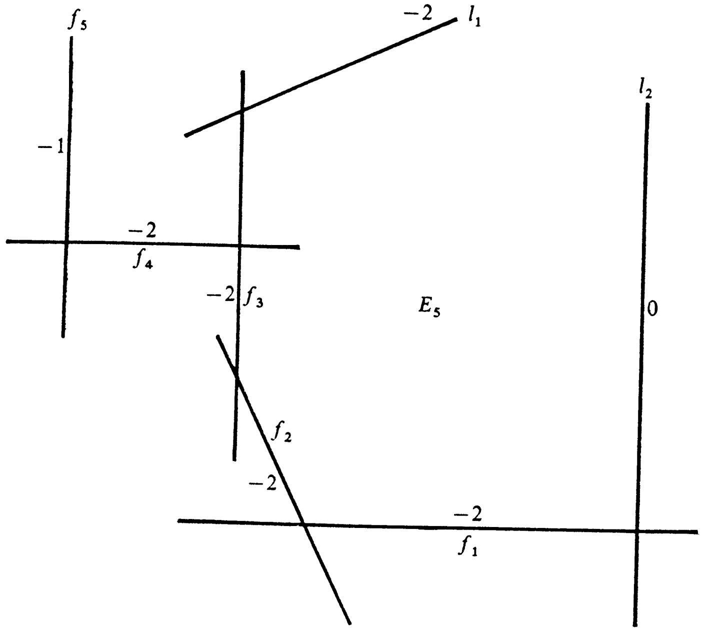


Figure 5

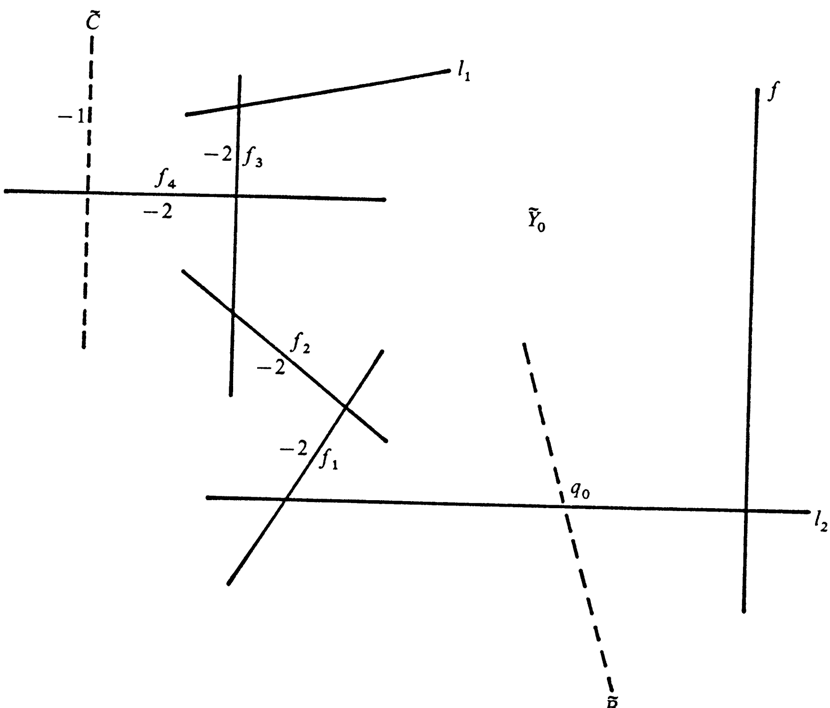


Figure 6

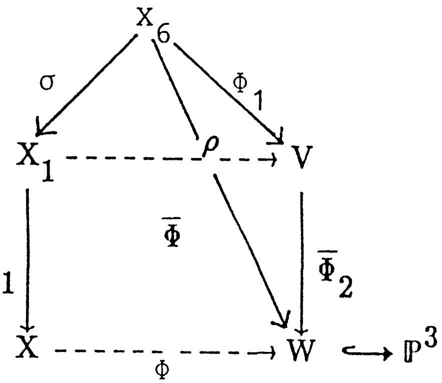
system $|\mathcal{O}_{Y_6}(Y_6)|$ has no fixed component and no base point. Therefore it defines a morphism $\nu_0\colon Y_6 = \tilde{Y}_0 \to Q$ of Y_6 onto a smooth conic $Q \cong \mathbb{P}^1$ in $\mathbb{P}^2$. Since $H^1(X_6, \mathcal{O}_{X_6}) = 0$ and $\mathrm{Pic}\, X \cong \mathbb{Z}$, the linear system $|Y_6| = |\mathcal{O}_{X_6}(Y_6)|$ has no base locus. Therefore we have a morphism $\bar{\Phi} := \bar{\Phi}_{|Y_6|}\colon X_6 \to W \hookrightarrow \mathbb{P}^3$ defined by the linear system $|Y_6|$, and have the diagram (D-1), which is desired. It is easy to see that

$$\bar{\Phi}(Y_6) = \Phi(Y) = \nu_0(Y_6) = Q \cong l_2 \qquad (3.23)$$

$$\bar{\Phi}\left(\bigcup_{j=1}^{4} \bar{C}_j' \bigcup C_5'\right) = \bar{\Phi}(f_5) \quad \text{(a line in } \mathbb{P}^3).$$

5. Since $N_{C_5|X_5} \cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1)$, by Reid [20], C_5' can be blown down along C, and then the blowing downs can be done step-by-step. Finally, we have a smooth projective 3-fold V with $b_2(V) = 2$, morphisms $\bar{\Phi}_1\colon X_6 \to V$, $\bar{\Phi}_2\colon V \to W$, and a birational map $\rho\colon X_1 \to V$, called a flip, such that
(i) $\bar{\Phi} = \bar{\Phi}_2 \circ \bar{\Phi}_1$,
(ii) $X_1 - C_1 \overset{\rho}{=} (V - \bar{f}_3$, where $\bar{f}_3 := \Phi_1(f_3)$, (see (D-2)).

(D-2)



Since $-K_{X_1} = Y_1 + E_1$, by (ii) above, we have $-K_V = A + \Sigma$, where $A := \bar{\Phi}(Y_6)$ and $\Sigma := \bar{\Phi}_1(E_6)$. For a general fiber F of $\bar{\Phi}_2\colon V \to W$, since $\deg(K_F) = (K_V \cdot F) = -(\Sigma \cdot F) \leqslant -1$, we have $F \cong \mathbb{P}^1$ and $(\Sigma \cdot F) = 2$. Since $\bar{\Phi}_1(l_1)$ is a smooth rational curve contained in Σ, and since $\bar{\Phi}_2 \circ \bar{\Phi}_1(l_1) = \bar{\Phi}(l_1)$ is a point, Σ is a meromorphic double section of $\bar{\Phi}_2\colon V \to W$.

Let G be a scheme-theoric fiber. Then we have $(G \cdot \Sigma) = 2$. Since $V - (\Sigma \cup A) \cong X_1 - (Y_1 \cup E_1) \cong \mathbb{C}^3$ by assumption, it contains no compact analytic curve. Thus $\bar{\Phi}_2\colon V \to W$ is a conic bundle over W, and $\bar{\Phi}_2$ is the contraction of an extremal ray on V. Thus, W is smooth by Mori [12]. Since $\deg W = 2$, $W \cong \mathbb{P}^1 \times \mathbb{P}^1$, hence, $b_2(V) = 3$. This is a contradiction, since $b_2(V) = b_2(X_1) = 2$. Therefore we have:

Conclusion

The case (a) of Proposition 2.5 can not occur.

4. Non-existence of the case (b)

1. Assume that there is a compactification $(X, Y) = (V_{22}, H_{22})$ of the case (b) in Proposition 2.5. Then we have Sing $Y = \{x, y\}$, where x is a minimally elliptic singularity of *Cu*-type (Table L-3, (7)), and y is a rational double point of A_{10}-type. Let $\pi: \tilde{Y} \to Y$ be the minimal resolution of the singularities of Y and put $\tilde{C} := \pi^{-1}(x), \pi^{-1}(y) =: \bigcup_{j=1}^{10} B_j$. Then $\tilde{C}$ is an irreducible rational curve with a cusp, and $K_{\tilde{Y}} = -\tilde{C}, (\tilde{C}\cdot\tilde{C})_{\tilde{Y}} = -2$. We can easily see that $\tilde{Y}$ can be obtained from $\mathbb{P}^2$ by succession of 11 blowing ups at a smooth point p on a cubic curve $C_0 \hookrightarrow \mathbb{P}^2$ with a cusp (infinitely near points allowed). Let $\mu: \tilde{Y} \to \mathbb{P}^2$ be the projection. Then $\tilde{C}$ is the proper transform of C_0 in $\tilde{Y}$ and $\mu^{-1}(p) = \bigcup_{j=1}^{11} B_j$, where B_{11} is the exceptional curve of the first kind (see Fig. 7).

We take sufficiently general hyperplane section H such that $D := H \cdot Y$ does not pass through the points x and y. Then D is a canonical curve of the genus $g = 12$ with $\deg D = 22$ in Y. Let $\tilde{D}$ be the proper transform of D in $\tilde{Y}$. Then we have

$$(\tilde{D}\cdot\tilde{D})_{\tilde{Y}} = \deg D = 22 \tag{4.1}$$

$$\tilde{D} = 3\tilde{C} + 2\tilde{G}, \tag{4.2}$$

where $\tilde{G}$ is the proper transform of a line $G \hookrightarrow \mathbb{P}^2$ with $p \notin G$ in $\tilde{Y}$. In particular, $(\tilde{G}\cdot\tilde{G})_{\tilde{Y}} = 1$.

LEMMA 4.1. *There is no line in X through the point x.*

Proof. Since the multiplicity $m(\mathcal{O}_{Y,x})$ is equal to two, any line through the point x is contained in Y. Let g be such a line in X, and $\tilde{g}$ be the proper transform of Y in $g \hookrightarrow \tilde{Y}$. Since $(D\cdot\tilde{g})_{\tilde{Y}} = (D\cdot g)_Y = (H\cdot g)_X = 1$, by (4.2), we have

$$3(\tilde{C}\cdot\tilde{g}) + 2(\tilde{G}\cdot\tilde{g}) = 1. \tag{4.3}$$

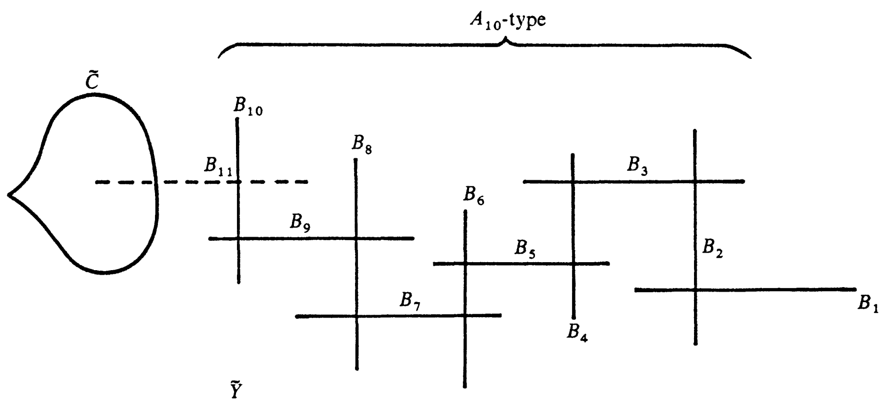


Figure 7

This is a contradiction. Q.E.D.

2. Let $\sigma: X_1 \to X$ be the blowing up of X at the point x, and put $E := \sigma^{-1}(x) \cong \mathbb{P}^2$. Let $Y_1 = \sigma^* H - 2E$ be the proper transform of Y in X. Then we have:

(i) $Y_1 \cdot E = 2l$, where l is a line in $E \cong \mathbb{P}^2$,

(ii) $\operatorname{Sing} Y_1 = l$,

(iii) $N_{l|X} \cong \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)$.

Let $\tau: X_2 \to X_1$ be the blowing up of X_1 along $l \cong \mathbb{P}^1$. By (iii) above, we have $L' := \tau^{-1}(l) \cong \mathbb{F}_2$. Let us denote the negative section (resp. a fiber) by s (resp. f). Let Y_2 be the proper transform of Y_1 in X_2. Then we have $Y_2 = \tau^* Y_1 - 2L'$. Let $\tilde{Y}_0$ be the contraction of the exceptional curve $\bigcup_{j=1}^{10} B_j$ in $\tilde{Y}$. Then $\tilde{Y}_0$ has a rational double point of A_{10}-type. By an easy computation, we have an isomorphism $\tilde{Y}_0 \overset{\nu}{\to} Y_2$. We identify Y_2 with $\tilde{Y}_0$ *via* ν. For simplicity, we put $\tilde{C} := \nu(\tilde{C}), \tilde{D} := \nu(\tilde{D}), \tilde{G} := \nu(\tilde{G})$. Then we have

$$L' \cdot Y_2 = \tilde{C}, \tag{4.4}$$

$$(\tilde{C} \cdot f)_{L'} = 2, (\tilde{C} \cdot s)_{L'} = 0, \tag{4.5}$$

(see Fig. 8).

3. We will study the linear system $|\mathscr{L}| := |\tau^* Y_1 - L'|$ on X_2.
Let us consider the exact sequence

$$0 \to \mathcal{O}_{X_2}(L') \to \mathcal{O}_{X_2}(\mathscr{L}) \to \mathcal{O}_{Y_2}(\mathscr{L}) \to 0. \tag{4.6}$$

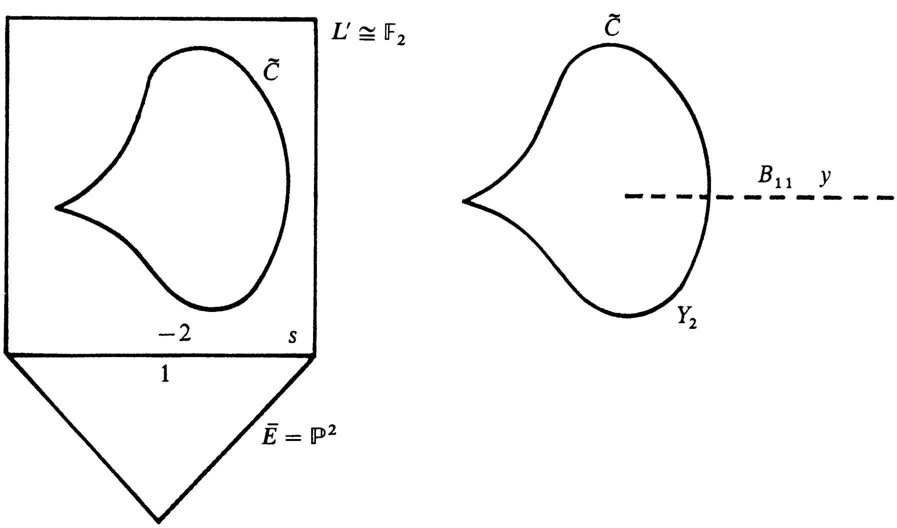


Figure 8

Since $\tau^* Y_1 - L' = \tau^*\sigma^* H - 2\bar{E} - 3L'$, we have

$$\mathcal{O}_{Y_2}(\mathcal{L}) = \mathcal{O}_{Y_2}(\tau^* Y_1 - L') = \mathcal{O}_{Y_2}(\tilde{D} - 3\tilde{C}) = \mathcal{O}_{Y_2}(2\tilde{G})$$

by (4.2), where $\bar{E} \cong \mathbb{P}^2$ is the proper transform of $E \cong \mathbb{P}^2$ in X_2. Since $H^i(Y_2, \mathcal{O}_{Y_2}(2\tilde{G})) = 0$ for $i > 0$, by the Riemann–Roch theorem, we have $H^0(Y_2, \mathcal{O}_{Y_2}(\mathcal{L})) \cong \mathbb{C}^6$. Since $H^0(X_2, \mathcal{O}(X_2, \mathcal{O}_{Y_2}(L')) \cong \mathbb{C}$ and $H^1(X_2, \mathcal{O}_{X_2}(L')) = 0$, we have finally the following exact sequence:

$$\begin{array}{ccccccccc} 0 \to & H^0(X_2, \mathcal{O}_{X_2}(L')) & \to & H^0(X_2, \mathcal{O}_{X_2}(\mathcal{L})) & \to & H^0(Y_2, \mathcal{O}_{Y_2}(\mathcal{L})) & \to 0 \\ & \wr\| & & \wr\| & & \wr\| & \\ & \mathbb{C} \cong H^0(X_2, \mathcal{O}_{X_2}) & & \mathbb{C}^7 & & \mathbb{C}^6 & \end{array} \tag{4.6}$$

Since $\dim|\mathcal{L}| = 6$, we have a rational map $\Phi := \Phi_{|\mathcal{L}|}: X_2 \to \mathbb{P}^6$ defined by the linear system $|\mathcal{L}|$.

Since the linear system $|\mathcal{O}_{Y_2}(2\tilde{G})|$ has no base locus on Y_2, neither does $|\mathcal{L}|$ by (4.6). Therefore $\Phi: X_2' \to \mathbb{P}^6$ is a morphism X_2 to $\mathbb{P}^6$ with

$$\Phi|_{Y_2} = \varphi := \varphi_{|\mathcal{O}_{Y_2}(2\tilde{G})|}: Y_2 \to \mathbb{P}^5,$$

where $\varphi_{|\mathcal{O}_{Y_2}(2\tilde{G})|}$ is a morphism defined by $|\mathcal{O}_{Y_2}(2\tilde{G})|$.

Thus we have the following:

LEMMA 4.2. *$\Phi: X_2 \to \Phi(X_2) \hookrightarrow \mathbb{P}^6$ is a morphism of X_2 onto a 3-fold $\Phi(X_2)$ of degree 4 in $\mathbb{P}^6$. Moreover, the restriction $\Phi|_{Y_2}: Y_2 \to \Phi(Y_2) = \varphi(Y_2) \hookrightarrow \mathbb{P}^5$ gives an birational morphism of Y_2 onto a surface $\varphi(Y_2)$ of degree 4 in $\mathbb{P}^5$.*

Proof. Since $(\tau^* Y_1 - L')^3 = 4$, we have $\deg \Phi(X_2) = 4$. Q.E.D.

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_{Y_2}(\tau^* Y_1 - 2L')) & \to \mathcal{O}_{X_2}(\mathcal{L}) \to & \mathcal{O}_{L'}(\mathcal{L}) & \to 0 \\ & \wr\| & & \wr\| & \\ & \mathcal{O}_{X_2}(Y_2) & & \mathcal{O}_{L'}(s + 3f) & \end{array} \tag{4.7}$$

Since $H^1(Y_2, \mathcal{O}_{Y_2}, \mathcal{O}_{Y_2}(Y_2)) \cong H^1(Y_2, \mathcal{O}_{Y_2}(2\tilde{G} - \tilde{C})) = 0$, we have a surjection

$$H^0(X_2 \mathcal{O}_{X_2}(\mathcal{L})) \twoheadrightarrow H^0(L', \mathcal{O}_{L'}(s + 3f)) \cong \mathbb{C}^6. \tag{4.8}$$

Since $\mathcal{O}_{L'}(s + 3f)$ is very ample on L', the morphism $\rho := \rho_{|s+3f|}: L' \to \rho(L') \hookrightarrow \mathbb{P}^5$ is an isomorphism of L' onto a smooth surface of degree 4 in $\mathbb{P}^5$. Thus we have the following

LEMMA 4.3. *The restriction* $\Phi|_{L'}: L' \to \Phi(L') \hookrightarrow \mathbb{P}^5$ *is an isomorphism of* L' *onto a smooth surface* $\Phi(L')$ *of degree* 4 *in* $\mathbb{P}^5$.

Finally, let us consider the exact sequence:

$$\begin{array}{ccccccc} 0 \to \mathcal{O}_{X_2}(\tau^*(Y_1 - E)) & \to & \mathcal{O}_{X_2}(\mathcal{L}) & \to & \mathcal{O}_E(\mathcal{L}) & \to 0 \\ \wr\| & & & & \wr\| & \\ \mathcal{O}_{X_2}(\tau^*(\sigma^*H - 3E)) & & & & \mathcal{O}_{\mathbb{P}^2}(1) & \end{array} \tag{4.9}$$

Then we have $H^1(X_2, \mathcal{O}_{X_2}(\tau^*(\sigma^*H - 3E))) = 0$, namely, we have a surjection

$$H^0(X_2, \mathcal{O}_{X_2}(\mathcal{L})) \twoheadrightarrow H^0(\bar{E}, \mathcal{O}_{\bar{E}}(\mathcal{L})) \cong \mathbb{C}^3. \tag{4.10}$$

Thus we have the following

LEMMA 4.4. *The restriction* $\Phi|_{\bar{E}}: \bar{E} \to \mathbb{P}^2$ *gives an isomorphism of* $\bar{E}$ *onto* $\mathbb{P}^2$.

5. Let γ be an irreducible curve in X_2 such that $(\tau^* Y_1 - L' \cdot \gamma) = 0$. Since $\tau^* Y_1 - L' = Y_2 + L'$, we have $(Y_2 \cdot \gamma) + (L' \cdot \gamma) = 0$. By Lemma 4.2, Lemma 4.3, Lemma 4.4, $\gamma \not\subset Y_2 \cup L'$. Thus $(Y_2 \cdot \gamma) = (L' \cdot \gamma) = 0$, namely, $(Y_1 \cdot \tau(\gamma)) = 0$. Hence, we have $Y_1 \cap \tau(\gamma) = \phi$ and $E \cap \tau(\gamma) \neq \phi$. This shows that there is no irreducible surface T in X_2 such that $\dim \Phi(T) \leqslant 1$. There are a finite numbers of conics in X through the point x (see [8]). Let γ be the proper transform of a conic in X through x. Then $\dim \Phi(\gamma) = 0$. In particular, there are a finite number of irreducible curves γ' in X_2 such that $\dim \Phi(\gamma') = 0$. Therefore we have the following

LEMMA 4.5. $\Phi: X_2 \to W := \Phi(X_2) \hookrightarrow \mathbb{P}^6$ *is a birational morphism of* X_2 *onto a* 3*-fold* W *of degree* 4 *in* $\mathbb{P}^6$. *In particular,* $b_2(X_2) = b_2(W) = 3$.

6. Since $\deg W = 4$ in $\mathbb{P}^6$, we have an equality

$$\deg W = \operatorname{codim} W + 1. \tag{4.11}$$

Since there is a smooth rational curve γ in X_2 such that $\dim \Phi(\gamma) = 0$ and $b_2(X_2) = b_2(W)$, the 3-fold W has a finite number of isolated singularities. Thus, W is a cone over a rational scroll or a cone over the Veronese surface. Hence, $b_2(W) = 1$. This is a contradiction, since $b_2(W) = 3$ by Lemma 4.5. Thus, we have:

Conclusion

The case (b) of Proposition 2.5 can not occur.

We have proved in Section 3 that the case (a) of Proposition 2.5 can not occur.

Therefore, in the case of the index $r = 1$, such a compactification of $\mathbb{C}^3$ does not exist. Thus we have the Theorem (see the Introduction).

5. Remarks and an example

1. Let (X, Y) be an analytic compactification of $\mathbb{C}^3$. Then we have (cf. [2], [3]):

Y has at most isolated singularities.

$\Updownarrow$

Y is normal. $\underset{\not\Leftarrow}{\Rightarrow}$ Y is projective.

$\Downarrow \;\; \not\Uparrow$

Y is irreducible. $\Leftrightarrow b_2(X) = 1$

In the case where Y is normal, we have determined the complete structure of such a (X, Y) (see Theorem in the Introduction).

On the other hand, we know that there is a non-normal hyperplane section E_5 of the Fano 3-fold V_5 such that $V_5 - E_5 \cong \mathbb{C}^3$([3]). This gives an example of a compactification (X, Y) of $\mathbb{C}^3$ with a non-normal irreducible boundary Y.

Recently, Peternell–Schneider [18] and Peternell [19] proved the following

THEOREM 5.1. *Let (X, Y) be a projective compactification of $\mathbb{C}^3$ with $b_2(X) = 1$. Assume that Y is non-normal. Then, X is a Fano 3-fold of the index $r(1 \leqslant r \leqslant 2)$, and*

(i) $r = 2 \Rightarrow (X, Y) \cong (V_5, E_5)$ *(up to isomorphism).*

(ii) $r = 1 \Rightarrow X \cong V_{22} \hookrightarrow \mathbb{P}^{13}$(or $V'_{22} \hookrightarrow \mathbb{P}^{12}$)
(Mukai–Umemura [14]).

2. Finally, we will prove that there is a non-normal hyperplane section H'_{22} of V'_{22} such that $V'_{22} - H'_{22} \cong \mathbb{C}^3$. Let $(a_0 : a_1 : \ldots : a_{12})$ be a homogeneous coordinate of $\mathbb{P}^{12}$. Then $V'_{22} \hookrightarrow \mathbb{P}^{12}$ can be written as follow (see p. 506 in [14]):

$$a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$$
$$a_0a_5 - 3a_1a_4 + 2a_2a_3 = 0$$
$$7a_0a_6 - 12a_1a_5 - 15a_2a_4 + 20a_3^2 = 0$$
$$a_0a_7 - 6a_2a_5 + 5a_3a_4 = 0$$
$$5a_0a_8 + 12a_1a_7 - 42a_2a_6 - 20a_3a_5 + 45a_4^2 = 0$$
$$a_0a_9 - 6a_1a_8 - 6a_2a_7 - 28a_3a_6 + 28a_4a_5 = 0$$
$$a_0a_{10} + 12a_1a_9 + 12a_2a_8 - 76a_3a_7 - 21a_4a_6 + 72a_5^2 = 0$$

$$a_0a_{11} + 24a_1a_{10} + 90a_2a_9 - 130a_3a_8 - 405a_4a_7 + 420a_5a_6 = 0$$
$$a_0a_{12} + 60a_1a_{11} + 534a_2a_{10} + 380a_3a_9 - 3195a_4a_8 - 720a_5a_7 + 2940a_6^2 = 0 \quad (*)$$
$$a_1a_{12} + 24a_2a_{11} + 90a_3a_{10} - 130a_4a_9 - 405a_5a_8 + 420a_6a_7 = 0$$
$$a_2a_{12} + 12a_3a_{11} + 12a_4a_{10} - 76a_5a_9 - 21a_6a_8 + 72a_7^2 = 0$$
$$a_3a_{12} - 6a_4a_{11} - 6a_5a_{10} - 28a_6a_9 + 28a_7a_8 = 0$$
$$5a_4a_{12} + 12a_5a_{11} - 42a_6a_{10} - 20a_7a_9 + 45a_8^2 = 0$$
$$a_5a_{12} - 6a_7a_{10} + 5a_8a_9 = 0$$
$$7a_6a_{12} - 12a_7a_{11} - 15a_8a_{10} + 20a_9^2 = 0$$
$$a_7a_{12} - 3a_8a_{11} + 2a_9a_{10} = 0$$
$$a_8a_{12} - 4a_9a_{11} + 3a_{10}^2 = 0$$

In the affine part $\{a_0 = 1\} \cong \mathbb{C}^{12}(a_1, \ldots, a_{12})$, let us consider the following coordinate transformation:

$$\begin{aligned}
x_1 &= a_1\\
x_2 &= a_2\\
x_3 &= a_3\\
x_4 &= a_4 - 4a_1a_3 + 3a_2^2\\
x_5 &= a_5 - 3a_1a_4 + 2a_2a_3\\
x_6 &= 7a_6 - 12a_1a_5 - 15a_2a_4 + 20a_3^2 \qquad (**)\\
x_7 &= a_7 - 6a_2a_5 + 5a_3a_4\\
x_8 &= 5a_8 + 12a_1a_7 - 42a_2a_6 - 20a_3a_5 + 45a_4^2\\
x_9 &= a_9 - 6a_1a_8 - 6a_2a_7 - 28a_3a_6 + 28a_4a_5\\
x_{10} &= a_{10} + 12a_1a_9 + 12a_2a_8 - 76a_3a_7 - 21a_4a_6 + 72a_5^2\\
x_{11} &= a_{11} + 24a_1a_{10} + 90a_2a_9 - 130a_3a_8 - 405a_4a_7 + 420a_5a_6\\
x_{12} &= a_{12} + 60a_1a_{11} + 534a_2a_{10} + 380a_3a_9 - 3150a_4a_8 - 720a_5a_7 + 2940a_6^2
\end{aligned}$$

Then the Jacobian $|\partial(x_1, \ldots, x_{12})/\partial(a_1, \ldots, a_{12})| = 35 \neq 0$, and further we have

$$V'_{22} \cap \{a_0 = 1\} \cong \{(x_1, \ldots, x_{12});\ x_j = 0\ (4 \leqslant j \leqslant 12)\} = \mathbb{C}^3(x_1, x_2, x_3).$$

We put $H'_{22} := V_{22} \cap \{a_0 = 0\}$. Then H'_{22} is non-normal.

Therefore the pair (V'_{22}, H'_{22}) is a compactification of $\mathbb{C}^3$ with a non-normal boundary.

One can easily see that the singular locus of H'_{22} is a line in V'_{22}.

Question 1. Is there a non-normal hyperplane section E_{22} in V_{22} ($\neq V'_{22}$) such that $V_{22} - E_{22} \cong \mathbb{C}^3$?

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