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Characterizing Hilbert space topology in terms of strong negligibility

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Abstract. Two decades ago R. D. Anderson showed that in Hilbert space manifolds the strongly negligible sets are precisely the σZ-sets. We investigate the conditions under which this property $SN = \sigma Z$ characterizes the l^2-manifolds among the complete ANRs. It is established that $SN = \sigma Z$ is characteristic for l^2-manifolds if every compact subset is a strong Z-set but not if every compactum is merely a Z-set.

Keywords: Strongly negligible set, σZ-set, strong Z-set, Hilbert space manifold, absolute retract, discrete disks property.

1. Introduction

In the late sixties R. D. Anderson introduced the concept of a strongly negligible set to infinite-dimensional topology. He showed in [1] that in Hilbert space manifolds the strongly negligible sets are precisely the σZ-sets. Let us denote this topological property by $SN = \sigma Z$. We investigate under what conditions this property characterizes the l^2-manifolds among the complete ANRs.

The property $SN = \sigma Z$ by itself is not sufficient. Consider for instance the spaces $\mathbb{R}^n$ which satisfy this condition, simply because they have no Z-sets. Obviously, we need to add a condition that guarantees the existence of enough Z-sets in the space. In an earlier publication [5] we proved the following:

THEOREM 1. *A complete ANR is an l^2-manifold if and only if $SN = \sigma Z$ and every compact subset is a strong Z-set.*

In this paper we show that this result is sharp:

THEOREM 2. *There exists a complete absolute retract X such that*

(a) *$SN = \sigma Z$.*
(b) *Every compact subset of X is a Z-set.*
(c) *X does not have the discrete disks property and hence X is not homeomorphic to l^2.*

So we have established that $SN = \sigma Z$ characterizes l^2-manifolds if every compactum is a strong Z-set but not if every compactum is merely a Z-set. The subtle distinction between Z-set and strong Z-set was discovered fairly recently by Bestvina, Bowers, Mogilski and Walsh [3]. It plays an essential role in the characterizations of incomplete manifolds, but has until now not shown up in characterizations of Hilbert space l^2.

2. Preliminaries

In this section we define the key notions and we present the basic ingredients for the construction of the example X. All topological spaces are assumed to be separable and metrizable.

If X is a space then the identity mapping on X is denoted by 1_X or simply by 1. We say that $h: X \to X$ is supported on $V \subset X$ if $h(V) \subset V$ and $h|_{X \setminus V} = 1$. Let $\mathcal{U}$ be a collection of subsets of X. Mappings $f, g: Y \to X$ are called $\mathcal{U}$-close if for each $y \in Y$ with $f(y) \neq g(y)$ there is a $U \in \mathcal{U}$ containing both $f(y)$ and $g(y)$. Note that if $h: X \to X$ is $\mathcal{U}$-close to 1 then h is supported on $\bigcup \mathcal{U}$.

DEFINITION 1. *A subset S of a space X is called strongly negligible if for every collection $\mathcal{U}$ of open subsets of X (not necessarily a cover of X) there is a homeomorphism h from X onto $X \setminus (S \cap \bigcup \mathcal{U})$ that is $\mathcal{U}$-close to* 1.

DEFINITION 2. *Let X be a space and let S be a closed subset of X. The set S is called a Z-set in X if for every open covering $\mathcal{U}$ of X there is a continuous $f: X \to X \setminus S$ that is $\mathcal{U}$-close to* 1. *The set S is called a strong Z-set if moreover f satisfies $\mathrm{Cl}_X(f(X)) \cap S = \varnothing$. A (strong) σZ-set is a countable union of (strong) Z-sets.*

DEFINITION 3. *Let $C(Y, X)$ denote the set of continuous functions from Y into X. Let I^2 denote the* 2*-cell. A space X is said to have the discrete disks property if for every sequence $(f_i)_{i=1}^{\infty}$ in $C(I^2, X)$ and every open covering $\mathcal{U}$ of X there exists a sequence $(g_i)_{i=1}^{\infty}$ in $C(I^2, X)$ such that each g_i is $\mathcal{U}$-close to f_i and the sequence of images $(g_i(I^2))_{i=1}^{\infty}$ has no cluster points in X.*

For a discussion of these concepts see [5].

There are two basic ingredients for the example. First, we have the comb space

$$K = \left(\left\{\frac{1}{n}\middle| n \in \mathbb{N}\right\} \times I\right) \cup (I \times \{0\}) \subset \mathbb{R}^2,$$

where I is the interval [0, 1] and the topology is Euclidean. This space was introduced by Bestvina et al. [3] to show that not every Z-set is a strong Z-set. Let α denote the point (0, 0) in K. The singleton $\{\alpha\}$ is a Z-set but not a strong Z-set. Both K and $K \setminus \{\alpha\}$ are easily seen to be complete absolute retracts.

The second ingredient is a homology cell Z; specifically a homologically trivial polyhedron that is not simply connected. In particular we shall use the cone $C(Z)$ and the suspension $S(Z)$ of Z*. Note that both $C(Z)$ and $S(Z)$ are compact absolute retracts. For $S(Z)$ this follows from the fact that the suspension of a homologically trivial space is contractible (see Spanier [8, p. 461]).

3. The example

We shall now construct the space X of Theorem 2. Consider the cone $C(Z)$ and assume that it is obtained by identifying $\{0\} \times Z$ to a point in the space $I \times Z$. Let z be a fixed point in Z. We attach the comb space K to $C(Z)$ by identifying the arc $I \times \{0\}$ from K with $I \times \{z\} \subset C(Z)$. Call the resulting space A. Note that the special point α of K is identified with the vertex of the cone. We shall continue to call this point α. Let $\pi: A \to I$ be the "projection" defined by $\pi(x, y) = x$ both for $(x, y) \in K$ and for $(x, y) \in C(Z)$. Observe that A consists of two absolute retracts meeting in an arc and hence it is also an AR. Define

$$B = ((A \backslash \{\alpha\}) \times l^2) \cup \{\alpha\}.$$

If ξ is the projection from B onto A then basic neighbourhoods of α in B are preimages of neighbourhoods of α in A. Furthermore, the set $(A \backslash \{\alpha\}) \times l^2$ is an open subset of B equipped with the product topology. Noting that the closed unit ball in l^2 is homeomorphic to l^2, an alternative definition of B would be the variable product

$$B' = \{(x, y) \in A \times l^2 \mid \|y\| \leqslant d(x, \alpha)\},$$

where d is some metric on A and $\|\cdot\|$ is the standard norm for l^2. Since B' is obviously a retract of $A \times l^2$ we have that B is an AR. Let $\tilde{X}$ be the complete AR $B \times \mathbb{R}$ and let $R = \{\alpha\} \times \mathbb{R} \subset \tilde{X}$. Note that $\tilde{X} \backslash R$ is the space $(A \backslash \{\alpha\}) \times l^2 \times \mathbb{R}$. Since $A \backslash \{\alpha\}$ is a complete ANR we have according to Toruńczyk [9] that $\tilde{X} \backslash R$ is an l^2-manifold.

Let S be a universal pseudo-boundary in $\mathbb{R}$, see Geoghegan and Summerhill [7]. Then S is a zero-dimensional σ-compactum in $\mathbb{R}$ such that for every zero-dimensional compactum C in $\mathbb{R}$ and every collection $\mathscr{U}$ of open subsets of $\mathbb{R}$, there is an autohomeomorphism h of $\mathbb{R}$ with h and 1 $\mathscr{U}$-close and

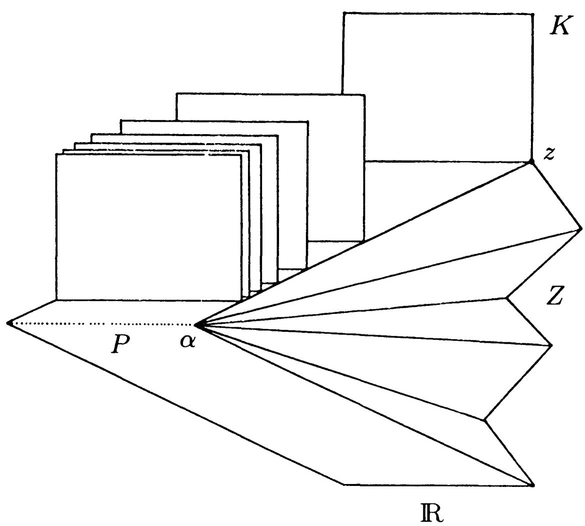
$$h(S) \cap \bigcup \mathscr{U} = (S \cup C) \cap \bigcup \mathscr{U}.$$

*Thanks are due to Jan van Mill for bringing these spaces to the attention of the author.

The set S is homeomorphic to the product of the cantor set and the space of rational numbers, in fact any countable dense union of cantor sets in $\mathbb{R}$ will do. The example X is given by

$$X = \tilde{X} \backslash (\{\alpha\} \times S).$$

The set $\{\alpha\} \times (\mathbb{R} \backslash S) \subset R$ is denoted by P and is homeomorphic to the space of irrational numbers. The following illustration shows the space X with the l^2 factor suppressed.



PROPOSITION 1. *A closed subset of* R *is a* Z*-set in* $\tilde{X}$ if and only if it is nowhere dense in R.

Proof. (i) Sufficiency. Let D be a Z-set in R that contains an interval $\{\alpha\} \times (a, b)$. Since being a Z-set is a local property this implies that $\{\alpha\} \times (a, b)$ is a Z-set in $B \times (a, b)$. Deleting a Z-set from the absolute retract $B \times (a, b)$ will result in another absolute retract so we may conclude that $(B \backslash \{\alpha\}) \times (a, b)$ is an AR. Note that $(B \backslash \{\alpha\}) \times (a, b)$, $B \backslash \{\alpha\} = A \backslash \{\alpha\} \times l^2$, $A \backslash \{\alpha\}$, $C(Z) \backslash \{\alpha\} = (0, 1] \times Z$ and Z all have the same homotopy type. Since the first space in this list is an AR and the last one is not simply connected we have a contradiction, proving sufficiency.

(ii) Necessity. Let D be a nowhere dense closed subset of $\mathbb{R}$. Since the Z-set property is σ-additive in complete spaces (see Bessaga and Pelczyński [2: prop. V.2.2]), we may assume that D is compact. Let n be an arbitrary natural number. Select a sequence $x_0 < x_1 < \cdots < x_m$ in $\mathbb{R} \backslash D$ such that $|x_i - x_{i+1}| < 1/n$ and

$D \subset [x_0, x_m]$. In order to keep the notation manageable we shall ignore the l^2-factor in B. It is easily seen that this does not essentially change the argument.

Let i be a fixed index such that the interval (x_i, x_{i+1}) meets D. Consider the set

$$U = \left(\pi^{-1}\left(\left[0, \frac{1}{n}\right]\right) \cap C(Z)\right) \times [x_i, x_{i+1}].$$

The boundary of U in $C(Z) \times \mathbb{R}$ is

$$\partial U = (\pi \times 1_{\mathbb{R}})^{-1}(\gamma) \cap (C(z) \times \mathbb{R}),$$

where γ is the boundary of $[0, 1/n] \times [x_i, x_{i+1}]$ in $I \times \mathbb{R}$, that is γ is the arc

$$\left(\left[0, \frac{1}{n}\right] \times \{x_i, x_{i+1}\}\right) \cup \left(\left\{\frac{1}{n}\right\} \times [x_i, x_{i+1}]\right).$$

Observe that ∂U is homeomorphic to the suspension $S(Z)$. As noted in section 2, $S(Z)$ is an absolute retract. Consequently there exists a retraction r_i of U onto ∂U. We may assume that r_i has the property $r_i([0, 1/n] \times \{z\} \times [x_i, x_{i+1}]) = \gamma \times \{z\}$, i.e. r_i preserves the plane where the cone part and the comb part of the space meet.

Now we extend r_i to the comb part of $A \times \mathbb{R}$. Consider the set

$$V = \left(K \cap \left[0, \frac{1}{n}\right] \times \left[0, \frac{1}{n}\right]\right) \times [x_i, x_{i+1}],$$

which meets U in the set $F = [0, 1/n] \times \{z\} \times [x_i, x_{i+1}]$. Note that V is homeomorphic to $K \times I$ and hence an absolute retract. Furthermore, $\{\alpha\} \times (x_i, x_{i+1})$ is a σZ-set in V and hence $\tilde{V} = V \backslash (\{\alpha\} \times (x_i, x_{i+1}))$ is an AR. Let $s_i\colon V \to \tilde{V}$ be an extension of $r_i|_F$ rhat fixes the boundary of V in $A \times \mathbb{R}$. Define $\bar{r}_i\colon U \cup V \to (U \cup V) \backslash (\{\alpha\} \times (x_i, x_{i+1}))$ by $\bar{r}_i = r_i \cup s_i$. Let $f\colon \tilde{X} \to \tilde{X} \backslash (\{\alpha\} \times D)$ be the union of the $\bar{r}_i$'s extended with the identity over $\tilde{X}$. By choosing n large we can get f arbitrarily close to the identity on $\tilde{X}$. This shows that $\{\alpha\} \times D$ is a Z-set in $\tilde{X}$. □

COROLLARY 1. *The space X is a complete AR.*

Proof. The pseudo-boundary S is a countable union of cantor sets in $\mathbb{R}$ and hence $\{\alpha\} \times S$ is a σZ-set in $\tilde{X}$. The space $\tilde{X}$ is a complete AR and consequently $X = \tilde{X} \backslash (\{\alpha\} \times S)$ is also a complete AR (Toruńczyk [10]). □

COROLLARY 2. *Every compact subset of X is a Z-set in X.*

Proof. Let D be a compact subset of X. Then $D \backslash R$ is a σ-compact subset of the l^2-manifold $\tilde{X} \backslash R$ and hence D/R is a σZ-set of $\tilde{X}$. The set $D \cap R$ is a compact subset of R that does not meet the dense set $\{\alpha\} \times S$. So $D \cap R$ is nowhere dense in R and a Z-set in $\tilde{X}$. Since a closed σZ-set is a Z-set in complete spaces we have established that D is a Z-set in $\tilde{X}$. Seeing that the difference between X and $\tilde{X}$ is just a σZ-set we find that D is also a Z-set in X. □

According to Toruńczyk [11] every l^2-manifold has the discrete approximation property which implies the discrete disks property. So the following proposition implies that X is not homeomorphic to Hilbert space.

PROPOSITION 2. *X does not have the discrete disks property.*

Proof. We shall prove that if $p \in P$ then there is an open covering $\mathscr{U}$ of X and a sequence $(g_i)_{i=1}^{\infty}$ in $C(I^2, X)$ such that for every sequence $(h_i)_{i=1}^{\infty}$ in $C(I^2, X)$ that is $\mathscr{U}$-close to $(g_i)_{i=1}^{\infty}$, the sequence of images $(h_i(\mathrm{I}^2))_{i=1}^{\infty}$ has p as a cluster point. Since basic neighbourhoods of α in B are preimages under ξ of neighbourhoods of α in A, we can ignore the l^2-factor in B and work entirely in $A \times \mathbb{R}$ rather than $B \times \mathbb{R}$.

Let (α, r) be an arbitrary point in P. Construct for every $i \in \mathbb{N}$ a homeomorphism $f_i \colon I \to J_i$ where J_i is the arc

$$\left(\left\{\frac{1}{i+1}, \frac{1}{i}\right\} \times I\right) \cup \left(\left[\frac{1}{i+1}, \frac{1}{i}\right] \times \{0\}\right) \subset K.$$

Define $g_i \colon I \times [-1, 1] \to K \times \mathbb{R}$ by

$$g_i(s, t) = (f_i(s), t + r).$$

Let $\varepsilon < \frac{1}{2}$ and consider the following coverings of A respectively $A \times \mathbb{R}$

$$\mathscr{V} = \left\{\left\{\frac{1}{n}\right\} \times (a, a + \varepsilon) \cap K \mid n \in \mathbb{N},\ a > 0\right\}$$
$$\cup \{((a, a + \varepsilon) \times [0, \varepsilon) \cap K) \cup ((a, a + \varepsilon) \times Z \cap C(Z)) \mid a \in \mathbb{R}\}$$

and

$$\mathscr{U} = \{V \times (a, a + \varepsilon) \mid a \in \mathbb{R} \text{ and } V \in \mathscr{V}\}.$$

Suppose that $h_i \colon I \times [-1, 1] \to A \times \mathbb{R}$ is $\mathscr{U}$-close to g_i. First let ρ_1 be the standard retraction of $A \times \mathbb{R}$ onto $A \times [r - 1, r + 1]$. Let $\rho_2 \colon A \to K$ be obtained by projecting $C(Z)$ onto $[0, 1] \times \{z\}$. Finally ρ_3 is the retraction from K onto J_i, that is obtained by mapping everything to the left of J_i onto the point

$(1/(i+1), 0)$ and everything to the right onto $(1/i, 0)$. Let $\tilde{h}_i$: $I \times [-1, 1] \to J_i \times [r-1, r+1]$ be defined by

$$\tilde{h}_i = ((\rho_3 \circ \rho_2) \times 1_{\mathbb{R}}) \circ \rho_1 \circ h_i.$$

It is easily verified that $\tilde{h}_i$ and g_i are still $\mathscr{U}$-close. Observe that g_i is a homeomorphism between the 2-cells $I \times [-1, 1]$ and $J_i \times [r-1, r+1]$ and that $\tilde{h}_i$ is ε-close to this homeomorphism. This implies that there is a $q_i \in I \times [-1, 1]$ which is mapped by $\tilde{h}_i$ onto a central point of the disk $j_i \times [r-1, r+1]$, say $((1/(i+\frac{1}{2}), 0), r)$. It follows from the properties of ρ_1, ρ_2 and ρ_3 that $\tilde{h}_i(q_i) = (\rho_2 \times 1) \circ h_i(q_i)$ and hence that $h_i(q_i)$ is an element of $\{1/(i+\frac{1}{2})\} \times Z \times \{r\} \subset C(Z) \times \{r\}$. Consequently, the sequence $(h_i(q_i))_{i=1}^{\infty}$ converges to (α, r). This proves that (α, r) is a cluster point of $(h_i(I \times [-1, 1]))_{i=1}^{\infty}$.

□

PROPOSITION 3. *In the space X the strongly negligible sets are precisely the σZ-sets.*

Proof. In a complete space every strongly negligible set is a σZ-set, see [5]. Let L be a σZ-set in X and consider $L \backslash P$. Since P is a closed subset of X, the set $L \backslash P$ can be written as a countable union of Z-sets L_i in X that do not meet P. Noting that $X \backslash P = \tilde{X} \backslash R$ is an l^2-manifold we find that every L_i is strongly negligible in $X \backslash P$. Since we may assume that the homeomorphisms witnessing this are supported on a set whose closure does not meet P we can extend these homeomorphisms with 1_P and conclude that L_i is strongly negligible in X. On the other hand, assume that we can show that every closed subset of P that is a Z-set in X is strongly negligible in X. Note that $L \cap P$ is a countable union of such sets and hence that L is a countable union of strongly negligible sets. This means that L itself is strongly negligible in X, see Cutler [4] or Dijkstra [6].

Let $\{\alpha\} \times M \subset P$ be a Z-set in X and let $\bar{M}$ stand for the closure of M in $\mathbb{R}$. We show that $\{\alpha\} \times M$ is strongly negligible in X. Since $\{\alpha\} \times S = \tilde{X} \backslash X$ is a σZ-set in $\tilde{X}$ we have that $\{\alpha\} \times \bar{M}$ is a Z-set in $\tilde{X}$. Using proposition 1 we find that $\bar{M}$ is nowhere dense in $\mathbb{R}$. Since strong negligibility is σ-additive we may assume that $\bar{M}$ is compact. Let $\mathscr{U}$ be a collection of open subsets of X. Since $\bar{M}$ is nowhere dense it is possible to select a sequence $O_1, O_2, O_3, \ldots$ of bounded, disjoint, open intervals in $\mathbb{R}$ and positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ such that

$$\mathscr{V} = \{\xi^{-1}(U_{\varepsilon_i}(\alpha)) \times O_i \mid i \in \mathbb{N}\}$$

is a refinement of $\mathscr{U}$ and $\bigcup \mathscr{V} \cap (\{\alpha\} \times \bar{M}) = \bigcup \mathscr{U} \cap (\{\alpha\} \times \bar{M})$, where $U_\varepsilon(\alpha)$ denotes the ε-neighbourhood of α in A with respect to some fixed metric on A.

Since S is a pseudo-boundary for zero-dimensional compacta there is a homeomorphism $f: \mathbb{R} \to \mathbb{R}$ that is $\{O_i \mid i \in \mathbb{N}\}$-close to $1_{\mathbb{R}}$ and that satisfies

$$f(S) \cap \bigcup_{i=1}^{\infty} O_i = (S \cup \bar{M}) \cap \bigcup_{i=1}^{\infty} O_i.$$

It is a straightforward but somewhat tedious exercise to show that f can be extended to a homeomorphism $h: \tilde{X} \to \tilde{X}$ such that h and 1 are $\mathscr{V}$-close. The details of this are completely analogous to the proof of claim 2 in Dijkstra [5]. Note that $h|_X$ is a homeomorphism from X onto $X \backslash (\bar{M} \cap \bigcup_{i=1}^{\infty} O_i) = X \backslash (M \cap \bigcup \mathscr{U})$. This proves that M is strongly negligible in X. □

References

1. Anderson, R. D., Strongly negligible sets in Fréchet manifolds. *Bull. Amer. Math. Soc.* 75, 64–67 (1969).
2. Bessaga, C., Pełczyński, A., *Selected topics in infinite dimensional topology*. Warsaw: PWN 1975.
3. Bestvina, M., Bowers, P. L., Mogilski, J., Walsh, J. J., Characterization of Hilbert space manifolds revisited. *Topology Appl.* 24, 53–69 (1986).
4. Cutler, W. H., Negligible subsets of infinite-dimensional Fréchet manifolds. *Proc. Amer. Math. Soc.* 23, 668–675 (1969).
5. Dijkstra, J. J., Strong negligibility of σ-compacta does not characterize Hilbert space. *Pacific J. Math.* 127, 19–30 (1987).
6. Dijkstra, J. J., Strongly negligible sets outside Fréchet manifolds. *Bull. London Math. Soc.* 19, 371–377 (1987).
7. Geoghegan, R., Summerhill, R. R., Pseudo-boundaries and pseudo-interiors in Euclidean spaces and topological manifolds. *Trans. Amer. Math. Soc.* 194, 141–165 (1974).
8. Spanier, E. H., *Algebraic Topology*. New York: McGraw-Hill, 1966.
9. Toruńczyk, H., Absolute retracts as factors of normed linear spaces. *Fund. Math.* 86, 53–67 (1974).
10. Toruńczyk, H., Concerning locally homotopy negligible sets and characterization of l^2-manifolds. *Fund. Math.* 101, 93–110 (1978).
11. Toruńczyk, H., Characterizing Hilbert space topology. *Fund. Math.* 111, 247–262 (1981).