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Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, II

Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, II

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Abstract. The paper is a continuation of my earlier paper on this subject. We prove analogous congruences, as in that paper, but modulo larger powers of 2.

0. Introduction

For the discriminant d of a real quadratic field, denote $k_2(d) = B_{2,(\frac{d}{\cdot})}$ if $d \neq 5$, 8 and $k_2(5) = k_2(8) = 4$. The Birch-Tate conjecture for real quadratic fields F with the discriminant d states that

$$|K_2 O_F| = k_2(d).$$

Here K_2, O_F, $(\frac{d}{\cdot})$ and $B_{k,\chi}$ denote the Milnor functor, the ring of integers of F, the Kronecker symbol and the kth Bernoulli number respectively. For the discriminant d of an imaginary quadratic field, let $h(d)$ denote the class number of this field.

We have found in [8] some connections between $k_2(d)$ and class numbers of appropriate imaginary quadratic fields of the Lerch-Berndt type (see [5] and [1]). From the obtained formulas we have got appropriate congruences for $k_2(d)$ modulo powers of 2 (4, 8, 16, 32 and 64). It is the purpose of this paper to prove analogous congruences modulo larger powers of 2.

For the discriminant D of a quadratic field and a square-free natural number $N > 1$ prime to D, let

$$\varphi_D(N) = \prod_{p \mid N} \left(p - \left(\frac{D}{p} \right) \right),$$

and

$$\psi_D(N) = \prod_{p \mid N} \left(1 - \left(\frac{D}{p} \right) \right),$$

where the products are taken over all prime factors of N. Set $\varphi_D(1) = \psi_D(1) = 1$. Note that $\varphi_D(N)$ and $\psi_D(N)$ are products of Euler's factors. Denote by φ Euler's totient function.

Let d and e, $e \mid d$ be the odd discriminants of quadratic fields. For $D \in \{-8e, -4e, e, 8e\}$, set

$$K(d, D) = -\left(\frac{e}{|d/e|}\right) \varphi_D(|d/e|)k_2(D), \quad \text{if } D > 0,$$

and

$$H(d, D) = \left(\frac{e}{|d/e|}\right) \psi_D(|d/e|)h(D), \quad \text{if } D < 0.$$

We shall prove the following:

THEOREM. *Let d be the odd discriminant of a quadratic field having n prime factors. We have in the above notation:*

$$\sum_{\substack{e \mid d \\ e > 1 \\ e \equiv 1 (\bmod 4)}} \left(K(d, 8e) - \left(34 - \left(\frac{e}{2}\right)\right) K(d, e)\right) +$$

$$+ \sum_{\substack{e \mid d \\ e < 0 \\ e \equiv 1 (\bmod 4)}} \left(K(d, -8e) + \left(\frac{e}{2}\right) K(d, -4e)\right) + K(d)$$

$$\equiv 2|d| \left(\sum_{\substack{e \mid d \\ e > 1 \\ e \equiv 1 (\bmod 4)}} \left(\left(\frac{e}{2}\right) H(d, -4e) + H(d, -8e)\right)\right.$$

$$+ \sum_{\substack{e \mid d \\ e < 0 \\ e \equiv 1 (\bmod 4)}} \left.\left(\left(1 - \left(\frac{e}{2}\right)\right) H(d, e) - H(d, 8e)\right) + H(d)\right)$$

$(\bmod \ 2^{n+6})$. *Here*

$$H(d) = \tfrac{1}{4}(-1)^n \varphi(|d|) + A(|d|) + \Theta(d),$$

where $A(|d|)$ is defined in Lemma 3 (see the section 2), and

$$\Theta(d) = \begin{cases} -\tfrac{13}{3}H(d, -3), & \text{if } 3 \mid d, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover

$$K(d) = \tfrac{11}{2}\varphi(|d|) - 2\varphi_8(|d|) + v(d),$$

where

$$v(d) = \begin{cases} 28K(d, 5), & \text{if } 5 \mid d, \\ 0, & \text{otherwise.} \end{cases}$$

For illustration we give the following:

COROLLARY. *Let p, $p > 5$ be a prime number and let $f_s(x)$ for $s \in \{1, 3, 5, 7\}$ be the polynomial defined as follows:*

$$f_s(x) = ax^2 + bx + c,$$

where

$$a = -\frac{1}{2},$$

$$b = -\left(\frac{-4}{s}\right)\left(1 + 2\left(\frac{8}{s}\right)\right)$$

$$c = \frac{1}{2}\left(11 - 4\left(\frac{8}{s}\right)\right).$$

We have for $p \equiv s(\mathrm{mod}\ 8)$, $s \in \{1, 3, 5, 7\}$:

(i) *if $p \equiv 1 \,(\mathrm{mod}\ 4)$, then:*

$$-k_2(8p) + \left(34 - \left(\frac{8}{p}\right)\right)k_2(p) \equiv 2p\left(\left(\frac{8}{p}\right)h(-4p) + h(-8p)\right) + f_s(p)\ (\mathrm{mod}\ 128),$$

(ii) *if $p \equiv 3 \,(mod\ 4)$, then:*

$$-k_2(8p) - \left(\frac{8}{p}\right)k_2(4p) \equiv 2p\left(\left(1 - \left(\frac{8}{p}\right)\right)h(-p) - h(-8p)\right) + f_s(p)\ (\mathrm{mod}\ 128).$$

This corollary (Theorem for $n = 1$) can be a consequence of Theorems 1, 2 [8], too. All the congruences obtained for $n \geq 2$ are new.

Let us note that in our notation Theorem of [4] states:

$$\sum_{\substack{e \mid d \\ e > 1 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(\left(\frac{e}{2} \right) H(d, -4e) + H(d, -8e) \right)$$

$$+ \sum_{\substack{e \mid d \\ e < 0 \\ e \equiv 1 (\mathrm{mod}\, 4)}} \left(\left(5 - \left(\frac{e}{2} \right) \right) H(d, e) - H(d, 8e) \right) + H_1(d) \equiv 0 \;(\mathrm{mod}\; 2^{n+2}),$$

where

$$H_1(d) = \tfrac{1}{2}(-1)^n \varphi(|d|) + A(|d|) + \tfrac{12}{13}\Theta(d).$$

In the present paper we extend the results of [4] and [3] using ideas of [4]. Similar problems were also dealt with in [2] and [7].

1. Lemmas of the Lerch-Berndt type

Let e be the odd discriminant of a quadratic field. Denote for $k = 1, 2, \ldots, 8$

$$I_k = ((k-1)|e|/8, \, k|e|/8).$$

Let

$$T_k = \sum_{l \in I_k} \left(\frac{e}{l} \right) \quad \text{and} \quad S_k = \sum_{l \in I_k} \left(\frac{e}{l} \right) l.$$

LEMMA 1 ([5], [1]). *We have:*

$$T_1 = \begin{cases} \tfrac{1}{4} \left(\dfrac{e}{2} \right) h(-4e) + \tfrac{1}{4}h(-8e), & \text{if } e > 0, \\[2mm] \tfrac{1}{4} \left(5 - \dfrac{e}{2} \right) h(e) - \tfrac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_2 = \begin{cases} \tfrac{1}{4} \left(2 - \dfrac{e}{2} \right) h(-4e) - \tfrac{1}{4}h(-8e), & \text{if } e > 0, \\[2mm] \tfrac{3}{4} \left(-1 + \dfrac{e}{2} \right) h(e) + \tfrac{1}{4}h(8e) + \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_3 = \begin{cases} \tfrac{1}{4} \left(-2 - \dfrac{e}{2} \right) h(-4e) + \tfrac{1}{4}h(-8e), & \text{if } e > 0, \\[2mm] \tfrac{3}{4} \left(1 - \dfrac{e}{2} \right) h(e) + \tfrac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

$$T_4 = \begin{cases} \frac{1}{4}\left(\frac{e}{2}\right) h(-4e) - \frac{1}{4}h(-8e), & \text{if } e > 0, \\ \frac{3}{4}\left(1 - \frac{e}{2}\right) h(e) - \frac{1}{4}h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

where $\lambda(e) = 1$, if $e = -3$, and $\lambda(e) = 0$, otherwise.

Moreover we have for $k = 5, 6, 7, 8$

$$T_k = \left(\frac{e}{-1}\right) T_{9-k}. \qquad\qquad\qquad \square$$

LEMMA 2 ([8]). *We have:*

$$S_1 = \begin{cases} \frac{1}{64} k_2(8e) - \frac{1}{64}\left(34 - \frac{e}{2}\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(\frac{e}{2}\right) h(-4e) + h(-8e)\right) + 7\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) + \frac{1}{64}\left(\frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(\left(1 - \frac{e}{2}\right) h(e) - h(8e)\right) - v(e), & \text{if } e < 0, \end{cases}$$

$$S_2 = \begin{cases} -\frac{1}{64}k_2(8e) + \frac{3}{64}\left(2 - 3\left(\frac{e}{2}\right)\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(4 - \frac{e}{2}\right)\right) h(-4e) - h(-8e)\right) - 3\omega(e), & \text{if } e > 0, \\ -\frac{1}{64}k_2(-8e) + \frac{1}{64}\left(4 - \frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(5\left(-1 + \frac{e}{2}\right) h(e) + h(8e)\right) + 5v(e), & \text{if } e < 0, \end{cases}$$

$$S_3 = \begin{cases} -\frac{1}{64}k_2(8e) - \frac{3}{64}\left(2 - 3\frac{e}{2}\right) k_2(e) \\ \quad + \frac{1}{32}e\left(\left(-4 - 3\frac{e}{2}\right) h(-4e) + 3h(-8e)\right) + 3\omega(e), & \text{if } e > 0, \\ \frac{1}{64}k_2(-8e) - \frac{1}{64}\left(4 + \frac{e}{2}\right) k_2(-4e) \\ \quad - \frac{1}{32}e\left(7\left(1 - \frac{e}{2}\right) h(e) + 3h(8e)\right) - 7v(e), & \text{if } e < 0, \end{cases}$$

$$S_4 = \begin{cases} \frac{1}{64}k_2(8e) - \frac{15}{64}\left(2 - \frac{e}{2}\right)k_2(e) \\[2mm] \quad + \frac{1}{32}e\left(3\left(\frac{e}{2}\right)h(-4e) - 3h(-8e)\right) + 9\omega(e), & \text{if } e > 0, \\[4mm] -\frac{1}{64}k_2(-8e) + \frac{1}{64}\left(\frac{e}{2}\right)k_2(-4e) \\[2mm] \quad - \frac{1}{32}e\left(13\left(1 - \frac{e}{2}\right)h(e) - 3h(8e)\right) - 13v(e), & \text{if } e < 0, \end{cases}$$

where $\omega(e) = \frac{1}{4}$, if $e = 5$, $\omega(e) = 0$, otherwise, and $v(e) = \frac{1}{8}$, if $e = -3$, $v(e) = 0$, otherwise. Moreover we have for $k = 5, 6, 7, 8$

$$S_k = eT_{9-k} - \left(\frac{e}{-1}\right)S_{9-k}. \qquad \Box$$

2. Lemma of the Nagell type

Let N be a natural number. In [6] (see also [4]) the explicit formulas for

$$\sum_{\substack{0 < k < N/8 \\ (k,N) = 1}} 1$$

are found. Now, we shall determine the sum

$$J(N) = \sum_{\substack{0 < k < N/8 \\ (k,N) = 1}} k$$

in the cases that are of our interest.

We shall prove the following:

LEMMA 3. *Take the notation of Introduction. Let x, y, z, u be respectively the numbers of prime divisors of an odd natural square-free number N, $N > 1$, of the form $8t + 1$, $8t - 1$, $8t + 3$, $8t - 3$. Set $n = x + y + z + u$. Then:*

$$J(N) = (-1)^n \frac{\varphi(N)}{128}((-1)^n N - 11) + (-1)^n \frac{\varphi_8(N)}{32} + (-1)^n \frac{N}{32} A(N),$$

where

$$A(N) = \begin{cases} 0, & \text{if } x > 0 \text{ or } x = 0, \ y \geq 0, \ z > 0, \ u > 0, \\ 2^n, & \text{if } x = 0, \ y \geq 0, \ z = 0, \ u > 0, \\ 2^{n-1}, & \text{if } x = 0, \ y \geq 0, \ z > 0, \ u = 0, \\ 3 \cdot 2^{n-1}, & \text{if } x = 0, \ y > 0, \ z = 0, \ u = 0. \end{cases}$$

Proof. We have

$$J(N) = \sum_{0 < k < N/8} k \sum_{f \mid (k,N)} \mu(f) = \sum_{f \mid N} \mu(f) \sum_{\substack{0 < k < N/8 \\ f \mid k}} k = \sum_{f \mid N} \mu(f) f \sum_{k=1}^{[N/8f]} k$$

$$= \sum_{f \mid N} \mu(f) f \frac{[N/8f]([N/8f] + 1)}{2} = \frac{1}{2}(-1)^n N \sum_{f \mid N} \frac{\mu(f)}{f} [f/8]([f/8] + 1)$$

$$= \frac{(-1)^n N}{128} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \mu(f) \frac{(f-s)(f-s+8)}{f}$$

$$= \frac{(-1)^n N}{128} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \mu(f)\left(f + 2(4 - s) + \frac{s^2 - 8s}{f}\right)$$

$$= \frac{(-1)^n N}{128} \sum_{f \mid N} \mu(f) f + \frac{(-1)^n N}{128} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} 2(4 - s) \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \mu(f) +$$

$$+ \frac{(-1)^n N}{128} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} (s^2 - 8s) \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \frac{\mu(f)}{f}.$$

Therefore by the table

s	$4-s$	$s^2 - 8s$
1	3	-7
3	1	-15
5	-1	-15
7	-3	-7

we obtain

$$J(N) = \frac{(-1)^n N}{128} \sum_{f \mid N} \mu(f) f + \frac{(-1)^n N}{64} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \mu(f) \left(\frac{-8}{s}\right) +$$

$$+ \frac{(-1)^n N}{64} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \mu(f) \left(\left(\frac{-8}{s}\right) + \left(\frac{-4}{s}\right)\right) - \frac{7(-1)^n N}{128} \sum_{f \mid N} \frac{\mu(f)}{f} -$$

$$- \frac{(-1)^n N}{32} \sum_{\substack{s=1 \\ s \text{ odd}}}^{7} \sum_{\substack{f \mid N \\ f \equiv s(\bmod 8)}} \frac{\mu(f)}{f} \left(1 - \left(\frac{8}{s}\right)\right) = \frac{(-1)^n N}{128} \sum_{f \mid N} \mu(f) f +$$

$$+ \frac{(-1)^n N}{64} \sum_{f \mid N} \mu(f) \left(\frac{-8}{f}\right) + \frac{(-1)^n N}{64} \sum_{f \mid N} \mu(f) \left(\left(\frac{-8}{f}\right) + \left(\frac{-4}{f}\right)\right) -$$

$$- \frac{7(-1)^n N}{128} \sum_{f \mid N} \frac{\mu(f)}{f} - \frac{(-1)^n N}{32} \sum_{f \mid N} \frac{\mu(f)}{f} \left(1 - \left(\frac{8}{f}\right)\right).$$

Now we shall use the well-known formula

$$\sum_{d \mid N} \mu(d)F(d) = \prod_{p \mid N} (1 - F(p)) \tag{2.1}$$

that is true for a multiplicative arithmetical function F. In this formula the sum and the product are extended over all divisors and over all prime factors of N respectively.

We get

$$J(N) = \frac{(-1)^n N}{128} \prod_{p \mid N} (1 - p) + \frac{(-1)^n N}{32} \prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right)$$

$$+ \frac{(-1)^n N}{64} \prod_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right) - \frac{11(-1)^n N}{128} \prod_{p \mid N} \left(1 - \frac{1}{p}\right)$$

$$+ \frac{(-1)^n N}{32} \prod_{p \mid N} \left(1 - \left(\frac{8}{p}\right) p^{-1}\right)$$

$$= \frac{(-1)^n \varphi(N)}{128} ((-1)^n N - 11) + \frac{(-1)^n \varphi_8(N)}{32} + \frac{(-1)^n N}{32} A(N),$$

where

$$A(N) = \prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right) + \frac{1}{2} \sum_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right).$$

Finally, it suffices to observe that

$$\prod_{p \mid N} \left(1 - \left(\frac{-8}{p}\right)\right) = \begin{cases} 2^n, & \text{if } x = z = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\prod_{p \mid N} \left(1 - \left(\frac{-4}{p}\right)\right) = \begin{cases} 2^n, & \text{if } x = u = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and Lemma 3 follows. $\square$

3. Proof of Theorem

We shall start with the following congruence (see (2.3) of [4]):

$$\sum_{e \mid d}^{*} (-1)^{\tau(e)} \left(\sum_{\substack{0 < k < |d|/8 \\ (k,d) = 1}} \left(\frac{e}{k}\right) k \right) \equiv 0 \pmod{2^n}, \tag{3.1}$$

where the asterisk over the sum means that it is taken over all divisors, negative or positive, of d such that $e \equiv 1 \pmod 4$. Here $\tau(e)$ denotes the number of distinct prime factors of e.

Denote for $e \mid d$, $e \equiv 1 \pmod 4$, $e \neq 1$

$$S = S(d, e) = \sum_{\substack{0 < k < |d|/8 \\ (k,d)=1}} \left(\frac{e}{k}\right) k.$$

We have (for details see (2.7), (2.8), (2.9) and the page 268 of [4]):

$$S = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \sum_{\substack{0 < k < |d|/8 \\ f \mid k}} \left(\frac{e}{k}\right) k = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{0 < 1 < |d|/8f} \left(\frac{e}{l}\right) l.$$

Hence we obtain

$$S = R + U, \tag{3.2}$$

where for $t = \left[\dfrac{|d/e|}{8f}\right]$, $t' = \left[\dfrac{|d|}{8f}\right]$

$$R = R(d, e) = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{l=1}^{t|e|} \left(\frac{e}{l}\right) l$$

$$= -\rho(e)h(e) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) ft|e|,$$

$$\rho(e) = \begin{cases} 0, & \text{if } e > 0, \\ \frac{1}{3}, & \text{if } e = -3, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$U = U(d, e) = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{l=t|e|+1}^{t'} \left(\frac{e}{l}\right) l,$$

because

$$\sum_{l=1}^{t|e|} \left(\frac{e}{l}\right) l = \begin{cases} 0, & \text{if } e > 0, \\ -t, & \text{if } e = -3, \\ teh(e), & \text{otherwise.} \end{cases}$$

First, we shall deal with $U(d, e)$. We find

$$\sum_{l=t|e|+1}^{t'} \left(\frac{e}{l}\right) l = \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) (m + t|e|)$$

$$= \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) m + t|e| \sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right).$$

Moreover we have

$$\sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) = \sum_{k=1}^{s} T_k,$$

where $(|d/e|/f) \equiv s(\mod 8)$, $s \in \{1, 3, 5, 7\}$ (for details see p. 268 of [4]) and

$$\sum_{m=1}^{t'-|e|t} \left(\frac{e}{m}\right) m = \sum_{k=1}^{s} S_k,$$

where T_k and S_k are defined in the section 2.

Therefore

$$U = \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\sum_{k=1}^{s} S_k + t|e| \sum_{k=1}^{s} T_k\right)$$

$$= \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \sum_{k=1}^{s} S_k$$

$$+ \frac{1}{8} \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| - s|e|f) \sum_{k=1}^{s} T_k$$

$$= \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\sum_{k=1}^{s} S_k - \frac{s|e|}{8} \sum_{k=1}^{s} T_k\right)$$

$$+ \frac{1}{8} |d| \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) \sum_{k=1}^{s} T_k. \tag{3.3}$$

As for $R(d, e)$ for $e < 0$ we have

$$R = -\frac{1}{8} \rho(e) h(e) \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| - s|e|f). \tag{3.4}$$

Now we shall use Lemmas 1 and 2. We have for $s = 1, 3, 5$ and 7

$$\sum_{k=1}^{s} T_k = \begin{cases} \frac{1}{4}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right)h(-4e) + \frac{1}{4}\left(\frac{-8}{s}\right)h(-8e), & \text{if } e > 0, \\[2ex] \frac{1}{4}\left(5 - \left(\frac{e}{2}\right)\right)h(e) - \frac{1}{4}\left(\frac{8}{s}\right)h(8e) - \lambda(e), & \text{if } e < 0, \end{cases}$$

(see (2.22) of [4] or Lemma 1), and from Lemma 2

$$\sum_{k=1}^{s} S_k = \begin{cases} \frac{1}{64}\left(\frac{8}{s}\right)k_2(8e) - \frac{1}{64}\left(34 - \left(\frac{e}{2}\right)\right)k_2(e) + \\[2ex] \quad + \frac{se}{32}\left(\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right)h(-4e) + \left(\frac{-8}{s}\right)h(-8e)\right) + 7\omega(e), & \text{if } e > 0, \\[2ex] \frac{1}{64}\left(\frac{-8}{s}\right)k_2(-8e) + \frac{1}{64}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right)k_2(-4e) \\[2ex] \quad - \frac{se}{32}\left(\left(1 - \left(\frac{e}{2}\right)\right)h(e) - \left(\frac{e}{2}\right)h(8e)\right) - sv(e), & \text{if } e < 0. \end{cases}$$

From the last two formulas we get that

$$\sum_{k=1}^{s} S_k - \frac{s|e|}{8}\sum_{k=1}^{s} T_k$$

$$= \begin{cases} \frac{1}{64}\left(\frac{8}{s}\right)k_2(8e) - \frac{1}{64}\left(34 - \left(\frac{e}{2}\right)\right)k_2(e) + 7\omega(e), & \text{if } e > 0, \\[2ex] \frac{1}{64}\left(\frac{-8}{s}\right)k_2(-8e) + \frac{1}{64}\left(\frac{-4}{s}\right)\left(\frac{e}{2}\right)k_2(-4e) + \frac{se}{8}h(e) + 2sv(e), & \text{if } e < 0. \end{cases}$$

Hence, from (3.2) and (3.3) we obtain for $e > 0$:

$$64(-1)^{\tau(d/e)}S = (-1)^{\tau(d/e)}\sum_{f \mid |d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)f\left(\left(\frac{8}{|d/e|/f}\right)k_2(8e) - \right.$$

$$\left. -\left(34 - \left(\frac{e}{2}\right)\right)k_2(e) + 64 \cdot 7\omega(e)\right) + 2|d|(-1)^{\tau(d/e)}\sum_{f \mid |d/e|}(-1)^{\tau(f)}\left(\frac{e}{f}\right)$$

$$\times \left(\left(\frac{-4}{|d/e|/f}\right)\left(\frac{e}{2}\right)h(-4e) + \left(\frac{-8}{|d/e|/f}\right)h(-8e)\right).$$

Therefore we have for $e > 0$:

$$64(-1)^{\tau(d/e)}S = b_1(d, e)k_2(8e) + b_2(d, e)k_2(e) + b_5(d, e) +$$

$$+ 2|d|(c_1(d, e)h(-4e) + c_2(d, e)h(-8e)), \tag{3.5}$$

where in view of (2.1)

$$b_1(d, e) = (-1)^{\tau(d/e)} \left(\frac{8}{|d/e|}\right) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{8e}{f}\right) f$$

$$= (-1)^{\tau(d/e)} \left(\frac{8}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{8e}{p}\right) p\right) = \left(\frac{e}{|d/e|}\right) \varphi_{8e}(|d/e|),$$

$$b_2(d, e) = -(-1)^{\tau(d/e)} \left(34 - \left(\frac{e}{2}\right)\right) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f$$

$$= -(-1)^{\tau(d/e)} \left(34 - \left(\frac{e}{2}\right)\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{e}{p}\right) p\right)$$

$$= -\left(34 - \left(\frac{e}{2}\right)\right) \left(\frac{e}{|d/e|}\right) \varphi_e(|d/e|),$$

$$b_5(d, e) = 0 \quad \text{if } e \neq 5, \quad \text{and} \quad b_5(d, 5) = 16 \cdot 7 \left(\frac{5}{|d/5|}\right) \varphi_5(|d/5|).$$

$c_1(d, e)$ and $c_2(d, e)$ are the same as in [4] i.e.

$$c_1(d, e) = \left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{-4e}{p}\right)\right)$$

$$= \left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right) \psi_{4e}(|d/e|),$$

$$c_2(d, e) = \left(\frac{e}{|d/e|}\right) \prod_{p \mid |d/e|} \left(1 - \left(\frac{-8e}{p}\right)\right) = \left(\frac{e}{|d/e|}\right) \psi_{8e}(|d/e|).$$

Similarly, from (3.2), (3.3) and (4.3) we find for $e < 0$:

$$S = U + R = \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f \left(\frac{1}{64} \left(\frac{-8}{|d/e|/f}\right) k_2(-8e) + \right.$$

$$+ \frac{1}{64} \left(\frac{-4}{|d/e|/f}\right)\left(\frac{e}{2}\right) k_2(-4e) + \frac{se}{8} h(e) + 2sv(e)\bigg) +$$

$$+ \frac{1}{32} |d| \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right)\left(\left(5 - \left(\frac{e}{2}\right)\right) h(e) - \right.$$

$$- \left(\frac{8}{|d/e|/f}\right) h(8e) - 4\lambda(e)\bigg) -$$

$$- \frac{1}{8} \rho(e)h(e) \sum_{f \mid |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) (|d| + sef)$$

$$= \frac{1}{64} k_2(-8e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f\left(\frac{-8}{|d/e|/f}\right) +$$

$$+ \frac{1}{64} \left(\frac{e}{2}\right) k_2(-4e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f\left(\frac{-4}{|d/e|/f}\right) +$$

$$+ \frac{1}{8} eh(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) fs + 2v(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) fs +$$

$$+ \frac{1}{32} |d| \left(\left(5 - \left(\frac{e}{2}\right)\right) h(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) - \right.$$

$$- h(8e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) \left(\frac{8}{|d/e|/f}\right) -$$

$$- 4\lambda(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) - 4\rho(e)h(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) \right) -$$

$$- \frac{1}{8} e\rho(e)h(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) fs.$$

Since for any $e < 0$

$$\frac{1}{8} h(e)e \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) fs + 2v(e) \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) fs -$$

$$- \frac{1}{8} \rho(e)h(e)e \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) sf = 0$$

we obtain for $e < 0$

$$64(-1)^{\tau(d/e)}S = b_3(d, e)k_2(-8e) + b_4(d, e)k_2(-4e) +$$

$$+ 2|d|(c_3(d, e)h(e) + c_4(d, e)h(8e) + c_5(d, e)), \tag{3.6}$$

where in virtue of (2.1)

$$b_3(d, e) = (-1)^{\tau(d/e)} \sum_{f \mid \mid \overline{d}/e \mid} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f\left(\frac{-8}{|d/e|/f}\right)$$

$$= (-1)^{\tau(d/e)} \left(\frac{-8}{|d/e|}\right) \prod_{p \mid \mid \overline{d}/e \mid} \left(1 - \left(\frac{-8e}{p}\right) p\right)$$

$$= \left(\frac{e}{|d/e|}\right) \varphi_{-8e}(|d/e|),$$

$$b_4(d, e) = (-1)^{\tau(d/e)} \left(\frac{e}{2}\right) \sum_{f \,|\, |d/e|} (-1)^{\tau(f)} \left(\frac{e}{f}\right) f\left(\frac{-4}{|d/e|/f}\right)$$

$$= (-1)^{\tau(d/e)} \left(\frac{e}{2}\right)\left(\frac{-4}{|d/e|}\right) \prod_{p \,|\, |d/e|} \left(1 - \left(\frac{-4e}{p}\right) p\right)$$

$$= \left(\frac{e}{2}\right)\left(\frac{e}{|d/e|}\right) \varphi_{-4e}(|d/e|).$$

Moreover similarly as in [4]

$$c_3(d, e) = \left(1 - \left(\frac{e}{2}\right)\right)\left(\frac{e}{|d/e|}\right) \prod_{p \,|\, |d/e|} \left(1 - \left(\frac{e}{p}\right)\right) = \left(1 - \left(\frac{e}{2}\right)\right)\left(\frac{e}{|d/e|}\right) \psi_e(|d/e|),$$

$$c_4(d, e) = -\left(\frac{e}{|d/e|}\right) \prod_{p \,|\, |d/e|} \left(1 - \left(\frac{8e}{p}\right)\right) = -\left(\frac{e}{|d/e|}\right) \psi_{8e}(|d/e|),$$

$$c_5(d, e) = 0 \quad \text{if } e \neq -3, \quad \text{and} \quad c_5(d, -3) = -\frac{13}{3}\left(\frac{-3}{|d/3|}\right) \psi_{-3}(|d/e|).$$

Now we get from (3.1)

$$(-1)^n \sum_{\substack{e \,|\, d \\ e > 1 \\ e \equiv 1 (\text{mod } 4)}} (-1)^{\tau(e)} 64 S(d, e) + (-1)^n \sum_{\substack{e \,|\, d \\ e < 0 \\ e \equiv 1 (\text{mod } 4)}} (-1)^{\tau(e)} 64 S(d, e) +$$

$$+ (-1)^n 64 J(|d|) \equiv 0 \ (\text{mod } 2^{n+6}).$$

Hence, from (3.5), (3.6) and Lemma 3, Theorem follows. □

Acknowledgement

References

[1] B. C. Berndt: Classical theorems on quadratic residues, *L'Enseign. Math.* 22 (1976), 261–304.

[2] G. Gras: Pseudo-mesures *p*-adiques associées aux fonctions *L* de *Q*, *Manuscripta Math.* 57 (1987), 373–415.

[3] G. Gras: Relations congruentielles linéaires entre nombres de classes de corps quadratiques, *Acta Arithm.* 52 (1989), 147–162.

[4] K. Hardy and K. S. Williams: A congruence relating the class numbers of complex quadratic fields, *Acta Arithm.* 47 (1986), 263–276.

[5] M. Lerch, Essai sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers, *Acta Math.* 29 (1905), 333–424.

[6] T. Nagell, Sur la distribution des nombres qui sont premiers avec un nombre entier donné, *Archiv for Math. og Naturvidenskab B.XXXVIII*, Nr 2 (1923).

[7] J. Urbanowicz: On the 2-primary part of a conjecture of Birch-Tate, *Acta Arithm.* 43 (1983), 69–81.

[8] J. Urbanowicz: Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, I, *Compositio Math.* 75 (1990), 247–270.