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Connections between $B_{2,\chi}$ for even quadratic Dirichlet characters χ and class numbers of appropriate imaginary quadratic fields, I

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Abstract. The paper gives some connections between the second generalized Bernoulli numbers of even quadratic Dirichlet characters and class numbers of appropriate imaginary quadratic fields. There are applied formulas of an old paper of M. Lerch of 1905.

0. Introduction

Let K_2 be the functor of Milnor. The Birch-Tate conjecture for real quadratic fields F with the discriminant d takes the form:

$$|K_2 O_F| = B_{2,(\frac{d}{\cdot})} \quad \text{(apart from } d = 5 \text{ and } 8).$$

Here O_F and $(\frac{d}{\cdot})$ denote the ring of integers and the character (the Kronecker symbol) of F respectively. $B_{2,(\frac{d}{\cdot})}$ denotes the second Bernoulli number belonging to the character $(\frac{d}{\cdot})$ (for information on the numbers $B_{k,\chi}$, see [6]).

B. Mazur and A. Wiles [4] have proved the conjecture up to 2-torsion.

Let $h(d)$ denote the class number of a quadratic field with the discriminant d. It is known that for $d < 0$:

$$h(d) = -B_{1,(\frac{d}{\cdot})} \quad \text{(apart from } d = -3 \text{ and } -4).$$

Here $B_{1,(\frac{d}{\cdot})}$ denotes the first Bernoulli number belonging to $(\frac{d}{\cdot})$. Denote $\bar{k}_2(d) = B_{2,(\frac{d}{\cdot})}$. Let D and Δ, $D, \Delta > 0$, $D \equiv 1 \pmod 4$, $\Delta \equiv 3 \pmod 4$ be natural numbers and let D and $-\Delta$ be the discriminants of quadratic fields. Then

$$D,\ -4D,\ -8D,\ 8D \text{ and } -\Delta,\ 4\Delta,\ 8\Delta,\ -8\Delta$$

are all the discriminants of quadratic fields except

$$-4,\ 8,\ -8.$$

All the results of this paper are consequences of two following theorems:

THEOREM 1. *Let for* $k = 0, 1, 2$ *and* 3

$$s_k = \sum_{l \in [kD/8, (k+1)D/8)} \left(\frac{D}{l}\right) l.$$

Then for $D \neq 5$:

(i) $k_2(D) = \frac{16}{45}\left(2\frac{D}{2} - 7\right)(s_0 + s_1) - \frac{2}{45}\left(2\frac{D}{2} - 7\right)Dh(-4D)$,

(ii) $$k_2(D) = -\frac{32}{75}\left(\frac{D}{2} + 4\right)(s_0 + s_2) + \frac{2}{75}\left(\frac{D}{2} + 4\right)D\left(-\left(\frac{D}{2} + 2\right)h(-4D) + 2h(-8D)\right),$$

(iii) $k_2(8D) = -32(s_1 + s_2) - 2D\left(2\frac{D}{2}h(-4D) - h(-8D)\right)$,

(iv) $$k_2(8D) + \left(\frac{D}{2} - 34\right)k_2(D) = 64s_0 - 2D\left(\frac{D}{2}h(-4D) + h(-8D)\right),$$

$$k_2(8D) + 3\left(3\frac{D}{2} - 2\right)k_2(D) = -64s_1 - 2D\left(\left(\frac{D}{2} - 4\right)h(-4D) + h(-8D)\right),$$

$$k_2(8D) - 3\left(3\frac{D}{2} - 2\right)k_2(D) = -64s_2 - 2D\left(\left(3\frac{D}{2} + 4\right)h(-4D) - 3h(-8D)\right),$$

$$k_2(8D) + 15\left(\frac{D}{2} - 2\right)k_2(D) = 64s_3 - 6D\left(\frac{D}{2}h(-4D) - h(-8D)\right).$$

THEOREM 2. *Let for* $k = 0, 1, 2$ *and* 3

$$s_k = \sum_{l \in [k\Delta/8, (k+1)\Delta/8)} \left(\frac{-\Delta}{l}\right) l.$$

Then for $\Delta \neq 3$:

(i) $k_2(4\Delta) = 16(s_0 + s_1) - 2\Delta\left(\frac{-\Delta}{2} - 1\right)h(-\Delta)$ (*see* [5], *too*),

(ii) $k_2(4\Delta) = 32\left(\frac{-\Delta}{2}\right)(s_0 + s_3) + 2\Delta\left(\frac{-\Delta}{2}\right)\left(7\left(\frac{-\Delta}{2} - 1\right)h(-\Delta) + 2h(-8\Delta)\right)$,

(iii) $k_2(8\Delta) = 32(s_0 - s_3) - 2\Delta\left(6\left(\frac{-\Delta}{2} - 1\right)h(-\Delta) + h(-8\Delta)\right)$,

(iv) $$k_2(8\Delta) + \left(\frac{-\Delta}{2}\right)k_2(4\Delta) = 64s_0 + 2\Delta\left(\left(\frac{-\Delta}{2} - 1\right)h(-\Delta) + h(-8\Delta)\right),$$

$$k_2(8\Delta) + \left(\frac{-\Delta}{2} - 4\right)k_2(4\Delta) = -64s_1 + 2\Delta\left(5\left(\frac{-\Delta}{2} - 1\right)h(-\Delta) + h(-8\Delta)\right),$$

$$k_2(8\Delta) - \left(\frac{-\Delta}{2}+4\right)k_2(4\Delta) = 64s_2 + 2\Delta\left(7\left(\frac{-\Delta}{2}-1\right)h(-\Delta)-3h(-8\Delta)\right),$$
$$k_2(8\Delta) - \left(\frac{-\Delta}{2}\right)k_2(4\Delta) = -64s_3 - 2\Delta\left(13\left(\frac{-\Delta}{2}-1\right)h(-\Delta)+3h(-8\Delta)\right).$$

We prove these theorems using the methods of an old paper of Lerch [3]. The theorems give us some congruences for $k_2(d)$, $d > 0$ and $h(d')$, $d' < 0$ modulo powers of 2, where d, d' belong to

$$\{D, -4D, -8D, 8D\} \quad \text{or} \quad \{-\Delta, 4\Delta, 8\Delta, -8\Delta\}.$$

We obtain from these congruences some relations between the exact divisibility of $k_2(d)$ and $h(d')$ by some powers of 2 (4, 8, 16, 32 and 64). In view of these results one may expect some corresponding conjectures for $|K_2O_F|$ (where F is a real quadratic field with the discriminant d) are true. On the other hand our Corollary 2(iv) to Theorem 1 proves a conjecture about values of zeta-functions implied by the Birch-Tate conjecture made by K. Kramer and A. Candiotti in [2].

Similar problems were dealt with in [5] and [1]. The results in the present paper are some further generalizations of those ones.

I would like to thank A. Schinzel for pointing out the paper [3] to me and to J. Browkin for his advice.

1. Notation

Let d be the discriminant of a quadratic field. It is well known that for $d > 0$

$$k_2(d) = \frac{d\sqrt{d}}{\pi^2} L(2, d) \tag{1.1}$$

and for $d < 0$

$$h(d) = \frac{w\sqrt{|d|}}{2\pi} L(1, d). \tag{1.2}$$

Here w is the number of the roots of unity in the quadratic field with the discriminant d, and $L(s, d) = L(s, (\frac{d}{\cdot}))$, where

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for any Dirichlet character χ (for details, see [6]).

Let $[x]$ denote the integral part of x. Let

$$R(x) = x - [x + \tfrac{1}{2}].$$

It is well known that

$$|R(x)| = \frac{1}{4} - 2 \sum_{2 \nmid n} \frac{\cos 2\pi n x}{\pi^2 n^2}, \tag{1.3}$$

where the summation is taken over all odd natural numbers (see [3]).

Let $\tau(d)$ denote the Gaussian sum belonging to $(\frac{d}{\cdot})$. It is well known that

$$\tau(d) = \begin{cases} \sqrt{d}, & \text{if } d > 0, \\ i\sqrt{|d|}, & \text{if } d < 0, \end{cases}$$

(see [6]).

Hence and from (1.3) we easily get that the following formulas hold for natural m prime to d:

$$\sum_{l=1}^{|d|-1} \left(\frac{d}{l}\right) \left| R\left(x + \frac{ml}{|d|}\right) \right| = -2 \left(\frac{d}{-m}\right) \sqrt{|d|}\, U(x, d), \tag{1.4}$$

where

$$U(x, d) = \begin{cases} \displaystyle\sum_{2 \nmid n} \frac{\left(\frac{d}{n}\right)}{\pi^2 n^2} \cos 2\pi n x, & \text{if } d > 0, \\ \displaystyle\sum_{2 \nmid n} \frac{\left(\frac{d}{n}\right)}{\pi^2 n^2} \sin 2\pi n x, & \text{if } d < 0, \end{cases}$$

(for details, see [3]). Let $R(x, d)$ denote for fixed m the left hand side of (1.4).

We are also going to use the following formulas:

$$k_2(d) = \frac{1}{d_1} \sum_{l=1}^{d} \left(\frac{d}{l}\right) l^2 \tag{1.5}$$

(see the exercise 4.2(a), [6]), and

$$k_2(d) = -\frac{4}{4 - (d/2)} \sum_{l=1}^{[d/2]} \left(\frac{d}{l}\right) l \tag{1.6}$$

for $d \neq 5, 8$. (1.6) follows from (1.4) (for $x = 0$ and $m = 1$) and (1.5) (see [3]).

2. Formulas for $R(x, d)$

Let d be the discriminant of a quadratic field and let m be a fixed natural number prime to d. We are going to define a partition of the interval $[0, |d|)$ into disjoint parts:

$$[0, |d|) = \bigcup_{k \in K} I_k$$

($I_k \cap I_{k'} = \emptyset$ for any $k, k' \in K$, $k \neq k'$).

These intervals I_k will depend on x.

Let in the case $x = 0$:

$$K = \{0, 1, \ldots, 2m-1\} \quad \text{and} \quad I_k = \left[k \frac{|d|}{2m}, (k+1)\frac{|d|}{2m}\right).$$

Put in the case $x = \frac{1}{4}$:

$$K = \{0, 1, \ldots, 2m\},$$

$$I_0 = \left[0, \frac{|d|}{4m}\right), \; I_k = \left[(2k-1)\frac{|d|}{4m}, (2k+1)\frac{|d|}{4m}\right) \text{ for } 1 \leqq k \leqq 2m-1$$

and

$$I_{2m} = \left[(4m-1)\frac{|d|}{4m}, |d|\right).$$

Set in the case $x = \frac{1}{8}$:

$$K = \{0, 1, \ldots, 2m\},$$

$$I_0 = \left[0, 3\frac{|d|}{8m}\right), \quad I_k = \left[(4k-1)\frac{|d|}{8m}, (4k+3)\frac{|d|}{8m}\right) \quad \text{for} \quad 1 \leqq k \leqq 2m-1 \quad \text{and}$$

$$I_{2m} = \left[(8m-1)\frac{|d|}{8m}, |d|\right).$$

Denote

$$s_k = \sum_{l \in I_k} \left(\frac{d}{l}\right) l, \quad \text{and} \quad t_k = \sum_{l \in I_k} \left(\frac{d}{l}\right).$$

Note that for $l \in I_k$

$$\left|R\left(x + \frac{ml}{|d|}\right)\right| = (-1)^k \left(x + \frac{ml}{|d|} - \left[\frac{k+1}{2}\right]\right).$$

Hence we get

$$R(x, d) = \frac{m}{|d|} \sum_{k \in K} (-1)^k s_k + \sum_{k \in K} (-1)^k \left(x - \left[\frac{k+1}{2} \right] \right) t_k. \tag{2.1}$$

Moreover, for $1 \leqq l \leqq |d| - 1$ we have for $x = 0$:

$$l \in I_k \Leftrightarrow |d| - l \in I_{2m-1-k},$$

and for $x = \frac{1}{4}$:

$$l \in I_k \Leftrightarrow |d| - l \in I_{2m-k}.$$

In the case $x = \frac{1}{8}$ the situation is more complicated. Denote in this case:

$$I''_{-1} = \varnothing, \quad I'_0 = \left[0, \frac{|d|}{8m}\right), \quad I''_0 = \left[\frac{|d|}{8m}, 3\frac{|d|}{8m}\right),$$

and for $1 \leqq k \leqq 2m - 1$

$$I'_k = \left[(4k-1)\frac{|d|}{8m}, (4k+1)\frac{|d|}{8m}\right), \quad I''_k = \left[(4k+1)\frac{|d|}{8m}, (4k+3)\frac{|d|}{8m}\right),$$

and

$$I'_{2m} = I_{2m}, \quad I''_{2m} = \varnothing.$$

Then

$$I_k = I'_k \cup I''_k \quad \text{and} \quad I'_k \cap I''_k = \varnothing.$$

Now, we see that for $1 \leqq l \leqq |d| - 1$ in the case $x = \frac{1}{8}$:

$$l \in I'_k \Leftrightarrow |d| - l \in I'_{2m-k},$$
$$l \in I''_k \Leftrightarrow |d| - l \in I''_{2m-1-k}.$$

Let in the case $x = \frac{1}{8}$:

$$s'_k = \sum_{l \in I'_k} \left(\frac{d}{l}\right) l, \qquad t'_k = \sum_{l \in I'_k} \left(\frac{d}{l}\right),$$

and

$$s_k'' = \sum_{l \in I_k''} \left(\frac{d}{l}\right) l, \quad t_k'' = \sum_{l \in I_k''} \left(\frac{d}{l}\right).$$

Then

$$s_k = s_k' + s_k'' \quad \text{and} \quad t_k = t_k' + t_k''.$$

Therefore we obtain in the case $x = 0$, $d > 0$:

$$s_k = -s_{2m-1-k} + dt_k, \quad t_k = t_{2m-k},$$

in the case $x = \frac{1}{4}$, $d < 0$:

$$s_k = s_{2m-k} - dt_k, \qquad t_k = -t_{2m-k} \quad (\text{hence } t_m = 0),$$

in the case $x = \frac{1}{8}$, $d > 0$:

$$s_k' = -s_{2m-k}' + dt_k' \quad (\text{hence } s_m' = \tfrac{1}{2}dt_m'), \qquad t_k' = t_{2m-k}',$$

$$s_k'' = -s_{2m-1-k}'' + dt_k'', \qquad t_k'' = t_{2m-1-k}'' \qquad (\text{hence } t_{2m}'' = 0 \text{ so } s_{2m}'' = 0),$$

and in the case $x = \frac{1}{8}$, $d < 0$:

$$s_k' = s_{2m-k}' - dt_k', \qquad t_k' = -t_{2m-k}' \quad (\text{hence } t_m' = 0),$$

$$s_k'' = s_{2m-1-k}'' - dt_k'', \qquad t_k'' = -t_{2m-1-k}'' \qquad (\text{hence } t_{2m}'' = 0 \text{ so } s_{2m}'' = 0).$$

Hence and from (2.1) we get in the case $x = 0$, $d > 0$:

$$\begin{aligned}
R(x, d) &= \frac{m}{d}\left(\sum_{k=0}^{m-1} (-1)^k s_k + \sum_{k=0}^{m-1} (-1)^{2m-1-k} s_{2m-1-k}\right) - \\
&\quad -\left(\sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t_k + \sum_{k=0}^{m-1} (-1)^{2m-1-k}\left[m - \frac{k}{2}\right] t_{2m-1-k}\right) \\
&= \frac{m}{d}\left(\sum_{k=0}^{m-1} (-1)^k s_k - \sum_{k=0}^{m-1} (-1)^k(-s_k + \mathrm{d}t_k)\right) - \\
&\quad -\left(\sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t_k - \sum_{k=0}^{m-1} (-1)^k \left[m - \frac{k}{2}\right] t_k\right) \\
&= \frac{2m}{d} \sum_{k=0}^{m-1} (-1)^k s_k + \sum_{k=0}^{m-1} (-1)^k \left(-m - \left[\frac{k+1}{2}\right] + \left[m - \frac{k}{2}\right]\right) t_k \\
&= \frac{2m}{d} \sum_{k=0}^{m-1} (-1)^k s_k - 2\sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t_k,
\end{aligned}$$

because

$$-m-\left[\frac{k+1}{2}\right]+\left[m-\frac{k}{2}\right]=-2\left[\frac{k+1}{2}\right].$$

On the other hand from (1.6) we obtain for $d>8$

$$\sum_{k=0}^{m-1} s_k = -\frac{1}{4}\left(4-\left(\frac{d}{2}\right)\right) k_2(d).$$

Therefore in the case $x=0$, $d>8$ we get:

$$R(x,d)=\frac{4m}{d}\sum_{\substack{0\leqq k\leqq m-1\\ 2\mid k}} s_k - 2\sum_{k=0}^{m-1}(-1)^k\left[\frac{k+1}{2}\right]t_k+\frac{m}{2d}\left(4-\left(\frac{d}{2}\right)\right)k_2(d). \tag{2.2}$$

Next, in the case $x=\frac{1}{4}$, $d<0$:

$$\begin{aligned}
R(x,d) &= \frac{m}{|d|}\left(\sum_{k=0}^{m-1}(-1)^k s_k + (-1)^m s_m + \sum_{k=0}^{m-1}(-1)^{2m-k}s_{2m-k}\right)+\\
&\quad+\left(\sum_{k=0}^{m-1}(-1)^k\left(\frac{1}{4}-\left[\frac{k+1}{2}\right]\right)t_k+(-1)^m\left(\frac{1}{4}-\left[\frac{m+1}{2}\right]\right)t_m+\right.\\
&\quad\left.+\sum_{k=0}^{m-1}(-1)^{2m-k}\left(\frac{1}{4}-\left[m-\frac{k-1}{2}\right]\right)t_{2m-k}\right)\\
&=\frac{m}{|d|}\left(\sum_{k=0}^{m-1}(-1)^k s_k+(-1)^m(2\bar{s}_m+d\bar{t}_m)+\sum_{k=0}^{m-1}(-1)^k(s_k+dt_k)\right)+\\
&\quad+\left(\sum_{k=0}^{m-1}(-1)^k\left(\frac{1}{4}-\left[\frac{k+1}{2}\right]\right)t_k+\sum_{k=0}^{m-1}(-1)^k\left(\frac{1}{4}-\left[m-\frac{k-1}{2}\right]\right)t_k\right)\\
&=\frac{2m}{|d|}\sum_{k=0}^{m}(-1)^k\bar{s}_k-(-1)^m m\bar{t}_m+\\
&\quad+\sum_{k=0}^{m}(-1)^k\left(-m-\left[\frac{k+1}{2}\right]+\left[m-\frac{k-1}{2}\right]\right)t_k\\
&=\frac{2m}{|d|}\sum_{k=0}^{m}(-1)^k\bar{s}_k-\sum_{k=0}^{m}(-1)^k k\bar{t}_k,
\end{aligned}$$

where

$$\bar{s}_m=\sum_{l\in I_m\cap[0,|d|/2)}\left(\frac{d}{l}\right)l,\qquad \bar{t}_m=\sum_{l\in I_m\cap[0,|d|/2)}\left(\frac{d}{l}\right),$$

and

$$\bar{s}_k = s_k, \qquad \bar{t}_k = t_k \quad \text{for } k \neq m.$$

We have used this notation because

$$s_m = 2\bar{s}_m + d\bar{t}_m.$$

On the other hand we have

$$\sum_{k=0}^{m} \bar{s}_k = \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) l = \frac{1}{2}\left(\left(\frac{d}{2}\right) - 1\right) dh(d)$$

because

$$\begin{aligned} dh(d) &= \sum_{l=1}^{|d|-1} \left(\frac{d}{l}\right) l = \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) l + \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{|d|-l}\right) (|d| - l) \\ &= 2 \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) l + d \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) \\ &= 2 \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) l + \left(2 - \left(\frac{d}{2}\right)\right) dh(d). \end{aligned}$$

We have used the well known formula

$$h(d) = \frac{1}{2 - (d/2)} \sum_{1 \leqq l \leqq |d|/2} \left(\frac{d}{l}\right) \quad \text{for } d < -4. \tag{2.3}$$

Therefore in the case $x = \frac{1}{4}$, $d < -4$:

$$R(x, d) = \frac{4m}{|d|} \sum_{\substack{0 \leqq k \leqq m \\ 2 | k}} \bar{s}_k - \sum_{k=0}^{m} (-1)^k k \bar{t}_k + m \left(\left(\frac{d}{2}\right) - 1\right) h(d). \tag{2.4}$$

Now, we consider the case $x = \frac{1}{8}$. We have from (2.1)

$$R(x, d) = R' + R'',$$

where

$$R' = \frac{m}{|d|} \sum_{k\in K} (-1)^k s'_k + \sum_{k\in K} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t'_k$$

$$= \frac{m}{|d|} \left(\sum_{k=0}^{m-1} (-1)^k s'_k + (-1)^m s'_m + \sum_{k=0}^{m-1} (-1)^{2m-k} s'_{2m-k}\right) +$$

$$+ \left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t'_k + (-1)^m \left(x - \left[\frac{m+1}{2}\right]\right) t'_m +\right.$$

$$\left. + \sum_{k=0}^{m-1} (-1)^{2m-k} \left(x - \left[m - \frac{k-1}{2}\right]\right) t'_{2m-k}\right),$$

and

$$R'' = \frac{m}{|d|} \sum_{k\in K} (-1)^k s''_k + \sum_{k\in K} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t''_k$$

$$= \frac{m}{|d|} \left(\sum_{k=0}^{m-1} (-1)^k s''_k + \sum_{k=0}^{m-1} (-1)^{2m-1-k} s''_{2m-1-k} + s''_{2m}\right) +$$

$$+ \left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t''_k +\right.$$

$$\left. + \sum_{k=0}^{m-1} (-1)^{2m-1-k} \left(x - \left[m - \frac{k}{2}\right]\right) t''_{2m-1-k} + (x-m) t''_{2m}\right).$$

Hence in the case $x = \frac{1}{8}$, $d > 0$:

$$R' = \frac{m}{d} \left(\sum_{k=0}^{m-1} (-1)^k s'_k + (-1)^m d\bar{t}'_m + \sum_{k=0}^{m-1} (-1)^k (-s'_k + dt'_k)\right) +$$

$$+ \left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t'_k + (-1)^m \left(x - \left[\frac{m+1}{2}\right]\right) 2\bar{t}'_m +\right.$$

$$\left. + \sum_{k=0}^{m-1} (-1)^k \left(x - \left[m - \frac{k-1}{2}\right]\right) t'_k\right)$$

$$= \frac{1}{4} (2-(-1)^m)\bar{t}'_m + \sum_{k=0}^{m-1} (-1)^k \left(m + x - \left[\frac{k+1}{2}\right] + x - \left[m - \frac{k-1}{2}\right]\right) t'_k$$

$$= \frac{1}{4} \sum_{k=0}^{m} (2-(-1)^k)\bar{t}'_k,$$

where

$$\bar{t}'_m = \sum_{l\in I'_m \cap [0,d/2)} \left(\frac{d}{l}\right) \quad \text{and} \quad \bar{t}'_k = t'_k \quad \text{for } k \neq m,$$

because

$$\bar{t}'_m = \tfrac{1}{2}t'_m$$

and

$$m + x - \left[\frac{k+1}{2}\right] + x - \left[m - \frac{k-1}{2}\right] = \frac{2(-1)^k - 1}{4}.$$

Next

$$\begin{aligned} R'' &= \frac{m}{d}\left(\sum_{k=0}^{m-1} (-1)^k s''_k - \sum_{k=0}^{m-1} (-1)^k(-s''_k + dt''_k)\right) + \\ &\quad + \left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t''_k - \sum_{k=0}^{m-1} (-1)^k \left(x - \left[m - \frac{k}{2}\right]\right) t''_k\right) \\ &= \frac{2m}{d} \sum_{k=0}^{m-1} (-1)^k s''_k + \sum_{k=0}^{m-1} (-1)^k \left(-m + x - \left[m - \frac{k}{2}\right]\right) t''_k \\ &= \frac{2m}{d} \sum_{k=0}^{m-1} (-1)^k s''_k - 2\sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t''_k. \end{aligned}$$

Therefore in the case $x = \frac{1}{8}$, $d > 0$ we get:

$$R(x, d) = \frac{2m}{d} \sum_{k=0}^{m-1} (-1)^k s''_k + \frac{1}{4}\sum_{k=0}^{m-1} (2-(-1)^k)\bar{t}'_k - 2\sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t''_k. \tag{2.5}$$

Now, in the case $x = \frac{1}{8}$, $d < 0$:

$$\begin{aligned} R' &= \frac{m}{|d|}\left(\sum_{k=0}^{m-1} (-1)^k s'_k + (-1)^m(2\bar{s}'_m + d\bar{t}'_m) + \sum_{k=0}^{m-1} (-1)^k(s'_k + dt'_k)\right) + \\ &\quad + \left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t'_k - \sum_{k=0}^{m-1} (-1)^k \left(x - \left[m - \frac{k-1}{2}\right]\right) t'_k\right) \\ &= \frac{2m}{|d|} \sum_{k=0}^{m} (-1)^k \bar{s}'_k - (-1)^m m\bar{t}'_m + \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \left(-m + x - \left[\frac{k+1}{2}\right] - x + \left[m - \frac{k-1}{2}\right]\right) t'_k \\ &= \frac{2m}{|d|} \sum_{k=0}^{m} (-1)^k \bar{s}'_k - \sum_{k=0}^{m} (-1)^k k\bar{t}'_k, \end{aligned}$$

where

$$\bar{s}'_m = \sum_{l \in I'_m \cap [0,|d|/2)} \left(\frac{d}{l}\right) l, \qquad t_m = \sum_{l \in I'_m \cap [0,|d|/2)} \left(\frac{d}{l}\right),$$

and

$$\bar{s}'_k = s'_k, \qquad \bar{t}'_k = t'_k \quad \text{for } k \neq m.$$

We have used this notation because

$$s'_m = 2\bar{s}'_m + d\bar{t}'_m.$$

Next

$$R'' = \frac{m}{|d|}\left(\sum_{k=0}^{m-1} (-1)^k s''_k - \sum_{k=0}^{m-1} (-1)^k (s''_k + dt''_k)\right) +$$
$$+\left(\sum_{k=0}^{m-1} (-1)^k \left(x - \left[\frac{k+1}{2}\right]\right) t''_k + \sum_{k=0}^{m-1} (-1)^k \left(x - \left[m - \frac{k}{2}\right]\right) t''_k\right)$$
$$= \sum_{k=0}^{m-1} (-1)^k \left(m + x - \left[\frac{k+1}{2}\right] + x - \left[m - \frac{k}{2}\right]\right) t''_k = \frac{1}{4}\sum_{k=0}^{m-1} (-1)^k t''_k,$$

because

$$m - \left[\frac{k+1}{2}\right] - \left[m - \frac{k}{2}\right] = 0.$$

Therefore in the case $x = \frac{1}{8}$, $d < 0$ we get:

$$R(x, d) = \frac{2m}{|d|}\sum_{k=0}^{m} (-1)^k \bar{s}'_k - \sum_{k=0}^{m} (-1)^k k \bar{t}'_k + \frac{1}{4}\sum_{k=0}^{m-1} (-1)^k t''_k. \tag{2.6}$$

3. Formulas for $U(x, d)$

Let d be the discriminant of a quadratic field and let m be a fixed natural number prime to d. We have from (1.1) in the cases $x = 0$, $d > 0$, $x = \frac{1}{4}$, $d < 0$, and $x = \frac{1}{8}$, $d \gtrless 0$

$$U(x, d) = \frac{1}{\sqrt{\rho}} \sum_{2 \nmid n} \frac{\left(\frac{d^*}{n}\right)}{\pi^2 n^2} = \left(1 - \frac{1}{4}\left(\frac{d^*}{2}\right)\right) \frac{k_2(d^*)}{d^* \sqrt{\rho d^*}},$$

where

$$\rho = \begin{cases} 2, & \text{if } x = \frac{1}{8}, \\ 1, & \text{otherwise,} \end{cases}$$

$d^* = d$ in the case $x = 0$, $d > 0$, and in the remaining considered cases d^* is the discriminant (of a real quadratic field) defined by the following equalities:

$$\left(\frac{d^*}{\cdot}\right) = \left(\frac{d}{\cdot}\right)\left(\frac{-4}{\cdot}\right), \quad \text{if } x = \tfrac{1}{4},\ d < 0,$$

$$\left(\frac{d^*}{\cdot}\right) = \left(\frac{d}{\cdot}\right)\left(\frac{8}{\cdot}\right), \quad \text{if } x = \tfrac{1}{8},\ d > 0, \quad \text{and}$$

$$\left(\frac{d^*}{\cdot}\right) = \left(\frac{d}{\cdot}\right)\left(\frac{-8}{\cdot}\right), \quad \text{if } x = \tfrac{1}{8},\ d < 0.$$

We have defined d^* as above because of

$$\sin\frac{\pi n}{2} = \left(\frac{-4}{n}\right) \quad \text{for } n \in \mathbb{Z}, \tag{3.1}$$

$$\cos\frac{\pi n}{4} = \frac{\sqrt{2}}{2}\left(\frac{8}{n}\right) \quad \text{for } n \in \mathbb{Z},\ n \text{ odd}, \tag{3.2}$$

$$\sin\frac{\pi n}{4} = \frac{\sqrt{2}}{2}\left(\frac{-8}{n}\right) \quad \text{for } n \in \mathbb{Z},\ n \text{ odd}. \tag{3.3}$$

Finally, we obtain from (1.4) and the above formula for $U(x, d)$

$$k_2(d^*) = \frac{-2\left(\frac{d}{-m}\right)}{4 - \left(\frac{d^*}{n}\right)} d^* \sqrt{\rho \frac{d^*}{|d|}}\, R(x, d). \tag{3.4}$$

4. Proof of Theorem 1

Let $D \equiv 1 (\mathrm{mod}\, 4)$, $D > 5$ be the discriminant of a quadratic field. We shall use the formula (3.4) for $x = 0$ and $d = D$. We get from (3.4) and (2.2) for natural m prime to D:

$$k_2(D) = -\frac{8m}{15} \frac{4 + \left(\frac{D}{2}\right)}{m + \left(\frac{D}{2}\right)} \sum_{\substack{0 \leq k \leq m-1 \\ 2|k}} s_k + \frac{4}{15} \frac{4 + \left(\frac{D}{2}\right)}{m + \left(\frac{D}{2}\right)} D \sum_{k=0}^{m-1} (-1)^{k+1} k T_k,$$

where

$$T_k = \sum_{l=0}^{[kD/2m]} \left(\frac{D}{l}\right),$$

because

$$\begin{aligned}\sum_{k=0}^{m} (-1)^k \left[\frac{k+1}{2}\right] t_k &= \sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] (T_{k+1} - T_k)\\ &= \sum_{k=1}^{m-1} \left((-1)^{k-1} \left[\frac{k}{2}\right] - (-1)^k \left[\frac{k+1}{2}\right]\right) T_k\end{aligned}$$

and

$$(-1)^{k-1} \left[\frac{k}{2}\right] - (-1)^k \left[\frac{k+1}{2}\right] = (-1)^{k+1} k.$$

Here $T_m = 0$.

Now, to prove the part (i) of the theorem it suffices to put $m = 2$ in the above formula for $k_2(D)$. Then

$$\sum_{k=0}^{m-1} (-1)^{k+1} k T_k = T_1 = \sum_{l=1}^{[D/4]} \left(\frac{D}{l}\right) = \tfrac{1}{2} h(-4D).$$

We have used the following formula:

$$h(-4d) = 2 \sum_{l=1}^{[D/4]} \left(\frac{D}{l}\right) \quad \text{(see [3])}. \tag{4.1}$$

To prove (ii) it is sufficient to put $m = 4$ in the formula for $k_2(D)$. Then

$$\begin{aligned}\sum_{k=0}^{m-1} (-1)^{k+1} k T_k &= T_1 - 2T_2 + 3T_3\\ &= -\frac{1}{2}\left(\left(\frac{D}{2}\right) + 2\right) h(-4D) + h(-8D)\end{aligned}$$

because of (4.1) (for T_2) and of

$$T_1 = \frac{1}{4}\left(\frac{D}{2}\right) h(-4D) + \frac{1}{4} h(-8D), \tag{4.2}$$

$$T_3 = -\frac{1}{4}\left(\frac{D}{2}\right) h(-4D) + \frac{1}{4} h(-8D). \tag{4.3}$$

The last two formulas follow immediately from the formula

$$\sum_{l=1}^{[Dx]} \left(\frac{D}{l}\right) = D \sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right)}{\pi n} \sin 2\pi n x$$

(here we must assume $x + (1/D) \notin \mathbb{Z}$ for $1 \leqq l \leqq D - 1$, see [3]) for $x = \frac{1}{8}$ and $\frac{3}{8}$ because of (1.2), (3.1) and (3.3).

To prove the parts (iii) and (iv) of Theorem 1 we shall use the formula (3.4) for $x = \frac{1}{8}$ and $d = D$. We get from (3.4) and (2.5) for natural m prime to D:

$$\begin{aligned} k_2(8D) = {} & -32 \left(\frac{D}{m}\right) \sum_{k=0}^{m-1} (-1)^k s_k'' - 4D \left(\frac{D}{m}\right) \sum_{k=0}^{m} (2-(-1)^k)\bar{t}_k' \\ & + 32D \left(\frac{D}{m}\right) \sum_{k=0}^{m-1} (-1)^k \left[\frac{k+1}{2}\right] t_k''. \end{aligned}$$

Now, to prove (iii) it suffices to put $m = 1$ in the above formula. Then

$$k_2(8D) = -32 \sum_{l \in [D/8, 3D/8)} \left(\frac{D}{l}\right) l - 4D \left(\sum_{l \in [0, D/8)} \left(\frac{D}{l}\right) - 3 \sum_{l \in [0, 3D/8)} \left(\frac{D}{l}\right) \right).$$

Therefore by (4.2) and (4.3) (iii) follows. To prove (iv) it is sufficient to apply (i), (ii), (iii) and (1.6). $\square$

5. Proof of Theorem 2

Let $\Delta \equiv 3 \pmod 4$, $\Delta > 3$ and let $-\Delta$ be the discriminant of a quadratic field. We shall use the formula (3.4) for $x = \frac{1}{4}$ and $d = -\Delta$. From (3.4) and (2.4) for natural m prime to Δ we get:

$$\begin{aligned} k_2(4\Delta) = {} & 16m \left(\frac{-\Delta}{m}\right) \sum_{\substack{0 \leqq k \leqq m \\ 2 \mid k}} \bar{s}_k - 4\Delta \left(\frac{-\Delta}{m}\right) \sum_{k=1}^{m} (-1)^{k+1}(2k-1)T_k \\ & -4\Delta \left(\frac{-\Delta}{m}\right) \mu m h(-\Delta), \end{aligned}$$

where

$$T_0 = 0, \ T_k = \sum_{l \in [0, (2k-1)\Delta/4m)} \left(\frac{-\Delta}{l}\right) \quad \text{for } 1 \leqq k \leqq m,$$

and

$$\mu = \begin{cases} 3-2\left(\frac{-\Delta}{2}\right), & \text{if } 2 \mid m, \\ -1, & \text{if } 2 \nmid m. \end{cases}$$

In fact, putting

$$T_{m+1} = \sum_{l=1}^{[\Delta/2]} \left(\frac{-\Delta}{l}\right)\left(= \left(2-\left(\frac{-\Delta}{2}\right)\right)h(-\Delta), \quad \text{see (2.3)}\right)$$

we get

$$\begin{aligned} \sum_{k=9}^{m} (-1)^k k\bar{t}_k &= \sum_{k=0}^{m} (-1)^k k(T_{k+1} - T_k) \\ &= -\sum_{k=1}^{m+1} (-1)^k (k-1)T_k - \sum_{k=1}^{m} (-1)^k k T_k \\ &= \sum_{k=1}^{m} (-1)^{k+1}(2k-1)T_k + (-1)^m m T_{m+1}. \end{aligned}$$

Now, to prove the part (i) of the theorem it is sufficient to put $m = 1$ in the formula for $k_2(4\Delta)$. Then

$$\sum_{k=1}^{m} (-1)^{k+1}(2k-1)T_k = T_1 = \sum_{l=1}^{[\Delta/4]} \left(\frac{-\Delta}{l}\right) = \tfrac{1}{2}h(-\Delta)\left(1+\left(\frac{-\Delta}{2}\right)\right).$$

Indeed, for $\Delta \neq 3$ we have the following formula:

$$\sum_{l=1}^{[\Delta x]} \left(\frac{-\Delta}{l}\right) = h(-\Delta) - \sqrt{\Delta} \sum_{n=1}^{\infty} \frac{\left(\frac{-\Delta}{n}\right)}{\pi n} \cos 2\pi n x \tag{5.1}$$

(here we must assume $x + (l/\Delta) \notin \mathbb{Z}$ for $1 \leqq l \leqq \Delta - 1$, see [3]). Hence for $x = \frac{1}{4}$ we get the formula for T_1.

To prove (ii) it suffices to put $m = 2$ in the formula for $k_2(4\Delta)$. Then

$$\sum_{k=1}^{m} (-1)^{k+1}(2k-1)T_k = T_1 - 3T_2 = -\tfrac{1}{2}\left(5-\left(\frac{-\Delta}{2}\right)\right)h(-\Delta) - h(-8\Delta).$$

Indeed, the last equality follows immediately from (5.1) for $x = \frac{1}{8}$ and $\frac{3}{8}$ respectively. Namely from (1.2) and (3.2) we get:

$$T_1 = \frac{1}{4}\left(5 - \left(\frac{-\Delta}{2}\right)\right) h(-\Delta) - \frac{1}{4} h(-8\Delta), \tag{5.2}$$

$$T_2 = \frac{1}{4}\left(5 - \left(\frac{-\Delta}{2}\right)\right) h(-\Delta) + \frac{1}{4} h(-8\Delta). \tag{5.3}$$

To prove (iii) and (iv) we shall use the formula (3.4) for $x = \frac{1}{8}$ and $d = -\Delta$. From (3.4) and (2.6) for $\Delta \neq 7$ and for natural m prime to Δ we get:

$$k_2(8\Delta) = 32m\left(\frac{-\Delta}{m}\right)\sum_{k=0}^{m}(-1)^k \bar{s}'_k - 16\Delta\left(\frac{-\Delta}{m}\right)\sum_{k=0}^{m}(-1)^k k\bar{t}'_k +$$
$$+4\Delta\left(\frac{-\Delta}{m}\right)\sum_{k=0}^{m-1}(-1)^k t''_k.$$

To prove (iii) it is sufficient to put $m = 1$ in the above formula. Then

$$k_2(8\Delta) = 32\left(\sum_{l\in[0,\Delta/8)}\left(\frac{-\Delta}{l}\right) l - \sum_{l\in[3\Delta/8,\Delta/2)}\left(\frac{-\Delta}{l}\right) l\right) -$$
$$-4\Delta\left(\sum_{l\in[0,\Delta/8)}\left(\frac{-\Delta}{l}\right) + 3\sum_{l\in[0,3\Delta/8)}\left(\frac{-\Delta}{l}\right) - 4\sum_{l\in[0,\Delta/2)}\left(\frac{-\Delta}{l}\right)\right).$$

Therefore by (5.2), (5.3) and (2.3) (iii) follows. To prove (iv) it suffices to apply the parts (i), (ii) and (iii) of this theorem. □

6. Corollaries to Theorem 1

Let $D \equiv 1 (\mathrm{mod}\, 4)$, $D > 5$ be the discriminant of a quadratic field.

COROLLARY 1. *Let φ denote Euler's totient function.*

(i) $k_2(D) \equiv 2h(-4D) + 2\varphi(D) + \varepsilon \ (\mathrm{mod}\, 32)$,

where $\varepsilon = 0$ unless $D = p \equiv -3 (\mathrm{mod}\, 8)$ a prime or $D = pq$, where $p \equiv q \not\equiv 1 (\mathrm{mod}\, 8)$ or $p \equiv q + 4 \equiv 3 (\mathrm{mod}\, 8)$, p,q-primes. In these cases $\varepsilon = 16$ if $p \equiv q \equiv -3 (\mathrm{mod}\, 8)$, $\varepsilon = -8$ if $p \equiv q \equiv -1 (\mathrm{mod}\, 8)$ and $\varepsilon = 8$ otherwise.

(ii) $k_2(D) \equiv 6h(-4D) - 4\left(2 - \frac{D}{2}\right) h(-8D) \ (\mathrm{mod}\, 32)$,

(iii) $k_2(D) \equiv -2\left(2 - \frac{D}{2}\right)\left(2h(-4D) - \left(\frac{D}{2}\right) h(-8D)\right) (\mathrm{mod}\, 32)$,

(iv) $k_2(8D) + \left(\frac{D}{2} - 34\right)k_2(D)$

$$\equiv -2\left(2\frac{D}{2} - 1\right)\left(\frac{D}{2}h(-4D) + h(-8D)\right)(\operatorname{mod} 64),$$

$k_2(8D + 3\left(3\frac{D}{2} - 2\right)k_2(D)$

$$\equiv -2\left(2\frac{D}{2} - 1\right)\left(\left(\frac{D}{2} - 4\right)h(-4D) + h(-8D)\right) (\operatorname{mod} 64),$$

$k_2(8D) - 3\left(3\frac{D}{2} - 2\right)k_2(D)$

$$\equiv -2\left(2\frac{D}{2} - 1\right)\left(\left(3\frac{D}{2} + 4\right)h(-4D) - 3h(-8D)\right) (\operatorname{mod} 64),$$

$k_2(8D) + 15\left(\frac{D}{2} - 2\right)k_2(D)$

$$\equiv -6\left(2\frac{D}{2} - 1\right)\left(\frac{D}{2}h(-4D) - h(-8D)\right) (\operatorname{mod} 64).$$

(v) *If* $D = p = 8t + 1$ *or* $8t - 3$ *a prime then*:

$k_2(D) \equiv 2h(-4D) + 16t \ (\operatorname{mod} 32)$,

$k_2(D) \equiv 32\alpha + 2\beta\left(-\left(2 + \frac{D}{2}\right)h(-4D) + 2h(-8D)\right) (\operatorname{mod} 64)$,

where $\alpha = 1$ *if* $p \equiv -3 \ (\operatorname{mod} 16)$ *and* $\alpha = 0$ *otherwise, and* $\beta = -1, -3$, *resp.* 5 *if* $p \equiv 1 \ (\operatorname{mod} 8)$, $p \equiv 5 \ (\operatorname{mod} 16)$, *resp.* $p \equiv -3 \ (\operatorname{mod} 16)$,

$$k_2(8D) \equiv 32\alpha + 2\beta\left(2\frac{D}{2}h(-4D) - h(-8D)\right) (\operatorname{mod} 64),$$

where $\alpha = 0$ *if* $p \equiv 1 \ (\operatorname{mod} 16)$ *and* $\alpha = 1$ *otherwise, and* $\beta = -1, -3$, *resp.* 5 *if* $p \equiv 1 \ (\operatorname{mod} 8)$, $p \equiv -3 \ (\operatorname{mod} 16)$, *resp.* $p \equiv 5 \ (\operatorname{mod} 16)$.

Proof. For (i), note that by the theorem on genera and (4.1) $4 \mid h(-4D)$ unless $D = p \equiv -3 \ (\operatorname{mod} 8)$ a prime, in which case $2 \parallel h(-4D)$. Therefore always

$$-\frac{2}{45}\left(2\left(\frac{D}{2}\right) - 7\right)Dh(-4D) \equiv 2h(-4D) \ (\operatorname{mod} 32).$$

For a positive number x and a positive integer n let $A(x, n)$ be the number of positive integers $\leq x$ that are prime to n. We have

$$s_0 + s_1 \equiv \sum_{l=1}^{[D/4]}\left(\frac{D}{l}\right) - \sum_{l=1}^{[D/8]}\left(\frac{D}{l}\right) \equiv A(D/4, D) - A(D/8, D) \ (\operatorname{mod} 2).$$

To prove (i) it suffices to use (i) of Theorem 1 and Nagell's formulas (2), (3) [5].

Since for $D \equiv 1 \pmod 4$

$$\frac{1}{75}\left(\left(\frac{D}{2}\right)+4\right)D \equiv \left(\frac{D}{2}\right) - 2 \pmod 8 \tag{6.1}$$

the part (iii) of the corollary follows. The part (iv) is a consequence of (iv) of Theorem 1 in view of

$$4 \,\Big|\, \pm\left(\frac{D}{2}\right) h(-4D) \pm h(-8D). \tag{6.2}$$

Indeed, from (2.3) (see the formulas for $h(-4D)$ and $h(-8D)$ given in [5]) we get

$$\begin{aligned}\pm\left(\frac{D}{2}\right) h(-4D) \pm h(-8D) &= \pm 2\left(\frac{D}{2}\right) \sum_{\substack{1 \leq l \leq D\\ l \equiv 0 \pmod 4}} \left(\frac{D}{l}\right) \pm 2 \sum_{\substack{1 \leq l \leq D\\ l \equiv 1 \pmod 4}} \left(\frac{8D}{l}\right)\\ &= 4\left(\frac{D}{2}\right) \sum_{\substack{1 \leq l \leq D\\ l \equiv 0 \pmod 4}} \tfrac{1}{2}(\pm 1 \pm (-1)^{l/4}) \left(\frac{D}{l}\right).\end{aligned}$$

The first part of (v) is a particular case of (i) of the corollary. Since

$s_0 + s_2$ is even except $p \equiv -3 \pmod{16}$,

and

$s_1 + s_2$ is odd except $p \equiv 1 \pmod{16}$

the remaining cases of (v) follow from (6.1) and from the divisibility

$4 \mid h(-4D)$ for $D \equiv 1 \pmod 8$. □

COROLLARY 2.

(i) $4 \parallel k_2(D) \Leftrightarrow 2 \parallel h(-4D) \Leftrightarrow 2 \parallel h(-8D) \Leftrightarrow 4 \parallel k_2(8D) \Leftrightarrow D = p \equiv -3 \pmod 8$ *a prime*,

(ii) $8 \parallel k_2(D) \Leftrightarrow 4 \parallel h(-4D)$,
$8 \parallel k_2(8D) \Leftrightarrow 4 \parallel h(-8D)$,
(*for* (i) *and* (ii) *see also* [5]),

(iii) $16 \parallel k_2(D) \Leftrightarrow (8 \parallel h(-4D)$ *and* $16 \mid \varphi(D) + \varepsilon/2)$ *or* $(16 \mid h(-4D)$ *and* $8 \parallel \varphi(D) + \varepsilon/2) \Leftrightarrow (8 \parallel h(-4D)$ *and* $8 \mid h(-8D))$ *or* $(16 \mid h(-4D)$ *and* $4 \parallel h(-8D))$, *where* ε *is defined in Corollary* 1(i),
$16 \parallel k_2(8D) \Leftrightarrow (8 \parallel h(-8D)$ *and* $8 \mid h(-4D))$ *or* $(16 \mid h(-8D)$ *and* $4 \parallel h(-4D))$,
$32 \mid k_2(D), k_2(8D)$ *otherwise*,

(iv) *If* $D = p \equiv 1 \pmod 8$ *a prime then*
$16 \parallel k_2(D) \Leftrightarrow (8 \parallel h(-4D)$ *and* $p \equiv 1 \pmod{16})$ *or* $(16 \mid h(-4D)$ *and* $p \equiv 9 \pmod{16})$,
$32 \parallel k_2(D) \Leftrightarrow (8 \mid h(-4D)$ *and* $(h(-4D)/8) + (h(-8D)/4) \equiv 2 \pmod 4) \Leftrightarrow (8 \parallel h(-4D)$ *and* $4 \parallel h(-8D)$ *and* $(h(-4D)/8) \equiv (h(-8D)/4) \pmod 4)$ *or* $(16 \parallel h(-4D)$ *and* $16 \mid h(-8D))$ *or* $(32 \mid h(-4D)$ *and* $8 \parallel h(-8D))$,
$64 \mid k_2(D) \Leftrightarrow (8 \mid h(-4D)$ *and* $(h(-4D)/8) + (h(-8D)/4) \equiv 0 \pmod 4)$.

Proof. To prove (i), (ii) of the corollary it is sufficient to use the congruence (i) of Corollary 1 modulo 16 i.e.

$$k_2(D) \equiv 2h(-4D) \pmod{16},$$

and the congruence (iii), and (6.2). To prove (iii) of Corollary 2, suppose $8 \mid h(-4D)$. Then, it suffices to apply (i) of Corollary 1. The second part of (iii) for $k_2(D)$ is an immediate consequence of (ii). The exact divisibility of $k_2(8D)$ by 16 follows from (iii) of Corollary 1. (iv) follows from (v) of that corollary. □

REMARK. J. Browkin has proved (unpublished) the first proposition of (iv) of Corollary 2 for $D = p \equiv 1 \pmod 8$ a prime. He has used the formula (i) of Theorem 1 for $D = p \equiv 1 \pmod 8$ and the first congruence of (v) of Corollary 1 that he has got with the methods from [5].

7. Corollaries to Theorem 2

Let $\Delta \equiv 3 \pmod 4$, $\Delta > 3$ and let $-\Delta$ be the discriminant of a quadratic field.

COROLLARY 1.

(i) $k_2(4\Delta) \equiv -6h(-\Delta)\left(\left(\frac{-\Delta}{2}\right) - 1\right) + 2\varphi(\Delta) + \varepsilon \pmod{32}$,

where $\varepsilon = 0$ *unless* $\Delta = p \equiv 3 \pmod 4$ *a prime, or* $\Delta = pq$, *where* $p \equiv q + 2 \equiv -1 \pmod 8$, *p, q-primes, or* $\Delta = pqr$, *where* $p \equiv q \equiv r \equiv -1, 3 \pmod 8$, *or* $p \equiv q \equiv -1$, *resp.* $3 \pmod 8$ *and* $r \equiv 3$, *resp.* $-1 \pmod 8$, *p, q, r-primes. In these cases* $\varepsilon = 4$ *if* $\Delta = p \equiv -1 \pmod 8$, $\varepsilon = -4$ *if* $\Delta = p \equiv 3 \pmod 8$ *and* $\varepsilon = 16$ *otherwise.*

(ii) $k_2(4\Delta) \equiv 6\left(\frac{-\Delta}{2}\right)\left(7\left(\frac{-\Delta}{2} - 1\right)h(-\Delta) + 2h(-8\Delta)\right) \pmod{32}$,
$k_2(4\Delta) \equiv -4h(-8\Delta) \pmod{32}$ *if* $\Delta \equiv -1 \pmod 8$, *in particular,*

(iii) $k_2(8\Delta) \equiv 2\left(1-2\frac{-\Delta}{2}\right)\left(6\left(1 - \frac{-\Delta}{2}\right)h(-\Delta) - h(-8\Delta)\right) \pmod{32}$,
$k_2(8\Delta) \equiv 2h(-8\Delta) \pmod{32}$ *if* $\Delta \equiv -1 \pmod 8$, *in particular,*

(iv) $k_2(8\Delta) + \left(\frac{-\Delta}{2}\right) k_2(4\Delta)$

$$\equiv -2\left(2\frac{-\Delta}{2} - 1\right)\left(\left(\frac{-\Delta}{2} - 1\right) h(-\Delta) + h(-8\Delta)\right) \pmod{64},$$

$k_2(8\Delta) + \left(\frac{-\Delta}{2} - 4\right) k_2(4\Delta)$

$$\equiv -2\left(2\left(\frac{-\Delta}{2} - 1\right)\right)\left(5\left(\frac{-\Delta}{2} - 1\right) h(-\Delta) + h(-8\Delta)\right) \pmod{64},$$

$k_2(8\Delta) - \left(\frac{-\Delta}{2} + 4\right) k_2(4\Delta)$

$$\equiv -2\left(2\frac{-\Delta}{2} - 1\right)\left(7\left(\frac{-\Delta}{2} - 1\right) h(-\Delta) - 3h(-8\Delta)\right) \pmod{64},$$

$k_2(8\Delta) - \left(\frac{-\Delta}{2}\right) k_2(4\Delta)$

$$\equiv 2\left(2\frac{-\Delta}{2} - 1\right)\left(13\left(\frac{-\Delta}{2} - 1\right) h(-\Delta) + 3h(-8\Delta)\right) \pmod{64}.$$

(v) *If* $\Delta = p = 8t - 1$ *or* $8t + 3$ *a prime then*

$$k_2(4\Delta) \equiv -6h(-\Delta)\left(\frac{-\Delta}{2} - 1\right) + 16t \pmod{32},$$

$$k_2(4\Delta) \equiv 32\alpha + 2\beta\left(\frac{-\Delta}{2}\right)\left(7h(-\Delta)\left(\frac{-\Delta}{2} - 1\right) + 2h(-8\Delta)\right) \pmod{64},$$

$$k_2(8\Delta) \equiv 32\alpha + 2\beta\left(13\left(1 - \frac{-\Delta}{2}\right) h(-\Delta) + h(-8\Delta)\right) \pmod{64},$$

where $\alpha = 1$ *if* $p \equiv 7 \pmod{16}$ *and* $\alpha = 0$ *otherwise, and* $\beta = -1, 3,$ *resp.* 11 *if* $p \equiv -1 \pmod 8$, $p \equiv 3 \pmod{16}$, *resp.* $p \equiv 11 \pmod{16}$.

Proof. For (i), we have

$$s_0 + s_1 \equiv A(\Delta/4, \Delta) - A(\Delta/8, \Delta) \pmod 2.$$

Now it suffices to use the part (i) of Theorem 2 and Nagell's formulas (2), (3) [5].

The part (ii) of the corollary is a consequence of (ii) of Theorem 2. Indeed

$$4 \mid 7h(-\Delta)\left(\left(\frac{-\Delta}{2}\right) - 1\right) + 2h(-8\Delta) \tag{7.1}$$

unless $2 \nmid h(-\Delta)$, $\Delta \equiv 3 \pmod 8$. Hence (ii) follows. (iii) is an immediate corollary from the part (iii) of Theorem 2 by

$$\Delta \equiv 1 - 2\left(\frac{-\Delta}{2}\right) \pmod 8 \quad \text{and} \quad 2 \mid h(-8\Delta).$$

(iv) is a consequence of (iv) of Theorem 2 in view of

$$4 \mid h(-8\Delta) \quad \text{for } \Delta \equiv -1 \pmod 8, \tag{7.2}$$

and

$$4 \mid 2h(-\Delta) \pm h(-8\Delta) \quad \text{for } \Delta \equiv 3 \pmod 8. \tag{7.3}$$

In fact, by the theorem on genera (7.3) holds unless $\Delta = p \equiv 3 \pmod 8$ a prime. In this case $2 \nmid h(-\Delta)$ and $2 \parallel h(-8\Delta)$ so (7.3) is also true.

The first part of (v) is a particular case of (i) of the corollary. Since $s_0 \pm s_3$ is even except $p \equiv 7 (\mathrm{mod}\ 16)$ the remaining cases of (v) follow from (7.1), (7.2) and (7.3). □

COROLLARY 2.

(i) *If* $\Delta \equiv -1 (\mathrm{mod}\ 8)$ *then*

$$16 \mid k_2(4\Delta).$$

Moreover in this case:

$$16 \parallel k_2(4\Delta) \Leftrightarrow 8 \parallel \varphi(\Delta) + \frac{\varepsilon}{2} \Leftrightarrow 4 \parallel h(-8\Delta),$$

$$32 \mid k_2(4\Delta) \Leftrightarrow 16 \mid \varphi(\Delta) + \frac{\varepsilon}{2} \Leftrightarrow 8 \mid h(-8\Delta),$$

where ε *is defined in the part* (i) *of Corollary* 1,

$$16 \parallel k_2(8\Delta) \Leftrightarrow 8 \parallel h(-8\Delta),$$
$$32 \mid k_2(8\Delta) \Leftrightarrow 16 \mid h(-8\Delta).$$

(ii) *If* $\Delta = p \equiv -1 (\mathrm{mod}\ 8)$ *a prime then*:

$16 \parallel k_2(4\Delta) \Leftrightarrow p \equiv 7 (\mathrm{mod}\ 16)$,
$32 \parallel k_2(4\Delta) \Leftrightarrow p \equiv -1 (\mathrm{mod}\ 16)$ *and* $8 \parallel h(-8\Delta)$,
$64 \mid k_2(4\Delta) \Leftrightarrow p = -1 (\mathrm{mod}\ 16)$ *and* $16 \mid h(-8\Delta)$,
$32 \parallel k_2(8\Delta) \Leftrightarrow (p \equiv 7 (\mathrm{mod}\ 16)$ *and* $32 \mid h(-8\Delta))$ *or* $(p \equiv -1 (\mathrm{mod}\ 16)$ *and* $16 \parallel h(-8\Delta))$,
$64 \mid k_2(8\Delta) \Leftrightarrow (p \equiv 7 (\mathrm{mod}\ 16)$ *and* $16 \parallel h(-8\Delta))$ *or* $(p \equiv -1 (\mathrm{mod}\ 16)$ *and* $32 \mid h(-8\Delta))$.

(iii) *If* $\Delta \equiv 3 \pmod 8$ *then*:

$4 \parallel k_2(4\Delta) \Leftrightarrow 2 \nmid h(-\Delta) \Leftrightarrow 2 \parallel h(-8\Delta) \Leftrightarrow 4 \parallel k_2(8\Delta) \Leftrightarrow \Delta = p \equiv 3 \pmod 8)$ *a prime*,

$$8 \parallel k_2(4\Delta) \Leftrightarrow 2 \parallel h(-\Delta),$$

$$8 \parallel k_2(8\Delta) \Leftrightarrow 4 \parallel h(-8\Delta),$$

$$16 \parallel k_2\left((4\Delta) \Leftrightarrow (4 \parallel h(-\Delta) \text{ and } 16 \mid \varphi(\Delta) + \frac{\varepsilon}{2}\right) \text{ or}$$

$$\left(8 \mid h(-\Delta) \text{ and } 8 \parallel \varphi(\Delta) + \frac{\varepsilon}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow (4 \parallel h(-\Delta) \text{ and } 8 \mid h(-8\Delta)) \text{ or } (8 \mid h(-\Delta) \text{ and } 4 \parallel h(-8\Delta)),$$

$$32 \mid k_2(4\Delta) \Leftrightarrow \left(4 \parallel h(-\Delta) \text{ and } 8 \parallel \varphi(\Delta) + \frac{\varepsilon}{2}\right) \text{ or } \left(8 \mid h(-\Delta) \text{ and } 16 \mid \varphi(\Delta) + \frac{\varepsilon}{2}\right),$$

where ε *is defined in the part* (i) *of Corollary* 1,

$$16 \parallel k_2(8\Delta) \Leftrightarrow (16 \mid h(-8\Delta) \text{ and } 2 \parallel h(-\Delta)) \text{ or } (8 \parallel h(-8\Delta) \text{ and } 4 \mid h(-\Delta)),$$

$$32 \mid k_2(8\Delta) \Leftrightarrow (16 \mid h(-8\Delta) \text{ and } 4 \mid h(-\Delta)) \text{ or } (8 \parallel h(-8\Delta) \text{ and } 2 \parallel h(-\Delta)).$$

Proof. If $\Delta \equiv -1 \pmod 8$ then we get from (i) of Corollary 1

$$k_2(4\Delta) \equiv 2\varphi(\Delta) + \varepsilon \pmod{32}.$$

Hence and from (ii), (iii) of Corollary 1 the part (i) of Corollary 2 follows. The first equivalence of (ii) follows from the first one of (i). The remaining ones are consequences of the last two congruences of the part (v) of Corollary 1. If $\Delta \equiv 3 \pmod 8$ then we get from (i) of Corollary 1

$$k_2(4\Delta) \equiv 12h(-\Delta) \pmod{16}.$$

Hence and from (ii), (iii) of Corollary 1 in the case $\Delta \equiv 3 \pmod 8$ the part (iii) of Corollary 2 follows.

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