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Trilinear forms for representations of GL(2) and local ε-factors

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1. Introduction

In this paper we study trilinear forms on irreducible, admissible representations of $\mathrm{GL}_2(k)$ for k a local field and relate the existence of such forms to local ε-factors attached to representations of the Deligne-Weil group of k. We recall that according to Jacquet-Langlands [J-L], there is a one-to-one correspondence between discrete series representations (π, V) of GL_2 of a local field $k \neq \mathbf{C}$ and irreducible representations (π', V') of the group of invertible elements D_k^* of the quaternion division algebra D_k over k. The correspondence is characterized by the character identity $\mathrm{ch}\pi(x) = -\mathrm{ch}\pi'(x)$ for all regular elliptic conjugacy classes x. In this introduction, and in fact up to Section 8 of this paper, we will confine ourselves only to non-archimedean fields. The results in the archimedean case are taken up in Section 9.

THEOREM 1.1. *For $G = \mathrm{GL}_2(k)$ or D_k^*, let V_1, V_2, V_3 be three irreducible, admissible representations of G. Then, up to scalars, there exists at most one non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$.*

THEOREM 1.2. *Let V_1, V_2, V_3 be three infinite dimensional, irreducible, admissible representations of $\mathrm{GL}_2(k)$ such that the product of their central characters is trivial. Then either there exists a $\mathrm{GL}_2(k)$-invariant, non-zero, linear form on $V_1 \otimes V_2 \otimes V_3$, or all the V_i, for $i = 1$ to 3, are discrete series representations and there is a non-zero D_k^*-invariant linear form on $V'_1 \otimes V'_2 \otimes V'_3$. Moreover, only one of the two possibilities occurs.*

THEOREM 1.3. *Let V_1, V_2, V_3 be three unramified principal series representations such that the product of their central characters is trivial. Let v_1, v_2, v_3 be non-zero vectors in V_1, V_2, V_3 respectively, invariant under $\mathrm{GL}_2(\mathcal{O}_k)$, with $\mathcal{O}_k$ the ring of integers in k. Then the $\mathrm{GL}_2(k)$-invariant, non-zero, linear form on $V_1 \otimes V_2 \otimes V_3$ takes a non-zero value on $v_1 \otimes v_2 \otimes v_3$.*

Our next theorem relates existence of trilinear forms to ε-factors. We recall that to a representation σ of the Deligne-Weil group of k, an additive character ψ of k, there is associated the ε-factor $\varepsilon(\sigma, \psi)$ (the ε-factor used in this paper is, in Tate's notation in Corvallis, $\varepsilon_L(\sigma, \psi) = \varepsilon_D(\sigma \cdot \| \ \|^{1/2}, \psi, dx)$ where dx is the Haar measure on k, self-dual with respect to the character ψ of k). If $\det \sigma = 1$ then the ε-factor does not depend on the choice of ψ. In such a situation we write the ε-factor as $\varepsilon(\sigma)$. According to local Langlands correspondence for GL_2 (proved in [J-L] for odd residue characteristic and by Kutzko [Ku2] in general) there exists a one-to-one correspondence between irreducible admissible representations of $GL_2(k)$ and two-dimensional F-semisimple representations of the Deligne-Weil group of the local field k.

THEOREM 1.4. *Let V_1, V_2, V_3 be three irreducible, admissible, infinite dimensional representations of* $GL_2(k)$ *such that the product of their central characters is trivial. If all the representations V_i, for $i = 1$ to 3, are supercuspidal, assume that the residue characteristic of k is $\neq 2$. Let σ_i be the representations of the Deligne-Weil group of k associated to V_i. Then $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$. It is $(+1)$ iff there exists a $GL_2(k)$-invariant linear form on $V_1 \otimes V_2 \otimes V_3$, and (-1) iff all the V_i are discrete series representations of* $GL_2(k)$ *and there exists a D_k^*-invariant linear form on $V'_1 \otimes V'_2 \otimes V'_3$.*

REMARK 1.5. The local L-factor associated to the 8-dimensional representation $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of the Deligne-Weil group of k is the one which appears in the Rankin triple product L-function, cf. [Ps-R].

REMARK 1.6. The results in this paper were motivated by the work of J. Repka and others on tensor product of unitary representations of $SL_2(\mathbf{R})$, cf.[Re]. We prove some of these results in the category of $(\mathfrak{g}, K)$-modules for $GL_2(\mathbf{R})$ in Section 9.

C. Asmuth and J. Repka also have some partial results, using Weil representations associated to quadratic and quaternionic algebras, about tensor products of unitary representations of SL_2 of a non-archimedean field of odd residue characteristic, cf.[A-R1], [A-R2]. These results are hard to interpret as they worked with SL_2, rather than GL_2, where even multiplicity one fails.

We now give a summary of the contents of the various sections. In Section 2 we fix the notation and other preliminaries that we will be using in this paper.

In Section 3 we prove that for any three irreducible admissible representations of D_k^* there exists, up to scalars, at most one invariant trilinear form on them. The proof is by the method of Gelfand-pairs. In Section 4 we prove a similar statement for $GL_2(k)$. The method of proof is due to Gelfand-Kazhdan, and Bernstein.

The proof of Theorem 1.2 is divided into many cases. Section 5 deals with those cases in which at least two of the representations are either principal series or are special representations, and also the case when two of the representations are supercuspidal and the third one is a principal series. The proof in the first two cases basically amounts to Mackey's theory about restriction of an induced representation to a subgroup in terms of the orbits of the subgroup, and a result of Waldspurger (Lemma 5.6(a)). The proof in the third case depends on the Kirillov model of supercuspidal representations. This section also contains a proof of Theorem 1.3 which appears as Theorem 5.10.

Section 6 deals with the case when two of the representations are discrete series, at least one of which is supercuspidal. The proof of Theorem 1.2 in this case depends on realising a supercuspidal representation as an induced representation from a finite dimensional representation of a maximal compact-modulo-centre subgroup of $\mathrm{GL}_2(k)$, as well as identities connecting the character of this finite dimensional representation to the character of the induced representation which enable us to reduce this case to one of the cases already considered. We should remark here that with some more effort the proof of Theorem 1.2 given here could be refined to give Theorem 1.1 about multiplicity one of the trilinear form. We have preferred instead to give an independent proof as the proof of Theorem 1.2 given here depends on explicit realization of the representations of $\mathrm{GL}_2(k)$, whereas the proof of Theorem 1.1 rests on general methods which use only the geometrical structure of $\mathrm{GL}_2(k)$.

Section 7 contains character formulae for representations of D_k^*, and some information about the tensor product of representations of D_k^*.

In Section 8 we prove Theorem 1.4 about ε-factors. The proof uses on the one hand explicit knowledge of when there is a trilinear form for representations of $\mathrm{GL}_2(k)$ based on Theorem 1.2, together with the information about tensor product of representations of D_k^* obtained in Section 7, and on the other hand calculation of the ε-factor for the tensor product of representations of Deligne-Weil group of k based on a theorem of Tunnell [Tu].

Finally, in Section 9 we prove the analogues of Theorems 1.1, 1.2 and 1.4 for $\mathrm{GL}_2(\mathbf{R})$.

We end this introduction by mentioning that the results of this paper have been used by M. Harris and S. Kudla to prove a conjecture of H. Jacquet about vanishing of central critical value of Rankin triple product L-functions, cf. [H-K].

2. Notation and other Preliminaries

In this paper k will always denote a fixed non-archimedean local field with $\mathcal{O}_k$ as the ring of integers, π as a uniformizing parameter, v the valuation, and q the cardinality of the residue field. The absolute value on k will be $\|x\| = q^{-v(x)}$.

A locally compact totally disconnected topological space X will be called an l-space. For such a space, $\mathscr{S}(X)$ will denote the space of locally constant compactly supported functions on X. The space of linear forms on $\mathscr{S}(X)$ will be called the space of distributions on X.

For H, a closed subgroup of an l-group G, and a smooth representation (ρ, V) of H, we denote by $\mathrm{Ind}_H^G V$, the space of functions from G to V satisfying the following conditions:

(1) $f(hg) = (\Delta_G^{1/2}/\Delta_H^{1/2})(h)\rho(h)f(g)\ \forall h \in H, g \in G$, with Δ_G(resp. Δ_H) denoting the modular function on G(resp. H).

(2) There exists an open compact subgroup $K_f \subset G$ such that $f(gg_0) = f(g)$ for all $g \in G$ and $g_0 \in K_f$.

The group G acts on this space of functions through right translation $(\pi(g_0)f)(g) = f(gg_0)$.

If we take the subspace of functions satisfying (1) and (2) above and which, moreover, are compactly supported modulo H, then the representation defined on this subspace of functions is denoted by $\mathrm{ind}_H^G V$, and is called the compact induction. Of course, if G/H is compact then $\mathrm{Ind}_H^G V = \mathrm{ind}_H^G V$.

For a smooth representation V of G, V^* will denote the space of linear forms on V. The space of smooth vectors in V^* will be denoted by $\tilde{V}$. Note that for a subgroup H of G, $\tilde{V} \subset \widetilde{V|_H} \subset V^*$.

For ρ a smooth representation of H and π one of G, we have the following version of Frobenius reciprocity (cf. [B-Z], but they use "un-normalised" induction!). Both the isomorphisms are functorial.

(1) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) \cong \mathrm{Hom}_H(\pi|_H, \rho \cdot (\Delta_G^{1/2}/\Delta_H^{1/2}))$

(2) $\mathrm{Hom}_G(\mathrm{ind}_H^G \rho, \tilde{\pi}) \cong \mathrm{Hom}_H(\rho \cdot (\Delta_H^{1/2}/\Delta_G^{1/2}), \widetilde{\pi|_H})$.

The infinite dimensional, irreducible, admissible representations of $\mathrm{GL}_2(k)$ fall into 3 types. The principal series of representations arise from inducing characters

$$\psi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \psi_1(a) \cdot \psi_2(d)$$

of the Borel subgroup B. We will denote this representation by $V_{(\psi_1, \psi_2)}$. We will use δ for Δ_G/Δ_B, given by: $\delta\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \|a\|/\|d\|$. If the characters ψ_1 and ψ_2 are trivial on $\mathcal{O}_k^*$, then the principal series is called unramified (or, spherical).

The principal series $V_{(\psi_1, \psi_2)}$ is irreducible iff $\psi_1(x) \cdot \psi_2(x)^{-1} \neq \|x\|^{\pm 1}$. If $\psi_1(x) \cdot \psi_2(x)^{-1} = \|x\|$, then $V_{(\psi_1, \psi_2)}$ has an infinite dimensional, irreducible sub-representation, and the quotient is a one-dimensional representation. If $\psi_1(x) \cdot \psi_2(x)^{-1} = \|x\|^{-1}$ then $V_{(\psi_1, \psi_2)}$ has a one-dimensional sub-representation and the quotient is irreducible. The infinite dimensional sub-quotients of principal series are called special representations. We will use the notation Sp to denote the special representation defined by the natural action of $\mathrm{GL}_2(k)$ on locally constant functions on $\mathbf{P}^1(k)$ modulo the constant functions on $\mathbf{P}^1(k)$.

Finally, there is a third class – the supercuspidal representations – which do not appear as sub-quotients of principal series. These will be discussed in more detail in Section 6.

For a representation (π, V) of $GL_2(k)$, and a character η of k^*, define a representation $\pi \otimes \eta$ of $GL_2(k)$ on V by $(\pi \otimes \eta)(g) = \pi(g)\cdot\eta(\det g)$, for $g \in GL_2(k)$. The representation $\pi \otimes \eta$ is called the twist of π by η. It is easy to see that any two special representations are twists of each other by a character.

On an irreducible, admissible representation of $GL_2(k)$, scalar matrices ($\cong k^*$) act by a character called the central character. For such a representation V, with central character ω, $\tilde{V} \cong V \otimes \omega^{-1}$. The central character of the principal series $V_{(\psi_1, \psi_2)}$ is $\psi_1 \cdot \psi_2$.

3. Multiplicity one for D_k^*

In this section we prove that for three irreducible representations V_1, V_2, V_3 of the group of invertible elements of the quaternion division algebra, there is at most one dimensional space of invariant linear form on $V_1 \otimes V_2 \otimes V_3$. The proof is based on the concept of Gelfand pairs which we recall.

DEFINITION. (G, H) with H a subgroup of a group G is called a Gelfand pair if there exists an anti-automorphism i of order 2 of G (i.e. $i(xy) = i(y)i(x)$, $i(i(x)) = x$, $\forall x, y \in G$) such that $i(HxH) = HxH$, $\forall x \in G$.

LEMMA 3.1. *If (G, H), with G a finite group, is a Gelfand pair then the trivial representation of H appears with multiplicity at most one in any irreducible representation of G.*

Proof. Well-known, cf. S. Lang's $SL_2(\mathbf{R})$, Chapter IV, Theorem 1 and Theorem 3. □

REMARK 3.2. More generally, if $H \subseteq G$ is a subgroup, then it is easy to see that the restriction to H of irreducible representations of G is multiplicity free if $H \hookrightarrow H \times G$ is a Gelfand pair.

It follows from Lemma 3.1 that to prove Theorem 1.1 for D_k^*, it suffices to show that $(D_k^* \times D_k^* \times D_k^*, \Delta D_k^*)$, with ΔD_k^* the diagonal subgroup of $D_k^* \times D_k^* \times D_k^*$, is a Gelfand pair. We do this now.

PROPOSITION 3.3. *Let D_k be a quaternion division algebra over any field k and let $x \to \bar{x} = tr(x) - x$ be the standard anti-automorphism of the quaternion division algebra. Then $(D_k^* \times D_k^* \times D_k^*, \Delta D_k^*)$ is a Gelfand pair with $i(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$.*

Proof. We need to check that i preserves all the double cosets of ΔD_k^* in $D_k^* \times D_k^* \times D_k^*$. This is easily seen to be equivalent to checking that given $x, y \in D_k^*$, there exists z such that $zxz^{-1} = \bar{x}$, $zyz^{-1} = \bar{y}$. For this it suffices to show that for any two subfields of D_k of degree 2 over k, there exists an element of D_k^* such that the inner conjugation by it fixes both the subfields and acts by the non-trivial automorphism over k, if the field has one. If K/k is a degree 2 field extension with a non-trivial automorphism then by the Skolem-Noether theorem there exists $j \in D_k^*$ such that the inner conjugation by j acts by the non-trivial automorphism. In fact, any non-zero element of $j \cdot K$ will have this property. Clearly $j \cdot K$ consists of trace zero elements. If the field K had no automorphism over k, all the elements of K would be of trace zero. Therefore the sought for element of D_k^* is any non-zero element in the intersection of two subspaces of the form $j \cdot K$ or K as the case may be, on each of which the trace map is zero. Since trace zero elements form a 3-dimensional space, the intersection is non-zero and such an element exists. □

REMARK 3.4. Multiplicity one is not true for $SL_1(D_k)$: It is easy to see that the structure of $G = SL_1(D_k)/\pm 1/\{X \equiv 1(\pi^2)\}$ is

$$0 \to \mathbf{F}_{q^2} \to G \to N^1/\pm 1 \to 0.$$

with N^1 = Norm one elements in $\mathbf{F}_{q^2}^*$ and the action of $N^1/\pm 1$ on $\mathbf{F}_{q^2}$ is:

$$\begin{aligned} N^1/\pm 1 \times \mathbf{F}_{q^2} &\to \mathbf{F}_{q^2} \\ (u, X) &\to u^2 X. \end{aligned}$$

The irreducible representations V of G which do not factor through $N^1/\pm 1$ are given by an orbit of $N^1/\pm 1$ on the non-trivial elements of the character group of $\mathbf{F}_{q^2}$, a non-trivial character of $\mathbf{F}_{q^2}$ appearing in only one irreducible representation. But clearly in $V \otimes V$ there are non-trivial characters of $\mathbf{F}_{q^2}$ appearing with multiplicity two.

REMARK 3.5. Using Proposition 3.3, it is easy to see that for a quadratic subfield K of D_k, $K^* \hookrightarrow K^* \times D_k^*$ is a Gelfand pair for the involution $i(x, y) = (x, j\bar{y}j^{-1})$ for $(x, y) \in K^* \times D_k^*$, with j any element of the normaliser of K^* which does not lie in K^* if K is separable over k, and with j any element of K^* if K is inseparable over k. It follows from Remark 3.2 that the characters of K^* appearing in an irreducible representation of D_k^* appear with multiplicity one.

4. Multiplicity one for GL(2)

In this section we prove that for any three irreducible, admissible representations V_1, V_2, V_3 of $GL_2(k)$ there is at most one dimensional space of $GL_2(k)$-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$. The proof again depends on the idea of

Gelfand-pairs, but there are problems as the groups involved are not compact. Also, in this case the involution does not preserve all the double cosets but only 'almost-all' double cosets. The method of proof that we give was initiated by Gelfand-Kazhdan in [G-K], for their proof of the uniqueness of the Whittaker model in the p-adic case and later developed by Bernstein [Be].

The following well-known lemma will often be used in what follows, sometimes without explicit mention.

LEMMA 4.1. *Let G be an l-group and H a closed subgroup. Then the pull back of distributions from G/H to G (by integration along the fibre) induces a canonical identification of distributions on G/H to H-invariant distributions on G.* □

LEMMA 4.2. *Let G be an l-group and H a closed subgroup such that G/H carries a G-invariant measure. Suppose $x \to \bar{x}$ is an anti-automorphism which leaves H invariant and acts trivially on those distributions on G which are H bi-invariant. Then for any smooth irreducible representation V of G,* $\dim(V^{*H})\cdot\dim([\tilde{V}]^{*H}) \leqslant 1$.

Proof. Let

$$l: V \to \mathbf{C}, \quad m: \tilde{V} \to \mathbf{C}$$

be H-invariant linear functionals. By Frobenius reciprocity, this is equivalent to G-linear maps

$$l': \mathscr{S}(G/H) \to V, \quad m': \mathscr{S}(G/H) \to \tilde{V}.$$

This gives rise to

$$B: \mathscr{S}(G \times G/H \times H) \cong \mathscr{S}(G/H) \otimes \mathscr{S}(G/H) \to V \otimes \tilde{V} \to \mathbf{C}.$$

Therefore we get a distribution on $G \times G$ which is $H \times H$-invariant on the right and G-invariant on the left.

Claim. $B(f, g) = B(i(g), i(f)) \,\forall f, g \in \mathscr{S}(G/H)$, where $i(f)(x) = f(\bar{x}^{-1})$.

Assuming the claim, we prove the lemma. Clearly,

$\mathrm{Ker}(l') =$ left kernel of B, i.e., $f \in \mathscr{S}(G/H)$ such that $B(f, g) = 0 \,\forall g \in \mathscr{S}(G/H)$,

$\mathrm{Ker}(m') =$ right kernel of B, i.e., $g \in \mathscr{S}(G/H)$ such that $B(f, g) = 0 \,\forall f \in \mathscr{S}(G/H)$.

Therefore by the above claim, $\mathrm{Ker}(l')$ and $\mathrm{Ker}(m')$ determine each other. By Schur's lemma $\mathrm{Ker}(l')$ determines l' and hence l; similarly $\mathrm{Ker}(m')$ determines m. But since l and m were arbitrary, it proves the lemma.

We now return to the proof of the claim. The mapping $(x, y) \to x^{-1}y$ from $G \times G$ to G identifies left-G-invariant distributions on $G \times G$ with distributions on G. Under this identification, distributions on $G \times G$ which are right invariant under $H \times H$ and left invariant under G are identified to distributions on G bi-invariant under H. Since by hypothesis any distribution on G bi-invariant under H is also invariant under the involution $x \to \bar{x}$, the following commutative diagram completes the proof of the claim.

$$\begin{array}{ccc} G \times G \ni (x, y) & \longrightarrow & x^{-1}y \in G \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ G \times G \ni (\bar{y}^{-1}, \bar{x}^{-1}) & \rightarrow & \bar{y}\bar{x}^{-1} \in G. \end{array}$$

□

If there is a $\mathrm{GL}_2(k)$-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ then clearly the product of central characters of V_i is trivial, and therefore we have an isomorphism of $\mathrm{GL}_2(k)$-modules $V_1 \otimes V_2 \otimes V_3 \cong \tilde{V}_1 \otimes \tilde{V}_2 \otimes \tilde{V}_3$. Therefore to prove that there is at most one-dimensional space of $\mathrm{GL}_2(k)$-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$, it suffices, because of the previous lemma, to check that any distribution on $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ which is bi-invariant under $\mathrm{GL}_2(k)$, embedded diagonally, is invariant under $(X, Y, Z) \to (\bar{X}, \bar{Y}, \bar{Z})$ with $\bar{M} = \mathrm{tr}M - M$ for any matrix M. The following lemma reduces this to a calculation on $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$.

LEMMA 4.3. *Let $x \to \bar{x}$ be an anti-automorphism on a unimodular group G. Then distributions on $G \times G \times G$ which are G-bi-invariant can be identified to distributions on $G \times G$ which are invariant under the inner conjugation action of G. Under this identification, distributions on $G \times G$ which are invariant under the involution $(x, y) \to (\bar{x}, \bar{y})$ go to distributions on $G \times G \times G$ which are invariant under $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$.*

Proof. The map

$$\begin{array}{rcl} \pi: G \times G \times G & \longrightarrow & G \times G \\ (x, y, z) & \longrightarrow & (x^{-1}y, x^{-1}z) \end{array}$$

identifies distributions on $G \times G \times G$, bi-invariant under diagonally embedded G, to distributions on $G \times G$, invariant under inner conjugation by G, acting diagonally. We write down this identification explicitly. The pull back $\pi^*\phi$ of a distribution ϕ on $G \times G$ is defined by

$$\pi^*\phi(f) = \phi\left[\int_G f(x, xy, xz)\,\mathrm{d}x\right] = \int_G \phi(f_x)\,\mathrm{d}x$$

where f a function on $G \times G \times G$ and for $x \in G$, f_x denotes the function on $G \times G$ defined by $f_x(y, z) = f(x, xy, xz)$ (the second equality is clear for functions of the type $f(x, y, z) = g(x) \cdot h(x^{-1}y, x^{-1}z)$, and hence for all functions f as any function is a linear combination of functions of this type). Assume now that ϕ is invariant under the involution $(x, y) \to (\bar{x}, \bar{y})$ and let us check that $\pi^*\phi$ is invariant under $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$. Let $\bar{f}$ denote the function $\bar{f}(x, y, z) = f(\bar{x}, \bar{y}, \bar{z})$. Then we have

$$\pi^*\phi(\bar{f}) = \phi\left[\int_G f(\bar{x}, \bar{y}\bar{x}, \bar{z}\bar{x})\,dx\right] = \int_G \phi(f_{\bar{x}}(\bar{x}^{-1}\bar{y}\bar{x}, \bar{x}^{-1}\bar{z}\bar{x}))\,dx.$$

As ϕ is invariant under conjugation $\phi(f_{\bar{x}}(\bar{x}^{-1}\bar{y}\bar{x}, \bar{x}^{-1}\bar{z}\bar{x})) = \phi(f_{\bar{x}}(\bar{y}, \bar{z}))$, and since ϕ is invariant under $(y, z) \to (\bar{y}, \bar{z})$, we get that

$$\pi^*\phi(\bar{f}) = \int_G \phi(f_{\bar{x}})\,dx = \int_G \phi(f_x)\,dx = \pi^*\phi(f). \qquad \square$$

The proof of Theorem 1.1, in the case of $\mathrm{GL}_2(k)$, is therefore reduced to proving that any $\mathrm{GL}_2(k)$-invariant distribution on $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ is invariant under $(X, Y) \to (\bar{X}, \bar{Y})$. This will be proved using the following theorem due to Bernstein [Be].

THEOREM 4.4. *Let $p: X \to Y$ be a continuous map of l-spaces. Suppose that an l-group $\hat{G}$ acts on X preserving all the fibers of $p: X \to Y$, and that P is a subgroup of $\hat{G}$ such that every P-invariant distribution on any fibre is $\hat{G}$-invariant. Then any P-invariant distribution on X is $\hat{G}$-invariant.* □

PROPOSITION 4.5. *A distribution on* $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$, *invariant under the inner conjugation action of* $\mathrm{GL}_2(k)$, *acting diagonally, is invariant under the involution* $(X, Y) \to (\bar{X}, \bar{Y})$.

Proof. Consider

$$\pi: \mathrm{GL}_2(k) \times \mathrm{GL}_2(k) \to \mathbf{A}^5$$
$$(X, Y) \longrightarrow (\mathrm{tr}(X), \mathrm{tr}(Y), \det(X), \det(Y), \mathrm{tr}(XY)).$$

Clearly the fibers of π are invariant under the action of $\mathrm{GL}_2(k)$, and also under the involution $(X, Y) \to (\bar{X}, \bar{Y})$. Since for $g \in \mathrm{GL}_2(k), \bar{g} = \det g \cdot g^{-1}$, the action of $\mathrm{GL}_2(k)$ and the involution $(X, Y) \to (\bar{X}, \bar{Y})$ commute:

$$g(\bar{X}, \bar{Y})g^{-1} = (\overline{gXg^{-1}}, \overline{gYg^{-1}}),$$

let $\hat{G} = \mathrm{GL}_2(k) \times \mathbf{Z}/2\mathbf{Z}$. By the above theorem of Bernstein it suffices to prove that any $\mathrm{GL}_2(k)$-invariant distribution supported on a fiber of the map π is invariant under $\hat{G}$. We will recall the structure of the fibres (only the case 1 needs a non-trivial checking which can be done over an algebraically closed field) and check for each case that a $\mathrm{GL}_2(k)$-invariant distribution supported on that fiber is invariant under $\hat{G}$. We will denote by $F_{(A,B)}$ the fibre of the mapping π passing through (A, B).

Case 1. A and B can't simultaneously be triangulated over the algebraic closure, k^a, of k. In this case $F_{(A,B)}$ consists of a single $\mathrm{GL}_2(k)$-orbit on which $\mathrm{GL}_2(k)/k^*$ acts simply transitively. So there is a one-dimensional space of distributions on this orbit which are $\mathrm{GL}_2(k)$-invariant and these are invariant under the involution $(X, Y) \to (\bar{X}, \bar{Y})$ as the involution commutes with the $\mathrm{GL}_2(k)$ action.

Case 2. A and B can simultaneously be triangulated over k. Assume that in a suitable basis A and B are given by $\begin{pmatrix} a_1 & s \\ 0 & a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 & t \\ 0 & b_2 \end{pmatrix}$, respectively.

Case 2_A. $a_1 \neq a_2$. After conjugation, we can assume that A and B look like $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ and $\begin{pmatrix} b_1 & t \\ 0 & b_2 \end{pmatrix}$, respectively. In this case the fibre $F_{(A,B)}$ consists of three $\mathrm{GL}_2(k)$-orbits passing through the points $\left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & 1 \\ 0 & b_2 \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ 1 & b_2 \end{pmatrix}\right\}$, and $\left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\right\}$.

For any $(X, Y) \in F_{(A,B)}$, we see from the above description of the fibre that there are two distinct lines, say L_1 and L_2, in the two-dimensional space V on which X acts by a_1 and a_2, respectively. Define the map

$$q: F_{(A,B)} \to \frac{\mathbf{P}(V) \times \mathbf{P}(V) - \Delta\mathbf{P}(V)}{\sigma}$$

$$(X, Y) \to [L_1], [L_2]$$

where

$$\sigma(l, m) = (m, l); \quad l, m \in \mathbf{P}(V).$$

With the natural action of $\mathrm{GL}_2(k)$ on both sides of the arrow, this mapping is $\mathrm{GL}_2(k)$-equivariant. Also

$$\left.\begin{array}{l} Xe_1 = a_1 e_1 \\ Xe_2 = a_2 e_2 \end{array}\right\} \Rightarrow \begin{array}{l} \bar{X}e_1 = a_2 e_1 \\ \bar{X}e_2 = a_1 e_2. \end{array}$$

Therefore the fibres of the map q are preserved under the involution $(X, Y) \to (\bar{X}, \bar{Y})$. Since we have a surjective map from $F_{(A,B)}$ onto the $\mathrm{GL}_2(k)$-homogeneous space $[\mathbf{P}(V) \times \mathbf{P}(V) - \Delta\mathbf{P}(V)]/\sigma \cong \mathrm{GL}_2(k)/N(T)$, with $N(T)$ the normaliser of the diagonal torus, the $\mathrm{GL}_2(k)$-invariant distributions on $F_{(A,B)}$ are isomorphic to $N(T)$-invariant distributions on the fibre of q passing through (A, B), cf. lemma on page 60 in [Be]. The fibre of the map q passing through the point (A, B) consists of the matrices $\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix}\}$, $\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ * & b_2 \end{pmatrix}\}$, $\{\begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} b_2 & * \\ 0 & b_1 \end{pmatrix}\}$, $\{\begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} b_2 & 0 \\ * & b_1 \end{pmatrix}\}$, where $*$ can take arbitrary values from k.

It follows that the fibre is the disjoint union of two copies of the union of the co-ordinate axes in the (x, y)-plane. The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ permutes the two disjoint copies. The group generated by the involution $(X, Y) \to (\bar{X}, \bar{Y})$ and the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the fibre is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and acts on the fibre by interchanging the x and y axes and permuting the two disjoint copies of the co-ordinate axes. Therefore an $N(T)$-invariant distribution on the fibre is the same as a T-invariant distribution on the union of the co-ordinate axes in the (x, y)-plane and to check that such a distribution is invariant under $(X, Y) \to (\bar{X}, \bar{Y})$ it suffices to prove the following lemma (whose proof is easy and is omitted).

LEMMA 4.6. *A distribution on the union of the two co-ordinate axes in the* (x, y) *plane, which is invariant under the action of* k^* *given by* $\{t, (x, y)\} \to (tx, t^{-1}y)$, *with* $t \in k^*$ *and* $x, y \in k$, *is invariant under the involution* $(x, y) \to (y, x)$. □

Case 2_B. $b_1 \neq b_2$. This is similar to 2_A.

Case 2_C. A and B are given by the matrices $\begin{pmatrix} a & s \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} b & t \\ 0 & b \end{pmatrix}$, respectively.

The fibre $F_{(A,B)}$ consists of $GL_2(k)$-orbits parametrized by $\lambda \in \mathbf{P}^1$ passing through the points $\{\begin{pmatrix} a & s \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & t \\ 0 & b \end{pmatrix}\}$ such that both s and t are not simultaneously zero and $[s, t] = \lambda \in \mathbf{P}^1$, and also the $GL_2(k)$-orbit passing through the point $\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\}$. In this case the involution $(X, Y) \to (\bar{X}, \bar{Y})$ clearly takes an orbit into itself and therefore a $GL_2(k)$-invariant distribution on such a fibre is invariant under the involution $(X, Y) \to (\bar{X}, \bar{Y})$ by Theorem 6.13 of [B-Z].

Case 3. A and B can simultaneously be triangulated over the algebraic closure, k^a, of k, but not over k. In this case one of the matrices, say A, does not have its eigenvalues over k, but only over a quadratic extension K of k. Since A and B can simultaneously be triangulated over k^a, let $v \in V \otimes_k k^a$ be a common eigenvector of A and B. We now split this case into two cases according to whether K is a separable extension or not.

Case 3_A. K/k is separable with $x \to \bar{x}$ denoting the non-trivial automorphism of K over k. Since A and B are defined over k, both v and $\bar{v}$ are common eigenvectors for A and B and $v \neq \bar{v}$, as eigenvalues of A are not defined over k. In particular, A and B can simultaneously be diagonalized over k^a. Combining with the knowledge of the fibres in case 2, it follows that the fibre of π passing through such an (A, B) is a single $GL_2(k)$-orbit isomorphic to $GL_2(k)/K^*$. Again this has a one-dimensional space of $GL_2(k)$-invariant distributions which are invariant under $(X, Y) \to (\bar{X}, \bar{Y})$.

Case 3_B. K/k is inseparable. Since v is not defined over k, eigenvalues of A are inseparable, and eigenvalues of B are either inseparable or B is a scalar matrix.

Therefore $\bar{X} = \mathrm{tr}X - X = X$ and $\bar{Y} = \mathrm{tr}Y - Y = Y$. So the involution $(X, Y) \to (\bar{X}, \bar{Y})$ acts by identity on this fibre and therefore there is nothing to prove. □

5. Trilinear forms I

We split the problem of constructing trilinear forms on $V_1 \times V_2 \times V_3$ into four cases:

Case 1. Two of the representations are special.

Case 2. Two of the representations belong to the principal series.

Case 3. Two of the representations are supercuspidal and one is principal series.

Case 4. Two of the representations are discrete series, at least one of which is supercuspidal.

In this section we take up cases 1 to 3. Case 4 will be taken up in the next section. We start with two general lemmas.

LEMMA 5.1. *Let L be a complex line bundle on an l-space X and Z a closed subspace. Let $\Gamma_c(X, L)$ denote the space of locally constant compactly supported sections of L. Then we have an exact sequence*:

$$0 \to \Gamma_c(X - Z, L|_{X-Z}) \to \Gamma_c(X, L) \to \Gamma_c(Z, L|_Z) \to 0.$$

Proof. For L a trivial line bundle, this is Proposition 1.8 in [B-Z]. For arbitrary L, one only needs to prove the surjectivity of the restriction map

$$\Gamma_c(X, L) \to \Gamma_c(Z, L|_Z) \to 0.$$

But since any section in $\Gamma_c(Z, L|_Z)$ can be written as a sum of sections with arbitrary small support, surjectivity follows. □

LEMMA 5.2. *For an l-group G, let W be a smooth G-module with a positive-definite, G-invariant, inner product. Assume that $V \subset W$ is an admissible G-submodule. Then V has a G-invariant complement, i.e., there exists a G-submodule $V' \subset W$ such that $W = V \oplus V'$.*

Proof. Let $\bar{W}$ be the Hilbert space obtained by completing W with respect to the inner product, and $\bar{V}$ the closure of V in $\bar{W}$. Since now we are in a Hilbert space, $\bar{W} = \bar{V} \oplus \bar{V}^{\perp}$, where $\bar{V}^{\perp}$ denotes the space of vectors in $\bar{W}$ perpendicular to $\bar{V}$. Define $V' = W \cap \bar{V}^{\perp}$.

Claim: $W = V \oplus V'$. For any vector $w \in W$ write $w = v + v'$ with $v \in \bar{V}$ and

$v' \in \bar{V}^{\perp}$. Since W is a smooth G-module, there exists a compact open subgroup K of G fixing w. By uniqueness of the expression $w = v + v'$ with $v \in \bar{V}$ and $v' \in \bar{V}^{\perp}$, this K also fixes v. We show that this implies that $v \in V$: write V as a direct sum of isotypical spaces for K, $V = \oplus_{\alpha \in \hat{K}} V_\alpha$, with V_α finite dimensional and distinct V_α orthogonal. Therefore $\bar{V} = \hat{\oplus}_{\alpha \in \hat{K}} V_\alpha$, where '$\hat{\oplus}$' denotes the Hilbert space direct sum. It is now clear that if a vector of $\bar{V}$ is invariant under K, it belongs to V. So $v \in V$ and therefore $v' \in W$ and hence $v' \in V' = W \cap \bar{V}^{\perp}$, and we are done. □

REMARK 5.3. The lemma is of course false without the admissibility condition on V.

LEMMA 5.4. *Let T be the diagonal torus in* $\mathrm{GL}_2(k)$ *and* Sp *the special representation of* $\mathrm{GL}_2(k)$. *Then* $\mathrm{Sp} \subset \mathscr{S}(\mathrm{GL}_2(k)/T)$, *in fact as a direct summand by the previous lemma, and we have an isomorphism*

$$\mathrm{Sp} \otimes \mathrm{Sp} \cong \mathscr{S}(\mathrm{GL}_2(k)/T)/\mathrm{Sp}.$$

Proof. By definition, the representation Sp sits in the exact sequence:

$$0 \to 1 \to \mathscr{S}(\mathbf{P}^1) \to \mathrm{Sp} \to 0. \qquad (*)$$

Therefore we have an exact sequence,

$$0 \to \mathscr{S}(\mathbf{P}^1) \otimes 1 + 1 \otimes \mathscr{S}(\mathbf{P}^1) \to \mathscr{S}(\mathbf{P}^1) \otimes \mathscr{S}(\mathbf{P}^1) \to \mathrm{Sp} \otimes \mathrm{Sp} \to 0. \qquad (**)$$

Now $\mathscr{S}(\mathbf{P}^1) \otimes \mathscr{S}(\mathbf{P}^1) \cong \mathscr{S}(\mathbf{P}^1 \times \mathbf{P}^1)$ and the action of $\mathrm{GL}_2(k)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ has two orbits, one open ($x \neq y \in \mathbf{P}^1 \times \mathbf{P}^1$) and the other closed ($\Delta\mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$). The open orbit can be identified with $\mathrm{GL}_2(k)/T$ with T the diagonal torus. Therefore we have an exact sequence:

$$0 \to \mathscr{S}(\mathrm{GL}_2(k)/T) \to \mathscr{S}(\mathbf{P}^1 \times \mathbf{P}^1) \to \mathscr{S}(\Delta\mathbf{P}^1) \to 0.$$

Since $\mathscr{S}(\mathbf{P}^1) \otimes 1 \subset \mathscr{S}(\mathbf{P}^1 \times \mathbf{P}^1)$ goes surjectively (in fact, isomorphically) to $\mathscr{S}(\Delta\mathbf{P}^1)$ under the restriction map to the diagonal, it follows that the subspaces $\mathscr{S}(\mathrm{GL}_2(k)/T)$ and $\mathscr{S}(\mathbf{P}^1) \otimes 1 + 1 \otimes \mathscr{S}(\mathbf{P}^1)$ span $\mathscr{S}(\mathbf{P}^1 \times \mathbf{P}^1)$. Also, the intersection of $\mathscr{S}(\mathrm{GL}_2(k)/T)$ and $\mathscr{S}(\mathbf{P}^1) \otimes 1 + 1 \otimes \mathscr{S}(\mathbf{P}^1)$ is the space of functions on $\mathbf{P}^1 \times \mathbf{P}^1$ of the form $F(x, y) = f(x) - f(y)$, with f a locally constant function on $\mathbf{P}^1$, therefore isomorphic to Sp. It therefore follows from the exact sequences (*) and (**) that $\mathrm{Sp} \otimes \mathrm{Sp} \cong \mathscr{S}(\mathrm{GL}_2(k)/T)/\mathrm{Sp}$. □

We next recall the following basic result about the Kirillov model of an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(k)$, cf. [Go] Theorem 1, formula 140, and Theorem 11.

THEOREM 5.5. (*a*) *Let π be an infinite dimensional irreducible admissible representation of* $\mathrm{GL}_2(k)$ *and let ψ be a non-trivial additive character of k. Then there is a unique representation $\mathscr{K}_\pi = \mathscr{K}_\pi(\psi)$ on a space of locally constant functions on k^* which is equivalent to π and which is such that*

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f(x) = \psi(bx) f(ax), \quad \text{for } a \in k^*, b \in k.$$

$\mathscr{K}_\pi$ equals $\mathscr{S}(k^*)$ if π is supercuspidal, contains $\mathscr{S}(k^*)$ as a subspace of codimension 1 *if π is special, and codimension* 2 *if π is a principal series.*

(*b*) *Locally constant functions on k^* which are zero outside a compact set in k and which look like $a\sqrt{\|x\|}\mu_1(x)$ around $0 \in k$, for $a \in \mathbf{C}$, form a codimension one k^*-submodule, for the inclusion of k^* in* $\mathrm{GL}_2(k)$ *as the subgroup $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, in the Kirillov model $\mathscr{K}_{\mu_1,\mu_2}$ of the principal series V_{μ_1,μ_2}, the quotient being the character $\sqrt{\|x\|}\mu_2$ of k^*.*

(*c*) *For an unramified principal series V_{μ_1,μ_2}, the following function, $W_{\mu_1,\mu_2} \in \mathscr{K}_{\mu_1,\mu_2}$ is invariant under* $\mathrm{GL}_2(\mathcal{O}_k)$.

$$W_{\mu_1,\mu_2}(x) = \sqrt{\|x\|} \sum_{i \geqslant 0, j \geqslant 0, i+j = v(x)} \mu_1(\pi^i)\mu_2(\pi^j) \quad \text{if } x \in \mathcal{O}_k,$$
$$= 0 \quad \text{if } x \notin \mathcal{O}_k. \qquad \square$$

LEMMA 5.6. (*a*) *For an infinite dimensional irreducible admissible representation V of* $\mathrm{GL}_2(k)$ *with central character ω, and any character χ of the split torus T which restricts to ω on the centre, there is a non-zero linear form l on V, unique up to scalars, on which T acts via the character χ, i.e., $l(t^{-1}v) = \chi(t)l(v)$ for $v \in V$.*

(*b*) *If, moreover, $V = V_{\mu_1,\mu_2}$ is an unramified principal series and χ is an unramified character of T, i.e., χ equals* 1 *on the maximal compact subgroup of T, then for $v \in V_{\mu_1,\mu_2}$ invariant under* $\mathrm{GL}_2(\mathcal{O}_k)$, $l(v) \neq 0$.

Proof. (a) This is due to Waldspurger, Lemmas 8 and 9 in [Wa].

(b) For $x \in k^*$, let $\chi(x)$ denote the value of the character χ at $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We split the proof of this part in two cases.

Case 1. $\chi^{-1} = \sqrt{\|x\|}\mu_1$ or $\sqrt{\|x\|}\mu_2$. In this case taking quotient by the codimension 1 subspace of Theorem 5.5(b) is the desired linear form. It is easy to see that W_{μ_1,μ_2} does not lie in this codimension one subspace, therefore the proof in this case is complete.

Case 2. $\chi^{-1} \neq \sqrt{\|x\|}\mu_1$ or $\sqrt{\|x\|}\mu_2$. In this case the linear form l is necessarily non-zero on $\mathscr{S}(k^*)$, and therefore l restricted to $\mathscr{S}(k^*)$ must be the linear form $f \to \int_{k^*} f(x)\chi(x)(dx/\|x\|)$. The proof in this case will be by contradiction.

If $l(W_{\mu_1,\mu_2}) = 0$ then clearly

$$l\left(\frac{\mu_1(\pi)\mu_2(\pi)}{q} W_{\mu_1,\mu_2} - \left\{\frac{\mu_1(\pi)}{\sqrt{q}} + \frac{\mu_2(\pi)}{\sqrt{q}}\right\} R_\pi W_{\mu_1,\mu_2} + R_{\pi^2} W_{\mu_1,\mu_2}\right) = 0,$$

where for $a \in k^*$ and f a function on k^*, $R_a f$ denotes the function $R_a f(x) = f(xa)$ on k^*. But it is easy to verify that the function

$$\frac{\mu_1(\pi)\mu_2(\pi)}{q} W_{\mu_1,\mu_2} - \left\{\frac{\mu_1(\pi)}{\sqrt{q}} + \frac{\mu_2(\pi)}{\sqrt{q}}\right\} R_\pi W_{\mu_1,\mu_2} + R_{\pi^2} W_{\mu_1,\mu_2}$$

is the characteristic function of $\pi^{-2}\mathcal{O}_k^*$, in particular belongs to $\mathcal{S}(k^*)$. Since χ is unramified, χ restricted to $\pi^{-2}\mathcal{O}_k^*$ is a constant function. Therefore

$$l\left(\frac{\mu_1(\pi)\mu_2(\pi)}{q} - \left\{\frac{\mu_1(\pi)}{\sqrt{q}} + \frac{\mu_2(\pi)}{\sqrt{q}}\right\} R_\pi W_{\mu_1,\mu_2} + R_{\pi^2} W_{\mu_1,\mu_2}\right)$$

$$= \int_{k^*}\left(\frac{\mu_1(\pi)\mu_2(\pi)}{q} - \left\{\frac{\mu_1(\pi)}{\sqrt{q}} + \frac{\mu_2(\pi)}{\sqrt{q}}\right\} R_\pi W_{\mu_1,\mu_2} + R_{\pi^2} W_{\mu_1,\mu_2}\right)\chi(x)\frac{dx}{\|x\|} \neq 0.$$

The proof is therefore complete by contradiction. □

Proof of Theorem 1.2 in Case 1. Let V_1 and V_2 be special representations. Since any special representation is a twist of Sp by a character, it suffices to prove the statement about trilinear form, in this case, when V_1 and V_2 are both isomorphic to Sp. Let V_3 be an irreducible admissible representation of $\mathrm{GL}_2(k)$. From Lemma 5.4, existence of a linear form on $\mathrm{Sp} \otimes \mathrm{Sp} \otimes V_3$ is equivalent to the existence of a $\mathrm{GL}_2(k)$-linear form on $[\mathcal{S}(\mathrm{GL}_2(k)/T)/\mathrm{Sp}] \otimes V_3$. From Frobenius reciprocity and Lemma 5.6(a), we know that there is a unique $\mathrm{GL}_2(k)$-linear form on $\mathcal{S}(\mathrm{GL}_2(k)/T) \otimes V_3$. If V_3 is not isomorphic to Sp, then clearly this linear form will be zero on $\mathrm{Sp} \otimes V_3 \subseteq \mathcal{S}(\mathrm{GL}_2(k)/T) \otimes V_3$, giving us a non-zero linear form on $[\mathcal{S}(\mathrm{GL}_2(k)/T)/\mathrm{Sp}] \otimes V_3$. If V_3 is isomorphic to Sp, then by Lemma 5.2, $\mathrm{Sp} \otimes \mathrm{Sp}$ is a direct summand in $\mathcal{S}(\mathrm{GL}_2(k)/T) \otimes \mathrm{Sp}$ and therefore the unique $\mathrm{GL}_2(k)$-linear form on $\mathcal{S}(\mathrm{GL}_2(k)/T) \otimes \mathrm{Sp}$ is the extension of the unique $\mathrm{GL}_2(k)$-linear form on $\mathrm{Sp} \otimes \mathrm{Sp}$ and is, in particular, non-zero on $\mathrm{Sp} \otimes \mathrm{Sp}$, completing the proof that there is a $\mathrm{GL}_2(k)$-linear form on $\mathrm{Sp} \otimes \mathrm{Sp} \otimes V_3$ iff V_3 is not isomorphic to Sp. □

In the construction of a $\mathrm{GL}_2(k)$-linear form on $V_1 \otimes V_2 \otimes V_3$, when at least two of the representations are principal series, we will need to know that certain extension of an admissible $\mathrm{GL}_2(k)$-module by another is split. We study the extension group in the next lemma. We will denote by

$$V \to V_N = \frac{V}{\{n\cdot v - v \mid n \in N, v \in V\}},$$

the standard Jacquet functor, which, as is well-known, takes admissible representations of $\mathrm{GL}_2(k)$ to admissible, i.e., finite dimensional representations of T, cf. [B-Z].

LEMMA 5.7 *For any smooth representation V of $\mathrm{GL}_2(k)$ and character χ of T, we have*

$$\mathrm{Ext}^i_{\mathrm{GL}_2(k)}[V, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi] = \mathrm{Ext}^i_T[V_N, \chi\cdot\delta^{1/2}], \quad \forall i \geqslant 0.$$

Proof. From Frobenius reciprocity,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_2(k)}[V, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi] &= \mathrm{Hom}_B[V, \chi\cdot\delta^{1/2}] \\ &= \mathrm{Hom}_T[V_N, \chi\cdot\delta^{1/2}].\end{aligned}$$

Therefore the lemma follows for $i=0$. Since the Jacquet functor is well known to be exact, it suffices to prove that the Jacquet functor takes a projective object in the category of smooth $\mathrm{GL}_2(k)$-modules to a projective object in the category of smooth T-modules. So let P be a projective $\mathrm{GL}_2(k)$-module, and suppose we have a surjective map of T-modules $E_1 \to E_2 \to 0$. We have to prove that the induced mapping from $\mathrm{Hom}_T[P_N, E_1]$ to $\mathrm{Hom}_T[P_N, E_2]$ is surjective. Since $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}$ is an exact functor, cf. [B-Z], we have a surjection

$$\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(E_1\cdot\delta^{-1/2}) \to \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(E_2\cdot\delta^{-1/2}) \to 0.$$

Therefore from the projectivity of the $\mathrm{GL}_2(k)$-module P, we have the surjection

$$\mathrm{Hom}_{\mathrm{GL}_2(k)}[P, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(E_1\cdot\delta^{-1/2})] \to \mathrm{Hom}_{\mathrm{GL}_2(k)}[P, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(E_1\cdot\delta^{-1/2})] \to 0.$$

Finally, the Frobenius reciprocity gives the desired surjection

$$\mathrm{Hom}_T[P_N, E_1] \to \mathrm{Hom}_T[P_N, E_2] \to 0. \qquad \square$$

Next, let us observe that for a finite dimensional representation E of T,

$\mathrm{Hom}_T[E, \chi\cdot\delta^{1/2}]$ is zero iff $\mathrm{Hom}_T[\chi\cdot\delta^{1/2}, E]$ is zero, and this is so iff $\mathrm{Ext}_T^1[E, \chi\cdot\delta^{1/2}]$ is zero. Therefore, we obtain the following corollary of the previous lemma.

COROLLARY 5.8. *For an admissible representation* V *of* $\mathrm{GL}_2(k)$,

$$\mathrm{Ext}^1_{\mathrm{GL}_2(k)}[V, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi] = 0 \quad \text{iff} \quad \mathrm{Hom}_{\mathrm{GL}_2(k)}[V, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi] = 0. \qquad \square$$

COROLLARY 5.9. *For an admissible representation* V *of* $\mathrm{GL}_2(k)$,

$$\mathrm{Ext}^1_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi, V] = 0 \quad \text{iff} \quad \mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi, V] = 0.$$

Proof. Given a short exact sequence, taking its admissible dual we get another short exact sequence. Since the dual of a (not-necessarily irreducible) principal series is a principal series, this proves the corollary. □

Proof of Theorem 1.2 in Case 2. V_1 and V_2 are irreducible principal series representations, say $V_1 = \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_1$ and $V_2 = \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_2$. Then

$$V_1 \otimes V_2 = \mathrm{Res}_{\mathrm{GL}_2(k)}\mathrm{Ind}_{B\times B}^{\mathrm{GL}_2(k)\times\mathrm{GL}_2(k)}(\chi_1 \times \chi_2).$$

The action of $\mathrm{GL}_2(k)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ has two orbits, one open ($x \neq y \in \mathbf{P}^1 \times \mathbf{P}^1$) and the other closed ($\Delta\mathbf{P}^1 \subseteq \mathbf{P}^1 \times \mathbf{P}^1$). The open orbit can be identified to $\mathrm{GL}_2(k)/T$ with T the diagonal torus. Therefore from Lemma 5.1, we get an exact sequence of $\mathrm{GL}_2(k)$-modules:

$$0 \to \mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2') \to V_1 \otimes V_2 \to \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}) \to 0,$$

where $\chi_2'\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \chi_2\begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$. The shift by $\delta^{1/2}$ in $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2})$ is because of the way principal series is defined. Applying $\mathrm{Hom}_{\mathrm{GL}_2(k)}[-, \tilde{V}_3]$ to this exact sequence, we get:

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}), \tilde{V}_3] \to \mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2, \tilde{V}_3] \\ &\to \mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2'), \tilde{V}_3] \to \mathrm{Ext}^1_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}), \tilde{V}_3] \to \cdots \end{aligned}$$

Since $\mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2, \tilde{V}_3]$ is the space of $\mathrm{GL}_2(k)$-linear forms on $V_1 \otimes V_2 \otimes V_3$, if $\mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}), \tilde{V}_3]$ is non-zero, we have a non-zero $\mathrm{GL}_2(k)$-linear form on $V_1 \otimes V_2 \otimes V_3$. On the other hand if

$$\mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}), \tilde{V}_3]$$

is zero, we know from Corollary 5.9 that $\mathrm{Ext}^1_{\mathrm{GL}_2(k)}[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}), \tilde{V}_3]$ is zero. By Frobenius reciprocity:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2'), \tilde{V}_3] &= \mathrm{Hom}_T[\chi_1\chi_2', \widetilde{V_3|_T}] \\ &= \text{space of linear forms on } V_3 \text{ on which } T \text{ acts} \\ &\quad \text{via the character } \chi_1\chi_2'.\end{aligned}$$

By Lemma 5.6(a), this space is one-dimensional. Therefore the space of $\mathrm{GL}_2(k)$-linear forms on $V_1 \otimes V_2 \otimes V_3$, which equals $\mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2, \tilde{V}_3]$, is non-zero. □

Proof of Theorem 1.2 in Case 3. Suppose that V_1 and V_2 are supercuspidal representations, and $V_3 = \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3$ is a principal series representation. Let ω_i, for $i = 1$ to 3, denote the central character of V_i. Since the smooth dual of the principal series $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3$ is $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3^{-1}$, we have

$$\mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2 \otimes \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3, \mathbf{C}] = \mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2, \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3^{-1}].$$

Therefore by Frobenius reciprocity, $\mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2 \otimes \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3, \mathbf{C}] = \mathrm{Hom}_B[V_1 \otimes V_2, \delta^{1/2}\cdot\chi_3^{-1}]$. From Theorem 5.5(a), the restriction to B of a supercuspidal representation of $\mathrm{GL}_2(k)$ with central character ω is $\mathrm{ind}_{Z\cdot N}^B(\omega \otimes \psi)$, where $Z \cong k^*$ is the centre of $\mathrm{GL}_2(k)$, ψ is a non-trivial additive character of N, and $\omega \otimes \psi$ is the character of $Z\cdot N$ given by $\omega \otimes \psi(z\cdot n) = \omega(z)\cdot\psi(n)$ for $z \in Z$, $n \in N$. Since for any character η of B, $[\mathrm{Ind}_{Z\cdot N}^B(\omega \otimes \psi)] \otimes \eta \cong \mathrm{Ind}_{Z\cdot N}^B[(\omega\cdot\eta|_Z) \otimes \psi]$, and since the smooth dual of $\mathrm{ind}_{Z\cdot N}^B(\omega \otimes \psi)$ is $\mathrm{ind}_{Z\cdot N}^B(\omega^{-1} \otimes \psi)$ we find that $\mathrm{Hom}_B[V_1 \otimes V_2, \delta^{1/2}\cdot\chi_3^{-1}] = \mathrm{Hom}_B[\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\omega_1 \otimes \psi), \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\omega_2^{-1}\omega_3^{-1} \otimes \psi)] \cong \mathbf{C}$, as $\omega_1 = \omega_2^{-1}\omega_3^{-1}$ by the condition on the central characters. □

The next theorem appeared as Theorem 1.3 in the introduction.

THEOREM 5.10. *If V_1, V_2, V_3 are irreducible, unramified principal series with vectors $v_i \in V_i$, invariant under $\mathscr{K} = \mathrm{GL}_2(\mathcal{O}_k)$, then the unique $\mathrm{GL}_2(k)$-invariant form on $V_1 \otimes V_2 \otimes V_3$ takes a non-zero value on $v_1 \otimes v_2 \otimes v_3$.*

Proof. Since an unramified principal series is a twist of an unramified principal series with trivial central character, we will assume that V_1, V_2, V_3 have trivial central character. Identifying the invariant linear form with an element of $\mathrm{Hom}_{\mathrm{GL}_2(k)}[V_1 \otimes V_2, \tilde{V}_3]$, it suffices to prove that $v_1 \otimes v_2$ goes to a non-zero vector in $\tilde{V}_3$, as $v_1 \otimes v_2$ and hence its image is clearly invariant under $\mathscr{K}$. Since $B\mathscr{K} = \mathrm{GL}_2(k)$, the unramified principal series representation V_i can be realised on functions on $\mathscr{K} \cap B\backslash\mathscr{K} = \mathbf{P}^1(k)$. For $V_i = \mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_i$, the spherical vector $v_i \in V_i$ is given by functions f_i on $\mathrm{GL}_2(k)$, defined by $f_i(bk) = \chi_i(b)\cdot\delta^{1/2}(b)$, with $b \in B$ and $k \in \mathscr{K}$, or by the constant function $\mathbf{1}$ on $\mathbf{P}^1(k)$. From this it is clear that in the exact sequence of $\mathrm{GL}_2(k)$-modules

$$0 \to \mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2^{-1}) \to V_1 \otimes V_2 \to \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}) \to 0 \qquad (*)$$

$v_1 \otimes v_2$ goes to a non-zero vector in $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\delta^{1/2})$. Therefore if $\chi_1\chi_2\delta^{1/2} = \chi_3^{\pm 1}$, the unique linear form on $V_1 \otimes V_2 \otimes V_3$, obtained from the surjection $V_1 \otimes V_2 \to \mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\cdot\delta^{1/2}) \to 0$, and the isomorphism of $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}(\chi_1\chi_2\delta^{1/2})$ with $\mathrm{Ind}_B^{\mathrm{GL}_2(k)}\chi_3$, takes non-zero value on $v_1 \otimes v_2 \otimes v_3$. It remains to treat the case when $\chi_1\chi_2\chi_3^{\pm 1}\delta^{1/2} \neq 1$, which we assume in the remainder of the proof (which will be by contradiction). Suppose that the image of $v_1 \otimes v_2$ in $\tilde{V}_3$ is zero. Then the image in $\tilde{V}_3$ of $T_\pi(v_1 \otimes v_2) - [\chi_1(\pi)\cdot\chi_2(\pi) + q\chi_1(\pi)^{-1}\cdot\chi_2(\pi)^{-1}](v_1 \otimes v_2)$, with $T_\pi = \mathscr{K}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)\mathscr{K}$ and $\chi_i(\pi)$ the value of χ_i at $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$, will also be zero.

Claim:

$$T_\pi(v_1 \otimes v_2) - [\chi_1(\pi)\cdot\chi_2(\pi) + q\chi_1(\pi)^{-1}\cdot\chi_2(\pi)^{-1}](v_1 \otimes v_2)$$

lies in the $\mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2^{-1})$ part of $V_1 \otimes V_2$ in the exact sequence (*), and in fact this function is non-zero, with constant value, on a *single* orbit of $\mathrm{GL}_2(\mathscr{O}_k)$ on $T\backslash\mathrm{GL}_2(k)$. Assuming the claim, we complete the proof of the theorem. In the Frobenius reciprocity isomorphism,

$$\mathrm{Hom}_T[\chi_1\chi_2^{-1}, \widetilde{V_3|_T}] \cong \mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2^{-1}), \tilde{V}_3],$$

an element $l \in \mathrm{Hom}_T[\chi_1\chi_2^{-1}, \widetilde{V_3|_T}]$ corresponding to a linear form l on V_3 such that $l(t^{-1}v) = \chi_1\chi_2^{-1}(t)v$ goes to $\phi_l \in \mathrm{Hom}_{\mathrm{GL}_2(k)}[\mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2^{-1}), V_3]$ given by

$$\phi_l(f)(v) = \int_{T\backslash\mathrm{GL}_2(k)} f(g)l(gv), \qquad \text{for } f \in \mathrm{ind}_T^{\mathrm{GL}_2(k)}(\chi_1\chi_2^{-1}), \quad \text{and } v \in V_3.$$

Since $T_\pi(v_1 \otimes v_2) - [\chi_1(\pi)\cdot\chi_2(\pi) + q\chi_1(\pi)^{-1}\cdot\chi_2(\pi)^{-1}](v_1 \otimes v_2)$ is non-zero, with constant value, on a single $\mathrm{GL}_2(\mathscr{O}_k)$-orbit, and since $gv_3 = v_3$ for $g \in \mathrm{GL}_2(\mathscr{O}_k)$, and since $l(v_3) \neq 0$ by Lemma 5.6(b), it follows that

$$\phi_l\{T_\pi(v_1 \otimes v_2) - [\chi_1(\pi)\cdot\chi_2(\pi) + q\chi_1(\pi)^{-1}\cdot\chi_2(\pi)^{-1}](v_1 \otimes v_2)\}(v_3) \neq 0.$$

This completes the proof of the theorem by contradiction.

Coming back to the claim, we need to calculate the action of

$$T_\pi = \mathscr{K}\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\mathscr{K} = \bigcup_{i \in \mathscr{O}/\pi}\begin{pmatrix}\pi & i\\ 0 & 1\end{pmatrix}\mathscr{K} \cup \begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\mathscr{K},$$

on the constant function **1** on $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$. For this purpose we need to write the product of a matrix in $\mathscr{K}$ with $\begin{pmatrix}\pi & i\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}$ as an element of the upper triangular matrix times an element of $\mathscr{K}$. We simply write the result. For $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathscr{K}$, we have

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\pi & i\\ 0 & 1\end{pmatrix} = \begin{cases} \begin{pmatrix}\dfrac{\pi(ad-bc)}{d+ic} & b+ia\\ 0 & d+ic\end{pmatrix}\begin{pmatrix}1 & 0\\ \dfrac{c\pi}{d+ic} & 1\end{pmatrix} & \text{if } d+ic \text{ is a unit}\\[2ex] \begin{pmatrix}\dfrac{bc-ad}{c} & \pi a\\ 0 & \pi c\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & \dfrac{d+ic}{\pi c}\end{pmatrix} & \text{otherwise.}\end{cases}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix} = \begin{cases} \begin{pmatrix}\dfrac{\pi(bc-ad)}{c} & a\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & \dfrac{\pi d}{c}\end{pmatrix} & \text{if } \pi^2 \nmid c\\[2ex] \begin{pmatrix}\dfrac{ad-bc}{d} & \pi b\\ 0 & \pi d\end{pmatrix}\begin{pmatrix}1 & 0\\ \dfrac{c}{\pi d} & 1\end{pmatrix} & \text{otherwise.}\end{cases}$$

After some calculation we find that the function

$$-\frac{T_\pi - [\chi_1(\pi)\chi_2(\pi) + q\chi_1^{-1}(\pi)\chi_2^{-1}(\pi)]}{(q^{-1/2}\chi_1(\pi) - q^{1/2}\chi_1(\pi)^{-1})(q^{-1/2}\chi_2(\pi) - q^{1/2}\chi_2(\pi)^{-1})}\mathbf{1}$$

(since for an irreducible principal series $V_{(\chi,\chi^{-1})}$, $\chi(\pi) \neq \pm\sqrt{q}$, the denominator is non-zero) on $\mathbf{P}^1(k) \times \mathbf{P}^1(k)$ equals 1 on $(x, y) \in \mathbf{P}^1(k) \times \mathbf{P}^1(k)$ whose reduction modulo π does not lie on the diagonal of $\mathbf{P}^1(\mathbf{F}_q) \times \mathbf{P}^1(\mathbf{F}_q)$, and 0 otherwise. It is easy to see that the set of elements $(x, y) \in \mathbf{P}^1(k) \times \mathbf{P}^1(k)$ whose reduction modulo π does not lie on the diagonal of $\mathbf{P}^1(\mathbf{F}_q) \times \mathbf{P}^1(\mathbf{F}_q)$ is a single $\mathrm{GL}_2(\mathcal{O}_k)$-orbit, substantiating the claim made before. □

6. Trilinear forms II

The proof of Theorem 1.2 in the case when two of the representations are discrete series, at least one of which is supercuspidal, depends on realizing a supercuspidal representation of $\mathrm{GL}_2(k)$ as an induced representation from a finite dimensional representation of a maximal compact-modulo-centre subgroup of $\mathrm{GL}_2(k)$ and certain character identities. Before stating these character identities, we recall that there are two conjugacy classes of maximal compact-modulo-centre subgroups of

$\mathrm{GL}_2(k)$ (these correspond to conjugacy classes of maximal compact subgroups of $\mathrm{PGL}_2(k)$); one of the conjugacy classes is represented by $k^*\cdot \mathrm{GL}_2(\mathcal{O}_k)$ and the other one by $J = k^*\cdot\Gamma_0(\pi) \cup k^*\cdot\left(\begin{smallmatrix}0 & 1\\ \pi & 0\end{smallmatrix}\right)\Gamma_0(\pi)$, where for $n \geqslant 0$, $\Gamma_0(\pi^n)$ denotes the group

$$\Gamma_0(\pi^n) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_k) : c \equiv 0 \pmod{\pi^n}\right\}.$$

Define a decreasing filtration $\mathrm{GL}_2(\mathcal{O}_k)(n)$, for $n \geqslant 1$, on $k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$ by

$$\mathrm{GL}_2(\mathcal{O}_k)(n) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_k) \,\middle|\, \begin{matrix} a, d \equiv 1 \pmod{\pi^n} \ \text{ and}\\ b, c \equiv 0 \pmod{\pi^n} \end{matrix}\right\},$$

and a decreasing filtration $J(n)$, for $n \geqslant 1$, on J by

$$J(n) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_k) \,\middle|\, \begin{matrix} a, d \equiv 1 \pmod{\pi^n}, \quad b \equiv 0 \pmod{\pi^n}\\ \text{and } c \equiv 0 \pmod{\pi^{n+1}} \end{matrix}\right\}.$$

We will use $\mathcal{K}$ to denote either of the conjugacy class of maximal compact-modulo-centre subgroup, and $\mathcal{K}(n)$ to denote the filtration defined above on $\mathcal{K}$.

DEFINITION. A finite dimensional irreducible representation W of $\mathcal{K}/\mathcal{K}(n)$, for $n \geqslant 1$, is called *very cuspidal* of *level* n if the representation W does not contain the trivial character of the subgroup $\left(\begin{smallmatrix}1 & \pi^{n-1}\mathcal{O}_k\\ 0 & 1\end{smallmatrix}\right) \subset \mathcal{K}/\mathcal{K}(n)$.

We now recall the definition of the *conductor* of a representation of $\mathrm{GL}_2(k)$. The conductor of a representation Π of $\mathrm{GL}_2(k)$, with central character ω, is the smallest integer n such that $\Pi\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) v = \omega(a)v$ for some $v \neq 0$ in the representation space of Π and all matrices $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(\pi^n)$. The conductor of the principal series $V_{(\psi_1,\psi_2)}$ is $\mathrm{cond}\psi_1 + \mathrm{cond}\psi_2$.

The *level* of a representation V of D_k^* is the minimum n such that the representation V is trivial on $D_k^*(n)$, where for $n = 0$, $D_k^*(n) = \mathcal{O}_{D_k}^*$ with $\mathcal{O}_{D_k}$ the ring of integers in D_k, and for $n > 0$, $D_k^*(n) = \{x \in \mathcal{O}_{D_k} | \pi_{D_k}^n \text{ divides } x - 1\}$ with π_{D_k} a uniformizing parameter of $\mathcal{O}_{D_k}$. The *conductor* of an irreducible representation V of D_k^* with level n is defined to be $n + 1$.

For a representation V of $\mathrm{GL}_2(k)$, and the representation V' of D_k^* associated to V by the Jacquet-Langlands correspondence, we have $\mathrm{cond}\, V = \mathrm{cond}\, V'$.

The *minimal conductor* of a representation V of $\mathrm{GL}_2(k)$, or of D_k^*, is the minimum of the conductors of the representations $V \otimes \chi$, where χ runs over the characters of k^*. The representation V will be called *minimal* if $\mathrm{cond}\, V \leqslant \mathrm{cond}\, V \otimes \chi$, for χ any character of k^*.

The following theorem, in this precise form, is due to Kutzko, Theorem 4.3 in [Ku1].

THEOREM 6.1. *There exists a bijective correspondence, obtained by compact induction, between very cuspidal representations of a set of conjugacy classes of maximal compact-modulo-centre subgroups of* $\mathrm{GL}_2(k)$ *and minimal irreducible supercuspidal representations of* $\mathrm{GL}_2(k)$. *A minimal representation of* $\mathrm{GL}_2(k)$ *of even conductor* $2n$ *is compactly induced from a very cuspidal representation of* $k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$ *of level n, and a minimal representation of odd conductor* $2n+1$ *is compactly induced from a very cuspidal representation of J of level n.* □

For a function f on D_k^*, invariant under conjugation, we define a class function $\hat{f}$ on $\mathrm{GL}_2(k)$ by defining the value of the function $\hat{f}$ on a regular elliptic conjugacy class in $\mathrm{GL}_2(k)$ to be the value of f on the corresponding conjugacy class in D_k^*, and defining $\hat{f}$ to be zero on all the other conjugacy classes of $\mathrm{GL}_2(k)$.

We recall from [J-L] Theorem 7.7, that the character of an irreducible, admissible representation V of $\mathrm{GL}_2(k)$, in the sense of distributions, is represented by a locally-L^1 function, locally constant on the set of regular semi-simple elements of $\mathrm{GL}_2(k)$. We let $\mathrm{ch}(V)$ denote this function on the regular semi-simple elements of $\mathrm{GL}_2(k)$, and undefined at the other conjugacy classes.

In the following lemma, the character of a supercuspidal representation of $\mathrm{GL}_2(k)$ on regular elliptic elements is obtained from [J-L] Proposition 15.5, and on split elements, from Proposition 5.5 and Proposition 6.11 of [Ku3].

LEMMA 6.2. *For a minimal supercuspidal representation V of* $\mathrm{GL}_2(k)$ *of conductor* $2n$ *or* $2n+1$, *and of central character* ω, *the distribution* $\mathrm{ch}(V)+\widehat{\mathrm{ch}}(V')$ *on* $\mathrm{GL}_2(k)$ *is represented by the class function*

$$\mathrm{ch}(V)+\widehat{\mathrm{ch}}(V')=\omega(\alpha)\left[\frac{2\|\beta\|}{\|\alpha-\beta\|}-\dim(V')\right]\quad at\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\quad for\ v\left(\frac{\alpha}{\beta}-1\right)\geqslant n,$$

$$=0\quad at\ all\ the\ other\ conjugacy\ classes\ in\ \mathrm{GL}_2(k).$$ □

Before stating the next lemma, we have to introduce the concept of $\mathcal{K}$*-generic* element, due to [Ku3]. For $\mathcal{K}=k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$, $K_u^*\subset k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$, for K_u the quadratic unramified extension of k. Any element of $k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$ conjugate by an element of $k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$ to an element of $K_u^*-k^*$ will be called $k^*\cdot\mathrm{GL}_2(\mathcal{O}_k)$-generic. Similarly, $K_r^*\subset J$, for K_r any separable quadratic ramified extension of k and any element of J conjugate by an element of J to an element of $K_r^*-k^*$ will be called J-generic.

The following lemma is from [Ku3] Propositions 5.5 and 6.11.

LEMMA 6.3. *For a supercuspidal representation V induced from a very cuspidal representation W of level n, of a maximal compact-modulo-centre subgroup* $\mathcal{K}$, *we have the following character identity:*

$$\mathrm{ch}(W)(x) = \mathrm{ch}(V)(x) \text{ if } x \notin k^*\mathcal{K}(n), \quad \text{but is } \mathcal{K}\text{-generic},$$

$$\mathrm{ch}(W)(\lambda) = \omega(\lambda)\cdot[q^n - q^{n-1}] \quad \text{if } \lambda \in k^*,$$

$$\mathrm{ch}(W)\left[\lambda\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix}\right] = -\omega(\lambda)\cdot q^{n-1} \quad \text{if } \lambda \in k^*,$$

$$\mathrm{ch}(W)(x) = 0 \quad \text{otherwise.}$$ □

We will also need to know the character of a principal series. The following lemma is Proposition 7.6 in [J-L].

LEMMA 6.4. *The character of the (not necessarily irreducible) principal series $V_{(\psi_1,\psi_2)}$ is concentrated on split elements, and its value at a split element $x \in \mathrm{GL}_2(k)$ with eigenvalues α, β is given by*

$$\mathrm{ch}\, V_{(\psi_1,\psi_2)}(x) = [\psi_1(\alpha)\psi_2(\beta) + \psi_1(\beta)\psi_2(\alpha)]\, \frac{\|\alpha\beta\|^{1/2}}{\|\alpha - \beta\|}.$$ □

The following lemma was proved by Gelfand and Graev in odd residue characteristic, and can be deduced from Section IV of [Ho] in general (for a proof, see also [Ca] Proposition 6.5).

LEMMA 6.5. *The dimension of a finite dimensional, irreducible representation of D_k^* depends only on the minimal conductor of the representation. If the minimal conductor is $2n + 1 > 1$, then the dimension of the representation is $q^{n-1}(q + 1)$, and if the minimal conductor is $2n > 0$, then the dimension of the representation is $2q^{n-1}$. For representations of minimal conductor 1, the dimension is 1.* □

Proof of Theorem 1.2 in case 4 (unequal conductor case)

Since it clearly suffices to prove Theorem 1.2 of the introduction under the additional hypothesis that V_1 and V_2 are minimal, we will assume in the rest of the section that V_1 and V_2 are minimal representations of the discrete series with V_1 supercuspidal. As we have already treated case 1 of Theorem 1.2, we will assume, possibly after renumbering, that if V_3 is discrete series, it is supercuspidal. Denote the central characters of V_i by ω_i, for $i = 1$ to 3. In this subsection we will assume that cond $V_1 >$ cond V_2; equal conductor case will be taken up in the next subsection. Write $V_1 = \mathrm{ind}_{\mathcal{K}}^{\mathrm{GL}_2(k)} W_1$, with W_1 a very cuspidal representation of $\mathcal{K}$ of level n. Since cond $V_1 >$ cond V_2, and therefore cond $V'_1 >$ cond V'_2, it is clear that all the representations of D_k^* appearing in $V'_1 \otimes V'_2$ have the same conductor as that of V'_1, and are minimal. From Lemma 6.5 it follows that

$V'_1 \otimes V'_2$ is a sum of $\dim(V'_2)$ many irreducible representations of D_k^*. Let $(V'_1 \otimes V'_2)'$ denote the representation of $\mathrm{GL}_2(k)$ associated to the representation $V'_1 \otimes V'_2$ of D_k^* by the Jacquet-Langlands correspondence (extended by linearity, to not necessarily irreducible representations), and write

$$(V'_1 \otimes V'_2)' = \mathrm{ind}_{\mathscr{K}}^{\mathrm{GL}_2(k)} W_{12}.$$

We have $V_1 \otimes V_2 = \mathrm{ind}_{\mathscr{K}}^{\mathrm{GL}_2(k)}[W_1 \otimes V_2|_{\mathscr{K}}]$ and therefore

$$V_1 \otimes V_2 \oplus (V'_1 \otimes V'_2)' = \mathrm{ind}_{\mathscr{K}}^{\mathrm{GL}_2(k)}[W_1 \otimes V_2|_{\mathscr{K}} \otimes W_{12}].$$

PROPOSITION 6.6. *With the notation as above, the representation* $[W_1 \otimes V_2|_{\mathscr{K}} \oplus W_{12}]$ *of* $\mathscr{K}$, *is the same as the representation* $W_1 \otimes V|_{\mathscr{K}}$ *of* $\mathscr{K}$, *where* V *is any principal series representation of* $\mathrm{GL}_2(k)$ *with the same central character as* V_2, *and* cond $V \leqslant$ cond V_2.

Proof. Observe that the tensor product of a finite dimensional representation of $\mathscr{K}$ with an admissible representation of $\mathscr{K}$, is an admissible representation, and that the character, in the sense of distributions, of the tensor product, is the product of characters. Also, observe that, two admissible representations are isomorphic iff their characters (in the sense of distributions) are the same. Therefore it suffices to prove that the character of $[W_1 \otimes V_2|_{\mathscr{K}} \oplus W_{12}]$ is the same as the character of $W_1 \otimes V|_{\mathscr{K}}$. As the character of an irreducible admissible representation of $\mathrm{GL}_2(k)$ is represented by a locally-L^1 function, locally constant on the set of regular semi-simple elements, it suffices to prove that the characters of these two representations of $\mathscr{K}$, on the set of regular semi-simple elements of $\mathrm{GL}_2(k)$ contained in $\mathscr{K}$, are the same. We do this now.

Case 1. x is $\mathscr{K}$-generic but does not belong to $k^* \cdot \mathscr{K}(n)$. From Lemmas 6.2 and 6.3, $\mathrm{ch}(W_1) = -\widehat{\mathrm{ch}}(V'_1)$, and $\mathrm{ch}(W_{12}) = -\widehat{\mathrm{ch}}(V'_1) \cdot \widehat{\mathrm{ch}}(V'_2)$ at such elements. Therefore from Lemma 6.2, $\mathrm{ch}(W_1) \cdot \mathrm{ch}(V_2) + \mathrm{ch}(W_{12}) = 0$, i.e., $\mathrm{ch}[W_1 \otimes V_2|_{\mathscr{K}} \oplus W_{12}] = 0$. Since the character of a principal series is concentrated on split elements, the character of $W_1 \otimes V|_{\mathscr{K}}$ at such an element is also 0.

Case 2. x is an elliptic element and is represented by $\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix}$ mod $\mathscr{K}(n)$. From Lemma 6.3, $\mathrm{ch}(W_1) = -q^{n-1}$ and $\mathrm{ch}(W_{12}) = -q^{n-1} \cdot \dim(V'_2)$ at such an element. Since cond $V_1 >$ cond V_2, $\mathrm{ch}(V_2) = -\widehat{\mathrm{ch}}(V'_2) = -\dim(V'_2)$. It follows that $[\mathrm{ch}(W_1) \cdot \mathrm{ch}(V_2) + \mathrm{ch}(W_{12})] = -q^{n-1}[\mathrm{ch}(V_2) + \dim V'_2] = 0$. Again, character of $W_1 \otimes V|_{\mathscr{K}}$ at such an element is also 0.

Case 3. $x \in \mathscr{K}$ is diagonalisable in $GL_2(k)$ and is represented by $\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix}$ modulo $\mathscr{K}(n)$. It is easy to see that the eigenvalues, say α, β of such an element belong to $\mathcal{O}_k$ and are congruent to 1 modulo π^n. As the conductor of V_2 is $\leqslant 2n$, the central character of V_2, and therefore also of V, is $\leqslant n$. It follows from

Lemma 6.2 that $\mathrm{ch}(V_2)(x) = \frac{2}{\|\alpha-\beta\|} - \dim V'_2$, and from Lemma 6.4 that $\mathrm{ch}(V)(x) = \frac{2}{\|\alpha-\beta\|}$. From Lemma 6.3, $\mathrm{ch}(W_1)(x) = -q^{n-1}$ and $\mathrm{ch}(W_{12})(x) = -q^{n-1}\cdot\dim V'_2$. It follows that

$$\begin{aligned}\mathrm{ch}(W_1)\cdot\mathrm{ch}(V_2) + \mathrm{ch}(W_{12}) &= -q^{n-1}[\mathrm{ch}(V_2) + \dim(V'_2)] \\ &= -q^{n-1}\cdot\frac{2}{\|\alpha-\beta\|} \\ &= \mathrm{ch}(V_1)\cdot\mathrm{ch}(V)\end{aligned}$$

Case 4. x does not belong to $k^*\cdot\mathscr{K}(n)$, and is also not contained in cases 1, 2 or 3. From Lemma 6.3, characters W_1 and W_{12} are 0 at such an element. Therefore $[\mathrm{ch}(W_1)\cdot\mathrm{ch}(V_2) + \mathrm{ch}(W_{12})] = 0$. Since $\mathrm{ch}(W_1) = 0$, the character of $W_1 \otimes V|_{\mathscr{K}}$ at such an element is also 0.

Case 5. $x \in \mathscr{K}(n)$. In this case $\mathrm{ch}W_1 = \dim W_1$, and $\mathrm{ch}W_{12} = \dim W_{12} = \dim W_1\cdot\dim V'_2$. Therefore $[\mathrm{ch}(W_1)\cdot\mathrm{ch}(V_2) + \mathrm{ch}(W_{12})] = \dim W_1\cdot[\mathrm{ch}(V_2) + \dim(V'_2)]$. From Lemma 6.2 if V_2 is supercuspidal, and Lemma 6.5 if V_2 is the special representation Sp, we know that for $x \in \mathscr{K}(n)$, $[\mathrm{ch}(V_2) + \dim(V'_2)]$ is supported only on split semi-simple elements of $\mathscr{K}(n)$, the value being $2/\|\alpha-\beta\|$ at $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$. From Lemma 6.4, $\mathrm{ch}(V)$ at such an element is also $2/\|\alpha-\beta\|$. □

Coming back to the proof of Theorem 1.2 in the unequal conductor case of case 4, let us first observe that we can always find a principal series representation V as in Proposition 6.6. Therefore it follows from Proposition 6.6 that for a supercuspidal representation V_1 and a discrete series representation V_2 with $\mathrm{cond}\, V_1 > \mathrm{cond}\, V_2$, $V_1 \otimes V_2 \oplus (V'_1 \otimes V'_2)' = V_1 \otimes V$. But we know from the proof of Theorem 1.2 in case 3 if V_3 is supercuspidal, and from the proof of Theorem 1.2 in case 1 if V_1 is a principal series, that the space of $\mathrm{GL}_2(k)$-invariant forms on $[V_1 \otimes V] \otimes V_3$ is one-dimensional. Therefore the space of $\mathrm{GL}_2(k)$-invariant forms on $[V_1 \otimes V_2 \oplus (V'_1 \otimes V'_2)'] \otimes V_3$ is one-dimensional, i.e., either there is a $\mathrm{GL}_2(k)$-invariant form on $V_1 \otimes V_2 \otimes V_3$ or there is a $\mathrm{GL}_2(k)$-invariant form on $(V'_1 \otimes V'_2)' \otimes V_3$, i.e., either there is a $\mathrm{GL}_2(k)$-invariant form on $V_1 \otimes V_2 \otimes V_3$ or there is a D_k^*-invariant form on $V'_1 \otimes V'_2 \otimes V'_3$, proving Theorem 1.2 in the unequal conductor case of case 4. □

Proof of Theorem 1.2 in case 4 (equal conductor case)

In this subsection we consider the case when V_1 and V_2 are supercuspidal representations of equal conductor (assumed, as before, to be minimal). As the other cases can be reduced to one of the cases already considered, it suffices to treat the case when V_3 is a supercuspidal representation of conductor greater than or equal to the conductor of V_1 (and V_2). We split the proof into two cases depending on whether the conductor of V_3 is equal to the conductor of V_1 (and V_2) or not.

Case A. All the V_i have equal conductor, in which case we can also assume V_3 to be minimal. Write $V_i = \mathrm{ind}_{\mathscr{K}}^{\mathscr{G}} W_i$, with W_i, for $i = 1$ to 3, very cuspidal representations of $\mathscr{K}$ of level n.

PROPOSITION 6.7. *The dimension, $m(W_1 \otimes W_2 \otimes W_3)$, of $\mathscr{K}$-linear form on $W_1 \otimes W_2 \otimes W_3$ plus the dimension, $m(V'_1 \otimes V'_2 \otimes V'_3)$, of D_k^*-linear form on $V'_1 \otimes V'_2 \otimes V'_3$ is one.*

Proof. We carry out the proof of this proposition for $\mathscr{K} = J$; the proof in the other case is exactly similar. Since the dimension of $\mathscr{G}$-invariant forms on a representation U of a finite group $\mathscr{G}$, with character $\mathrm{ch}U(x)$, for $x \in \mathscr{G}$, is $\frac{1}{\mathscr{G}} \cdot \Sigma_x \mathrm{ch}U(x)$,

$$m(W_1 \otimes W_2 \otimes W_3) = \frac{1}{[J: k^* \cdot J(n)]} \sum_{x \in J/k^* J(n)} \mathrm{ch}W_1 \cdot \mathrm{ch}W_2 \cdot \mathrm{ch}W_3(x).$$

Since $\mathrm{ch}W_i(x)$ is non-zero only if either x is J-generic, or if x is conjugate to the unipotent element $\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix}$, or if x is the trivial element of $J/k^* J(n)$, we have to do the above summation at only such elements. For a $\mathscr{K}$-generic element x in $J/k^* \cdot J(n)$, let Z_x denote the cardinality of the centraliser of x in $J/k^* \cdot J(n)$. Since the centraliser of $\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix}$ is easily seen to be of cardinality q^{3n-1}, and since $[J: k^* J(n)] = 2(q-1)q^{3n-1}$ we have

$$m(W_1 \otimes W_2 \otimes W_3) = \frac{1}{2} \sum_K \sum_{x \in \frac{K - k^* K(2n)}{k^* K(2n)}} \frac{1}{Z_x} \mathrm{ch}W_1 \cdot \mathrm{ch}W_2 \cdot \mathrm{ch}W_3(x) + $$

$$+ \frac{(-q^{n-1})^3}{q^{3n-1}} + \frac{q^{3(n-1)}(q-1)^3}{2q^{3n-1}(q-1)}.$$

Here K runs over all separable, ramified quadratic extensions of k.

Similarly, let Z'_x denote the cardinality of the centraliser of x in $D_k^*/k^* D_k^*(2n)$. From Satz 2 in [Ko] and Lemma 3.5 in [Ca], where they calculate the centraliser of an element $x \in D_k^*/k^* D_k^*(2n)$, and a J-generic element x in $J/k^* \cdot J(n)$ respectively, it follows that $Z_x = Z'_x$. Since $\mathrm{ch}W_i(x) = -\mathrm{ch}V'_i(x)$, for $x \in K^* - k^* K(2n)$ with K as above, and $\mathrm{ch}W_i(x) = 0$ for x a non-trivial element of $D_k^*/k^* D_k^*(2n)$ and not coming from a separable ramified quadratic extension of k, it follows that

$$m(V'_1 \otimes V'_2 \otimes V'_3) = -\frac{1}{2}\sum_K \sum_{x \in \frac{K - k^*K(2n)}{k^*K(2n)}} \frac{1}{Z_x} \operatorname{ch} W_1 \cdot \operatorname{ch} W_2 \cdot \operatorname{ch} W_3(x) + $$
$$+ \frac{q^{3(n-1)}(q+1)^3}{2q^{3n-1}(q+1)}$$

Therefore

$$m(W_1 \otimes W_2 \otimes W_3) + m(V'_1 \otimes V'_2 \otimes V'_3) = 1. \qquad \square$$

Since

$$\begin{aligned} V_1 \otimes V_2 &= \operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes V_2|_{\mathscr{K}}] \\ &= \operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes W_2] \oplus \operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes (V_2|_{\mathscr{K}} - W_2)], \end{aligned}$$

the proof of Theorem 1.2, in this case, will be completed if we could show that $\operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes (V_2|_{\mathscr{K}} - W_2)] \otimes V_3$ has no $\mathrm{GL}_2(k)$-invariant linear form. This will be done in two steps. The first step consists in showing that the representation $W_1 \otimes (V_2|_{\mathscr{K}} - W_2)$ of $\mathscr{K}$ is the same as the representation $W_1 \otimes (V|_{\mathscr{K}} - W)$ of $\mathscr{K}$, where $V = \operatorname{Ind}_B^G \chi$ is a principal series representation as in Proposition 6.6, i.e., with the same central character as V_2 and with $\operatorname{cond} V \leqslant \operatorname{cond} V_2$, and $W = \operatorname{Ind}_{B_{\mathscr{K}/\mathscr{K}(n)}}^{\mathscr{K}/\mathscr{K}(n)} \chi$, where $B_{\mathscr{K}/\mathscr{K}(n)}$ denotes the group of upper triangular matrices in $\mathscr{K}/\mathscr{K}(n)$. This is done by looking at the characters as in the proof of Proposition 6.6, we omit the simple calculation. The second step consists in showing that there exists a $\mathrm{GL}_2(k)$-invariant linear form on $\operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes W] \otimes V_3$, and therefore by Theorem 1.1, for $\mathrm{GL}_2(k)$, there does not exist a $\mathrm{GL}_2(k)$-invariant linear form on

$$\operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes (V|_{\mathscr{K}} - W)] \otimes V_3 = \operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes (V_2|_{\mathscr{K}} - W_2)] \otimes V_3.$$

We now carry out the proof of the second step, i.e., show that there exists a $\mathrm{GL}_2(k)$-invariant linear form on $\operatorname{ind}_{\mathscr{K}}^G[W_1 \otimes W] \otimes V_3$. Since

$$W_1 \otimes W = W_1 \otimes \operatorname{Ind}_{B_{\mathscr{K}/\mathscr{K}(n)}}^{\mathscr{K}/\mathscr{K}(n)} \chi = \operatorname{Ind}_{B_{\mathscr{K}/\mathscr{K}(n)}}^{\mathscr{K}/\mathscr{K}(n)}[W_1|_{B_{\mathscr{K}/\mathscr{K}(n)}} \otimes \chi],$$

and since W_1 is very cuspidal, $W_1|_{B_{\mathscr{K}/\mathscr{K}(n)}} = \operatorname{Ind}_{k^*N}^{B_{\mathscr{K}/\mathscr{K}(n)}}[\omega_1 \otimes \psi]$ where N is the group of upper triangular unipotent matrices with entries in $\mathcal{O}_k$, ψ is a character of $N/N(n)$, non-trivial on $N(n-1)/N(n)$, and $\omega_1 \otimes \psi$ is the character of k^*N

defined by $\omega_1 \otimes \psi(xn) = \omega_1(x)\psi(n)$, for $x \in k^*, n \in N$. It follows that

$$W_1 \otimes W = \mathrm{Ind}_{k^*N}^{\mathscr{K}/\mathscr{K}(n)}[\omega_1\omega_2 \otimes \psi].$$

Since W_3 is very cuspidal of level n, and of central character ω_3 such that $\omega_1\omega_2\omega_3 = 1$, it is clear that there exists a $\mathscr{K}$-invariant linear form on $W_1 \otimes W \otimes W_3 = \mathrm{Ind}_{k^*N}^{\mathscr{K}/\mathscr{K}(n)}[\omega_1\omega_2 \otimes \psi] \otimes W_3$. Therefore by Frobenius reciprocity there exists a $\mathrm{GL}_2(k)$-invariant linear form on $\mathrm{ind}_{\mathscr{K}}^{G}[W_1 \otimes W] \otimes V_3$.

Case B. The conductor of V_3 is larger than the conductor of V_1 (and V_2). In this case it is clear that there is no D_k^*-invariant trilinear form on $V_1' \otimes V_2' \otimes V_3'$. So we need to prove that there is a $\mathrm{GL}_2(k)$-invariant form on $V_1 \otimes V_2 \otimes V_3$. With the notation of case A, it is clear that $\mathrm{ind}_{\mathscr{K}}^{G}[W_1 \otimes W] \otimes V_3$ has no $\mathrm{GL}_2(k)$-invariant linear form, because otherwise from Frobenius reciprocity V_3 will be of conductor $\leqslant 2n + 1$, in contradiction to our assumption on the conductor of V_3. Therefore by the proof of Theorem 1.2 in case 3, there exists a $\mathrm{GL}_2(k)$-invariant linear form on

$$\mathrm{ind}_{\mathscr{K}}^{G}[W_1 \otimes (V|_{\mathscr{K}} - W)] \otimes V_3 = \mathrm{ind}_{\mathscr{K}}^{G}[W_1 \otimes (V_2|_{\mathscr{K}} - W_2)] \otimes V_3. \qquad \square$$

7. Tensor product of representations of division algebra

The aim of this section is to recall the character formulae for the representations of D_k^* for D_k the quaternion division algebra over k, and to obtain some information about the tensor products of representations of D_k^* that will be needed in the next section. We will assume in this section that the residue characteristic of k is $\neq 2$.

Representations of D_k^*, of dimension >1, are parametrised by characters of K^*, for K a quadratic extension of k, which do not factor through the norm map $K^* \xrightarrow{N} k^*$. We will use V_χ to denote the representation of D_k^* corresponding to a character χ of K^*. The central character of V_χ is $\chi|_{k^*} \cdot \omega_{K/k}$ where $\omega_{K/k}$ is the unique non-trivial character of k^*/NK^*. We let $x \to \bar{x}$ denote the non-trivial automorphism of K/k, and for a character χ of K^*, let $\bar{\chi}$ denote the character $\bar{\chi}(x) = \chi(\bar{x})$, and *not* the complex conjugate character. Representations of D_k^* parametrised by characters of the unramified quadratic extension of k will be called unramified representations, and representations parametrised by characters of a ramified extension will be called ramified representations.

We will assume that the representations of D_k^* of dimension >1 have central character of conductor $\leqslant 1$. This can be achieved by twisting by a character. It can be checked that such representations are minimal.

The following character information has been obtained from the paper of Sally and Shalika [S-S] and the character identity between finite dimensional irreducible representations of D_k^* and the discrete series representations of $GL_2(k)$, see also [Si1] pages 50 and 51, where he tabulates the results for $PGL_2(k)$.

Unramified Representations. For K/k, the quadratic unramified extension and χ a character of K^* of conductor n, the conductor of the representation V_χ is $2n$ and has dimension $2 \cdot q^{n-1}$. The character of V_χ at an element of $L^* - k^* \cdot L^*(2n-1)$, for L a quadratic ramified extension of k, is zero. The character of V_χ at an element of $K^* - k^* \cdot K^*(2n-1)$ is given by

$$\mathrm{ch}(V_\chi)(x) = (-1)^{n+1} \frac{[\chi(x) + \chi(\bar{x})]}{\|x - \bar{x}\|_K^{1/2}} q^{-v(x)} (-1)^{v(x-\bar{x})}.$$

Ramified Representations. For $L_1 = k(\sqrt{\pi})$ and χ a character of L_1^* of conductor $2n$, the conductor of the representation V_χ is $2n+1$, and has dimension $q^n + q^{n-1}$. The character of V_χ at an element of $K^* - k^* \cdot K^*(2n)$, for K the quadratic unramified extension of k, is zero. The character of V_χ at an element of $L_2^* - k^* \cdot L_2^*(2n-2)$, for $L_2 = k(\sqrt{\xi\pi})$, a quadratic ramified extension of k different from L_1, is zero (equivalently, the condition under which a character η of L_2^* appears in V_χ depends only on η restricted to $k^* \cdot L_2^*(2n-2)$), and its value at an element x of $L_2^*(2n-2)$ is given by

$$\mathrm{ch}(V_\chi)(x) = -q^{n-1} \sum_{t \in \mathbf{F}_q} \chi(1 + \pi^{n-1}\sqrt{\pi}t) \cdot \omega(t^2\xi - y^2) \quad \text{for } x = 1 + \pi^{n-1}\sqrt{\xi\pi}y,$$

where ω is the unique quadratic character of $\mathbf{F}_q^*$.

The character of V_χ at an element of $L_1^* - k^* \cdot L_1^*(2n)$ is given by

$$\mathrm{ch}(V_\chi)(x) = -G_\chi \cdot \omega_{K/k}\left(\frac{x - \bar{x}}{\sqrt{\pi}}\right) \cdot \frac{[\chi(x) + \omega(-1)\chi(\bar{x})]}{\|x - \bar{x}\|_K^{1/2}} q^{-v(x)/2}$$

$$\text{for } x \notin L_1^*(2n-1) \cdot k^*,$$

$$\mathrm{ch}(V_\chi)(x) = -q^{n-1} \sum_{\mathbf{F}_q \ni t \neq \pm y} \chi(1 + \pi^{2n-1}t) \cdot \omega(t^2 - y^2)$$

$$\text{for } x = 1 + \pi^{n-1}\sqrt{\pi}y,$$

where G_χ is the Gauss sum

$$G_\chi = \frac{1}{\sqrt{q}} \sum_{x \in \mathbf{F}_q^*} \chi(1 + \pi^{n-1}\sqrt{\pi}x)\omega(x). \qquad 7.1.1$$

PROPOSITION 7.2. *For K a quadratic extension of k, let χ be a character of K^*, and let V be a representations of D_k^* such that the conductor of V is* less *than the conductor of V_χ. Then $V \otimes V_\chi$ is a sum of representations of D_k^* coming from the same quadratic field K and of the same conductor as V_χ. If the character of V on K^* is given by* $\mathrm{ch}(V) = \Sigma\eta$ *where the η are 1-dimensional representations of K^*, then $V \otimes V_\chi = \Sigma_\eta V_{\chi\eta}$.*

Proof. The proof follows easily by comparing the characters of $V \otimes V_\chi$ and $\Sigma_\eta V_{\chi\eta}$. We omit the details. □

COROLLARY 7.3. *Let V_1 and V_2 be two representations of D_k^* of the same odd conductor $2n + 1$. If V_1 and V_2 come from distinct ramified field extensions then $V_1 \otimes V_2$ is a sum of representations of conductor $2n + 1$.*

Proof. It suffices to observe that $\tilde{V}_3 \subseteq V_1 \otimes V_2$ iff $\tilde{V}_2 \subseteq V_1 \otimes V_3$. □

COROLLARY 7.4. *Let* V_{χ_1} and V_{χ_2} *be two representations of D_k^* of the same conductor and belonging to the same quadratic field extension. Suppose* $\mathrm{cond}(\chi_1\chi_2) = \mathrm{cond}(\chi_1\chi_2^{-1}) = \mathrm{cond}(\chi_1)$. *Then $V_{\chi_1} \otimes V_{\chi_2}$ consists of representations of the same conductor as V_{χ_1} and V_{χ_2}.* □

8. Local ε-factors

We begin by fixing some notation for representations of the Weil group, W_k, of k, and of representations of the Deligne-Weil group, DW_k, of k, and refer to [Ta] as a general reference to this section. For K a finite extension of k, and σ a representation of W_k, we will use the notation $\sigma|_K$ to denote the restriction of σ to $W_K \subset W_k$. Similarly for η a representation of W_K, we will denote $\mathrm{Ind}_{W_K}^{W_k} \eta$, the induced representation of W_k, by $\mathrm{Ind}_K^k \eta$. By local class field theory, the abelianization of W_k is isomorphic to k^*; we will use the isomorphism for which a geometric Frobenius of W_k (i.e., whose action on the residue field extension is the inverse of the usual Frobenius) corresponds, under this isomorphism, to a uniformizing parameter in k. Via this isomorphism, characters of k^* will be identified to characters of W_k; in particular the norm $\|\cdot\|$ on k^* will be thought of as a character on W_k.

For K a quadratic extension of k and χ a character of K^*, the two-dimensional representation $\mathrm{Ind}_K^k \chi$ of W_k, will simply be denoted by σ_χ. σ_χ is irreducible iff χ does not factor through the norm map from K^* to k^*. We will say that σ_χ is associated to the quadratic field K. The determinant of σ_χ is $\chi|_{k^*} \cdot \omega_{K/k}$, where $\omega_{K/k}$ is the character of k^* associated to the quadratic field extension K/k.

A representation of DW_k is by definition a representation σ of W_k on a vector space V and a nilpotent operator N on V, such that $\sigma(w)N\sigma(w)^{-1} = \|w\|N$, for all $w \in W_k$. The tensor product of two representation of DW_k with nilpotent operators N_1, N_2 is defined to be the usual tensor product for the representations of the Weil group, and the nilpotent operator being $N_1 \otimes \mathbf{1} + \mathbf{1} \otimes N_2$. A representation of DW_k is called F-semisimple if the action of W_k is semisimple. Let sp(n) be the representation of DW_k on an n-dimensional vector space with basis $e_0, e_1, \ldots, e_{n-1}$ such that $Ne_i = e_{i+1}$ for $i < n-1$, $Ne_{n-1} = 0$, and $we_i = \|w\|^{(2i+1-n)/2} e_i$ for $w \in W_k$. Representations of DW_k, for which the nilpotent operator is trivial, will be identified to representations of W_k.

For π an irreducible, admissible representation of $\mathrm{GL}_2(k)$, $\sigma(\pi)$ will denote the two-dimensional representation of DW_k associated to π by the local Langlands correspondence, cf. [Ku2]. For σ a 2-dimensional F-semisimple representation of DW_k, $\pi(\sigma)$ will denote the corresponding representation of $\mathrm{GL}_2(k)$, and if $\pi(\sigma)$ is a discrete series then $\pi'(\sigma)$ will denote the corresponding representation of D_k^*. For a representation σ of W_k which is a sum of two characters ψ_1, ψ_2 of k^*, $\pi(\sigma)$ is the principal series $V_{(\psi_1,\psi_2)}$, if the principal series is irreducible, and is the one-dimensional sub-quotient of it, if the principal series is reducible. For the two-dimensional irreducible representation σ_χ of W_k, obtained by inducing a character χ of a separable quadratic extension K, $\pi(\sigma_\chi)$ is a supercuspidal representation of $\mathrm{GL}_2(k)$, and $\pi'(\sigma_\chi)$ is the representation of D_k^* which was denoted by V_χ in the previous section. For k of residue characteristic $\neq 2$, these are the only two-dimensional semisimple representations of W_k. For an irreducible admissible representation π of $\mathrm{GL}_2(k)$, $\sigma(\pi)$ has a non-trivial N iff π is a special representation. The representation Sp(2) of $\mathrm{GL}_2(k)$ corresponds to the representation sp(2) of DW_k.

To a non-trivial additive character ψ of k, and a virtual representation σ of DW_k, there is associated the local ε-factor $\varepsilon(\sigma, \psi) \in \mathbf{C}^*$. The ε-factor used in this paper is, in Tate's notation in [Ta], $\varepsilon_L(\sigma, \psi) = \varepsilon_D(\sigma \cdot \| \|^{1/2}, \psi, \mathrm{d}x)$ where $\mathrm{d}x$ is the Haar measure on k, self-dual with respect to the character ψ of k. For an unramified character χ of k^*, $\varepsilon(\chi, \psi) = \chi(\pi)^{n(\psi)}$, where $n(\psi)$ is the largest integer such that ψ is trivial on $\pi^{-n(\psi)}\mathcal{O}_k$. For a ramified character χ of k^*

$$\varepsilon(\chi, \psi) = \int_{k^*} \chi(x^{-1})\psi(x)\,\mathrm{d}x \overset{\mathrm{def}}{=} \sum_{n=-\infty}^{\infty} \int_{v(x)=n} \chi(x^{-1})\psi(x)\,\mathrm{d}x.$$

The ε-factors satisfy the following basic properties.

8.1.1 $\varepsilon(\sigma_1 \oplus \sigma_2, \psi) = \varepsilon(\sigma_1, \psi)\varepsilon(\sigma_2, \psi)$.

8.1.2 $\varepsilon(\mathrm{Ind}_K^k \sigma, \psi) = \varepsilon(\sigma, \psi_K)$ where σ is a degree zero representation of DW_k and $\psi_K(x) = \psi(\mathrm{tr}_{K/k} x)$.

8.1.3 $\varepsilon(\sigma, \psi) \cdot \varepsilon(\sigma^*, \psi) = \det \sigma(-1)$ where σ^* is the contragredient of σ.

8.1.4 $\varepsilon(\sigma \otimes \mathrm{Ind}_K^k \chi, \psi) = \varepsilon(\sigma|_K \otimes \chi, \psi_K) \cdot \omega_{K/k}^{\dim \sigma/2}(-1)$, where σ is an even dimensional representation of the Weil group of k, χ a character of K^*, for K a quadratic extension field of k and ψ_K is as in 8.1.2.

8.1.5 $\varepsilon(\sigma, \psi_a) = (\det \sigma)^{\dim(\sigma)}(a)\varepsilon(\sigma, \psi)$ where $\psi_a(x) = \psi(ax)$.

8.1.6 $\varepsilon(\sigma, \psi_K) = \varepsilon(\sigma^\tau, \psi_K)$, with ψ_K as in 8.1.2, and $\tau \in \mathrm{Gal}(K/k)$ for a Galois extension K/k, acting on the representations σ of W_K, denoted by $\sigma \mapsto \sigma^\tau$.

8.1.7 $\varepsilon(\mathrm{sp}(n) \otimes \rho, \psi) = \varepsilon(\rho, \psi)^n \cdot \det(-F, \rho^I)^{n-1}$, where ρ is a representation of W_k, F is a geometric Frobenius of W_k, and I is the inertia subgroup of W_k.

Before we state Tunnell's theorem about ε-factors, let us remark that according to a theorem of A. Silberger, cf. [Si2], any character of K^*, for K a quadratic extension of k, appears in an irreducible admissible representation of $\mathrm{GL}_2(k)$ with multiplicity $\leqslant 1$. Also, for π a discrete series representation of $\mathrm{GL}_2(k)$ and π' the corresponding representation of D_k^*, a character of K^* whose restriction to k^* equals the central character of π, appears in precisely one of the representations π or π'. This follows easily from the above theorem of Silberger and the character identity $\mathrm{ch}\pi(x) = -\mathrm{ch}\pi'(x)$. The following theorem is due to Tunnell, cf. [Tu].

THEOREM 8.2. *Let π be an infinite dimensional, irreducible, admissible representation of $\mathrm{GL}_2(k)$ with central character ω_π. For a separable quadratic field extension K/k, let χ be a character of K^* which restricts to ω_π on k^* and ψ a non-trivial character of k. Then $\varepsilon(\sigma(\pi)|_K \otimes \chi^{-1}, \psi_K)$ is independent of ψ, and χ appears in π iff $\varepsilon(\sigma(\pi)|_K \otimes \chi^{-1}) \cdot \omega_\pi(-1) = 1$, or equivalently iff $\varepsilon(\sigma(\pi) \otimes \mathrm{Ind}_K^k \chi^{-1}) \cdot \omega_{K/k}(-1) \cdot \omega_\pi(-1) = 1$.* □

We have the following simple statements about induced representations of representations of Weil groups.

8.3.1. For K_1, K_2 two distinct quadratic extensions of k, let τ_1 be the non-trivial automorphism of K_1K_2/K_1 and χ a character of K_1^*, thought of as a character of W_{K_1}. Then we have

$$\mathrm{Res}_{K_2} \mathrm{Ind}_{K_1}^k \chi = \mathrm{Ind}_{K_1K_2}^{K_2} \chi(x \cdot \tau_1 x).$$

8.3.2. For K/k a quadratic extension with τ the non-trivial automorphism and α (resp. χ) a character of k^* (resp. K^*) thought of as character of W_k (resp. W_K), we have

$$\alpha \otimes \mathrm{Ind}_K^k \chi = \mathrm{Ind}_K^k[\chi \otimes \alpha(x \cdot \tau x)].$$

Let us observe that if σ_1, σ_2 and σ_3 are three two-dimensional representations of W_k such that the product of their determinants is trivial, then the determinant of $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is also trivial and therefore from 8.1.5, $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3, \psi)$ is independent of ψ, which will therefore be denoted by $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$. Also, $(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)^* \cong \sigma_1 \otimes \sigma_2 \otimes \sigma_3$, and therefore from 8.1.3, $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = +1$.

The proof of Theorem 1.4 will be broken up in various cases and we take these cases up one-by-one. We begin with the case when one of the representations, say σ_1, is reducible. Since in this case we know from Theorem 1.2 that there always is a trilinear form on $\pi(\sigma_1) \otimes \pi(\sigma_2) \otimes \pi(\sigma_3)$, we need to prove that the ε-factor is 1. In other cases we will omit to mention that the conclusion about ε-factor that we draw is in conformity with Theorem 1.4.

PROPOSITION 8.4. *Let $\sigma_1, \sigma_2, \sigma_3$ be three two-dimensional representations of the Weil group of k such that the product of their determinants is* 1 *and assume that σ_1 is a sum of two characters. Then $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$.*

Proof. Write σ_1 as sum of two characters, say θ_1, θ_2. Since the product of the determinants of $\sigma_1, \sigma_2, \sigma_3$ is trivial, the contragredient of $\theta_1 \otimes \sigma_2 \otimes \sigma_3$ is $\theta_2 \otimes \sigma_2 \otimes \sigma_3$. Therefore by 8.1.3,

$$\varepsilon(\theta_1 \otimes \sigma_2 \otimes \sigma_3)\varepsilon(\theta_2 \otimes \sigma_2 \otimes \sigma_3) = \det(\theta_1 \otimes \sigma_2 \otimes \sigma_3)(-1).$$

Now $\det(\theta_1 \otimes \sigma_2 \otimes \sigma_3) = \theta_1^4 \det(\sigma_2)^2 \det(\sigma_3)^2$, therefore $\det(\theta_1 \otimes \sigma_2 \otimes \sigma_3) \times (-1) = 1$. Therefore,

$$\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \varepsilon(\theta_1 \otimes \sigma_2 \otimes \sigma_3)\varepsilon(\theta_2 \otimes \sigma_2 \otimes \sigma_3) = 1. \qquad \square$$

We now do the calculation of the ε-factor $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$ when one of the representations, say σ_3, is the special representation sp(2) but σ_1 and σ_2 are not special. Suppose that the determinant of σ_1 (resp. σ_2) is ω_1 (resp. ω_1^{-1}). From 8.1.7, $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \mathrm{sp}(2)) = \varepsilon(\sigma_1 \otimes \sigma_2)^2 \cdot \det(-F, (\sigma_1 \otimes \sigma_2)^I)$. Since $\sigma_1 \otimes \sigma_2$ is self-dual with determinant 1, it follows from 8.1.3 that $\varepsilon(\sigma_1 \otimes \sigma_2)^2 = 1$.

PROPOSITION 8.5. $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \mathrm{sp}(2)) = \det(-F, (\sigma_1 \otimes \sigma_2)^I)$ *is equal to* (-1) *iff σ_1 and σ_2 are irreducible representations of W_k and $\sigma_1^* \cong \sigma_2$.*

Proof. Since $(\sigma_1 \otimes \sigma_2)^I = \mathrm{Hom}_I(\sigma_1^*, \sigma_2)$, it follows that for the ε-factor to be (-1), it is necessary that there is a non-trivial intertwining between σ_1^* and σ_2 as an I-module. If σ_1 is reducible as a W_k-module, and therefore sum of two characters, it is easy to see that a non-trivial intertwining as an I-module between σ_1^* and σ_2 forces σ_2 also to be reducible as a W_k-module. Writing $\sigma_1 = \chi_1 \oplus \omega_1\chi_1^{-1}, \sigma_2 = \chi_2 \oplus \omega_1^{-1}\chi_2^{-1}$, we have, $\sigma_1 \otimes \sigma_2 = \chi_1\chi_2 \oplus \omega_1^{-1}\chi_1\chi_2^{-1} \oplus \omega_1\chi_1^{-1}\chi_2 \oplus \chi_1^{-1}\chi_2^{-1}$. It follows that $(\sigma_1 \otimes \sigma_2)^I$, if non-zero, is either two or four-dimensional and in either case $\det(-F, (\sigma_1 \otimes \sigma_2)^I) = 1$.

Now assume that both σ_1 and σ_2 are irreducible W_k-modules. If σ_1 (equivalently, σ_1^*) is reducible as an I-module then clearly in the representation space of σ_1^* there is a basis e_1, e_2 such that $Fe_1 = e_2$, $Fe_2 = -\omega_1^{-1}(F)e_1$ and I acts by a character θ_1 (resp. $\omega_1^{-1}\theta_1^{-1}$) on e_1 (resp. e_2). Therefore if σ_1 is irreducible as a W_k-module but reducible as an I-module and $\mathrm{Hom}_I(\sigma_1^*, \sigma_2) \neq 0$ then $\sigma_1^* \cong \sigma_2$, as W_k-modules. Moreover, $(\sigma_1 \otimes \sigma_2)^I$ is a two-dimensional space and the determinant of $-F$ on it is -1. If σ_1^* remains irreducible when restricted to I, and $\mathrm{Hom}_I(\sigma_1^*, \sigma_2) \neq 0$, then σ_2 must also be the same irreducible representation when restricted to I. By Schur's lemma, this irreducible representation of I extends in only two ways to a representation of W_k with a given determinant. If σ_1^* is isomorphic (resp. not isomorphic) to σ_2 as a W_k-module then the action of the geometric Frobenius on $\mathrm{Hom}_I(\sigma_1^*, \sigma_2) \cong \mathbf{C}$ is through 1 (resp. -1). This completes the proof of the proposition. □

For the ε-factor, $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$, when two of the representations, say σ_2 and σ_3 are the special representation sp(2), we have:

PROPOSITION 8.6. *For a two-dimensional representation σ_1 of the Deligne-Weil group with trivial determinant,* $\varepsilon(\sigma_1 \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2)) = 1$ *iff σ_1 is not isomorphic to* sp(2).

Proof. The proof is a trivial consequence of 8.1.7, and will be omitted. □

In the rest of the paper we will be considering only (tensor products of) two-dimensional *irreducible* representations of the Weil group W_k for k of residue characteristic $\neq 2$. Also, as in Section 7, we will assume that the determinants of these representations have conductor $\leqslant 1$.

PROPOSITION 8.7. *If $\sigma_{\chi_1}, \sigma_{\chi_2}, \sigma_{\chi_3}$ are three two dimensional representations of the Weil group of k, corresponding to characters χ_1, χ_2, χ_3 of pairwise distinct quadratic field extensions K_1, K_2, K_3 of k such that the product of the determinants of σ_{χ_i}, for $i = 1$ to 3, is trivial. Then $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = 1$.*

Proof. Let L be the composite of K_1, K_2, K_3. Then, $\mathrm{Gal}(L/k) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Let τ_1 (resp. $\tau_2, \tau_3 = \tau_1 \cdot \tau_2$) be the non-trivial element of $\mathrm{Gal}(L/K_1)$ (resp. $\mathrm{Gal}(L/K_2)$, $\mathrm{Gal}(L/K_3)$). We have

$$\begin{aligned}
&\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) \\
&\quad = \varepsilon((\sigma_{\chi_1} \otimes \sigma_{\chi_2})|_{K_3} \otimes \chi_3, \psi_{K_3}) && \text{(by 8.1.4)} \\
&\quad = \varepsilon(\sigma_{\chi_1(x \cdot \tau_1 x)} \otimes \sigma_{\chi_2(x \cdot \tau_2 x)} \otimes \chi_3, \psi_{K_3}) && \text{(by 8.3.1)} \\
&\quad = \varepsilon([\sigma_{\chi_1(x \cdot \tau_1 x)\chi_2(x \cdot \tau_2 x)} \oplus \sigma_{\chi_1(x \cdot \tau_1 x)\chi_2(\tau_3 x \cdot \tau_1 x)}] \otimes \chi_3, \psi_{K_3}) \\
&\quad = \varepsilon([\sigma_{\chi_1(x \cdot \tau_1 x)\chi_2(x \cdot \tau_2 x)\chi_3(x \cdot \tau_3 x)} \oplus \sigma_{\chi_1(x \cdot \tau_1 x)\chi_2(\tau_3 x \cdot \tau_1 x)\chi_3(x \cdot \tau_3 x)}], \psi_{K_3}) && \text{(by 8.3.2)} \\
&\quad = \varepsilon(\chi_1(x \cdot \tau_1 x)\chi_2(x \cdot \tau_2 x)\chi_3(x \cdot \tau_3 x), \psi_L) \cdot \\
&\qquad \cdot \varepsilon(\chi_1(x \cdot \tau_1 x)\chi_2(\tau_3 x \cdot \tau_1 x)\chi_3(x \cdot \tau_3 x), \psi_L) && (*)
\end{aligned}$$

In the last step we have used that $\omega_{L/K_3}(-1) = 1$ for K_3 any quadratic extension of k.

Since the product of the determinants of σ_{χ_i}, for $i = 1$ to 3, is trivial, it follows that $\Pi_i\chi_i(x\cdot\tau_1 x\cdot\tau_2 x\cdot\tau_3 x) = 1$. Therefore

$$\begin{aligned}&\varepsilon(\chi_1(x\cdot\tau_1 x)\chi_2(\tau_3 x\cdot\tau_1 x)\chi_3(x\cdot\tau_3 x), \psi_L)\\ &\quad = \varepsilon(\chi_1(\tau_2 x\cdot\tau_3 x)\chi_2(\tau_1 x\cdot\tau_3 x)\chi_3(\tau_2 x\cdot\tau_1 x), \psi_L) \qquad \text{(by 8.1.6 for } L/K_2)\\ &\quad = \varepsilon(\chi_1(x\cdot\tau_1 x)^{-1}\chi_2(x\cdot\tau_2 x)^{-1}\chi_3(x\cdot\tau_3 x)^{-1}, \psi_L)\end{aligned}$$

Therefore from (*) and 8.1.3, $\varepsilon(\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3}) = 1$. □

PROPOSITION 8.8. *For K a quadratic extension of k, let σ_{χ_1} and σ_{χ_2} be two representations of the Weil group of k, corresponding to characters χ_1 and χ_2 of K^* with* $\operatorname{cond}(\chi_1) > \operatorname{cond}(\chi_2)$. *Then for a two-dimensional representation σ_{χ_3} of the Weil group corresponding to the same field K, and such that the product of the determinants, ω_i, of σ_{χ_i}, for $i = 1$ to 3, is trivial, $\varepsilon(\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3}) = -1$ iff there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1})\otimes\pi'(\sigma_{\chi_2})\otimes\pi'(\sigma_{\chi_3})$.*

Proof. Since

$$\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3}\cong\sigma_{\chi_1}\otimes\sigma_{\chi_3}\otimes\sigma_{\chi_2}\cong[\sigma_{\chi_1\chi_3}\oplus\sigma_{\chi_1\bar{\chi}_3}]\otimes\sigma_{\chi_2},$$

we have

$$\begin{aligned}&\varepsilon(\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3})\\ &\quad = \varepsilon(\sigma_{\chi_1\chi_3}\otimes\sigma_{\chi_2})\cdot\varepsilon(\sigma_{\chi_1\bar{\chi}_3}\otimes\sigma_{\chi_2}) \qquad \text{(by 8.1.1)}\\ &\quad = \omega_{K/k}(-1)\varepsilon(\sigma_{\chi_2}|_K\otimes\chi_1\chi_3)\cdot\omega_{K/k}(-1)\varepsilon(\sigma_{\chi_2}|_K\otimes\chi_1\bar{\chi}_3) \qquad \text{(by 8.1.4)}\\ &\quad = \omega_2(-1)\varepsilon(\sigma_{\chi_2}|_K\otimes\chi_1\chi_3)\cdot\omega_2(-1)\varepsilon(\sigma_{\chi_2}|_K\otimes\chi_1\bar{\chi}_3).\end{aligned}$$

Since the product of the determinants of σ_{χ_i}, for $i = 1$ to 3, is trivial,

$$\omega_{K/k}\cdot(\chi_1\chi_2\chi_3)|_{k^*} = 1.$$

Therefore we can use Tunnell's theorem to conclude that $\omega_2(-1)\varepsilon(\sigma_{\chi_2}|_K\cdot\chi_1\chi_3) = -1$ iff $(\chi_1\chi_3)^{-1}$ is a weight of $\pi'(\sigma_{\chi_2})$. If $(\chi_1\chi_3)^{-1}$ is a weight of $\pi'(\sigma_{\chi_2})$, then $(\chi_1\bar{\chi}_3)^{-1}$ can't be a weight of $\pi'(\sigma_{\chi_2})$ because otherwise, $\operatorname{cond}(\chi_1\chi_3)\leqslant\operatorname{cond}(\chi_2)$ and $\operatorname{cond}(\chi_1\bar{\chi}_3)\leqslant\operatorname{cond}(\chi_2)$, therefore $\operatorname{cond}(\chi_1^2\chi_3\bar{\chi}_3)\leqslant\operatorname{cond}(\chi_2)$. A contradiction to our assumption that $\operatorname{cond}(\chi_1)$, and therefore cond (χ_1^2) is greater than $\operatorname{cond}(\chi_2)$, and that the $\operatorname{cond}(\chi_3\bar{\chi}_3)\leqslant 1$.

Therefore, $\varepsilon(\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3}) = -1$ iff either $(\chi_1\chi_3)^{-1}$ is a weight of $\pi'(\sigma_{\chi_2})$ or $(\chi_1\bar{\chi}_3)^{-1}$ is a weight of $\pi'(\sigma_{\chi_2})$, i.e., iff either $\chi_3^{-1} = \chi_1 +$ a weight of $\pi'(\sigma_{\chi_2})$, or $\bar{\chi}_3^{-1} = \chi_1 +$ a weight of $\pi'(\sigma_{\chi_2})$. In either case $\tilde{\pi}'(\sigma_{\chi_3}) = \pi'(\sigma_{(\chi_1+\text{a weight of }\pi'(\sigma_{\chi_2}))})$.

From Proposition 7.2, these are precisely the representations $\tilde{\pi}'(\sigma_{\chi_3})$ of D_k^* which appear in $\pi'(\sigma_{\chi_1})\otimes\pi'(\sigma_{\chi_2})$. Therefore $\varepsilon(\sigma_{\chi_1}\otimes\sigma_{\chi_2}\otimes\sigma_{\chi_3}) = -1$ iff there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1})\otimes\pi'(\sigma_{\chi_2})\otimes\pi'(\sigma_{\chi_3})$. □

The proof of the next proposition is analogous to the proof of the previous proposition and will therefore be omitted.

PROPOSITION 8.9. *For K a quadratic extension of k, let* σ_{χ_1} *and* σ_{χ_2} *be two representations of the Weil group of k, corresponding to characters* χ_1 *and* χ_2 *of* K^* *with* $\mathrm{cond}(\chi_1) = \mathrm{cond}(\chi_2)$. *Then for a two dimensional representation* σ_{χ_3} *of the Weil group such that the product of the determinants,* ω_i, *of* σ_{χ_i}, *for* $i = 1$ *to* 3, *is trivial, and such that* $\mathrm{cond}(\pi'(\sigma_{\chi_1})) \neq \mathrm{cond}(\pi'(\sigma_{\chi_3}))$, $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$ *iff there exists a* D_k^**-invariant linear form on* $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$. □

PROPOSITION 8.10. *For K a quadratic extension of k, let* σ_{χ_1} *and* σ_{χ_2} *be two representations of the Weil group of k, corresponding to characters* χ_1 *and* χ_2 *of* K^* *with* $\mathrm{cond}(\chi_1) > \mathrm{cond}(\chi_2)$. *Then for a two dimensional representation* σ_{χ_3} *of the Weil group corresponding to a character* χ_3 *of quadratic field* $L \neq K$, *and such that the product of the determinants,* ω_i, *of* σ_{χ_i}, *for* $i = 1$ *to* 3, *is trivial, we have* $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = 1$.

Proof. As in the previous proposition,

$$\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = \omega_3(-1)\varepsilon(\sigma_{\chi_3}|_K \otimes \chi_1\chi_2)\cdot\omega_3(-1)\varepsilon(\sigma_{\chi_3}|_K \otimes \chi_1\bar{\chi}_2).$$

We now split the proof into two cases.

Case 1. One of the fields K and L is unramified and the other one ramified.

In this case the condition under which a character λ of K^*, with $\lambda|_{k^*}$ = central character of $\pi'(\sigma_{\chi_3})$, appears in the representation $\pi'(\sigma_{\chi_3})$ of D_k^*, depends only on the conductor of λ. Since $\mathrm{cond}(\chi_1\chi_2) = \mathrm{cond}(\chi_1\bar{\chi}_2)$, it follows from Theorem 8.2 that $\omega_3(-1)\varepsilon(\sigma_{\chi_3}|_K \otimes \chi_1\chi_2)$ and $\omega_3(-1)\varepsilon(\sigma_{\chi_3}|_K \otimes \chi_1\bar{\chi}_2)$ are either both 1 or both (-1). Therefore $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = 1$.

Case 2. K and L are both ramified fields (of course, distinct).

Suppose that the character χ_3 of L^* is of conductor $2n$. If $\mathrm{cond}(\chi_1\chi_2) = \mathrm{cond}(\chi_1\bar{\chi}_2) > 2n$, then both $(\chi_1\chi_2)^{-1}$ and $(\chi_1\bar{\chi}_2)^{-1}$ belong to $\pi(\sigma_{\chi_3})$. If $\mathrm{cond}(\chi_1\chi_2) = \mathrm{cond}(\chi_1\bar{\chi}_2) \leqslant 2n$ then we know that the condition under which a character χ of K^* of conductor $\leqslant 2n$ appears in $\pi(\sigma_{\chi_3})$ depends only on the restriction of χ to $K^*(2n-1)/K^*(2n)$. Since $\mathrm{cond}(\chi_1) > \mathrm{cond}(\chi_2)$, $\chi_1\chi_2$ and $\chi_1\bar{\chi}_2$ restrict to the same character on $K^*(2n-1)/K^*(2n)$. And again as in case 1, $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = 1$. □

PROPOSITION 8.11. *Suppose that* $\pi'(\sigma_{\chi_1})$, $\pi'(\sigma_{\chi_2})$ *and* $\pi'(\sigma_{\chi_3})$ *are representations of* D_k^*/k^* *of conductor* $2n+1$, *with* χ_1, χ_2 *characters of a ramified quadratic field K, and* χ_3 *a character of a ramified quadratic field* $L \neq K$, *and such that the product* *of the determinants of* σ_{χ_i}, *for* $i = 1$ *to* 3, *is trivial. Then there exists a* D_k^**-invariant linear form on* $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ *iff* $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$.

Proof. It suffices to prove that there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff either $(\chi_1\chi_2)^{-1} \in \pi'(\sigma_{\chi_3})$ and $(\chi_1\bar{\chi}_2)^{-1} \notin \pi'(\sigma_{\chi_3})$, or $(\chi_1\chi_2)^{-1} \notin \pi'(\sigma_{\chi_3})$ and $(\chi_1\bar{\chi}_2)^{-1} \in \pi'(\sigma_{\chi_3})$. We start by fixing some notation for the division algebra. Let $D_k = k(\sqrt{\alpha}, \sqrt{\pi})$ with α a unit in k and $\sqrt{\alpha}\sqrt{\pi} = -\sqrt{\pi}\sqrt{\alpha}$; $K = k(\sqrt{\pi})$, $L = k(\xi\sqrt{\pi})$ with $\xi \in k[\sqrt{\alpha}]$. Clearly, L is not isomorphic to K iff the norm of ξ from $k(\sqrt{\alpha})$ to k is not a square in k. We can identify $D_k^*(2n-1)/D_k^*(2n)$ to $\mathbf{F}_{q^2}$ via the map: $1 + \pi^{n-1}(a\sqrt{\pi} + b\sqrt{\pi}\sqrt{\alpha}) \mapsto (a + b\sqrt{\alpha})$.

For ψ^0, a non-trivial character of $\mathbf{F}_q$, let ψ be the character of $\mathbf{F}_{q^2}$ given by $\psi(x) = \psi^0 \mathrm{tr}_{\mathbf{F}_{q^2}/\mathbf{F}_q}(x)$, $x \in \mathbf{F}_{q^2}$. Using ψ, we can identify characters of $\mathbf{F}_{q^2}$ to $\mathbf{F}_{q^2}$: $x \in \mathbf{F}_{q^2}$ corresponding to the character ψ_x: $\psi_x(z) = \psi(zx)$ for $z \in \mathbf{F}_{q^2}$. Under this identification, the inner conjugation action of $O_{D_k}^*/D_k^*(2n)$, where O_{D_k} is the ring of integers in D_k, on $D_k^*(2n-1)/D_k^*(2n)$ acting on the space of characters of $D_k^*(2n-1)/D_k^*(2n)$ is through the action of the norm one subgroup of $\mathbf{F}_{q^2}^*$ acting by multiplication on $\mathbf{F}_{q^2}$.

A character of $D_k^*(2n-1)/D_k^*(2n)$ is invariant under the inner conjugation action of $\sqrt{\pi}$ iff it is of the form ψ_x with $x \in \mathbf{F}_q$.

A non-trivial character of $D_k^*(2n-1)/D_k^*(2n)$ is invariant under the inner conjugation action of $\xi\sqrt{\pi}$ iff it is of the form ψ_x with $x \in \mathbf{F}_{q^2}$ such that its norm to $\mathbf{F}_q$ is not a square in $\mathbf{F}_q$.

The representations $\pi'(\sigma_{\chi_i})$, when restricted to $D_k^*(2n-1)/D_k^*(2n)$, are sum of characters of $\mathbf{F}_{q^2}$, the characters which appear forming a single orbit under the action of norm one subgroup of $\mathbf{F}_{q^2}^*$ and each character appearing q^{n-1} times. To know the relation of this orbit to χ_i, we recall from the theorem on page 370 of [G-Ku], that the character $\omega_{K/k}$ of k^* can be extended to a character $\bar{\omega}$ of K^* such that $\bar{\omega}|_{K^*(1)} \equiv 1$, and such that for any character χ of K^* of conductor $2n$, the unique extension of $\chi\bar{\omega}$ from K^* to $K^* \cdot D_k^*(n)$ when induced from $K^* \cdot D_k^*(n)$ to D_k^* gives the representation of $D_k^*/D_k^*(2n)$ associated to the character χ of K^*. It follows that $\pi'(\sigma_{\chi_1})$, $\pi'(\sigma_{\chi_2})$ correspond to the orbit of the characters ψ_{x_1}, ψ_{x_2} for $x_1, x_2 \in \mathbf{F}_q$, and $\pi'(\sigma_{\chi_3})$ corresponds to the character ψ_{x_3} with norm of x_3 not a square in $\mathbf{F}_q$. Moreover, the restriction of ψ_{x_1} (resp. ψ_{x_2}) to $K^*(2n-1) \subseteq D_k^*(2n-1)$ is the restriction of χ_1 (resp. χ_2) to $K^*(2n-1)$.

We begin by finding the condition under which there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$. Since $L \neq K$, we know that the product of characters of $\pi'(\sigma_{\chi_1})$, $\pi'(\sigma_{\chi_2})$, $\pi'(\sigma_{\chi_3})$ is supported only on elements of $D_k^*(2n-1)/D_k^*(2n)$. We, therefore, need to find the condition under which $x_1t_1 + x_2t_2 = x_3$ has a solution with t_1, t_2 belonging to norm one elements of $\mathbf{F}_{q^2}^*$. Let $t_1 = a_1 + \sqrt{\alpha}b_1$, $t_2 = a_2 + \sqrt{\alpha}b_2$, with $a_1^2 - \alpha b_1^2 = 1$, $a_2^2 - \alpha b_2^2 = 1$, and let $x_3 = a_3 + \sqrt{\alpha}b_3$. Clearly, $x_1t_1 + x_2t_2 = x_3$ has a solution iff the following system of equations has a solution in $\mathbf{F}_q$:

$$a_1x_1 + a_2x_2 = a_3$$
$$b_1x_1 + b_2x_2 = b_3$$
$$a_1^2 - \alpha b_1^2 = 1$$
$$a_2^2 - \alpha b_2^2 = 1.$$

Or, iff the following system of equations has a solution in $\mathbf{F}_q$:

$$(a_3 - a_2x_2)^2 - \alpha(b_3 - b_2x_2)^2 = x_1^2$$
$$a_2^2 - \alpha b_2^2 = 1.$$

After some manipulation it is seen that this system of equations has a solution iff $\alpha[(a_3^2 - \alpha b_3^2) - (x_1 + x_2)^2][(a_3^2 - \alpha b_3^2) - (x_1 - x_2)^2]$ is a square in $\mathbf{F}_q$.

We now find the condition under which $(\chi_1 \cdot \chi_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$. Since the character of $\pi'(\sigma_{\chi_3})$ is supported only on $K^*(2n-1)$ among the elements of K^*, it follows that the condition under which $(\chi_1 \cdot \chi_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ depends only on the restriction of $(\chi_1 \cdot \chi_2)^{-1}$ to $K^*(2n-1)/K^*(2n)$. Therefore for $(\chi_1 \cdot \chi_2)^{-1}$ to appear in $\pi'(\sigma_{\chi_3})$, we must have $\psi[(x_1 + x_2)z] = \psi[tx_3z]$, for some norm one element t of $\mathbf{F}_{q^2}$ and $\forall z \in \mathbf{F}_q$. Therefore $(\chi_1 \cdot \chi_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ iff $x_1 + x_2 - x_3 t \in \sqrt{\alpha} \cdot \mathbf{F}_q$. For $t = a + \sqrt{\alpha} b$, this condition is equivalent to the system of equations:

$$x_1 + x_2 - (a_3 a + \alpha b_3 b) = 0$$
$$a^2 - \alpha b^2 = 1.$$

After some manipulation it is seen that this system of equations has a solution iff $-\alpha[(a_3^2 - \alpha b_3^2) - (x_1 + x_2)^2]$ is a square in $\mathbf{F}_q$. Similarly, $(\chi_1 \cdot \bar{\chi}_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ iff $-\alpha[(a_3^2 - \alpha b_3^2) - (x_1 - x_2)^2]$ is a square in $\mathbf{F}_q$. Combining the results of the previous two paragraphs we find that if there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ then either $(\chi_1 \cdot \chi_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ and $(\chi_1 \cdot \bar{\chi}_2)^{-1}$ does not appear in $\pi'(\sigma_{\chi_3})$ or vice-versa. It follows that there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$. □

The following lemma is Proposition 2.10(a) in [Tu]. It can be derived from the character formulae for representations of D_k^* recalled in Section 7.

LEMMA 8.12. *For a ramified quadratic extension K/k, let θ, χ be characters of K^* such that $\chi|_{k^*} = \theta|_{k^*} \cdot \omega_{K/k}$. Let $\bar{\omega}$ denote an extension of the character $\omega_{K/k}$ of k^* to K^* such that $\bar{\omega}(x) = 1$ for $x \equiv 1 \bmod \pi_K$. Assume that* $\mathrm{cond}(\theta\chi\bar{\omega}) = \mathrm{cond}(\theta\bar{\chi}\bar{\omega}) = \mathrm{cond}(\theta) = 2n$, *and write $\chi(1 + \pi_K^{2n-1}x) = \theta(1 + \pi_K^{2n-1}tx)$ for some $t \in \mathcal{O}_k/\pi_k$. Then χ appears in $\pi'(\sigma_\theta)$ iff $(t^2 - 1)$ is not a square in $\mathcal{O}_k/\pi_k$. In particular, the condition under which χ appears in $\pi'(\sigma_\theta)$ depends only on $\chi|_{K^*(2n-1)}$.* □

PROPOSITION 8.13. *Let $\sigma_{\chi_1}, \sigma_{\chi_2}, \sigma_{\chi_3}$ be three representations of the Weil group corresponding to the same quadratic field K and of the same conductor, and such that the product of the determinants of σ_{χ_i}, for $i = 1$ to 3, is trivial. Then $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$ iff there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$.*

Proof. It suffices to prove that there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff either $(\chi_1\chi_2)^{-1} \in \pi'(\sigma_{\chi_3})$ and $(\chi_1\bar{\chi}_2)^{-1} \notin \pi'(\sigma_{\chi_3})$, or $(\chi_1\chi_2)^{-1} \notin \pi'(\sigma_{\chi_3})$ and $(\chi_1\bar{\chi}_2)^{-1} \in \pi'(\sigma_{\chi_3})$. As the unramified case is analogous and much simpler, we will give the details of the proof only for K a ramified quadratic extension. Assume that the conductors of $\pi'(\sigma_{\chi_i})$ is $2n + 1$ for all i. Then the dimension of D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ is

$$\frac{1}{[D_k^*:D_k^*(2n)\cdot k^*]} \sum_{x \in D_k^*/D_k^*(2n)\cdot k^*} \mathrm{ch}\pi'(\sigma_{\chi_1})(x)\cdot \mathrm{ch}\pi'(\sigma_{\chi_2})(x)\cdot \mathrm{ch}\pi'(\sigma_{\chi_3})(x).$$

We write this as a sum over the toral elements. For brevity of notation, let $f(x) = \mathrm{ch}\pi'(\sigma_{\chi_1})(x)\cdot \mathrm{ch}\pi'(\sigma_{\chi_2})(x)\cdot \mathrm{ch}\pi'(\sigma_{\chi_3})(x)$. As recalled in Section 7, the character of a representation $\pi'(\sigma_\chi)$, for χ a character of a ramified quadratic extension K/k, is zero on non-trivial elements of the unramified torus and for a quadratic ramified field $L \neq K$, it is non-zero only on $L^*(2n-1)$. The character on K^* is given by

$$\mathrm{ch}(\pi'(\sigma_\chi))(x) = G_\chi \cdot \frac{[\chi(x) + \omega(-1)\chi(\bar{x})]}{\|x - \bar{x}\|_K^{1/2}}\, q^{-v(x)/2} \quad \text{for } x \notin K^*(2n-1)\cdot k^*.$$

Here ω is the unique quadratic character of $\mathbf{F}_q^*$ and G_χ is the Gauss sum given by the formula in 7.1.1. From the standard properties of the Gauss sum, $G_{\chi_1}\cdot G_{\chi_2} = \omega(-x_1x_2)$ where $x_1$and x_2 belong to $\mathbf{F}_q^*$; these were defined in Proposition 8.11 and have the property that $\chi_1(1 + x_1t) = \chi_2(1 + x_2t)$, $\forall t \equiv 0 \bmod(\pi_K^{2n-1})$.

From [Ko] Satz 2, the number of conjugates of an element $x \in K^* - k^*K^*(2n)$, in $D_k^*/k^*D_k^*(2n)$, is $q^{2n-1}(q+1)\Delta(x)$, where $\Delta(x) = \|(x-\bar{x})^2/x\bar{x}\|_k$. Note that $\Delta(x)$ does not make sense on all of $K^*/K^*(2n)\cdot k^*$ but only on $[K^* - K^*(2n)\cdot k^*]/K^*(2n)\cdot k^*$.

It follows that

$$\sum_{x \in \frac{D_k^*}{D_k^*(2n)\cdot k^*}} f(x) = \sum_{x \in \frac{D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x) + \sum_{x \in \frac{D_k^* - D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x)$$

$$= \sum_{x \in \frac{D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x) + \frac{q^{2n-1}(q+1)}{2}\left[\sum_{x \in \frac{K^* - K^*(2n-1)}{K^*(2n)\cdot k^*}} \Delta(x)\cdot f(x)\right] \quad (*).$$

Using the character formula for $\mathrm{ch}(\pi'(\sigma_{\chi_1}))$ and $\mathrm{ch}(\pi'(\sigma_{\chi_2}))$ we obtain that $f(x)\Delta(x)$ is equal to

$$\{\omega(-x_1x_2)[\chi_1\chi_2(x) + \bar{\chi}_1\bar{\chi}_2(x)] + \omega(x_1x_2)[\chi_1\bar{\chi}_2(x) + \bar{\chi}_1\chi_2(x)]\}\mathrm{ch}\pi'(\sigma_{\chi_3})(x)$$

for $x \in K^* - K^*(2n-1)\cdot k^*$. Let f_+ and f_- be the functions on K^* defined by:

$$f_+(x) = \omega(-x_1x_2)[\chi_1\chi_2(x) + \bar{\chi}_1\bar{\chi}_2(x)]\cdot \mathrm{ch}\pi'(\sigma_{\chi_3})(x),$$
$$f_-(x) = \omega(x_1x_2)[\chi_1\bar{\chi}_2(x) + \bar{\chi}_1\chi_2(x)]\cdot \mathrm{ch}\pi'(\sigma_{\chi_3})(x).$$

From $(*)$ we obtain that:

$$\sum_{x\in\frac{D_k^*}{D_k^*(2n)\cdot k^*}} f(x) = \sum_{x\in\frac{D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x) +$$
$$+\frac{q^{2n-1}(q+1)}{2}\left[\sum_{x\in\frac{K^*}{K^*(2n)\cdot k^*}} f_+(x) - \sum_{x\in\frac{K^*(2n-1)}{K^*(2n)\cdot k^*}} f_+(x)\right]$$
$$+\frac{q^{2n-1}(q+1)}{2}\left[\sum_{x\in\frac{K^*}{K^*(2n)\cdot k^*}} f_-(x) - \sum_{x\in\frac{K^*(2n-1)}{K^*(2n)\cdot k^*}} f_-(x)\right]$$

We now evaluate the various terms of the above sum. Observe that the calculation of the previous proposition is valid in this case also, the only difference being that this time $x_3 \in \mathbf{F}_q$ instead of the condition of the previous proposition that the norm of x_3 is not a square in $\mathbf{F}_q$. Using the abbreviation $\lambda = -\alpha[x_3^2 - (x_1 + x_2)^2]$ and $\mu = -\alpha(x_3^2 - (x_1 - x_2)^2]$, we obtain from the proof of the previous proposition that

$$\sum_{x\in\frac{D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x) = (q^{3n} + q^{3n-1})[1 - \omega(\lambda)\cdot\omega(\mu)],$$

$$\sum_{x\in\frac{K^*(2n-1)\cdot k^*}{K^*(2n)\cdot k^*}} f_+(x) = 2q^n\cdot[1 + \omega(\lambda)]\cdot\omega(-x_1x_2),$$

$$\sum_{x\in\frac{K^*(2n-1)\cdot k^*}{K^*(2n)\cdot k^*}} f_-(x) = 2q^n\cdot[1 + \omega(\mu)]\cdot\omega(x_1x_2).$$

We have used the standard convention that $\omega(0) = 0$.

Also, denoting by $m[\chi, \pi'(\sigma_{\chi_3})]$ the multiplicity of the character χ of K^* in $\pi'(\sigma_{\chi_3})$, we have

$$\sum_{x \in \frac{K^*}{K^*(2n)\cdot k^*}} f_+(x) = 4q^n \cdot m[(\chi_1\chi_2)^{-1}, \pi'(\sigma_{\chi_3})] \cdot \omega(-x_1x_2),$$

and

$$\sum_{x \in \frac{K^*}{K^*(2n)\cdot k^*}} f_-(x) = 4q^n \cdot m[(\chi_1\bar{\chi}_2)^{-1}, \pi'(\sigma_{\chi_3})] \cdot \omega(x_1x_2).$$

We now split the proof into two cases.

Case 1. $x_3 \neq \pm x_1 \pm x_2$. In this case the condition under which a character of K^* appears in $\pi'(\sigma_{\chi_3})$ depends only on the restriction of the character to $K^*(2n-1)$. From the above calculation it follows that

$$\sum_{x \in \frac{K^*}{K^*(2n)\cdot k^*}} f_\pm(x) = \sum_{x \in \frac{K^*(2n-1)\cdot k^*}{K^*(2n)\cdot k^*}} f_\pm(x).$$

Therefore

$$\sum_{x \in \frac{D_k^*}{D_k^*(2n)\cdot k^*}} f(x) = \sum_{x \in \frac{D_k^*(2n-1)\cdot k^*}{D_k^*(2n)\cdot k^*}} f(x) = (q^{3n} + q^{3n-1})[1 - \omega(\lambda)\omega(\mu)].$$

It follows that there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff $\omega(\lambda)\cdot\omega(\mu) = -1$, i.e., if $(\chi_1 \cdot \chi_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ then $(\chi_1 \cdot \bar{\chi}_2)^{-1}$ does not appear in $\pi'(\sigma_{\chi_3})$ and vice-versa, i.e., iff $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$.

Case 2. By symmetry it suffices to consider the case when $x_3 = x_1 + x_2$. Therefore $\omega(\lambda) = 0$, $\omega(\lambda)\cdot\omega(\mu) = 0$ and since x_3 can't be $\pm(x_1 - x_2)$, $\mu \neq 0$. Therefore as in case 1

$$\sum_{x \in \frac{K^*}{K^*(2n)\cdot k^*}} f_-(x) = \sum_{x \in \frac{K^*(2n-1)\cdot k^*}{K^*(2n)\cdot k^*}} f_-(x).$$

It follows that

$$\sum_{x \in \frac{D_k^*}{D_k^*(2n)\cdot k^*}} f(x) = (q^{3n} + q^{3n-1}) + \\ + (q^{3n} + q^{3n-1})\{2m[(\chi_1\chi_2)^{-1}, \pi'(\sigma_{\chi_3})] - 1\}\omega(-x_1x_2).$$

Therefore there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff

$$\omega(-x_1x_2)\{2m[(\chi_1\chi_2)^{-1}, \pi'(\sigma_{\chi_3})] - 1\} = 1. \qquad (**)$$

Now by lemma 8.12, $(\chi_1\bar{\chi}_2)^{-1}$ appears in $\pi'(\sigma_{\chi_3})$ iff $(x_1 - x_2/x_1 + x_2)^2 - 1$ is not a square in $\mathbf{F}_q$, i.e., iff $(-x_1x_2)$ is not a square in $\mathbf{F}_q$. Therefore from (**) we obtain that there exists a D_k^*-invariant linear form on $\pi'(\sigma_{\chi_1}) \otimes \pi'(\sigma_{\chi_2}) \otimes \pi'(\sigma_{\chi_3})$ iff $\varepsilon(\sigma_{\chi_1} \otimes \sigma_{\chi_2} \otimes \sigma_{\chi_3}) = -1$. □

9. $\mathrm{GL}_2(\mathbf{R})$

In this section we prove the analogues of Theorems 1.1, 1.2, 1.4 for $\mathrm{GL}_2(\mathbf{R})$. We have not been able to treat the case when all the three representations are principal series. By a representation of $\mathrm{GL}_2(\mathbf{R})$ we will always understand a Harish-Chandra module.

To facilitate the arguments necessitated by the disconnectedness of $\mathrm{GL}_2(\mathbf{R})$, we make the following definition.

For a real Lie group G with Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$, closed subgroups $H' \subset H$ with the same Lie algebra $\mathfrak{h}_0$, and a $(\mathfrak{g}, H')$-module V, we define a $(\mathfrak{g}, H)$-module $\mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V$ as follows. As a vector space, $\mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V$ consists of all functions f on H with values in V satisfying $f(h'h) = h'f(h)$, $\forall h' \in H'$ and $h \in H$. H acts on this space of functions by right translation and $X \in \mathfrak{g}$ acts on $f \in \mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V$ by $(X \cdot f)(h) = X^h \cdot f(h)$, where X^h denotes the adjoint action of H on $\mathfrak{g}$. It is easy to check that the axioms for a $(\mathfrak{g}, H)$-module are satisfied. We have the following simple lemma whose proof will be omitted.

LEMMA 9.1. *For a $(\mathfrak{g}, H)$-module W, and the notation as above, we have*

(a) $[\mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V] \otimes W = \mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} [V \otimes W]$.

(b) $\mathrm{Hom}_{(\mathfrak{g},H)}[\mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V, W] = \mathrm{Hom}_{(\mathfrak{g},H')}[V, W]$.

In particular, $\mathrm{Ind}_{(\mathfrak{g},H')}^{(\mathfrak{g},H)} V$ *has a $(\mathfrak{g}, H)$-invariant linear form iff V has a $(\mathfrak{g}, H')$-invariant linear form.* □

We now fix some notation. From now on we will use $\mathfrak{g}$ (resp. $\mathfrak{gl}$) to denote the complexified Lie algebra of $\mathrm{SL}_2(\mathbf{R})$ (resp. $\mathrm{GL}_2(\mathbf{R})$). Let K' denote the subgroup of $\mathrm{SL}_2(\mathbf{R})$ consisting of the matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and K the normaliser of K' in $\mathrm{SL}_2(\mathbf{R})^{\pm}$, where $\mathrm{SL}_2(\mathbf{R})^{\pm}$ is the subgroup of those elements in $\mathrm{GL}_2(\mathbf{R})$ whose determinant lies in ± 1. Denote the Lie algebra of K' by $\mathfrak{k}_0$, and the two Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$ by $\mathfrak{b}_+$ and $\mathfrak{b}_-$. Let χ_n, for $n \in \mathbf{Z}$, denote the character of K' sending $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ to $e^{in\theta}$. We will also use χ_n to denote the corresponding character of $\mathfrak{k}$, or of $\mathfrak{b}_+$, or of $\mathfrak{b}_-$. Since $\mathrm{GL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})^{\pm} \times \mathbf{R}^+$ where $\mathbf{R}^+$ is the group of scalar matrices with positive entries, we will identify representations of $SL_2(\mathbf{R})^{\pm}$ to those representations of $GL_2(\mathbf{R})$ on which $\mathbf{R}^+$ acts trivially. Similarly, representations of SU_2 will be identified to those representation of $\mathbf{H}^*$, for $\mathbf{H}$ the quaternion division algebra over $\mathbf{R}$, on which $\mathbf{R}^+$ acts trivially.

The following lemma summarizes the information about representations of $SL_2(\mathbf{R})^{\pm}$ that we will need, cf. [J-L].

LEMMA 9.2. (a) *A principal series representation V has a basis $\{e_i\}$ consisting of eigenvectors of $\mathfrak{k}$ with eigenvalues χ_i, where i runs over all even integers (resp. odd integers) if $-\mathbf{1} \in SL_2(\mathbf{R})$ acts trivially on V (resp. non-trivially on V), such that for $i > j$, $N^{(i-j)/2}e_j = e_i$ where N is a non-zero element of the nilradical of $\mathfrak{b}_+$.*

(b) *The discrete series representations, D_n, of $SL_2(\mathbf{R})^{\pm}$ are parametrised by integers $n \geqslant 2$, and are given by $\mathrm{Ind}_{(\mathfrak{g},K')}^{(\mathfrak{g},K)}(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n)$, which as $(\mathfrak{g}, K')$-module is $(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n) \oplus (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_-)} \chi_{-n})$. The character χ_i of K' appears in D_n iff $|i| \geqslant n$ and $i \equiv n \bmod 2$.*

(c) *Under the Jacquet-Langlands correspondence, D_n corresponds to the $n - 1$ dimensional representation of SU_2 of highest weight $n - 2$.* □

THEOREM 9.3. *Suppose that V_1, V_2 and V_3 are three infinite dimensional irreducible $(\mathfrak{gl}, \tilde{K})$-modules, for $\tilde{K}$ a maximal compact-modulo-centre subgroup of $GL_2(\mathbf{R})$, such that the product of the central characters of V_i is trivial. Assume that at least one of the representations V_i is a discrete series. Then either there exists a non-zero $(\mathfrak{gl}, \tilde{K})$-invariant linear form on $V_1 \otimes V_2 \otimes V_3$, which is unique up to scalars, or all the representations V_i are discrete series representations and there exists a non-zero $\mathbf{H}^*$-invariant linear form on $V'_1 \otimes V'_2 \otimes V'_3$, which is also unique up to scalars. Moreover, only one of the two possibilities occurs.*

Proof. After twisting by a character, we can assume that positive scalar matrices act trivially on V_i, and therefore V_i will be thought of as representations of $SL_2(\mathbf{R})^{\pm}$, for $i = 1$ to 3, such that $-\mathbf{1} \in SL_2(\mathbf{R})^{\pm}$ acts trivially on $V_1 \otimes V_2 \otimes V_3$. The proof of this theorem will be divided in two cases.

Case I. V_1 and V_2 are discrete series representations.

From Lemma 9.1(a),

$$D_n \otimes D_m = \mathrm{Ind}_{(\mathfrak{g},K')}^{(\mathfrak{g},K)}[\{\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n\} \otimes_{\mathbf{C}} \{(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_m) \oplus (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_-)} \chi_{-m})\}].$$

From Poincare-Birkhoff-Witt theorem it is clear that

$$(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n) \otimes_{\mathbf{C}} (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_-)} \chi_{-m}) \cong \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{k})} \chi_{n-m}, \tag{$*$}$$

and it is easy to see that

$$(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n) \otimes_{\mathbf{C}} (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_m) \cong \sum_{i \geqslant 0} (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_{n+m+2i}). \tag{$**$}$$

Using isomorphisms (*) and (**), and Lemma 9.1(b), we conclude that $V_1 \otimes V_2 \otimes V_3$ has a $(\mathfrak{g}, K)$-invariant linear form iff either V_3 has a lowest weight vector of weight $n + m + 2i$ for some integer $i \geqslant 0$, or V_3 contains a vector on which $\mathfrak{k}$ acts by the character $n - m$, or equivalently, $V_1 \otimes V_2 \otimes V_3$ does not have a $(\mathfrak{g}, K)$-invariant linear form iff V_3 is a discrete series representation D_w with $n + m > w > n - m$. By the well-known Clebsch-Gordan theorem about tensor product of representations of SU_2, and Lemma 9.2(c), these are precisely the representations V_3 such that $V_1' \otimes V_2' \otimes V_3'$ has a SU_2-invariant linear form.

Case II. V_1 and V_2 are principal series representations, and V_3 is the discrete series representation D_n.

By Lemma 9.1, $V_1 \otimes V_2 \otimes V_3 = \mathrm{Ind}_{(\mathfrak{g}, K')}^{(\mathfrak{g}, K)}[(V_1 \otimes V_2) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n)]$ has a $(\mathfrak{g}, K)$-invariant linear form iff $[(V_1 \otimes V_2) \otimes (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b}_+)} \chi_n)]$ has a $(\mathfrak{g}, K')$-invariant linear form, i.e., iff there exists a linear form on $V_1 \otimes V_2$ which transforms by the character χ_n of $\mathfrak{b}_+$, i.e., a linear form on

$$(V_1 \otimes V_2)_{\mathfrak{n}^+} = \frac{V_1 \otimes V_2}{\{n \cdot v | n \in \mathfrak{n}^+, \text{ and } v \in V_1 \otimes V_2\}}$$

which transforms by the character χ_n of $\mathfrak{k}$. From Lemma 9.2(a), it is easy to see that the subspace $\Sigma_i [e_i \otimes f]$ of $V_1 \otimes V_2$, where f is any fixed eigenvector of $\mathfrak{k}$ in V_2, goes isomorphically to $(V_1 \otimes V_2)_{\mathfrak{n}^+}$, and therefore $(V_1 \otimes V_2)_{\mathfrak{n}^+}$ contains all the characters χ of K' such that $\chi(-1)$ equals 1 (resp. -1) if $-\mathbf{1}$ acts trivially on V_3 (resp. non-trivially). This completes the proof of case 2. □

REMARK 9.4. The proof of Theorem 9.3 given above works also when at least one of the representations V_i of $GL_2(\mathbf{R})$ is a principal series representation which is reducible when restricted to $SL_2(\mathbf{R})$. □

We now take up the question of ε-factors. For this, we fix some notation and recall some standard facts.

The Weil group $W_{\mathbf{C}/\mathbf{R}}$ of $\mathbf{R}$ is the normaliser of $\mathbf{C}^*$ in $\mathbf{H}^*$ and sits in the exact sequence:

$$0 \to \mathbf{C}^* \to W_{\mathbf{C}/\mathbf{R}} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

For $m \geqslant 0$, let σ_m be the two-dimensional representation $\mathrm{Ind}_{\mathbf{C}^*}^{W_{\mathbf{C}/\mathbf{R}}}(z/|z|)^m$ of $W_{\mathbf{C}/\mathbf{R}}$. For the character $\psi(x) = \exp(2\pi i x)$ of $\mathbf{R}$,

$$\varepsilon(\sigma_m, \psi) = i^{m+1} \quad \text{for} \quad m \geqslant 0 \tag{*}$$

Under the Langlands correspondence, the discrete series representation D_m of

$GL_2(\mathbf{R})$ for $m \geqslant 2$, corresponds to the representation σ_{m-1} of the Weil group $W_{\mathbf{C}/\mathbf{R}}$.

THEOREM 9.5. *Suppose that* V_1, V_2 *and* V_3 *are three infinite dimensional irreducible* $(\mathrm{gl}, \tilde{K})$*-modules, for* $\tilde{K}$ *a maximal compact-modulo-centre subgroup of* $GL_2(\mathbf{R})$, *such that the product of the central characters of* V_i *is trivial. Then* $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$. *It is equal to* -1 *iff all the representations* V_i *belong to discrete series, and there exists a* $\mathbf{H}^*$*-invariant linear form on* $V'_1 \otimes V'_2 \otimes V'_3$.

Proof. If one of the representations is a principal series, the proof of Proposition 8.4 remains valid in the archimedean case also. It therefore suffices to assume that all the representations V_i of $GL_2(\mathbf{R})$ are discrete series representations, which, after twisting by a character, can be assumed to be D_{m_i}, for $i = 1$ to 3. Assume that $m_1 \geqslant m_2 \geqslant m_3$. Since

$$\sigma_{m_1} \otimes \sigma_{m_2} \otimes \sigma_{m_3} = \sigma_{\{m_1+m_2+m_3\}} \oplus \sigma_{\{m_1+m_2-m_3\}} \oplus \sigma_{\{m_1-m_2+m_3\}} \oplus \sigma_{\{|m_1-m_2-m_3|\}},$$

Clebsch-Gordan's theorem about tensor product of representations of SU_2, Lemma 9.2(c), and (*) easily complete the proof of the theorem. □

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