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On relative amenability for von Neumann algebras

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Introduction

The concept of correspondence between two von Neumann algebras has been introduced by A. Connes ([8], [9]) as a very useful tool for the study of type II_1 factors. Recently, S. Popa has systematically developed this point of view to get some new insight in the domain [21]. Among many interesting results and remarks, he discussed Connes' classical work on the injective II_1 factor in the framework of correspondences, and he defined and studied a natural notion of amenability for a finite von Neumann algebra M relative to a von Neumann subalgebra N. When the Jones' index $[M:N]$ is finite or when M is injective the inclusion $N \subset M$ is amenable, but this situation occurs in many other examples. For instance, if M is the crossed product of a finite von Neumann algebra N by an action of a discrete group G preserving a faithful finite normal trace of N, then $N \subset M$ is amenable if and only if G is an amenable group ([21], Th. 3.2.4).

In [28], Zimmer considered a notion of amenable action in ergodic theory, which was extended in [1] to actions on arbitrary von Neumann algebras. We say that the G-action α on N is amenable if there exists an equivariant norm one projection from $L^\infty(G) \otimes N$ onto N, the G-action on $L^\infty(G) \otimes N$ being the tensor product of the action by left translation on $L^\infty(G)$ and the action α on N. When there exists a G-invariant state on the centre $Z(N)$ of N, the amenability of the action is equivalent to the amenability of the group ([1], Prop. 3.6). Otherwise, it is easy to construct amenable actions of non amenable groups. Since Popa's notion of amenable inclusion makes sense for arbitrary von Neumann algebras, he asked ([21], 3.4.2) whether the amenability of the G-action α was equivalent to the amenability of the inclusion $N \subset M = N \times_\alpha G$ in the case of a discrete group G acting on any von Neumann algebra N. In this paper we give a positive answer to this question (Prop. 3.4).

As far as we are concerned with non finite von Neumann algebras M and N, it seems more convenient to consider a correspondence between M and N as a self-dual right Hilbert N-module on which M acts to the left, since it avoids the choice of auxiliary weights. This point of view has been already systematically used in [4] for the general study of the index of conditional expectations. In the first section we recall the needed background on correspondences and Hilbert modules. In particular, to any inclusion $N \subset M$ is associated a correspondence Y_N (see 1.8) between M and M which gives informations about the embedding $N \subset M$. Popa has defined the inclusion to be amenable if the identity correspondence of M is weakly contained in Y_N.

In Section 2 we consider an action α of a discrete group G on a von Neumann algebra N, and we denote by M the crossed product $N \times_\alpha G$. The classical notions of positive type functions and group representations can respectively be extended in this context of dynamical systems to notions of positive type functions on G with respect to (N, G, α) and of cocycles (2.4 and 2.1). These two concepts are closely related, as in the usual case. For each cocycle T relative to $(Z(N), G, \alpha)$ we associate in a natural way a correspondence X between M and M (2.6). A positive type function corresponding to T gives rise to a normal completely positive map from M to M, which is a coefficient of the correspondence X (2.8). The positive type functions relative to $(Z(N), G, \alpha)$, having finite supports, yield coefficients of the correspondence Y_N associated to the inclusion $N \subset M$, and the constant positive type function equal to the unit of $Z(N)$ gives the identity automorphism of M, which is, of course, a coefficient of the identity correspondence of M. We proved in [3] that the G-action α on N is amenable if and only if this constant function is the limit, for the topology of the σ-weak pointwise convergence, of a net of positive type functions relative to $(Z(N), G, \alpha)$ with finite supports. Using this fact, we show in Section 3 the equivalence between the amenability of the action and the amenability of the inclusion $N \subset M$.

1. Preliminaries on correspondences

We recall here some facts on correspondences and Hilbert modules, mostly coming from [8], [9], [4], [21], [20], [22], [23], [24], [17], where the reader will find more details. For simplicity, in this paper we shall only consider σ-finite von Neumann algebras. Let M and N be two von Neumann algebras.

1.1. *A correspondence between M and N* is a Hilbert space H with a pair of commuting normal representations π_M and π_{N^0} of M and N^0 (the opposite of N) respectively [8]. Usually the triple (H, π_M, π_{N^0}) will be denoted by H, and for $x \in M$, $y \in N$ and $h \in H$, we shall write xhy instead of $\pi_M(x)\pi_{N^0}(y^0)h$.

Note that H gives rise to a representation of the binormal tensor product $M \otimes_{\text{bin}} N^0$ (see [11] for the definition of bin). Two correspondences H and H' are equivalent if they are (unitarily) equivalent when considered as representations of $M \otimes_{\text{bin}} N^0$.

We denote by $\text{Corr}(M, N)$ the set of equivalence classes of correspondences between M and N, and we shall use the same notation H for a correspondence and its class. We shall write Corr(M) for Corr(M, M). The standard form [13] of M yields an element $L^2(M)$ of Corr(M) called *the identity correspondence* of M. We shall sometimes write $L^2(M, \varphi)$ instead of $L^2(M)$, with a fixed faithful normal positive form φ on M.

1.2. Let us recall now another useful equivalent way of defining correspondences, which has been developed in [4]. Let X be a self-dual (right) Hilbert N-module (see [20]). We denote by $\langle\, ,\rangle$ (or $\langle\, ,\rangle_N$ in case of ambiguity) the N-valued inner product, and we suppose that it is conjugate linear in the first variable. The von Neumann algebra of all N-linear continuous operators from X to X will be denoted by $\mathscr{L}_N(X)$ (or $\mathscr{L}(X)$ when $N = \mathbb{C}$). Following ([4], Def. 2.1), by a M–N *correspondence* we mean a pair (X, π) where X is as above, and π is a unital normal homomorphism from M into $\mathscr{L}_N(X)$. More briefly, such a correspondence will be denoted by X, and we shall often write $x\xi$ instead of $\pi(x)\xi$ for $x \in M$ and $\xi \in X$. Let us remark that M–N correspondences are what Rieffel has called normal N-rigged M-modules in ([23], Def. 5.1). Two M–N correspondences X and X' are said to be equivalent if there exists a M–N linear isomorphism from X onto X' preserving the scalar products.

1.3. Let X be a self-dual Hilbert N-module. We call *s-topology* the topology defined on X by the family of semi-norms q_φ, where φ is any normal positive form on N and

$$q_\varphi(\eta) = \varphi(\langle \eta, \eta\rangle)^{1/2}, \quad \text{for } \eta \in X.$$

We say that a vector ξ in a M–N correspondence X is *cyclic* if the set $M\xi N = \{x\xi y, x \in M, y \in N\}$ is s-total in X.

The set of equivalence classes of M–N correspondences will be denoted by $C(M, N)$, and we shall not make any distinction between a correspondence and its class. We shall write $C(M)$ instead of $C(M, M)$. There is a natural bijection $\wedge$ between $C(M, N)$ and Corr(M, N), that will be described now.

1.4. Let $X \in C(M, N)$ and let $H_X = X \otimes_N L^2(N)$ be the Hilbert space obtained by inducing the standard representation of N up to M via X ([22], Th. 5.1). The induced representation of M in H_X and the right action of N on H_X defined by

$$(\xi \otimes h)y = \xi \otimes (hy), \quad \text{for } \xi \in X, \quad h \in L^2(N), \quad y \in N,$$

give rise to an element $\wedge(X) = H_X$ of Corr(M, N).

Conversely, given $H \in$ Corr(M, N), let X_H be the space $\mathrm{Hom}_{N^0}(L^2(N), H)$ of continuous N^0-linear operators from $L^2(N)$ into H. Let N acts on the right of X_H by composition of operators and define on X_H a N-valued inner product by $\langle r, s\rangle = r^*s$ for $r, s \in X_H$. Then X_H is a self-dual Hilbert N-module ([23], Th. 6.5).

Moreover, M acts on the left of X_H by composition of operators and we obtain in this way a M–N correspondence.

The maps $X \mapsto H_X$ and $H \mapsto X_H$ are inverse from each other ([4], Th. 2.2 and [23], Prop. 6.10). In fact, there is a natural isomorphism between the M–N correspondences X and $\mathrm{Hom}_{N^o}(L^2(N), X \otimes_N L^2(N))$, given by assigning to any $\xi \in X$ the element $\Theta_\xi : h \mapsto \xi \otimes h$ of $\mathrm{Hom}_{N^o}(L^2(N), X \otimes_N L^2(N))$.

1.5. Let M, N, P be von Neumann algebras, $X \in C(M, N)$ and $Y \in C(N, P)$. We denote by $X \otimes_N Y$ the self-dual completion (see [20], Th. 3.2) of the algebraic tensor product $X \odot Y$ endowed with the obvious right action of P and the P-valued inner product

$$\langle \xi \otimes \eta, \xi_1 \otimes \eta_1 \rangle = \langle \eta, \langle \xi, \xi_1 \rangle_N \eta_1 \rangle_P, \quad \text{for } \xi, \xi_1 \in X, \quad \eta, \eta_1 \in Y.$$

LEMMA. (i) *For* $x \in \mathscr{L}_N(X)$, *there is an element* $\rho(x)$ *in* $\mathscr{L}_P(X \otimes_N Y)$ *well defined by*

$$\rho(x)(\xi \otimes \eta) = (x\xi) \otimes \eta, \quad \text{for } \xi \in X, \eta \in Y.$$

We get in this way a normal homomorphism from $\mathscr{L}_N(X)$ *into* $\mathscr{L}_P(X \otimes_N Y)$.

(ii) *If the representation of* N *into* $\mathscr{L}_P(Y)$ *is faithful, then* ρ *is faithful.*

(iii) *If we take* $Y = L^2(N)$, *viewed as an element of* $C(N, \mathbb{C})$, *then* ρ *is an isomorphism of the von Neumann algebra* $\mathscr{L}_N(X)$ *onto the commutant* $\mathrm{Hom}_{N^o}(H_X, H_X)$ *of the right action of* N *on* $H_X = X \otimes_N L^2(N)$.

Proof. For the proof of (i) see ([22], Th. 5.9 and [4], Prop. 2.9). Let us show that ρ is isometric under the assumption of (ii). If $\xi \in X$ we define a continuous P-linear operator Θ_ξ from Y into $X \otimes_N Y$ by $\Theta_\xi(\eta) = \xi \otimes \eta$ for $\eta \in Y$. It is easily checked that $(\Theta_\xi)^*(\xi' \otimes \eta) = \langle \xi, \xi' \rangle_N \eta$ for $\xi' \in X$ and $\eta \in Y$, so that $\|\Theta_\xi\|^2 = \|\Theta_\xi^* \Theta_\xi\| = \|\langle \xi, \xi \rangle_N\| = \|\xi\|^2$.

Let $x \in \mathscr{L}_N(X)$ and $\varepsilon > 0$, and take $\xi \in X$ with $\|\xi\| = 1$ and $\|x\xi\| \geqslant \|x\| - \varepsilon$. Now choose $\eta \in Y$ with $\|\eta\| = 1$ and $\|\Theta_{x\xi}(\eta)\| \geqslant \|x\xi\| - \varepsilon$. Then we have

$$\|\rho(x)(\xi \otimes \eta)\| = \|x\xi \otimes \eta\| \geqslant \|x\xi\| - \varepsilon \geqslant \|x\| - 2\varepsilon$$

and

$$\|\xi \otimes \eta\| = \|\Theta_\xi(\eta)\| \leqslant \|\xi\| \, \|\eta\| = 1,$$

from which it follows that $\|\rho(x)\| = \|x\|$.

Let us prove (iii) now. Obviously the range of ρ is contained in $\mathrm{Hom}_{N^o}(H_X, H_X)$. Conversely, let $r \in \mathrm{Hom}_{N^o}(H_X, H_X)$ and consider the element $\bar{r}$ of $\mathscr{L}_N(X)$ such that

$\Theta_{\bar{r}\xi} = r \circ \Theta_\xi$ for $\xi \in X$. Then for $\xi \in X$ and $h \in L^2(N)$ we have

$$\rho(\bar{r})(\xi \otimes h) = \bar{r}(\xi) \otimes h = \Theta_{\bar{r}\xi}(h) = r \circ \Theta_\xi(h) = r(\xi \otimes h),$$

and thus $\rho(\bar{r}) = r$. □

1.6. Keeping the notations of 1.5, we say that the self-dual Hilbert P-module $X \otimes_N Y$ provided with the homomorphism of M into $\mathscr{L}_P(X \otimes_N Y)$ given by restricting ρ is the *composition correspondence of X by Y*. It is the version in the setting of Hilbert modules of the composition of correspondences defined in ([8], §II).

There are other classical operations on correspondences. We shall need the following ones. Let $H \in \mathrm{Corr}(M, N)$ be a correspondence between M and N. Let $\bar{H}$ be the conjugate Hilbert space. If $h \in H$, we denote by $\bar{h}$ the vector h when viewed as an element of $\bar{H}$. Then $\bar{H}$ has a natural structure of correspondence from N to M by

$$y\bar{h}x = \overline{x^*hy^*}, \quad \text{for } x \in M, \quad y \in N, \quad h \in H.$$

(see [21], 1.3.7). We call it the *adjoint correspondence* of H.

Thanks to the bijection $\wedge$ between $C(M, N)$ and $\mathrm{Corr}(M, N)$, we see that to each $X \in C(M, N)$ we can associate an element $\bar{X} \in C(N, M)$, also called the *adjoint correspondence of X*. In general we haven't an explicit description of $\bar{X}$ (see however 1.8 below).

A *subcorrespondence* of $X \in C(M, N)$ is a submodule Y of X closed for the s-topology and stable by the left action of M. There is a natural bijection between the set of subcorrespondences of X and the set of projections in $\mathscr{L}_N(X)$ which commute with the range of M in $\mathscr{L}_N(X)$ by the left action. If X and Y are two M–N correspondences, we say that Y is *contained in* X and we write $Y \subset X$ if Y is equivalent to a subcorrespondence of X.

1.7. We shall have to consider the following special case of composition of correspondences. Let H be a Hilbert space and N a von Neumann algebra. Then, in an obvious way, H is an element of $C(\mathscr{L}(H), \mathbb{C})$ and N is an element of $C(\mathbb{C}, N)$. Thus we may define the composition correspondence $H \otimes_{\mathbb{C}} N$, written $H \otimes N$ afterwards. The N-valued scalar product in $H \otimes N$ is given by

$$\langle h \otimes y, h_1 \otimes y_1 \rangle = \langle h, h_1 \rangle y^* y_1 \quad \text{for } h, h_1 \in H \quad \text{and} \quad y, y_1 \in N.$$

Take an orthonormal basis $(e_i)_{i \in I}$ in H. Denote by $l^2_w(I, N)$ the right N-module of nets $(y_i)_{i\in I}$ of elements of N such that $\sum_{i\in I} y_i^* y_i$ is σ-weakly convergent. Provided with the N-module inner product $\langle (x_i)_{i\in I}, (y_i)_{i\in I}\rangle = \sum_{i\in I} x_i^* y_i$, it is a self-dual Hilbert N-module, and the map which sends $(y_i)_{i\in I}$ on $\sum_{i\in I} e_i \otimes y_i$ is an isomorphism of Hilbert N-modules from $l^2_w(I, N)$ onto $H \otimes N$. (See [20], p. 457–459). We shall identify $l^2_w(I, N)$ and $l^2(I) \otimes N$. Remark that $\mathscr{L}_N(H \otimes N)$ may be identified to the von Neumann tensor product $\mathscr{L}(H) \otimes N$ in a natural way.

1.8. Next, we shall give fundamental examples of correspondences, related to completely positive maps. Let $X \in C(M, N)$ and $\xi \in X$. Then $\Phi\colon x \mapsto \langle \xi, x\xi\rangle$ is a completely positive normal map from M into N. We shall say that that Φ is a *coefficient* of X, or is associated to X.

Conversely, given a completely positive normal map Φ from M into N, by the Stinespring construction we get a M–N correspondence X_Φ. The self-dual Hilbert N-module is obtained by separation and self-dual completion of the right N-module $M \odot N$(algebraic tensor product) gifted with the N-module inner product

$$\langle m \otimes n, m_1 \otimes n_1\rangle = n^*\Phi(m^* m_1) n_1, \quad \text{for } m, m_1 \in M, \quad n, n_1 \in N.$$

The normal representation π_Φ of M into $\mathscr{L}_N(X_\Phi)$ is given by

$$\pi_\Phi(x)(m \otimes n) = xm \otimes n \quad \text{for } x, m \in M, \quad n \in N.$$

If ξ_Φ denotes the class of $1 \otimes 1$ in X_Φ, we have $\Phi(x) = \langle \xi_\Phi, x\xi_\Phi\rangle$ for each $x \in M$, and ξ_Φ is a cyclic vector for the correspondence X_Φ. We shall say that X_Φ *is the correspondence associated* to Φ.

If X is a M–N correspondence and ξ is a cyclic vector in X, then it is easily seen that X is equivalent to the correspondence X_Φ, where Φ is the coefficient of X given by ξ. Furthermore, every M–N correspondence is a direct sum of cyclic correspondences, so that, as pointed out by A. Connes in [8], the notions of completely positive maps and correspondences are closely related.

When Φ is a normal conditional expectation from M onto a von Neumann subalgebra N, it is easily checked that X_Φ is equivalent to the separated, self-dual completion of the right N-module M with N-valued inner product $(m, m_1) \mapsto \Phi(m^* m_1)$, endowed with the obvious left action of M. More generally, to every semi-finite normal operator valued weight Φ from M to N (see [14]), Def. 2.1), one can associate a M–N correspondence X_Φ which extends the classical Gelfand-Segal construction for usual normal semi-finite weights (see [4], Prop. 2.8).

The right M-module M endowed with its inner product $\langle m, m_1\rangle = m^* m_1$ is self-dual. Gifted with its natural left M-module structure, it is the M–M correspondence associated to the identity homomorphism of M. It will be called the *identity M–M correspondence*, and denoted by X_M or M; of course $\wedge(X_M) = L^2(M)$.

Let now ρ be a normal homomorphism from M into a von Neumann algebra N. It is straightforward to show that X_ρ is equivalent to the Hilbert N-subspace $\rho(1)N$ of the right Hilbert N-module N, with left action of M given by

$$x.n = \rho(x)n, \quad \text{for } x \in M, \quad n \in \rho(1)N.$$

Suppose next that N is a von Neumann subalgebra of M. The N–M correspondence associated to the inclusion $\iota: N \to M$ will be denoted by X_N. Note that X_N is obtained from $X_M = M$ by restricting to N the left action of M. Remark also that $\wedge(X_N)$ is $L^2(M)$ where we restrict to N the standard representation of M and keep the right action of M. Let E be a faithful normal conditional expectation from M onto N. It has been noticed in [4] that the (equivalence class of the) M–N correspondence X_E is the adjoint correspondence $\bar{X}_N$ of X_N. Indeed, it is shown in ([4], Corol. 2.14) that $\wedge(X_E)$ is equivalent to $L^2(M)$ considered as a M–N bimodule by restricting to N the right action of M, and this correspondence is easily seen to be equivalent to the adjoint of $\wedge(X_N)$, thanks to the antilinear involutive isometry J of $L^2(M)$. (In fact, this remark remains true when E is any faithful normal semi-finite operator valued weight from M to N).

Even if there doesn't exist any conditional expectation from M onto N, we may consider $\bar{X}_N$. Note that by Lemma 1.5(iii), $\mathscr{L}_N(\bar{X}_N)$ is isomorphic to the commutant of the right action of N on $L^2(M)$, since $\wedge(\bar{X}_N) = L^2(M)$ viewed as M–N bimodule. It follows that the normal homomorphism from M into $\mathscr{L}_N(\bar{X}_N)$ which appears in the definition of the M–N correspondence $\bar{X}_N$ is injective, because it comes from the standard representation of M.

The M–M correspondence $\bar{X}_N \otimes_N X_N$ will be denoted by Y_N. It has been introduced by Popa ([21], 1.2.4) in the finite case, as a very useful tool for the study of the inclusion $N \subset M$. When there exists a normal faithful conditional expectation E from M onto N, then $Y_N = X_E \otimes_N X_N$ and Y_N is also the M–M correspondence associated to E viewed as a completely positive map from M to M (see [4], Th. 2.12).

Let us remark that $Y_M = X_M = M$. For $N = \mathbb{C}$, the $\mathbb{C}$–M correspondence $X_\mathbb{C}$ is the Hilbert M-module M with obvious action of $\mathbb{C}$, and $\bar{X}_\mathbb{C}$ is the Hilbert space $L^2(M)$ with the standard representation of M. Thus $Y_\mathbb{C} = \bar{X}_\mathbb{C} \otimes_\mathbb{C} X_\mathbb{C} = L^2(M) \otimes M$ is the *coarse M–M correspondence* (see [8], Def. 3).

1.9. For later use, we prove the following result (see [21], Prop. 1.2.5.(ii)).

LEMMA. *Let M be a von Neumann algebra and N a finite dimensional von Neumann subalgebra of M. Then we have $Y_N \subset Y_\mathbb{C}$.*

Proof. Let $z_1, \ldots, z_k$ be the minimal projections of the centre $Z(N)$, and $(e^j_{pq})_{1 \leqslant p,q \leqslant n_j}$ a matrix units system for Nz_j where $j = 1, \ldots, k$. Let $u^j_p = e^j_{p1}$ for $p = 1, \ldots, n_j$ and $j = 1, \ldots, k$. We choose a normal faithful state φ on M and we put $\alpha_j = \varphi(e^j_{11})$ for $j = 1, \ldots, k$. Then one easily checks that the map E on M defined by

$$E(x) = \sum_{\substack{1 \leqslant p,q < n_j \\ j=1,\ldots,k}} (1/\alpha_j) u^j_p \varphi(u^{j*}_p x u^j_q) u^{j*}_q$$

is a normal faithful conditional expectation from M onto N.

We take for $L^2(M)$ the standard form $L^2(M, \varphi)$ of the identity correspondence given by φ, and we identify M to a subspace of $L^2(M, \varphi)$. Let

$$\xi = \sum_{\substack{1 \leqslant p \leqslant n_j \\ j=1,\ldots,k}} (1/\alpha_j^{1/2}) u^j_p \otimes u^{j*}_p \in Y_{\mathrm{C}} = L^2(M, \varphi) \otimes M.$$

We have, for $x \in M$,

$$\begin{aligned}\langle \xi, x\xi \rangle &= \sum_{\substack{p,q \\ i,j}} (1/\alpha_i^{1/2} \alpha_j^{1/2}) \langle u^i_p \otimes u^{i*}_p, x u^j_p \otimes u^{j*}_q \rangle \\ &= \sum_{\substack{p,q \\ i,j}} (1/\alpha_i^{1/2} \alpha_j^{1/2}) u^i_p \varphi(u^{i*}_p x u^j_q) u^{j*}_q \\ &= E(x) = \langle \xi_\Phi, x\xi_\Phi \rangle,\end{aligned}$$

where Φ is E considered as a completely positive map from M to M. Thus, $x\xi y \mapsto x\xi_\Phi y$, with $x, y \in M$, induces an equivalence between the subcorrespondence of $L^2(M, \varphi) \otimes M$ having ξ as cyclic vector and Y_N which is the M–M correspondence associated to Φ.

Notice that Φ appears as a completely positive map which is a finite sum of completely positive maps factored by φ in the sense of ([19], Def. 1).

1.10. LEMMA. *A correspondence X contains the identity correspondence M if and only if there exists a non zero central and separating vector ξ in X (i.e. $\xi x = x\xi$ for all $x \in M$ and if $\xi x = 0$ then $x = 0$).*

Proof. The necessity of the existence of ξ is obvious. Conversely suppose that there is a non zero separating central vector ξ in X. Then $\langle \xi, \xi \rangle$ belongs to $Z(M)$ and its support is 1. Consider the polar decomposition $\xi = \eta \langle \xi, \xi \rangle^{1/2}$ of ξ (see [20], Prop. 3.11). Then η is central and since $\langle \eta, \eta \rangle$ is the support of $\langle \xi, \xi \rangle$, we have $\langle \eta, \eta \rangle = 1$. Now it is easy to prove that ηM defines a subcorrespondence of X equivalent to M. $\square$

1.11. REMARK. In ([21], Prop. 1.2.5) Popa has shown that for type II_1 factors $N \subset M$ the properties $[M:N] < \infty$ and $M \subset Y_N$ are closely related, where $[M:N]$ denotes as usually the Jones' index. More generally, let E be a faithful normal conditional expectation from a von Neumann algebra M onto a von Neumann subalgebra N. In [4], the index of E has been defined to be finite if there exists $k > 0$ such that the map $\iota \circ E - k \,|\, d_M$ from M to M is completely positive (ι being the injection of N into M). This definition is equivalent to the one given by Kosaki [18] when M and N are factors, and extends Jones' definition. It follows easily from ([4] Th. 3.5) and Lemma 1.10 that $M \subset Y_N$ when the index of E is finite, and that, conversely, if $M \subset Y_N$ with $N' \cap M = \mathbb{C}$ then the index of E is finite. Thus, Popa's result remains true in general.

1.12. Recall that in [9] a topology has been defined on Corr (M, N), described by its neighbourhoods in the following way.

DEFINITION. Let $H_0 \in \mathrm{Corr}(M, N), \varepsilon > 0, E \subset M$ and $F \subset N$ two finite sets, and $S = \{h_1, \dots, h_n\}$ a finite subset of H_0. We denote by $U(H_0; \varepsilon, E, F, S)$ the set of $H \in \mathrm{Corr}(M, N)$ such that there exist $k_1, \dots, k_n \in H$ with $|\langle k_i, xk_jy\rangle - \langle h_i, xh_jy\rangle| < \varepsilon$ for all $x \in E, y \in F$ and $i, j = 1, \dots, n$. The we consider the well defined topology on $C(M, N)$ for which these sets U are basis of neighbourhoods.

Note that if we consider correspondences as representations of $M \otimes_{\mathrm{bin}} N^0$ (the binormal ones), then it is easily verified that the above topology on $\mathrm{Corr}(M, N)$ is induced by the quotient topology introduced in [11] on the set of (unitary equivalence classes of) representations of $M \otimes_{\mathrm{bin}} N^0$.

We shall now give an equivalent way of defining this topology on $C(M, N)$.

DEFINITION. Let $X_0 \in C(M, N)$, $\mathscr{V}$ a σ-weak neighbourhood of 0 in N, E a finite subset of M and $S = \{\xi_1, \dots, \xi_n]$ a finite subset of X_0. We denote by $V(X_0; \mathscr{V}, E, S)$ the set of $X \in C(M, N)$ such that there exist $\eta_1, \dots, \eta_n \in X$ with $\langle \eta_i, x\eta_j\rangle - \langle \xi_i, x\xi_j\rangle \in \mathscr{V}$ for all $x \in E$ and $i, j = 1, \dots, n$. We provide $C(M, N)$ with the topology having such sets as basis of neighbourhoods.

PROPOSITION. *The bijection* $\wedge : C(M, N) \to \mathrm{Corr}(M, N)$ *is an homeomorphism.*

Proof. Let $X_0 \in C(M, N)$ and $H_0 = X_0 \otimes_N L^2(N, \varphi)$, where φ is a fixed faithful normal state on N. Denote by h_φ the canonical cyclic vector in $L^2(N, \varphi)$. Consider a neighbourhood $U = U(H_0; \varepsilon, E, F, S)$ of H_0. Then we may suppose that $S = \{\xi_1 \otimes h_\varphi, \dots, \xi_n \otimes h_\varphi\}$ with $\xi_1, \dots, \xi_n$ in X_0, since the subspace $\{\xi \otimes h_\varphi, \xi \in X_0\}$ is dense in H_0. Let:

$$S' = \{\xi_1, \dots, \xi_n\} \quad \text{and } \mathscr{V} = \{x \in N, |\langle h_\varphi, xh_\varphi y\rangle| < \varepsilon \quad \text{for } y \in F\}.$$

Then we shall prove that the image of $V = V(X_0; \mathscr{V}, E, S')$ by $\wedge$ is contained in U.

Take $X \in V$ and let $H = X \otimes_N L^2(N, \varphi)$. There exist $\eta_1, \ldots, \eta_n \in X$ with

$$|\langle h_\varphi, (\langle \eta_i, x\eta_j \rangle - \langle \xi_i, x\xi_j \rangle) h_\varphi y \rangle| < \varepsilon \quad \text{for } x \in E,\ y \in F,\ 1 \leqslant i, j \leqslant n,$$

so that

$$|\langle \eta_i \otimes h_\varphi, x\eta_j \otimes h_\varphi y \rangle - \langle \xi_i \otimes h_\varphi, x\xi_j \otimes h_\varphi y \rangle| < \varepsilon$$

for $x \in E, y \in F, 1 \leqslant i, j \leqslant n$; hence $H \in U$.

Conversely, consider a neighbourhood $V = V(X_0; \mathscr{V}, E, S)$ of X_0, where $S = \{\xi_1, \ldots, \xi_n\} \subset X_0$ and $\mathscr{V} = \{x \in N, |\varphi_i(x)| < 1, 1 \leqslant i \leqslant p\}$, with $\varphi_1, \ldots, \varphi_p$ given normal positive forms on N. Let ψ be a faithful normal positive form on N with $\varphi_i \leqslant \psi$ for $1 \leqslant i \leqslant p$. By ([10], Prop. 2.5.1) there exist $y_i \in N$ such that

$$\varphi_i(x) = \langle h_\psi, x h_\psi y_i \rangle, \quad \text{for } x \in N.$$

We may suppose that $H_0 = X_0 \otimes_N L^2(N, \psi)$. Let $S' = \{\xi_1 \otimes h_\psi, \ldots, \xi_n \otimes h_\psi\}$ and $F = \{y_1, \ldots, y_p\}$, and let us show that the image of $U = U(H_0; 1, E, F, S')$ by Λ^{-1} is contained in V. Consider $H \in \mathrm{Corr}(M, N)$ such that there exist $h_1, \ldots, h_n \in H$ with

$$|\langle h_i, x h_j y \rangle - \langle \xi_i \otimes h_\psi, x\xi_j \otimes h_\psi y \rangle| < 1 \quad \text{for } x \in E, y \in F, 1 \leqslant i, j \leqslant n.$$

We may suppose that $H = X \otimes_N L^2(N, \psi)$ with $X = \Lambda^{-1}(H)$, and since the set $\{\eta \otimes h_\psi, \eta \in X\}$ is a dense subspace of H, we may take $h_i = \eta_i \otimes h_\psi$ with $\eta_i \in X$, for $i = 1, \ldots, n$. Then we have

$$\begin{aligned} |\varphi_k(\langle \eta_i, x\eta_j \rangle - \langle \xi_i, x\xi_j \rangle)| &= |\langle h_\psi, (\langle \eta_i, x\eta_j \rangle - \langle \xi_i, x\xi_j \rangle) h_\psi y_k \rangle| \\ &= |\langle \eta_i \otimes h_\psi, x\eta_j \otimes h_\psi y_k \rangle - \langle \xi_i \otimes h_\psi, x\xi_j \otimes h_\psi y_k \rangle| \\ &< 1 \end{aligned}$$

for $x \in E, 1 \leqslant i, j \leqslant n, k = 1, \ldots, p$, so that $X \in V$. □

1.13. REMARKS. (a) Let $X_0 \in C(M, N)$ with a cyclic vector ξ_0. Then it is easy to see that X_0 has a basis of neighbourhoods of the form $V(X_0; \mathscr{V}, E, \{\xi_0\})$. In particular, the identity correspondence $Y_M = M$ has $V(M; \mathscr{V}, E)$ as basis of neighbourhoods, where $\mathscr{V}$ is a σ-weak neighbourhood of O in M, E is a finite subset of M, and $V(M; \mathscr{V}, E)$ is the set of $X \in C(M, M)$ such that there exists $\eta \in X$ with $\langle \eta, x\eta \rangle - x \in \mathscr{V}$ for $x \in E$.

(b) Let (Φ_i) be a net of normal completely positive maps from M to N and let

$\Phi: M \to N$ be also a normal completely positive map. If $\Phi_i(x)$ converges σ-weakly to $\Phi(x)$ for all $x \in M$, then obviously X_{Φ_i} tends to X_Φ in $C(M, N)$.

1.14. As it has already been pointed out in [19] and [21], the notions of irreducibility, weak containment, type, still apply to M–N correspondences when the latter are regarded as representations of the C^*-algebra $M \otimes_{\text{bin}} N^0$.

DEFINITION. We say that a correspondence $X \in C(M, N)$ is *irreducible* if the commutant of M in $\mathscr{L}_N(X)$ is reduced to the scalar operators.

Thanks to the Lemma1.5 (iii), this means that the associated representation of $M \otimes_{\text{bin}} N^0$ is irreducible.

DEFINITION. We say that a correspondence $X \in C(M, N)$ *is weakly contained* in $Y \in C(M, N)$ if the associated representation π_X of $M \otimes_{\text{bin}} N^0$ is weakly contained in the representation π_Y, that is $\operatorname{Ker} \pi_X \supset \operatorname{Ker} \pi_Y$.

This means that π_X(resp. X) belongs to the closure of the set of finite direct sums of copies of π_Y(resp. Y) in the set of representations of $M \otimes_{\text{bin}} N^0$ gifted with the quotient topology of Fell (resp. in $C(M, N)$) ([12], Th. 1.1). When X is irreducible, this is equivalent to the fact that π_X belongs to the closure of $\{\pi_Y\}$, or to the fact that X is in the closure of $\{Y\}$ in $C(M, N)$ (see [12], or ([10], §3.4)).

2. Cocycles, positive type functions and correspondences

In this section we consider a (W^*-) dynamical system (N, G, α) where G is a discrete group and α is an homomorphism from G into the group of automorphisms of N.

2.1. DEFINITION. Let K be a Hilbert space. A map $g \to T_g$ from G into the unitary group of $\mathscr{L}(K) \otimes N = \mathscr{L}_N(K \otimes N)$ such that

$$T_{st} = T_s(I_K \otimes \alpha_s)(T_t), \quad \text{for } s, t \in G$$

will be called a *unitary cocycle* for (N, G, α).

We denote by $Z(N, G, \alpha)$ the set of such cocycles, where of course the Hilbert space K may vary.

To every unitary representation π of G in H_π, we can associate the cocycle $T: s \mapsto \pi(s) \otimes 1$, with values in the unitary group of $\mathscr{L}(H_\pi) \otimes N$. When π is the trivial representation of G we obtain the *identity cocycle* $I: s \mapsto 1 \in N$. The left regular representation of G is denoted by λ as well as the associated cocycle $s \mapsto \lambda(s) \otimes 1$, with values in the unitary group of $\mathscr{L}(l^2(G)) \otimes N$. It is called the (left) *regular cocycle* for (N, G, α).

Consider now the special case where N is an abelian von Neumann algebra.

Then there exist (in an essentially unique way) a probability space (X, μ) and a Borel G-action $(x, s) \mapsto xs$ leaving μ quasi-invariant such that

$$(\alpha_s f)(x) = f(xs) \ \mu\text{.a.e.,} \quad \text{for } f \in L^\infty(X, \mu).$$

Let T be a cocycle for (N, G, α) with values in the unitary group of $\mathscr{L}(K) \otimes N = L^\infty(X, \mathscr{L}(K))$. Let $T_s(x) = \beta(x, s)$ μ.a.e. for all $s \in G$. Then the cocycle equality becomes

$$\beta(x, st) = \beta(x, s)\beta(xs, t) \ \mu\text{.a.e.} \quad \text{for all } s, t \in G.$$

Thus the elements of $Z(N, G, \alpha)$ are the unitary cocycles considered by Zimmer in [27].

2.2. DEFINITION (see [6]). Let X be a Hilbert N-module. An homomorphism $v: s \mapsto v_s$ from G into the group of $\mathbb{C}$-linear, bijective, bicontinuous maps of X onto itself will be called an *action of G on X*. We say that the action is *α-equivariant* if

$$\alpha_t\langle \eta, \xi \rangle = \langle v_t\eta, v_t\xi \rangle, \quad \forall t \in G, \quad \xi, \eta \in X,$$
$$v_t(\xi x) = v_t(\xi)\alpha_t(x), \quad \forall t \in G, \quad \xi \in X, \quad x \in N.$$

2.3. Let K be a Hilbert space. It is easily checked that we can define an α-equivariant action $\hat{\alpha}^k$ (or more simply $\hat{\alpha}$) of G on $K \otimes N$ by

$$\hat{\alpha}_s(k \otimes x) = k \otimes \alpha_s(x), \quad \text{for } k \in K, \quad x \in N.$$

Consider now a cocycle T for (N, G, α), with values in the unitaries of $\mathscr{L}(K) \otimes N$. Then $s \mapsto T_s \circ \hat{\alpha}_s$ is an α-equivariant action of G on $K \otimes N$, since we have

$$(I_K \otimes \alpha_s)(S) = \hat{\alpha}_s \circ S \circ \hat{\alpha}_{s^{-1}}$$

for $S \in \mathscr{L}(K) \otimes N$ and $s \in G$. Conversely, if v is an α-equivariant action of G on $K \otimes N$, then $s \mapsto T_s = v_s \circ \hat{\alpha}_{s^{-1}} \in \mathscr{L}_N(K \otimes N) = \mathscr{L}(K) \otimes N$ is a unitary cocycle. In this way, we obtain a natural bijection between $Z(N, G, \alpha)$ and the set of α-equivariant G-actions on Hilbert N-modules of the form $K \otimes N$.

2.4. Recall from [3] that a map $s \mapsto f(s)$ from G into N is said to be of *positive type* (with respect to α) if for every $s_1, \ldots, s_n \in G$, the matrix $(\alpha_{s_i}(f(s_i^{-1}s_j))) \in M_n(N)$ is positive.

Let v be an α-equivariant action of G on a Hilbert N-module X, and take $\xi \in X$. Then $s \mapsto \langle \xi, v_s\xi \rangle$ is a positive type function with values in N. Conversely, every positive type function comes in such a way from an α-equivariant action ([3], Prop. 2.3). We may consider only self-dual modules, and even of the type $K \otimes N$:

LEMMA. *Let f be a positive type map from G into N. There exist an Hilbert space K, an α-equivariant action v of G on $K \otimes N$ and a vector $\xi \in K \otimes N$ such that $f(s) = \langle \xi, v_s \xi \rangle$ for $s \in G$.*

Proof. By ([3], Prop. 2.3), there exist a Hilbert N-module E, an α-equivariant action w on E, and a vector $\eta \in E$ such that $f(s) = \langle \eta, w_s \eta \rangle$ for $s \in G$. We denote by X the self-dual completion of E, which can be viewed as the set of N-module bounded maps of E into N (see [20], §3). Then it is easily shown that w may be extended to an α-equivariant action $\tilde{w}$ on X by

$$\tilde{w}_s(\tau)(\xi) = \alpha_s(\tau(w_{s^{-1}}\xi)) \quad \text{for } s \in G, \xi \in E, \tau \in X.$$

By ([20], Th. 3.12) X is isomorphic to a self-dual Hilbert N-submodule of $l^2(I) \otimes N$, where I is a well chosen infinite set of indices, and thus the Hilbert N-modules $X \oplus (l^2(I) \otimes N)$ and $l^2(I) \otimes N$ are isomorphic. Let v be the α-equivariant action on $l^2(I) \otimes N$ transfered by such an isomorphism from the action on $X \oplus (l^2(I) \otimes N)$ which is equal to w on X and to $\hat{\alpha}$ on $l^2(I) \otimes N$. If ξ is the vector in $l^2(I) \otimes N$ which corresponds to $\eta \in X \oplus (l^2(I) \otimes N)$, then we have $f(s) = \langle \xi, v_s \xi \rangle$ for $s \in G$. □

2.5. In the rest of Section 2, we denote by M the crossed product $N \times_\alpha G$. Recall that M is generated by N and by the range of an homomorphism $s \mapsto u_s$ from G into the unitary group of M such that $u_s x u_{s^{-1}} = \alpha_s(x)$ for $x \in N$ and $s \in G$. More precisely, every element of M may be written in a unique way as a σ-weakly convergent sum $\sum_{s \in G} u_s x_s$, where $x_s \in N$ for $s \in G$. We denote by E the faithful normal conditional expectation of M onto N such that $E(\sum_{s \in G} u_s x_s) = x_e$, where e is the neutral element of G. Let $(\varepsilon_s)_{s \in G}$ be the canonical orthonormal basis of $l^2(G)$. It is straightforward to check that the Hilbert N-modules X_E and $l^2(G) \otimes N$ are isomorphic by the map sending $\sum_{s \in G} u_s x_s \in M \subset X_E$ onto $\sum_{s \in G} \varepsilon_s \otimes x_s$. Hence, we may identify $\mathscr{L}_N(X_E)$ with $\mathscr{L}(l^2(G)) \otimes N$, and it is easy to see that when we make this identification, an element $x = \sum_{s \in G} u_s x_s \in M \subset \mathscr{L}_N(X_E)$ becomes the matrix $(x_{s,t})$ where $x_{s,t} = \alpha_{s^{-1}}(x_{ts^{-1}})$ for $s, t \in G$. In other words, the embedding $M \subset \mathscr{L}_N(X_E)$ is the well known embedding of $N \times_\alpha G$ in $\mathscr{L}(l^2(G)) \otimes N$ (see [25] for instance).

Take a normal faithful state φ on N and let $\psi = \varphi \circ E$. The Hilbert space $L^2(M, \psi)$ is isomorphic to $l_2(G) \otimes L^2(N, \varphi)$ by the map which sends $\sum_{s \in G} u_s x_s \in M \subset L^2(M, \psi)$ onto $\sum_{s \in G} \varepsilon_s \otimes x_s$ (where N is here viewed as a subspace of $L^2(N, \varphi)$). With this identification, $x \in N \subset M \subset \mathscr{L}(L^2(M, \psi))$ becomes the operator sending $\xi \in l^2(G) \otimes L^2(N, \varphi)$ onto $s \mapsto \alpha_{s^{-1}}(x)\xi(s)$ and $u_s \in M$ becomes the operator $\lambda_s \otimes 1$.

2.6. To each cocycle for $(Z(N), G, \alpha)$ we can associate in a natural way a M–M correspondence. This has already been noticed in ([2], Prop. 4.3), and extends a construction of ([9], proof of Th. 2) where $N = \mathbb{C}$.

PROPOSITION. *Let T be a cocycle for $(Z(N), G, \alpha)$ with values in the unitary group of $\mathscr{L}(K) \otimes Z(N)$ and let $X = K \otimes M$. There exists a normal homomorphism π of M into $\mathscr{L}_M(K \otimes M) = \mathscr{L}(K) \otimes M$ such that*

$$\pi(x) = 1_K \otimes x, \ \forall x \in N,$$
$$\pi(u_s) = T_s \circ (1_K \otimes u_s), \forall s \in G.$$

Thus (X, π) is a M–M correspondence, which will be said to be associated to T.

Proof. We identify $(K \otimes M) \otimes_M L^2(M)$ to the Hilbert space tensor product $K \otimes L^2(M)$ in the obvious way, and we denote by ρ the canonical injective normal homomorphism from $\mathscr{L}_M(X)$ into $\mathscr{L}(X \otimes_M L^2(M)) = \mathscr{L}(K \otimes L^2(M))$ (see 1.5). We shall prove that π comes from a normal homomorphism from M into $\mathscr{L}(K \otimes L^2(M))$ via ρ. For each $S \in \mathscr{L}_M(X) = \mathscr{L}(K) \otimes M$ we have $\rho(S) = S$ considered as acting on $K \otimes L^2(M)$ in the natural way, since this is clearly true for decomposable elements of $\mathscr{L}(K) \otimes M$.

We take $L^2(M) = l^2(G) \otimes L^2(N)$ (see 2.5) and we write the elements ξ of $K \otimes l^2(G) \otimes L^2(N)$ as maps from G into $K \otimes L^2(N)$. Then we have

$$(\rho(\pi(x))\xi)(s) = (1_K \otimes \alpha_{s^{-1}}(x))\xi(s) \qquad \text{for } x \in N,$$
$$(\rho(\pi(u_t))\xi)(s) = (I_K \otimes \alpha_{s^{-1}})(T_t))\xi(t^{-1}s) \quad \text{for } t \in G,$$

where 1_K is the unit of $\mathscr{L}(K)$ and I_K the identity automorphism of $\mathscr{L}(K)$. Denote by w the unitary operator on $K \otimes l^2(G) \otimes L^2(N)$ such that

$$(w\xi)(s) = (I_K \otimes \alpha_{s^{-1}})(T_s)\xi(s), \quad \text{for } s \in G.$$

Since $1_K \otimes \alpha_{s^{-1}}(x) = (I_K \otimes \alpha_{s^{-1}})(1_K \otimes x)$ and $(I_K \otimes \alpha_{s^{-1}})(T_s)$ commute for $x \in N$ and $s \in G$, we see that $w^*\rho(\pi(x))w = \rho(\pi(x))$. On the other hand, for $\xi \in K \otimes l^2(G) \otimes L^2(N)$ and $s, t \in G$ we have

$$\begin{aligned}(w^*\rho(\pi(u_t))w\xi)(s) &= [(I_K \otimes \alpha_{s^{-1}})(T_s^* T_t)]\,[(I_K \otimes \alpha_{s^{-1}t})(T_{t^{-1}s})]\xi(t^{-1}s)\\ &= \xi(t^{-1}s)\end{aligned}$$

by the cocycle property on T. Hence π is the normal homomorphism from M into $\mathscr{L}_M(X)$ such that $\rho(\pi(x)) = w(1_K \otimes x)w^* \in \mathscr{L}(K \otimes L^2(M))$ for each $x \in M \subset \mathscr{L}(L^2(M))$. □

2.7. PROPOSITION. (i) *If T is the identity cocycle for $(Z(N), G, \alpha)$, the associated M–M correspondence is the identity correspondence.*

(ii) *Y_N is the M–M correspondence associated to the regular cocycle for* $(Z(N), G, \alpha)$.

Proof. (i) is obvious. Let us prove (ii). The Hilbert M-module $Y_N = X_E \otimes_N X_N$ is isomorphic to $(l^2(G) \otimes N) \otimes_N X_N$ (see 2.5) and thus to $l^2(G) \otimes M$ by the map which sends $(\sum_{s\in G} u_s x_s) \otimes y \in X_E \otimes_N X_N$ onto $\sum_{s\in G} \varepsilon_s \otimes x_s y$. If we identify Y_N and $l^2(G) \otimes M$ thanks to this isomorphism we see that the left action π' of M on Y_N becomes the action on $l^2(G) \otimes M$ given by

$$(\pi'(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s)$$
$$(\pi'(u_t)\xi)(s) = \xi(t^{-1}s)$$

for $\xi \in l^2(G) \otimes M, x \in N$ and $s, t \in G$.

Let w be the automorphism of $l^2(G) \otimes M$ such that $(w\xi)(s) = u_s\xi(s)$. Then we have

$$w\pi'(x)w^* = 1_{l^2(G)} \otimes x, \quad \forall x \in N,$$
$$w\pi'(u_s)w^* = \lambda_s \otimes u_s, \quad \forall s \in G.$$

Therefore, Y_N is equivalent to the M–M correspondence associated to the regular cocycle for $(Z(N), G, \alpha)$. □

2.8. The following proposition extends the construction of completely positive maps carried out by Haagerup in ([15], Lemma 1.1).

PROPOSITION. *Let f be a positive type map from G into $Z(N)$ with respect to α. Then there exists a unique normal completely positive map Φ_f from M into M such that*

$$\Phi_f(u_s x) = f(s)u_s x \quad \textit{for } s \in G \quad \textit{and } x \in N,$$

and Φ_f is N-bilinear.

More precisely, suppose that f is given by $f(s) = \langle \xi, v_s \xi \rangle$ as in lemma 2.4 but with N replaced by $Z(N)$. Then, denoting by T the cocycle corresponding to v, Φ_f is the coefficient of the M–M correspondence associated to T, which is defined by $\xi \in K \otimes Z(N) \subset K \otimes M$.

Proof. The unicity of Φ_f is obvious. Let $(K \otimes M, \pi)$ be the M–M correspondence associated to T. For $x \in N$ and $t \in G$, we have

$$\begin{aligned}\langle \xi, \pi(u_t)x\xi \rangle_M &= \langle \xi, \pi(u_t)\xi x \rangle_M \text{ since } \xi \in K \otimes Z(N)\\ &= \langle \xi, T_t \circ (1_K \otimes u_t)\xi u_t^{-1} \rangle_M u_t x\\ &= \langle \xi, T_t \circ \hat{\alpha}_t(\xi) \rangle u_t x = f(t)u_t x.\end{aligned}$$

Thus $y \mapsto \langle \xi, \pi(y)\xi \rangle_M$ is a N-bilinear normal completely positive map with the required property. □

2.9. REMARK. Suppose that G is freely acting on N in the sense of [16] and let Φ be a N-bilinear normal completely positive map from M to M. For $s \in G$, put $f(s) = \Phi(u_s)u_s^*$. We easily check that $f(s) \in N' \cap M$, which is equal to $Z(N)$ since the action α is free. Now f is a positive type map because we have, for $a_1, \dots, a_n$ in $Z(N)$ and $s_1, \dots, s_n$ in G,

$$\sum_{i,j=1}^{n} a_i^* \alpha_{s_i}(f(s_i^{-1}s_j))a_j = \sum_{i,j=1}^{n} a_i^* u_{s_i}\Phi(u_{s_i}^* u_{s_j})u_{s_j}^* a_j \geqslant 0$$

by the complete positivity of Φ.

Thus, when the G-action α is free, every N-bilinear normal completely positive map Φ from M to M comes from a positive type function as indicated in 2.8.

2.10. Of course, if f is the constant map with value equal to the unit of $Z(N)$, the associated completely positive map is the identity automorphism of M.

PROPOSITION. *Let f be a positive type map from G to $Z(N)$ with finite support. Then the associated completely positive map Φ_f is a coefficient of the M–M correspondence Y_N.*

Proof. Let $\tilde{\alpha}$ be the α-equivariant action of G on $l^2(G) \otimes Z(N)$, associated to the regular cocycle λ, which means that $(\tilde{\alpha}_t h)(s) = \alpha_t(h(t^{-1}s))$ for $h \in l^2(G) \otimes Z(N)$ and $s, t \in G$. Since f has a finite support, by ([3], Prop. 2.5) there exists $h \in l^2(G) \otimes Z(N)$ such that $f(s) = \langle h, \tilde{\alpha}_s h \rangle$. Then the result follows from Propositions 2.8 and 2.7(ii). □

2.11. We denote by $PT_1(Z(N), G, \alpha)$ the set of positive type maps from G to $Z(N)$ with respect to α, such that $\sup_{s \in G} \|f(s)\| \leqslant 1$ (or, equivalently $f(e) \leqslant 1$ ([3], Prop. 2.4)), and we endow this set with the topology of pointwise σ-weak convergence. The space of normal completely positive maps from M to M will be denoted by $CP(M)$ and equipped similarly with the topology of pointwise σ-weak convergence.

PROPOSITION. *The map $f \mapsto \Phi_f$ from $PT_1(Z(N), G, \alpha)$ into $CP(M)$ is continuous.*

Proof. We show the continuity at $f_0 \in PT_1(Z(N), G, \alpha)$. Let $\mathscr{V}$ be a σ-weak neighbourhood of O in M and $\{x^1, \dots, x^n\}$ a finite subset of M. We write $x^i = \sum_{s \in G} u_s x_s^i$ for $1 \leqslant i \leqslant n$.

We choose a faithful normal state φ on N, and for $a \in N$ and $s \in G$, we denote by $\varphi_{a,s}$ the form $x \mapsto \varphi \circ E(au_s x)$ on M. When (a, s) describes $N \times G$, we get a total family of elements in the predual M_*, with respect to the norm. Hence, we may find $a_1, \dots, a_p$ in N and $s_1, \dots, s_p$ in G such that for every $y \in M$ satisfying

$$\|y\| \leqslant 2 \sup_{1 \leqslant j \leqslant n} \|x^j\| \quad \text{and} \quad |\varphi_{a_i, s_i}(y)| < 1, \quad \text{for } i = 1, \dots, p$$

we have $y \in \mathscr{V}$.

Let $\mathscr{W}$ be the σ-weak neighbourhood of O in $Z(N)$ given by

$$\mathscr{W} = \{x \in Z(N), |\varphi(a_i \alpha_{s_i}(x) x^j_{s_i^{-1}})| < 1 \text{ for } 1 \leqslant i \leqslant p \text{ and } 1 \leqslant j \leqslant n\}.$$

We shall show that if $f \in PT_1(Z(N), G, \alpha)$ satisfies

$$f(s_i^{-1}) - f_0(s_i^{-1}) \in \mathscr{W} \quad \text{for } i = 1, \dots, p,$$

then $\Phi_f(x^j) - \Phi_0(x^j) \in \mathscr{V}$ for $j = 1, \dots, n$ (where $\Phi_0 = \Phi_{f_0}$), and this will end the proof. We have

$$|\varphi_{a_i, s_i}(\Phi_f(x^j) - \Phi_0(x^j))| = \left| \sum_{t \in G} \varphi(a_i E(u_{s_i}(f(t) - f_0(t)) u_t x^j_t)) \right|$$
$$= |\varphi(a_i \alpha_{s_i}(f(s_i^{-1}) - f_0(s_i^{-1})) x^j_{s_i^{-1}})| < 1$$

for $i = 1, \dots, p$ and $j = 1, \dots, n$. As Φ_f and Φ_0 are contractions, we get $\|\Phi_f(x^j) - \Phi_0(x^j)\| \leqslant 2\|x^j\|$, and therefore we have $\Phi_f(x^j) - \Phi_0(x^j) \in \mathscr{V}$. □

3. Amenability

3.1. DEFINITION (see [21] Def. 3.1). Let $N \subset M$ be von Neumann algebras. We say that M is *amenable relative* to N (or that the *inclusion is amenable*) if the identity correspondence $Y_M = M$ is weakly contained in Y_N.

Note that when there exists a faithful normal conditional expectation from M onto N with finite index, the inclusion is amenable since Y_M is then contained in Y_N (see 1.11).

Consider now the case $N = \mathbb{C}$. The representation of $M \otimes_{\text{bin}} M^0$ defined by the identity correspondence is $x \otimes y^0 \mapsto xJy^*J$ acting on $L^2(M)$, where, as usual, J is the antilinear involution on $L^2(M)$ given by the Tomita–Takesaki theory. The representation of $M \otimes_{\text{bin}} M^0$ associated to the coarse correspondence is $x \otimes y^0 \mapsto x \otimes Jy^*J$ acting on $L^2(M) \otimes L^2(M)$. Thus the inclusion $\mathbb{C} \subset M$ is amenable if and only if the map $x \otimes y \mapsto xy$ from the algebraic tensor product $M \odot M'$ into the C^*-subalgebra of $\mathscr{L}(L^2(M))$ generated by M and M' is continuous when $M \odot M'$ is equipped with the minimal C^*-norm. It is proved in

([11]), Prop. 4.5) that this property is equivalent to semi-discreteness, and by [11], [7], [5] and [26] it is equivalent to injectivity.

The following result, which extends a part of Popa's Theorem 3.2.3 in [21], shows that relative amenability implies a relative injectivity property.

3.2. PROPOSITION. *Let $N \subset M$ be an amenable inclusion. Then there exists a norm one projection from $\mathscr{L}_N(\bar{X}_N)$ onto M (naturally identified to a von Neumann subalgebra of $\mathscr{L}_N(\bar{X}_N)$).*

Proof. By hypothesis, Y_M belongs to the closure in $C(M)$ of the set of finite direct sums of copies of Y_N. Hence there exists a net $(\eta_i)_{i\in I}$, where each η_i is a finite sequence $\eta_1^i, \ldots, \eta_{p_i}^i$ of elements of Y_N, such that for each $x \in M$

$$\sum_{1\leqslant j\leqslant p_i} \langle \eta_j^i, x\eta_j^i\rangle \text{ converges } \sigma\text{-weakly to } x.$$

Choose an ultrafilter $\mathscr{U}$ finer than the filter obtained from the directed set I. Let φ be a normal positive form on M and take $x \in \mathscr{L}_N(\bar{X}_N)$ (identified to the von Neumann subalgebra $\rho(\mathscr{L}_N(\bar{X}_N))$ of $\mathscr{L}_M(Y_N)$ by Lemma 1.5(ii)). We have

$$\left|\varphi\left(\sum_{1\leqslant j\leqslant p_i} \langle \eta_j^i, x\eta_j^i\rangle\right)\right| \leqslant \|x\|\varphi\left(\sum_{1\leqslant j\leqslant p_i} \langle \eta_j^i, \eta_j^i\rangle\right), \quad \text{for } i \in I.$$

This allows us to define

$$S(\varphi, x) = \lim_{\mathscr{U}} \varphi\left(\sum_{1\leqslant j\leqslant p_i} \langle \eta_j^i, x\eta_j^i\rangle\right)$$

and we get

$$|S(\varphi, x)| \leqslant \|x\| \lim_{\mathscr{U}} \varphi\left(\sum_{1\leqslant j\leqslant p_i} \langle \eta_j^i, \eta_j^i\rangle\right) = \|x\|\varphi(1) = \|x\|\ \|\varphi\|.$$

It follows that $(\varphi, x) \mapsto S(\varphi, x)$ is a bilinear continuous form on $M_* \times \mathscr{L}_N(\bar{X}_N)$. Thus, for each $x \in \mathscr{L}_N(\bar{X}_N)$ there is an element $\Phi(x)$ in M well defined by

$$\varphi(\Phi(x)) = S(\varphi, x), \quad \text{for } \varphi \in M_*.$$

Obviously Φ is positive with $\Phi(x) = x$ for all $x \in M$, and therefore it is a norm one projection from $\mathscr{L}_N(\bar{X}_N)$ onto M (see [24] Th. 3.1). □

3.3. REMARKS. (1) It follows from Proposition 3.2 that if $N \subset M$ is an amenable inclusion, and if N is an injective von Neumann algebra, then M is also injective, since it is the case for $\mathscr{L}_N(\bar{X}_N)$.

(2) The converse of the above proposition has been proved by Popa in ([21] Th. 3.2.3) when M is a finite factor. When $N = \mathbb{C}$, one has $\mathscr{L}_N(\bar{X}_N) = \mathscr{L}(L^2(M))$, and the converse of Proposition 3.2 is the fact that injectivity implies semi-discreteness. The following proposition gives another case where this converse is true.

3.4. Let (N, G, α) be a dynamical system as in Section 2. In [1] we have defined a notion of amenability for the action α, generalizing the corresponding notion introduced by Zimmer [28] in ergodic theory. For G discrete we have shown that the action α is amenable if and only if there exists a norm one projection from $\mathscr{L}(l^2(G)) \otimes N$ onto $N \times_\alpha G$ (canonically embedded into $\mathscr{L}(l^2(G)) \otimes N$) (see [1], Prop. 3.11).

PROPOSITION. *Let (N, G, α) be a dynamical system with G discrete. The following conditions are equivalent*:

(i) *the inclusion $N \subset M = N \times_\alpha G$ is amenable*;

(ii) *the action of G on N is amenable*;

(iii) *there is a norm one projection from $\mathscr{L}_N(\bar{X}_N) = \mathscr{L}(l^2(G)) \otimes N$ onto M.*

Proof. Remark that the embedding of M into $\mathscr{L}_N(\bar{X}_N)$ identified to $\mathscr{L}(l^2(G)) \otimes N$ is the usual embedding in the theory of crossed products (see 2.5). Then the equivalence between (ii) and (iii) follows from ([1] Prop. 3.11). The implication (i) $\Rightarrow$ (iii) has been proved in Proposition 3.2. So it remains to see that (ii) $\Rightarrow$ (i). By ([3], Th. 3.3) there exist a net $(f_i)_{i\in I}$ of elements of $PT_1(Z(N), G, \alpha)$ with finite support such that $f_i(s)$ converges to 1 σ-weakly for every $s \in G$. For $i \in I$, denote by Φ_i the completely positive map associated to f_i, and let $X_i = X_{\Phi_i}$. We have $X_i \subset Y_N$ since Φ_i is a coefficient of Y_N by Proposition 2.10. Furthermore, it follows from Proposition 2.11 that $\Phi_i(x)$ tends to x σ-weakly for all $x \in M$, and thus $\lim_i X_i = Y_M$ in $C(M)$. This proves that Y_M belongs to the closure of Y_N in $C(M)$. □

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