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Extension of mixed Hodge Modules

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Introduction

In [S1–2] we introduced the notion of mixed Hodge Modules on complex algebraic varieties X which corresponds philosophically to that of mixed perverse sheaves [BBD], and proved the stability of its bounded derived categories $D^b\mathrm{MHM}(X)$ by the standard functors f_*, $f_!$, $f^!$, ψ_g, $\phi_{g,1}$, $\mathbb{D}$, $\boxtimes$, $\otimes$, $\mathscr{H}om$ so that these functors are compatible with the corresponding functors on the underlying $\mathbb{Q}$-complexes. Here we applied simply the well-known formula

$$M \otimes N = \delta^*(M \boxtimes N),\ \mathscr{H}om(M, N) = \delta^!(\mathbb{D}M \boxtimes N) \tag{0.1}$$

for the last two functors, where $\delta\colon X \to X \times X$ is the diagonal immersion. But these definitions are justified if we have the following

(0.2) THEOREM. *For* $M, N \in D^b\mathrm{MHM}(X)$, *we have canonical isomorphisms*

$$\mathrm{Hom}(M \otimes N, \mathbb{D}^H_X) = \mathrm{Hom}(M, \mathbb{D}N)$$
$$\mathrm{Hom}(\mathbb{Q}^H_X, \mathscr{H}om(M, N)) = \mathrm{Hom}(M, N)$$

where Hom *is taken in the bounded derived category* $D^b\mathrm{MHM}(X)$.

Here $\mathbb{D}^H_X = \mathbb{D}\mathbb{Q}^H_X = a^!_X\mathbb{Q}^H$, $\mathbb{Q}^H_X = a^*_X\mathbb{Q}^H$ with $a_X\colon X \to pt$ and $\mathbb{Q}^H \in \mathrm{MHM}(pt)$ the trivial mixed Hodge structure of rank 1 and type (0, 0). The aim of this paper is to give the proof of this theorem. By duality it is enough to show the first assertion. Using the vanishing cycle functors we can construct the natural morphism from the left to the right for M, N mixed Hodge Modules by induction on the dimension of their supports (cf. 1). Then we get the assertion by calculating the effaceability (cf. 2). As a corollary we get a short exact sequence

$$0 \to \mathrm{Ext}^1(\mathbb{Q}^H, H^{i-1}(a_X)_* \mathscr{H}om(M, N)) \to \mathrm{Ext}^i(M, N) \to$$
$$\to \mathrm{Hom}(\mathbb{Q}^H, \mathrm{H}^i(\mathrm{a}_X)_* \mathscr{H}om(M, N)) \to 0$$

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for mixed Hodge Modules M, N on X, where the middle Ext is taken in the abelian category of mixed Hodge Modules, and the first Ext and the last Hom in polarizable mixed Hodge structures, cf. 2.10. For M, N admissible variations of mixed Hodge structures in the sense of Steenbrink-Zucker [SZ] and Kashiwara [K] on a smooth variety X, this implies

$$0 \to \mathrm{Ext}^1(\mathbb{Q}^H, H^0(X, M^* \otimes N)) \to \mathrm{Ext}^1(M, N) \to \mathrm{Hom}(\mathbb{Q}^H, H^1(X, M^* \otimes N)) \to 0$$

where $M^* = \mathcal{H}om(M, \mathbb{Q}_X^H) = (\mathbb{D}M)(-\dim X)[-2\dim X]$ is the dual variation of M, and the middle Ext is taken in the abelian category of admissible variations. We apply this to the extension class defined by the short exact sequence $0 \to W_{i-1}M \to W_iM \to Gr_i^W M \to 0$ inductively so that the admissible variations (whose definition is given locally on a compactification of X) can be understood globally on X by induction on the length of the weight filtration of M, where the polarizable variations of Hodge structures are considered to be well understood. In 3 we prove directly the second exact sequence for admissible variation of mixed $\mathbb{Z}$-Hodge structure in the case M is torsion-free without using the results in 1 and 2, see 3.6. (This argument can be applied to the analytic case if we use [KK].)

The idea of proof of Theorem (0.2) was inspired by the correspondence with Durfee during the preparation of the joint paper (see the remark after the proof of 3.4 in [DS]). I would like to thank him for a good question, and Institut des Hautes Etudes Scientifiques and Institute for Advanced Study for hospitality.

In this note variety means a reduced and separated algebraic variety over $\mathbb{C}$. For the underlying filtered $\mathcal{D}$-Modules of mixed Hodge Modules we use the analytic $\mathcal{D}$-Modules rather than the algebraic ones to simplify the calculation of Hodge filtration. This is allowed by GAGA and the extendability of mixed Hodge Modules to any compactification.

1. Vanishing cycle functors

In this section we construct the canonical morphism $\mathrm{Hom}(M \otimes N, \mathbb{D}_X^H) \to \mathrm{Hom}(M, \mathbb{D}N)$ for $M, N \in \mathrm{MHM}(X)$.

1.1. Let S denote the one-dimensional affine space $\mathrm{Spec}\,\mathbb{C}[t]$ with the coordinate t. Let E_k be the standard unipotent variation of $\mathbb{Q}$-mixed Hodge structures on $S^* = S\backslash\{0\}$ of rank $k+1$ such that its monodromy has one Jordan block and its stalk at $1 \in S$ splits naturally over $\mathbb{Q}$ and has weights $0, 2, \ldots, 2k$. Here standard means that the Hodge filtration is given like nilpotent orbits, i.e. the Hodge bundles are generated over $\mathcal{O}_{S^*}$ by the global sections annihilated by $(t\partial_t)^{k+1}$, and $Gr_{2i}^W(E_k)_1$ the graduation of the stalk at 1 is given an isomorphism with $\mathbb{Q}(-i)$ in a compatible way with the action of N the logarithm of the unipotent part of the monodromy tensored by $(2\pi i)^{-1}$. We have natural inclusions $E_k \to E_{k+1}$ compatible with the trivialization of $(E_k)_1$ so that E_k becomes an inductive system. Clearly E_k are admissible variations of $\mathbb{Q}$-mixed Hodge structures and $E_k[1] \in \mathrm{MHM}(S^*)$, cf. [S2]. By definition we have a natural isomorphism of mixed Hodge structures:

$$\psi_t E_k[1] = (E_k)_1 \quad \text{(the stalk at } 1 \in S\text{)}. \tag{1.1.1}$$

Here note that ψ, ϕ_1 on mixed Hodge Modules correspond to ${}^p\psi = \psi[-1]$, ${}^p\phi_1 = \phi_1[-1]$, cf. [S1–2]. For a subfield A of $\mathbb{C}$, let E_k^A and E_k^D denote the underlying A-local system and (analytic) $\mathscr{D}_{S^*}$-Module of E_k so that $E_k^D = \mathcal{O}_{S^*} \otimes_A E_k^A$. By the trivialization we have a canonical multivalued section e_j of $E_k^A(j)$ $(0 \leqslant j \leqslant k)$ compatible with the scalar extension of A and the inclusions $E_k \to E_{k+1}$ such that $e_j/j! \in Gr_{2j}^W \psi_t E_k^A(j)$ corresponds to $1 \in A$ by (1.1.1) and the trivialization of $Gr_{2j}^W(E_k)_1$, and $e_0, \ldots, e_k$ give the canonical splitting of $(E_k)_1$ as a mixed Hodge structure. Then $Ne_j = je_{j-1}$ where $e_{-1} := 0$. We denote by $\tilde{e}_j$ the corresponding section of E_k^D, i.e.

$$\begin{aligned}\tilde{e}_j &= \exp(-\log t \otimes N)e_j \\ &= \sum_{0 \leqslant i \leqslant j} (j!/i!(j-i)!)(-\log t)^i \otimes e_{j-i}\end{aligned} \tag{1.1.2}$$

so that $t\partial_t \tilde{e}_j = -j\tilde{e}_{j-1}$ and $F^p E_k^D = \otimes_{j \geqslant p} \mathcal{O}_{S^*}\tilde{e}_j$, where F is the Hodge filtration. Here we take a universal covering $\pi: \tilde{S}^* \to S^*$ with coordinate z of $\tilde{S}^*$ such that $\pi^* t = \exp(2\pi i z)$, and put $\log t = 2\pi i z$, where we choose $i = \sqrt{-1}$. We have a natural inclusion

$$E_k^A \to \pi_* A_{\tilde{S}^*} \tag{1.1.3}$$

so that by the natural isomorphism

$$\Gamma(\tilde{S}^*, \pi^*\pi_* A_{\tilde{S}^*}) = (\pi^*\pi_* A_{\tilde{S}^*})_0 = (\pi_* A_{\tilde{S}^*})_1 = \prod_{n \in \mathbb{Z}} A_{\{n\}} \tag{1.1.4}$$

e_j corresponds to $\{(2\pi i)^j \otimes n^j\}_{n \in \mathbb{Z}}$, and $\tilde{e}_j$ to $(-2\pi i z)^j = (-\log t)^j \in \Gamma(S^*, \pi_* \mathcal{O}_{\tilde{S}^*}) = \Gamma(\tilde{S}^*, \mathcal{O}_{\tilde{S}^*})$, cf. [S3,2.3]. We have a natural multiplication $E_j \otimes E_k \to E_{j+k}$ compatible with the natural multiplication of $\pi_* A_{\tilde{S}^*}$ (and the stalk wise multiplication of $\Pi_{n \in \mathbb{Z}} A_{\{n\}}$) by (1.1.3) (and (1.1.4)) so that $e_j \otimes e_k$ goes to e_{j+k}.

1.2. LEMMA. *Let $f: X \to S$ be a function on a variety X, and put $X_0 = f^{-1}(0), X^* = X \backslash X_0$ with $i: X_0 \to X, j: X^* \to X$ the natural inclusions. Then for*

$M \in \mathrm{MHM}(X)$ *we have a natural isomorphism*

$$\psi_{f,1}(j^* M \otimes f^* E_k) = \psi_{f,1} M \boxtimes \psi_t E_k[1] \tag{1.2.1}$$

compatible with that for underlying perverse sheaves.

Proof. The assertion is local and we may assume X smooth and $X = X_0 \times S$ using the graph of f. Then the underlying filtered $\mathscr{D}_X$-Module of $j_*(j^* M \otimes f^* E_k)$ is naturally isomorphic to $\bigoplus_{0 \leqslant j \leqslant k}(M', F[-j]) \otimes \tilde{e}_j$, where (M', F) denotes the underlying filtered $\mathscr{D}$-Module of $j_* j^* M$, and $P(m \otimes \tilde{e}_j) = (Pm \otimes \tilde{e}_j)$, $\partial_t(m \otimes \tilde{e}_j) = \partial_t m \otimes \tilde{e}_j - t^{-1} m \otimes j\tilde{e}_{j-1}$ for $P \in \mathscr{D}_{X_0}, m \in M'$. This implies the isomorphism (1.2.1) for the underlying filtered $\mathscr{D}$-Modules, and following the definition of the isomorphism $\psi DR \simeq DR\psi$ in [S1, 3.4], we can check the compatibility with that for perverse sheaves using the inverse of (1.1.2): $e_j = \sum (j!/i!(j-i)!)(\log t)^i \tilde{e}_{j-i}$. The compatibility of the isomorphism with the weight filtration W follows from the definition of the relative monodromy filtration, because the induced filtration on the left hand side of (1.2.1) by the weight filtration of $(j^* M \otimes f^* E_k)$ is the convolution of the induced filtration on $\psi_{f,1} M$ and the weight filtration of $\psi_t E_k[1]$, and its relative monodromy filtration is given by the convolution of the relative monodromy filtration of $\psi_{f,1} M$ and the weight filtration of $\psi_t E_k[1]$.

1.3. PROPOSITION. *With the above notation, let* $M \in \mathrm{MHM}(X)$ *and* $k' \in \mathbb{N}$, *and assume* $N^{k'+1}\psi_{f,1} M = 0$. *Then* $H^{-1} i^* j_*(j^* M \otimes f^* E_k)$ *is independent of* $k \geqslant k'$ *by* $E_k \to E_{k+1}$, *and we have a natural isomorphism*

$$\varinjlim_k H^{-1} i^* j_*(j^* M \otimes f^* E_k) = \psi_{f,1} M \tag{1.3.1}$$

compatible with the definition of $\psi_{f,1}$ *on the underlying perverse sheaf by the inclusion* (1.1.3), *cf.* [S1, 3.4.14] [S3, 2.3].

Proof. By the above lemma and [S2, 2.24] (using $\mathrm{id} \leftarrow \xi_f \to \phi_{f,1}$ in [S2, 2.23]) we have a canonical isomorphism

$$\begin{aligned} H^{-1} i^* j_*(j^* M \otimes f^* E_k) &= \mathrm{Ker}(j_!(j^* M \otimes f^* E_k) \to j_*(j^* M \otimes f^* E_k)) \\ &= \mathrm{Ker}(N\colon \psi_{f,1}(j^* M \otimes f^* E_k) \to \psi_{f,1}(j^* M \otimes f^* E_k)(-1)) \\ &= \mathrm{Ker}(N\colon \psi_{f,1} M \boxtimes \psi_t E_h[1] \to (\psi_{f,1} M \boxtimes \psi_t E_k[1])(-1)) \end{aligned} \tag{1.3.2}$$

where N in the last term is defined by $N \boxtimes id + id \boxtimes N$. Let L be a filtration of $M' := \psi_{f,1} M$ such that $N(L^i M') \subset L^{i+1} M'(-1), L^0 M' = M'$. Put

$$K_{i,k} = \mathrm{Ker}(N\colon M'/L^{i+1} M' \boxtimes \psi_t E_k[1] \to (M'/L^{i+1} M' \boxtimes \psi_t E_k[1])(-1)).$$

Then we can check inductively the surjectivity of $K_{i,k} \to K_{i,k+1}$ and $K_{i+1,k+1} \to K_{i,k+1}$ for $k \geqslant i$ using the short exact sequences:

$$0 \to Gr^i_L M' \to M'/L^{i+1}M' \to M'/L^iM' \to 0$$

and the functoriality of the snake lemma (for the morphisms $E_k \to E_{k+1}$), because $E_k \to E_{k+1}$ induces the zero morphism on the cokernel of N on $Gr^i_L M' \boxtimes \psi_t E_k[1]$ and an isomorphism on its kernel for $k \geqslant 0$. Therefore we get the first assertion, because $K_{i,k} \to K_{i,k+1}$ are always injective. We have a natural morphism from the last term of (1.3.2) to $\psi_{f,1}M$ induced by $\psi_t E_k[1] \to Gr^W_0 \psi_t E_k[1] = \mathbb{Q}^H$, and we can check that it is an isomorphism if $N^{k+1}\psi_{f,1}M = 0$ by a similar argument. For the compatibility with the definition of $\psi_{f,1}$ on the underlying perverse sheaf, the compatibility of (1.3.2) is reduced to the Lemma 1.4 below, because $\xi_f, {}^p\phi_{f,1}$ (cf. the remark after (1.1.1)) induce the identity on the perverse sheaves supported in X_0, cf. [loc. cit]. Here it is enough to show the compatibility up to sign, because we can change the sign of the isomorphism for mixed Hodge Modules. Put $S' = \tilde{S}^*, S'' = S' \times_{S^*} S'$ with $\pi'_1, \pi'_2 : S'' \to S'$ the first and second projection and $\tilde{\delta}: S' \to S''$ the diagonal, i.e. $S'' = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 - z_2 \in \mathbb{Z}\}$ and $\pi'_a(z_1, z_2) = z_a (a = 1, 2), \tilde{\delta}(z) = (z, z)$. Put $\pi'' := \pi_1\pi'_2 = \pi_2\pi'_1$ with $\pi_1 = \pi_2 = \pi : S' \to S^*$, cf. 1.1. We denote by the same symbols the base change of $\pi, \pi_a, \pi'_a, \tilde{\delta}$ by $X \to S$. Let K' be the underlying perverse sheaf of j^*M represented by an injective complex. We have natural isomorphisms

$$\begin{aligned}\psi_f K' &= i^*j_*(\pi_1)_*\pi_1^*K' = (\psi_f(\pi_1)_*\pi_1^*K')^T \\ &= i^*j_*(\pi''_*\pi''^*K')^{T_2} \xrightarrow[\sim]{\tilde{\delta}^*} i^*j_*(\pi_*\pi^*K') = \psi_f K' \end{aligned} \qquad (1.3.3)$$

where T_a is the monodromy induced by the automorphism of S'' defined by $z_a \mapsto z_a + 1$ using the pull-back. We check that the composition is the identity on $\psi_f K'$, and compatible with the above isomorphism by the natural morphisms

$$i^*j_*(K' \otimes f^*E_k^{\mathbb{Q}}) \to i^*j_*(\pi_1)_*\pi_1^*K'$$

$$\psi_f K' \boxtimes \psi_t E_k^{\mathbb{Q}} \to \psi_f(K' \otimes f^*E_k^{\mathbb{Q}}) \to \psi_f(\pi_1)_*\pi_1^*K' = i^*j_*\pi''_*\pi''^*K'.$$

Here the last composition is induced by the product of the pull-back of $\psi_f K'$ by π'_1 and $\psi_t(\pi_1)_*\mathbb{Q}_{S'} = i^*j_*\pi''_*\mathbb{Q}_{S''}$, and $\tilde{\delta}^*$ in (1.3.3) corresponds to the projection of the last term of (1.3.2) to $\psi_{f,1}M$, because $e_j = 0$ on $\operatorname{Im}\tilde{\delta}$ for $j > 0$ by definition, cf. (1.1.4).

1.4. LEMMA. *With the above notation, let K be a perverse sheaf on X^* represented by an injective complex. Then we have the natural isomorphism*

${}^p\phi_{f,1}C(j_!K \to j_*K) = C(N\colon {}^p\psi_{f,1}K \to {}^p\psi_{f,1}K(-1))$ *induced by can:* ${}^p\psi_{f,1}K \xrightarrow{\sim} {}^p\phi_{f,1}j_!K$, Var: ${}^p\phi_{f,1}j_*K \xrightarrow{\sim} {}^p\psi_{f,1}K(-1)$ *so that* ${}^p\phi_{f,1}$ *of the isomorphism*

$$C(j_!K \to j_*K) = i^*j_*K = C(N\colon {}^p\psi_{f,1}K \to {}^p\psi_{f,1}K(-1)) \tag{1.4.1}$$

coincides with the identity on $C(N\colon {}^p\psi_{f,1}K \to {}^p\psi_{f,1}K(-1))$ *in the derived category up to sign, where* ${}^p\psi_{f,1} = \psi_{f,1}[-1]$ (*same for* ϕ) *and the last isomorphism is induced by the natural inclusion* $i^*j_*K \to \psi_{f,1}K$ (cf. [S1, 3.4.14] *for the definition of* $\psi_{f,1}$).

Proof. Put $K' = \psi_{f,1}K$. We have a natural isomorphism

$$i^*j_*K = [K' \xrightarrow{N} K'(-1)] \tag{1.4.2}$$

induced by the natural inclusion $i^*j_*K \to \psi_{f,1}K$ so that the natural inclusion corresponds to the natural projection $[K' \to K'(-1)] \to K'$. Here $N\colon K' \to K'(-1)$ is surjective as a morphism of complex by assumption on K. By definition ${}^p\phi_{f,1} = [i^* \to \psi_{f,1}]$, and we get the isomorphism

$$\begin{aligned} {}^p\phi_{f,1}C(j_!K \to j_*K) &= \begin{bmatrix} 0 & \longrightarrow & K' & \xrightarrow{N} & K'(-1) \\ \downarrow & & \downarrow \scriptstyle\mathrm{id} & & \downarrow \\ K' & \xrightarrow{\mathrm{id}} & K' & \longrightarrow & 0 \end{bmatrix} \\ &= [K' \oplus K' \to K' \oplus K'(-1)] \end{aligned} \tag{1.4.3}$$

because $i^*j_!K = 0$, where the morphism in the last term is defined by $(x, y) \mapsto (x + y, Ny)$. Then ${}^p\phi_{f,1}$ of (1.4.1) is expressed by

$$[K' \oplus K' \to K' \oplus K'(-1)] \to [K' \xrightarrow{N} K'(-1)] \tag{1.4.4}$$

where the morphism is given by $(x, y; z, w) \mapsto (y, w)$, because ${}^p\phi_{f,1}$ is the identity on i^*j_*K and the first isomorphism of (1.4.1) corresponds to the natural projection of ${}^p\phi_{f,1}j_*K = [i^*j_*K \to \psi_{f,1}K]$ onto i^*j_*K. On the other hand the isomorphism can: ${}^p\psi_{f,1}K \to {}^p\phi_{f,1}j_!K$ corresponds to the identity on K', and Var: ${}^p\phi_{f,1}j_*K \to {}^p\psi_{f,1}K(-1)$ to the quasi isomorphism

$$[K' \xrightarrow{(\mathrm{id}, N)} K' \oplus K'(-1)] \to K'(-1)[-1] \tag{1.4.5}$$

defined by $(y; z, w) \to (Nz - w)$, because its restriction to $[\mathrm{Ker}\, N \to K']$ coincides with Var by definition. These morphisms are compatible with the morphism of

${}^p\phi_{f,1} j_! K$ to ${}^p\phi_{f,1} j_* K$, and induce a quasi isomorphism

$$[K' \oplus K' \to K' \oplus K'(-1)] \to [K' \xrightarrow{N} K'(-1)] \tag{1.4.6}$$

defined by $(x, y; z, w) \mapsto (x, Nz - w)$. The sum of (1.4.4) and (1.4.6) is $(x, y; z, w) \mapsto (x + y, Nz)$ and homotopic to zero, where the homotopy is given by $(x, y; z, w) \mapsto (z, 0)$.

1.5. PROPOSITION. *With the notation and assumption of* 1.3 *we have a natural morphism* $M \to j_*(j^*M \otimes f^*E_k)$ *induced by* $E_0 \to E_k$, *and a natural isomorphism*

$$\varinjlim_k H^0[i^*M \to i^*j_*(j^*M \otimes f^*E_k)] = \phi_{f,1} M \tag{1.5.1}$$

compatible with the definition of ${}^p\phi_{f,1}$ *on the underlying perverse sheaf.*

Proof. By $i^* = C(j_! j^* \to \mathrm{id}) = C(\mathrm{can}\colon \psi_{f,1} \to \phi_{f,1})$ in [S2, 2.23–24], $H^0[i^*M \to i^*j_*(j^*M \otimes f^*E_k)]$ is isomorphic to H^0 of

$$\left[\begin{array}{ccc} \psi_{f,1}M & \xrightarrow{\mathrm{can}} & \phi_{f,1}M \\ \downarrow & & \downarrow \mathrm{Var} \\ \psi_{f,1}(j^*M \otimes f^*E_k) & \xrightarrow{N} & \psi_{f,1}(j^*M \otimes f^*E_k)(-1) \end{array}\right] \to \left[\begin{array}{ccc} \psi_{f,1}M & \to & \phi_{f,1}M \\ \downarrow \mathrm{id} & & \downarrow \\ \psi_{f,1}M & \longrightarrow & 0 \end{array}\right]$$
$$= \phi_{f,1}M \tag{1.5.2}$$

if $N^{k+1}\psi_{f,1}M = 0$, where the first morphism induces an isomorphism of H^0 by 1.3. Then we can check that the isomorphism is compatible with the definition of ${}^p\phi_{f,1}$ on the underlying perverse sheaf up to sign by essentially the same argument as in 1.3–4 using the isomorphism $i^*K = [\psi_{f,1}K \xrightarrow{\mathrm{can}} \phi_{f,1}K]$ such that the natural morphism $i^*K \to \psi_{f,1}K$ is identified with the projection. The detail is left to the reader.

1.6. PROPOSITION. *With the notation of* 1.2, *let* M, M' *be mixed Hodge Modules on* X, *and* $\mathbb{S}\colon M \otimes M' \to \mathbb{D}_X^H$ a morphism in $D^b\mathrm{MHM}(X)$. *Then we have canonical morphisms*

$$\begin{aligned} \psi_{f,1}\mathbb{S}\colon \psi_{f,1}M \otimes \psi_{f,1}M' &\to \mathbb{D}_{X_0}^H(1) \\ \phi_{f,1}\mathbb{S}\colon \phi_{f,1}M \otimes \phi_{f,1}M' &\to \mathbb{D}_{X_0}^H \end{aligned} \tag{1.6.1}$$

compatible with ${}^p\psi_{f,1}\mathbb{S}^{\mathbb{Q}}, {}^p\phi_{f,1}\mathbb{S}^{\mathbb{Q}}$ *on the underlying* $\mathbb{Q}$-*complexes, where* $\mathbb{S}^{\mathbb{Q}}$ *denotes the underlying morphism of* $\mathbb{S}$ *on* $\mathbb{Q}$-*complexes.*

Proof. Let $\tilde{N}: E_k \to E_{k-1}(-1)$ be a morphism of variations of mixed Hodge structures such that e_j is sent to $-je_{j-1}$. Then by the isomorphism (1.3.1) the action of $\tilde{N}$ on $\psi_{f,1}M$ corresponds to $\mathrm{id} \otimes N$, cf. [S3, 2.3]. Put $i^*M = C(j_!j^*M \to M)$, $M_k = j^*M \otimes f^*E_k$, ${}_k\psi_1 M = C(j_!M_k \to j_*M_k)$, ${}_k\psi'_1 M = {}_k\psi_1 M[-1]$, ${}_k\phi'_1 M = [i^*M \to {}_k\psi_1 M]$ and

$$ {}_k i^! M = [i^*M \to {}_k\psi_1 M \xrightarrow{-N} {}_{k-1}\psi_1 M(-1)], $$

where $i^*M \to {}_k\psi_1 M$ and $N: {}_k\psi_1 M \to {}_{k-1}\psi_1 M(-1)$ are induced by $E_0 \to E_k$ and $\tilde{N}: E_k \to E_{k-1}(-1)$. We have a natural morphism ${}_{k-1}\psi_1 M(-1)[-2] \to {}_k i^! M$ induced by the natural inclusion and its mapping cone is naturally isomorphic to ${}_k\phi'_1 M$. By 1.5 we have a natural morphism

$$ \phi_{f,1}M = H^0[i^*M \to {}_k\psi_1 M] \to [i^*M \to {}_k\psi_1 M] = {}_k\phi'_1 M \tag{1.6.2} $$

for $k > 0$, because $H^{-1}[i^*M \to {}_k\psi_1 M] = 0$ by the injectivity of $\psi_{f,1}M \to \psi_{f,1}M_k$ in (1.5.2). We take a represent of $\mathbb{S}: M \otimes M' \to \mathbb{D}_X^H$ and choose a represent of the functor δ^* by choosing an affine open covering, cf. [S2, (4.4.1)], so that we have a commutative diagram in $C^b\mathrm{MHM}(X)$:

$$ \begin{array}{ccc} M \otimes M' & \longrightarrow & \mathbb{D}_X^H \\ \big\downarrow & & \big\downarrow \\ j_*j^*M \otimes j_*j^*M' & \to & j_*\mathbb{D}_{X^*}^H \end{array} $$

by adjunction for j^*, j_*. It induces also

$$ j_!j^*M \otimes M' \to j_!j^*M \otimes j_*j^*M' \to j_!\mathbb{D}_{X^*}^H, \text{ etc.} $$

compatible with the natural morphisms $j_!j^*M \to M \to j_*j^*M$, $j_!\mathbb{D}_{X^*}^H \to \mathbb{D}_X^H \to j_*\mathbb{D}_{X^*}^H$, etc., because $j_!j^*(M \otimes M')$ represents $j_!j^*M \otimes M'$ by $\delta^*(j \times \mathrm{id})_! = j_!\delta'^* = j_!\delta''^*(\mathrm{id} \times j)^*$, cf. [S2, (4.4.3)], where

$$ \begin{array}{ccccc} X^* & = & X^* & \xrightarrow{\;j\;} & X \\ \big\downarrow{\scriptstyle \delta''} & & \big\downarrow{\scriptstyle \delta'} & & \big\downarrow{\scriptstyle \delta} \\ X^* \times X^* & \xrightarrow{\mathrm{id} \times j} & X^* \times X & \xrightarrow{j \times \mathrm{id}} & X \times X \end{array} $$

Then using $M_a \otimes M'_b = j^*(M \otimes M') \otimes f^*(E_a \otimes E_b)$ and the multiplication $E_a \otimes$

$E_b \to E_k (a + b < k)$, cf. 1.1, we get the morphisms

$$_a\psi_1 M \otimes {}_b\psi_1 M' \to {}_k\psi_1 \mathbb{D}_X^H \to {}_{k+1}i^!\mathbb{D}_X^H(1)[2]$$

and

$$_a\psi'_1 M \otimes {}_b\psi'_1 M' \to {}_{k+1}i^!\mathbb{D}_X^H(1)$$

as in [S1, 5.2.3] (cf. also [S3, 2.1]), and this gives the desired $\psi_{f,1}\mathbb{S}$ by the natural morphism $\psi_{f,1}M \to {}_a\psi'_1 M$ for $a \gg 0$, etc. (cf. 1.3), because

$$_ki^!\mathbb{D}_X^H = [i^*\mathbb{D}_X^H \to i^*j_*\mathbb{D}_{X^*}^H] = i^!\mathbb{D}_X^H$$

by $\mathrm{Ker}(\tilde{N}: E_k \to E_{k-1}(-1)) = E_0$. The argument is similar for ϕ. We have a pairing $i^*M \otimes {}_{k+1}i^!M' \to {}_{k+1}i^!\mathbb{D}_X^H$ compatible with the above pairing by the natural morphisms $i^*M \to {}_a\psi_1 M, {}_b\psi_1 M'(-1) \to {}_{k+1}i^!M'[2]$ so that $\phi_{f,1}\mathbb{S}$ is obtained by the same argument as in [loc. cit].

1.7. THEOREM. *Let M, M' be mixed Hodge Modules on X, and K, K' their underlying perverse sheaves. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(M \otimes M', \mathbb{D}_X^H) & \longrightarrow & \mathrm{Hom}(M, \mathbb{D}M') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(K \otimes K', \mathbb{D}_X) & \xrightarrow{\sim} & \mathrm{Hom}(K, \mathbb{D}K') \end{array} \qquad (1.7.1)$$

where the vertical morphisms are induced by the forgetful functor $D^b\mathrm{MHM}(X) \to D^b\mathrm{Perv}(\mathbb{Q}_X) \xrightarrow{\sim} D_c^b(\mathbb{Q}_X)$, *and the last horizontal morphism by* $\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, \mathcal{H}om(B, C))$.

Proof. The assertion means that the last horizontal isomorphism preserves the subgroups of the morphisms of $D^b\mathrm{MHM}(X)$. Here the injectivity of the right vertical morphism is clear by definition (and the commutativity of the forgetful functor with the dual $\mathbb{D}$), and that of the left follows from the compatibility of the adjunction for $a_X^!, (a_X)_!$ with the forgetful functor, because

$$H^i(a_X)_!(K \otimes K') = 0 \quad \text{and} \quad H^i(a_X)_!(M \otimes M') = 0 \quad \text{for } i > 0 \qquad (1.7.2)$$

by the adjunction

$$\mathrm{Ext}^i(K, \mathbb{D}K') = \mathrm{Ext}^i(K \otimes K', \mathbb{D}_X) = \mathrm{Ext}^i((a_X)_!(K \otimes K'), \mathbb{Q}),$$

where the i^{th} extensions of perverse sheaves are zero for $i < 0$, cf. [BBD]. In particular the assertion is local on X, and we proceed by induction on dim X. The assertion is trivial if X is a point by the definition of the dual, cf. [B1][C]. In general we may assume that there is a function f such that the restrictions of M, M' to $X^* := f^{-1}(S^*)$ are variations of mixed Hodge structures, where the assertion holds on $X_0 := f^{-1}(0)$ by inductive hypothesis. Then for $u \in \mathrm{Hom}(M \otimes M', \mathbb{D}_X^H)$ the corresponding morphism $v^{\mathbb{Q}} \in \mathrm{Hom}(K, \mathbb{D}K')$ induces a morphism of mixed Hodge Modules $j^*M \to j^*\mathbb{D}M'$ on X^*, and it is enough to show that ${}^p\phi_{f,1}v^{\mathbb{Q}}: {}^p\phi_{f,1}K \to {}^p\phi_{f,1}\mathbb{D}K'$ underlies a morphism of mixed Hodge Modules $\phi_{f,1}M \to \phi_{f,1}\mathbb{D}M'$ by [S2, 2.28]. By inductive hypothesis it is reduced to that ${}^p\phi_{f,1}u^{\mathbb{Q}}: {}^p\phi_{f,1}K \otimes {}^p\phi_{f,1}K' \to \mathbb{D}_{X_0}$ underlies a morphism of $D^b\mathrm{MHM}(X_0)$: $\phi_{f,1}M \otimes \phi_{f,1}M' \to \mathbb{D}_{X_0}^H$, and follows from 1.6. (Here the isomorphism $\mathbb{D}\phi_{f,1} = \phi_{f,1}\mathbb{D}$ and its compatibility with the forgetful functor are also used, see [S2, 2.6] and [S3].)

REMARKS. (i) The forgetful functor $\mathrm{MHM}(X) \to \mathrm{Perv}(\mathbb{Q}_X)$ is faithful, but $D^b\mathrm{MHM}(X) \to D_c^b(\mathbb{Q}_X)$ is not (for example, consider Ext^1 of mixed Hodge structures, see also 3.5).

(ii) The bijectivity of the first horizontal morphism of (1.7.1) will be proved in the next section by using the effaceability condition.

2. Effaceability

In this section we prove Theorem (0.2).

2.1. We first review the elementary theory of cohomological functor. Let $\mathscr{A}$, $\mathscr{B}$ be abelian categories, and $K^b\mathscr{A}$, $D^b\mathscr{A}$ as in [V]. An additive functor $\mathbb{H}: D^b\mathscr{A} \to \mathscr{B}$ is called a *cohomological functor* if for a distinguished triangle $\to M' \to M \to M'' \xrightarrow{+1}$ of $D^b\mathscr{A}$, $\mathbb{H}M' \to \mathbb{H}M \to \mathbb{H}M''$ is exact. We put $\mathbb{H}^iM = \mathbb{H}(M[i])$ for $i \in \mathbb{Z}$ so that we have a long exact sequence

$$\to \mathbb{H}^iM' \to \mathbb{H}^iM \to \mathbb{H}^iM'' \to \mathbb{H}^{i+1}M' \to$$

for a triangle as above. The definition is similar for the contravariant functor where the definition of $\mathbb{H}^i$ is replaced by $\mathbb{H}^iM = \mathbb{H}_{-i}M = \mathbb{H}(M[-i])$. A cohomological functor $\mathbb{H}$ is called *left* (resp. *right*) *exact*, if $\mathbb{H}^iM = 0$ for $M \in \mathscr{A}$ and $i < 0$ (resp. $i > 0$). The restriction $\mathbb{F}$ of $\mathbb{H}$ to $\mathscr{A}$ is left (resp. right) exact in the usual sense, if $\mathbb{H}$ is a left (resp. right) exact cohomological functor. From now on we assume $\mathscr{B}$ to be the category of (sheaf of) abelian groups or modules. A covariant (resp. contravariant) functor $\mathbb{F}: \mathscr{A} \to \mathscr{B}$ is called *effaceable* if for any $M \in \mathscr{A}$ and $e \in \mathbb{F}(M)$, there exists an injection $M \to M'$ (resp. a surjection $M' \to M$) such that the image of e in $\mathbb{F}(M')$ is zero. Sometimes we shall call a cohomological functor effaceable if so is its restriction to $\mathscr{A}$.

2.2. With the above notation and assumption, let $\mathbb{F}^i: \mathscr{A} \to \mathscr{B}$ be additive covariant (resp. contravariant) functors of abelian categories with functor morphisms $d: \mathbb{F}^i \to \mathbb{F}^{i+1}$ such that $d^2 = 0$. We have an additive functor $\mathbb{F}: K^b\mathscr{A} \to K\mathscr{B}$ such that $\mathbb{F}(M)$ is the single complex associated with the double complex whose (p, q)-components are $\mathbb{F}^q(M^p)$ (resp. $\mathbb{F}^q(M^{-p})$). For $M \in K^b\mathscr{A}$, let $K(M/)$ (resp. $K(/M)$) be the category of quasi isomorphisms u of $K^b\mathscr{A}$ such that $S(u) = M$ (resp. $T(u) = M$) where $S(u)$ and $T(u)$ are the source and the target of u, and the morphisms of $K(M/)$ and $K(/M)$ are the obvious ones, cf. [V]. We define

$$\mathbb{R}^i\mathbb{F}M = \varinjlim_{u \in K(M/)} H^i(\mathbb{F}(T(u))) \text{ (resp. } \varinjlim_{u \in K(/M)} H^i(\mathbb{F}(S(u))))$$

so that $\mathbb{R}^i\mathbb{F}M = \mathbb{R}^0\mathbb{F}(M[i])$ (resp. $\mathbb{R}^0\mathbb{F}(M[-i])$, where $H^i: K^b\mathscr{B} \to \mathscr{B}$ is the natural cohomological functor. Then $\mathbb{H} := \mathbb{R}^0\mathbb{F}$ defines a cohomological functor. In fact the well-definedness of $\mathbb{H}(u)$ for the morphisms u of $D^b\mathscr{A}$ is standard, and we may assume $M' = C(M \to M'')[-1]$ for the exactness of $\mathbb{H}M' \to \mathbb{H}M \to \mathbb{H}M''$. Note that $\mathbb{R}^i\mathbb{F}: D^b\mathscr{A} \to \mathscr{B}$ is left exact if $\mathbb{F}^j = 0$ for $j < i$ by the canonical truncation τ of $K^b\mathscr{A}$, and effaceable if $\mathbb{F}^j = 0$ for $j \geqslant i$ by the triangle $\to \sigma_{\geqslant 1}M' \to M' \to M'^0 \to$, where $M \to M'$ is a quasi isomorphism such that $M'^i = 0$ for $i < 0$ and $M \in \mathscr{A}$, cf. [D2][V] for the definition of τ, σ (the argument is similar in the contravariant case). In particular, if $\mathbb{F} = \mathbb{F}^0$, i.e. $\mathbb{F}^i = 0$ for $i \neq 0$, $\mathbb{R}^0\mathbb{F}$ is left exact and $\mathbb{R}^i\mathbb{F}$ are effaceable for $i > 0$. In this case the restriction of $\mathbb{R}^0\mathbb{F}$ to $\mathscr{A}$ coincides with $\mathbb{F}$ iff $\mathbb{F}$ is left exact.

2.3. LEMMA. *With the above notation and assumption, let $\mathbb{H}: D^b\mathscr{A} \to \mathscr{B}$ be a left exact cohomological functor*, and $\mathbb{F}$ *its restriction to $\mathscr{A}$. Then we have canonical functor morphisms $\mathbb{R}^i\mathbb{F} \to \mathbb{H}^i$, and they are isomorphisms iff the restrictions of $\mathbb{H}^i$ to $\mathscr{A}$ are effaceable for any $i > 0$.*

Proof. We assume $\mathbb{H}$ covariant. The argument is similar in the contravariant case. For $M \in K^b\mathscr{A}$ we have a spectral sequence

$$E_1^{pq} = \mathbb{H}^q(M^p) \Rightarrow \mathbb{H}^{p+q}(M) \tag{2.3.1}$$

by Verdier, cf. also [S1, 5.2.18]. The left exactness means that $E_1^{pq} = 0$ for $q < 0$ and the differential d_1 is induced by that of M. By the edge morphism we get the functorial morphism $H^p(\mathbb{F}(M^\cdot)) = E_2^{p0} \to \mathbb{H}^p(M)$, and passing to the limit we get the desired morphism. For $M \in \mathscr{A}$, $\mathbb{R}^0\mathbb{F}(M) = \mathbb{F}(M) = \mathbb{H}^0(M)$ is clear by the left exactness, and the remaining assertion follows from the next:

2.4. LEMMA. Let $\mathbb{H} \to \mathbb{H}'$ *be a morphism of cohomological functors, and assume the effaceability of $\mathbb{H}^i$ and the bijectivity of $\mathbb{H}^{i-1}(M) \to \mathbb{H}'^{i-1}(M)$* for any $M \in \mathscr{A}$. *Then $\mathbb{H}^i(M) \to \mathbb{H}'^i(M)$ is injective for $M \in \mathscr{A}$, and the bijectivity is equivalent to the effaceability of $\mathbb{H}'^i$.*

Proof. By the effaceability of $e \in \mathbb{H}^i(M)$ we have a short exact sequence $0 \to M \to M' \to M'' \to 0$ such that the image of e in $\mathbb{H}^i(M')$ is zero. Then $e = 0$ if the image of e in $\mathbb{H}'^i(M)$ is zero by the commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & \mathbb{H}^{i-1}(M') & \longrightarrow & \mathbb{H}^{i-1}(M'') & \longrightarrow & \mathbb{H}^i(M) & \longrightarrow & \mathbb{H}^i(M') & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \mathbb{H}'^{i-1}(M') & \longrightarrow & \mathbb{H}'^{i-1}(M'') & \longrightarrow & \mathbb{H}'^i(M) & \longrightarrow & \mathbb{H}'^i(M') & \longrightarrow \end{array}$$

The argument is similar for the bijectivity.

2.5. With the notation of 2.1–2, let $\mathscr{A} = \mathrm{MHM}(X)$ for an algebraic variety X, and $\mathscr{B} = M(\mathbb{Q})$ the category of $\mathbb{Q}$-modules. For $M \in \mathscr{A}$ we define contravariant functors ${}_M\mathbb{H}^i, \mathbb{H}^i_M \colon D^b\mathscr{A} \to \mathscr{B}$ by

$$ {}_M\mathbb{H}^i(N) = \mathrm{Ext}^i(M \otimes N, \mathbb{D}^H_X), \mathbb{H}^i_M(N) = \mathrm{Ext}^i(N \otimes M, \mathbb{D}^H_X) \tag{2.5.1}$$

where $M \otimes N = N \otimes M$ by the involution of $X \times X$ inducing the identity on the diagonal. Then ${}_M\mathbb{H}^0, \mathbb{H}^0_M$ are left exact cohomological functors by the proof of 1.7, cf. (1.7.2). Let ${}_M\mathbb{F}, \mathbb{F}_M$ denote the restriction of ${}_M\mathbb{H}^0, \mathbb{H}^0_M$ to $\mathscr{A}$. We have functor morphisms $\mathbb{R}^i\,{}_M\mathbb{F} \to {}_M\mathbb{H}^i, \mathbb{R}^i\mathbb{F}_M \to \mathbb{H}^i_M$, and they are functor isomorphisms iff ${}_M\mathbb{H}^i, \mathbb{H}^i_M$ are effaceable for any $i > 0$ by 2.2–4. These arguments can be generalized to the case $M \in K^b\mathscr{A}$ by applying 2.2 to $\mathbb{F}^i = {}_{M^{-i}}\mathbb{F}, \mathbb{F}_{M^{-i}}$, i.e. $\mathbb{R}^i\,{}_M\mathbb{F}(N)$, $\mathbb{R}^i\mathbb{F}_M(N)$ are the inductive limit of the cohomology of the double complex whose (p, q)-component is

$$\mathrm{Ext}^0(M^{-p} \otimes \tilde{N}^{-q}, \mathbb{D}^H_X), \mathrm{Ext}^0(\tilde{N}^{-p} \otimes M^{-q}, \mathbb{D}^H_X)$$

where $\tilde{N} = S(u)$ with u running over the elements of $K(/N)$ in the notation of (2.2.1). For the construction of the morphisms $\mathbb{R}^i\,{}_M\mathbb{F} \to {}_M\mathbb{H}^i$, etc. we use the isomorphism

$$\mathrm{Ext}^i(M \otimes N, \mathbb{D}^H_X) = \mathrm{Ext}^i(M \boxtimes N, \delta_* \mathbb{D}^H_X), \text{ etc.}$$

with the filtration of $M \boxtimes N$ by the total degree, where $\delta \colon X \to X \times X$ is the diagonal. Then passing to the limit we get a commutative diagram

$$\begin{array}{ccccc} {}_M\mathbb{H}^i(N) & \longleftarrow & \mathbb{R}^i\,{}_M\mathbb{F}(N) & \searrow & \\ \| & & \downarrow & & \\ \mathrm{Ext}^i(M \otimes N, \mathbb{D}^H_X) & \longleftarrow & \mathbb{R}^i\mathbb{F}(M, N) & \longrightarrow & \mathrm{Ext}^i(M, \mathbb{D}N) \\ \| & & \uparrow & & \\ \mathbb{H}^i_N(M) & \longleftarrow & \mathbb{R}^i\mathbb{F}_N(M) & \nearrow & \end{array} \tag{2.5.2}$$

where $\mathbb{R}^i\mathbb{F}(M, N)$ is the inductive limit of the cohomology of the double complex whose (p, q)-component is $\mathrm{Ext}^0(\tilde{M}^{-p} \otimes \tilde{N}^{-q}, \mathbb{D}_X^H)$ for $\tilde{M} \to M, \tilde{N} \to N$ quasi isomorphisms of $K^b\mathscr{A}$. Here the morphisms to $\mathrm{Ext}^i(M, \mathbb{D}N)$ are induced by 1.7, and they are isomorphisms if so are they for $M, N \in \mathrm{MHM}(X)$ and $i = 0$, i.e. the first horizontal morphism in (1.7.1) is an isomorphism. Assume $N^i = 0$ for $i > 0$ and $H^iN = 0$ for $i < 0$. Put $N' = H^0N(=N \text{ in } D^b\mathscr{A})$. Then $\mathbb{H}_N^0 = \mathbb{H}_{N'}^0$ is left exact and we have

$$\mathbb{R}^0\mathbb{F}_N(M) = \mathbb{H}_N^0(M) = \mathrm{Ext}^0(M \otimes N', \mathbb{D}_X^H) \quad \text{for } M \in \mathscr{A} \tag{2.5.3}$$

by the left exactness of $\mathbb{H}_{N^i}^0, {}_M\mathbb{H}^0$. Therefore $\mathbb{R}^1\mathbb{F}_N(M) \to \mathbb{H}_N^1(M)$ is always injective for $M \in \mathscr{A}$, and $\mathbb{R}^i\mathbb{F}_N \to \mathbb{H}_N^i$ are isomorphisms iff $\mathbb{H}_N^i = \mathbb{H}_{N'}^i$ are effaceable for $i > 0$ by the same argument as in 2.4.

2.6. LEMMA. *Let X be a smooth variety, M, $N \in D^b\mathrm{MHM}(X)$, and L an admissible variation of mixed Hodge structure on X so that $L[d_X] \in \mathrm{MHM}(X)$ and $\otimes L\colon \mathrm{MHM}(X) \to \mathrm{MHM}(X)$ is an exact functor. Then we have a canonical isomorphism*

$$\mathrm{Hom}(M, L^* \otimes N) \to \mathrm{Hom}(M \otimes L, N) \tag{2.6.1}$$

induced by $\otimes L$ and the natural morphism $L^* \otimes L \to \mathbb{Q}_X^H$, where $L^* = \mathscr{H}om(L, \mathbb{Q}_X^H) = (\mathbb{D}L)(-d_X)[-2d_X]$ is the dual variation of L.

Proof. By definition each side of (2.6.1) is the inductive limit of the cohomology of the double complexes whose components are $\mathrm{Hom}(\tilde{M}^{-p} \otimes L, \tilde{N}^q)$, etc. where $\tilde{M} \to M, N \to \tilde{N}$ are quasi isomorphisms. Therefore the assertion is reduced to the case $M, N \in \mathrm{MHM}(X)$. We have the canonical morphism $\mathbb{Q}_X^H \to L \otimes L^*$ by duality, and this gives the inverse combined with the tensor of L^*. In fact the assertion is reduced to that for perverse sheaves by the faithfulness of the forgetful functor of the mixed Hodge Modules, and we may assume L trivial. The detail is left to the reader.

2.7. COROLLARY. *With the notation and assumption of* 2.6 *the morphisms*

$$\mathrm{Ext}^i(M \otimes N, \mathbb{D}_X^H) \leftarrow \mathbb{R}^i\mathbb{F}_N(\mathbb{M}) \to \mathrm{Ext}^i(M, \mathbb{D}N) \tag{2.7.1}$$

in (2.5.2) *are isomorphisms for $M, N \in \mathrm{MHM}(X)$ such that the underlying perverse sheaf of N is a local system up to shift.*

Proof. This is essentially the special case of 2.6 where L, N in 2.6 correspond to $N[-d_X]$, $\mathbb{D}_X^H[i - d_X] = \mathbb{Q}_X^H(d_X)[i + d_X]$, and it is enough to check the compatibility of the morphisms (or check (2.8.2–3) below). The detail is left to the reader.

2.8. THEOREM. *The morphisms in* (2.5.2) *are isomorphisms for any M, $N \in K^b\mathrm{MHM}(X)$, and induces the isomorphisms for $M, N \in D^b\mathrm{MHM}(X)$. In particular* *we get a functorial isomorphism*

$$\mathrm{Hom}(M \otimes N, \mathbb{D}_X^H) = \mathrm{Hom}(M, \mathbb{D}N) \tag{2.8.1}$$

compatible with the corresponding natural morphism for $\mathbb{Q}$*-complexes.*

Proof. By 2.5 it is enough to show:

the morphism in (1.7.1) induces the isomorphism (2.8.1) for $M, N \in \mathrm{MHM}(X)$ (2.8.2)

$\mathbb{H}_N^i$ is effaceable for any $N \in \mathrm{MHM}(X)$ and $i > 0$. (2.8.3)

In fact we may assume $N \in \mathrm{MHM}(X)$ in the proof of isomorphism for the left horizontal morphisms of (2.5.2), because (2.8.2) implies that the morphisms to $\mathrm{Ext}^i(M, \mathbb{D}N)$ are isomorphisms, and it is enough to show the first horizontal isomorphism of (2.5.2). As shown in the proof of 1.6 we have the adjunction

$$\mathrm{Hom}(j_!M \otimes N, \mathbb{D}_X^H) \xrightarrow{\sim} \mathrm{Hom}(M \otimes j^*N, \mathbb{D}_U^H) \tag{2.8.4}$$

for $M \in D^b\mathrm{MHM}(U)$, $N \in D^b\mathrm{MHM}(X)$, where $j: U \to X$ is an affine open immersion. In fact the case of mixed Hodge Module was shown and the general case follows from the exactness of $j_!$ using the morphism $j_!j^*\tilde{M} \to \tilde{M}$ for any quasi isomorphism $\tilde{M} \to j_!M$. In paticular the assertions (2.8.2–3) are local. For (2.8.3) we take an affine open covering $X = \cup U_i$ and use the surjection $\oplus j_{i!}j_i^*M \to M$ so that (2.8.3) is reduced to that for j_i^*N by (2.8.4), where $j_i: U_i \to X$. For (2.8.2) we use the spectral sequence (with the vanishing of E_1^{pq} for $q < 0$, cf. (1.7.2)):

$$E_1^{pq} = \bigoplus_{|I|=p+1} \mathrm{Ext}^q(j_I^*M \otimes j_I^*N, \mathbb{D}_{U_I}^H) \Rightarrow \mathrm{Ext}^{p+q}(M \otimes N, \mathbb{D}_X^H)$$

associated to the filtration σ of the co-Čech resolution of M whose components are $\bigoplus_{|I|=1-p} j_{I!}j_I^*M$ (using (2.8.4)), where $j_I: U_I := \bigcap_{i\in I} U_i \to X$, cf. also [BBD].

We prove the assertions by induction on $\dim X$. The case $\dim X = 0$ is the special case of 2.7, cf. also [C] [B1]. In general we may assume that there exists a function f such that with the notation of 1.2, X^* is smooth, the underlying perverse sheaf of j^*N is a local system up to shift, and the assertion is proved on X_0. Associated to the triangle $\to j_!j^*M \to M \to i_*i^*M \to$ we have the morphisms of long exact sequences by 2.5:

$$\begin{array}{ccccccc}
\longrightarrow \mathrm{Ext}^k(i_*i^*M \otimes N, \mathbb{D}_X^H) & \longrightarrow & \mathrm{Ext}^k(M \otimes N, \mathbb{D}_X^H) & \longrightarrow & \mathrm{Ext}^k(j^*M \otimes N, \mathbb{D}_{X^*}^H) & \longrightarrow \\
\uparrow \alpha' & & \uparrow \alpha & & \uparrow \alpha'' & \\
\longrightarrow \mathbb{R}^k\mathbb{F}_N(i_*i^*M) & \longrightarrow & \mathbb{R}^k\mathbb{F}_N(M) & \longrightarrow & \mathbb{R}^k\mathbb{F}_{j^*N}(j^*M) & \longrightarrow \\
\downarrow \beta' & & \downarrow \beta & & \downarrow \beta'' & \\
\longrightarrow \mathrm{Ext}^k(i_*i^*M, \mathbb{D}N) & \longrightarrow & \mathrm{Ext}^k(M, \mathbb{D}N) & \longrightarrow & \mathrm{Ext}^k(j^*M, \mathbb{D}j^*N) & \longrightarrow
\end{array} \tag{2.8.5}$$

where the compatibility of the adjunction (2.8.4) with the vertical morphisms follows from the same argument as in the proof of (2.8.4). Then α'', β'' are isomorphisms by 2.7. For the proof of (2.8.2), i.e. the bijectivity of β for $k = 0$, it is enough to show the bijectivity of β' for $k = 0$ and the injectivity of β' for $k = 1$ by the diagram (2.8.5). We have a canonical triangle

$$\to \psi_{f,1}M \xrightarrow{\mathrm{can}} \phi_{f,1}M \to i^*M \to \tag{2.8.6}$$

cf. [S2, 2.23–24], and using the associated long exact sequence (and the left exactness), the assertion is reduced to the case $\operatorname{supp} M \subset X_0$, if the injectivity of β for $k = 1$ is shown in this case. But this injectivity is reduced to the bijectivity of the morphisms in (2.5.2):

$$\mathrm{Ext}^1(M \otimes N, \mathbb{D}_X^H) \leftarrow \mathbb{R}^1{}_M\mathbb{F}(N) \to \mathrm{Ext}^1(M, \mathbb{D}N)$$

by the commutativity of (2.5.2) and the last remark of 2.5 (e.g. the injectivity of α for $i = 1$, etc.). Therefore by the duality and the symmetry of $M \otimes N$, (2.8.2) is reduced to (2.8.2–3) in the case $\operatorname{supp} N \subset X_0$. Then (2.8.3) is also reduced to (2.8.2–3) in this case by the similar argument, because the morphisms to $\mathrm{Ext}^i(M, \mathbb{D}N)$ in (2.5.2) are isomorphisms by (2.8.2).

Now we prove the case $\operatorname{supp} N \subset X_0$. By the same argument as above (i.e. using (2.8.5–6)), (2.8.2) is reduced to the case $\operatorname{supp} M$, $\operatorname{supp} N \subset X_0$, and follows from the inductive hypothesis using the adjunction for $i_*, i^!$ with $i^!\mathbb{D}_X^H = \mathbb{D}_{X_0}^H$, because i_* commutes with dual and tensor. Then (2.8.3) is also reduced to this case, and follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^k(M' \otimes N', \mathbb{D}^H_{X_0}) & \xrightarrow{\sim} & \mathrm{Ext}^k(i_*M' \otimes i_*N', \mathbb{D}^H_X) \\
\uparrow \wr & & \uparrow \\
\mathbb{R}^k\mathbb{F}_{N'}(M') & \longrightarrow & \mathbb{R}^k\mathbb{F}_{i_*N'}(i_*M') \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Ext}^k(M', \mathbb{D}N') & \xrightarrow{\sim} & \mathrm{Ext}^k(i_*M', \mathbb{D}i_*N')
\end{array}$$

for $M', N' \in \mathrm{MHM}(X_0)$, where the first two horizontal morphisms are induced by the trace morphism $i_*\mathbb{D}^H_{X_0} \to \mathbb{D}^H_X$, and the first and the last horizontal morphisms are isomorphisms by the adjunction and the fully faithfulness of $i_*: D^b\mathrm{MHM}(X_0) \to D^b\mathrm{MHM}(X)$, cf. [S2, 2.23].

Now it remains to show the compatibility with the corresponding isomorphism on the underlying $\mathbb{Q}$-complexes. For $(K', F), (L', F) \in \mathscr{D}^bF_{\text{bête}}$ in [BBD] corresponding to $K, L \in C^b\mathrm{Perv}(\mathbb{Q}_X)$, we have a commutative diagram

$$\begin{array}{ccc}
H^i(\mathrm{Ext}^0(K^{\cdot} \boxtimes L^{\cdot}, \delta_*\mathbb{D}_X)) & \longrightarrow & \mathrm{Ext}^i(K \boxtimes L, \delta_*\mathbb{D}_X) \\
\downarrow \text{real} & & \downarrow \text{real} \\
H^i(\mathrm{Ext}^0(Gr_FK' \boxtimes Gr_FL', \delta_*\mathbb{D}_X)) & \longrightarrow & \mathrm{Ext}^i(K' \boxtimes L', \delta_*\mathbb{D}_X) \\
\| & & \| \\
H^i(\mathrm{Ext}^0(Gr_FK', \mathbb{D}Gr_FL')) & \longrightarrow & \mathrm{Ext}^i(K', \mathbb{D}L')
\end{array} \tag{2.8.7}$$

by the edge morphism of the spectral sequence associated with the quasi filtration induced by F on $K \boxtimes L$ and $K' \boxtimes L'$, where the left hand sides are the cohomologies of the double complexes whose (p, q)-components are $\mathrm{Ext}^0(K^{-p} \boxtimes L^{-q}, \delta_*\mathbb{D}_X)$, $\mathrm{Ext}^0(Gr_F^{-p}K' \boxtimes Gr_F^{-q}L', \delta_*\mathbb{D}_X)$, $\mathrm{Ext}^0(Gr_F^{-p}K', \mathbb{D}Gr_F^{-q}L')$. Here the adjunction for δ^*, δ_* and the vanishing of Ext^i for $i < 0$ are used. Then passing to the limit we get the compatibility by [BBD] [B2], because the limit of the last horizontal morphism coincides with the morphism induced by the functor real, cf. [loc. cit].

2.9. COROLLARY. *With the notations of* 2.8 *we have functorial isomorphisms*

$$\mathrm{Hom}(\mathbb{Q}^H_X, \mathscr{H}om(M, N)) = \mathrm{Hom}(M, N) \tag{2.9.1}$$

$$\mathrm{Hom}(L, \mathcal{H}om(M, N)) = \mathrm{Hom}(L \otimes M, N) \tag{2.9.2}$$

for $L, M, N \in D^b\mathrm{MHM}(X)$ *compatible with the corresponding isomorphisms on the underlying* $\mathbb{Q}$*-complexes.*

Proof. It is enough to show (2.9.2), because $\mathbb{Q}_X^H \otimes M = M$. In fact this follows from $\mathbb{Q}_X^H \boxtimes M = (a_X \times \mathrm{id})^* M$ and the functoriality of the pull-backs for the compositions. For (2.9.2) it is enough to show

$$L \otimes \mathbb{D}\, \mathcal{H}om(M, N) = L \otimes M \otimes \mathbb{D}N \tag{2.9.3}$$

by 2.8 and $\mathbb{D}^2 = \mathrm{id}$. But it is clear by definition and $\mathbb{D}\delta^! = \delta^* \mathbb{D}$. For the compatibility with the corresponding isomorphism on the underlying $\mathbb{Q}$-complexes, we identify $\delta^!(\mathbb{D}K \boxtimes K')$ with $\mathcal{H}om(K, K')$ for $K, K' \in D_c^b(\mathbb{Q}_X)$ using the above construction and the natural isomorphism

$$\mathrm{Hom}(K'', \mathcal{H}om(K, K')) = \mathrm{Hom}(K'' \otimes K, K')$$

for any K''. Then the compatibility is clear.

As a corollary we get

2.10. THEOREM. *Let* X *be an algebraic variety, and* $M, N \in D^b\mathrm{MHM}(X)$ *with* $K, L \in D_c^b(\mathbb{Q}_X)$ *their underlying* $\mathbb{Q}$*-complexes. Then we have a canonical short exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1(\mathbb{Q}^H, H^{i-1}(a_X)_* \mathcal{H}om(M, N)) \xrightarrow{\alpha} \mathrm{Ext}^i(M, N) \\ \longrightarrow \mathrm{Hom}(\mathbb{Q}^H, H^i(a_X)_* \mathcal{H}om(M, N)) \longrightarrow 0 \end{aligned} \tag{2.10.1}$$

such that the morphism β *is identified with* $\mathrm{Ext}^i(M, N) \to \mathrm{Ext}^i(K, L) = H^i(X, \mathcal{H}om(K, L))$ *induced by the forgetful functor. Here the middle* Ext *in* (2.10.1) *is taken in* $D^b\mathrm{MHM}(X)$, *and the first* Ext *and the last* Hom *in* $\mathrm{MHM}(pt)$.

Proof. This is clear by 2.9 and the compatibility of the adjunction for $a_X^*, (a_X)_*$ with the forgetful functor, because Ext^i in $\mathrm{MHM}(pt)$ vanishes for $i > 1$ by [B1], cf. also 3.6 below for the case of admissible variation of mixed Hodge structure with $i = 1$.

3. Extension of admissible variations

In this section we assume X smooth and connected. We prove 2.10 directly in the case of admissible variation of mixed Hodge structure for $i = 1$.

3.1. Let A be a noetherian subring of $\mathbb{R}$. We assume

$$\text{any torsion free finite } A\text{-module is projective} \tag{3.1.1}$$

so that

$$\mathrm{Ext}^i(M, N) = 0 \quad \text{for } A\text{-modules } M,\ N \text{ and } i > 1 \tag{3.1.2}$$

i.e. A is a field or a Dedekind domain. Let B be the subfield of $\mathbb{R}$ generated by A. If A is a field or (a localization of) the ring of integers of an algebraic number field, we have $B = A \otimes_{\mathbb{Z}} \mathbb{Q}$, and the A-torsions of an A-module M (i.e. $\mathrm{Ker}(M \to M \otimes_A B)$) coincide with the $\mathbb{Z}$-torsions. We denote by $\mathrm{MHS}(A)$ the abelian category of A-mixed Hodge structures, and $\mathrm{MHS}(A)^p$ its full subcategory of polarizable objects, cf. [D2], where the weight filtration W is defined over B, and polarizable means that the graded Hodge structures are polarizable over B. Let $\mathrm{VMHS}(X, A)_{\mathrm{ad}}$ be the abelian category of admissible variations of A-mixed Hodge structures [SZ][K], where the definition in [SZ] is valid only in the unipotent local monodromy case, and the general case is reduced to this case in [K] cutting by curves and taking ramified coverings. By definition we have

$$\mathrm{VHS}(X, A, n)^p \subset \mathrm{VMHS}(X, A)_{\mathrm{ad}} \tag{3.1.3}$$

$$\mathrm{MHS}(A)^p = \mathrm{VMHS}(pt, A)_{\mathrm{ad}} \tag{3.1.4}$$

where $pt = \mathrm{Spec}\,\mathbb{C}$ and $\mathrm{VHS}(X, A, n)^p$ is the category of polarizable variations of A-Hodge structures of weight n on X.

3.2. If A is a field we can define $\mathrm{MHM}(X, A)$ the category of mixed Hodge Modules with A-structure as in [S1−2], and prove

$$\mathrm{MHM}(X, A)_{\mathrm{sm}} = \mathrm{VMHS}(X, A)_{\mathrm{ad}} \tag{3.2.1}$$

where the left is the full subcategory of smooth mixed Hodge Modules of $\mathrm{MHM}(X, A)$ (i.e. the underlying perverse sheaves are local systems on X up to shift), cf. [S2]. More precisely $\mathrm{VMHS}(X, A)_{\mathrm{ad}} \subset D^b\mathrm{MHM}(X, A)$ and $M \in \mathrm{MHM}(X, A)_{\mathrm{sm}}$ iff $M[-\dim X] \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$. In particular we have

$$\mathrm{MHM}(pt, A) = \mathrm{MHS}(A)^p \quad \text{if } A \text{ is a field.} \tag{3.2.2}$$

3.3. With the notation and assumption of 3.1, we denote by M_R the underlying R-local system of $M \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ for $R = A$, B, $\mathbb{C}$. We say that $M \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ is *torsion-free* (resp. *torsion*) if so are the stalks of M_A. For $M \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ we have a canonical exact sequence

$$0 \to M^T \to M \to M^F \to 0 \tag{3.3.1}$$

such that M^T (resp. M^F) is torsion (resp. torsion-free). By definition of the underlying A-structure of a variation of mixed Hodge structure we can check easily

$$\mathrm{Ext}^i(M, N) = \mathrm{Ext}^i(M_A, N_A) \quad \text{for } i \leqslant 1 \quad \text{if } M \text{ or } N \text{ is torsion} \tag{3.3.2}$$

For $M \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ we define the weight filtration W on M by $W_i M_A = \mathrm{Ker}(M_A \to M_B / W_i M_B)$ so that $M / W_i M$ are torsion-free and $(W_i M)_B = W_i M_B$. Then $W_i M$ are torsion for $i \ll 0$, and in general W is not separated and $M \mapsto W_i M$ is not exact unless A is a field. We are interested in the extension defined by the short exact sequence

$$0 \to W_{i-1} M \to W_i \to Gr_i^W M \to 0 \tag{3.3.3}$$

where $Gr_i^W M$ is torsion-free by definition. Let $A_X^H \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ be the constant variation of weight 0 whose underlying A-local system is A_X, and put $A^H = A_{pt}^H$, cf. (3.1.4). If $M \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ is torsion-free, its dual variation $M^* := \mathcal{H}om(M, A_X^H)$ can be defined naturally so that the restriction to the fibers commute with dual, cf. [C][B1] for the case $X = pt$. Here M^* is torsion-free by definition and $M^{**} = M$. By the same argument as in the proof of 2.6 we can show:

3.4. LEMMA. *We have a canonical isomorphism*

$$\mathrm{Ext}^i(N, M) = \mathrm{Ext}^i(A_X^H, N^* \otimes M) \tag{3.4.1}$$

for $M, N \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$ *such that* N *is torsion-free, where the* Ext^i *are taken in the derived category of* $\mathrm{VMHS}(X, A)_{\mathrm{ad}}$.

3.5. REMARKS. (i) By (3.4.1) for $i = 1$, the short exact sequences $0 \to M \to L \to N \to 0$ and $0 \to N^* \otimes M \to L' \to A_X^H \to 0$ correspond to each other by the diagram of nine lemma:

$$\begin{array}{ccccccccc}
0 \longrightarrow & N^* \otimes M & \longrightarrow & L' & \longrightarrow & A_X^H & \longrightarrow 0 \\
& \| & & \cap\downarrow & & \cap\downarrow & \\
0 \longrightarrow & N^* \otimes M & \longrightarrow & N^* \otimes L & \longrightarrow & N^* \otimes N & \longrightarrow 0 \\
& \downarrow & & \twoheaddownarrow & & \twoheaddownarrow & \\
& 0 & \longrightarrow & N^* \otimes N/A_X^H & = & N^* \otimes N/A_X^H & \longrightarrow 0
\end{array} \tag{3.5.1}$$

where the inverse functor is defined similarly.

(ii) If N is not torsion-free, we have a long exact sequence induced by (3.3.1):

$$\to \mathrm{Ext}^i(N^F, M) \to \mathrm{Ext}^i(N, M) \to \mathrm{Ext}^i(N^T, M) \xrightarrow{\partial} \mathrm{Ext}^{i+1}(N^F, M) \to \tag{3.5.2}$$

where ∂ is induced by the composition with $e \in \mathrm{Ext}^1(N^F, N^T)$ corresponding to (3.3.1), and $\mathrm{Ext}^i(N^T, M) = \mathrm{Ext}^i(N_A^T, M_A)$ for $i \leqslant 1$ by (3.3.2). Here $e = 0$ by (3.1.1)(3.3.2) if $X = pt$.

(iii) If A is a field, we have an exact sequence

$$0 \to \mathrm{Hom}(A_X^H, W_0 M) \to \mathrm{Hom}(A_X^H, Gr_0^W M) \xrightarrow{\partial} \mathrm{Ext}^1(A_X^H, W_{-1} M) \\ \to \mathrm{Ext}^1(A_X^H, W_0 M) \to 0 \tag{3.5.3}$$

where ∂ is given by the composition with $e \in \mathrm{Ext}^1(Gr_0^W M, W_{-1} M)$ corresponding to the short exact sequence (3.3.3), and $\mathrm{Ext}^1(A_X^H, Gr_0^W M) = 0$ by polarization.

(iv) In the case $X = pt$ we define as in [B1][C]:

$$J(M) = \mathrm{Coker}(M_A \oplus W_0 M_B \oplus F^0 W_0 M_{\mathbb{C}} \to M_B \oplus W_0 M_{\mathbb{C}}) \tag{3.5.4}$$

for $M \in \mathrm{MHS}(A)$, where the morphism is given by the alternating sum of four natural inclusions, cf. [loc. cit]. Then

$$J(M) = \mathrm{Coker}(M_A \oplus F^0 M_{\mathbb{C}} \to M_{\mathbb{C}}) \quad \text{if } M \text{ has weights } \leqslant 0 \tag{3.5.5}$$

$$J(M) = \mathrm{Coker}(M_A \to M_B) \quad \text{if } M \text{ has weights } > 0, \tag{3.5.6}$$

and for a short exact sequence $0 \to M' \to M \to M'' \to 0$ we have

$$0 \to \mathrm{Hom}(A^H, M') \to \mathrm{Hom}(A^H, M) \to \mathrm{Hom}(A^H, M'') \\ \longrightarrow J(M') \longrightarrow J(M) \longrightarrow J(M'') \longrightarrow 0 \tag{3.5.7}$$

by snake lemma, because $\mathrm{Hom}(A^H, M)$ is the kernel of the morphism in (3.5.4). We have a canonical isomorphism by [loc. cit]:

$$\mathrm{Ext}^1(A^H, M) = J(M) \quad \text{for } M \in \mathrm{MHS}(A), \tag{3.5.8}$$

where Ext^1 is taken in $\mathrm{MHS}(A)$ (here 2.6 holds also for $\mathrm{MHS}(A)$). In fact we may assume M torsion-free by (3.1.1), cf. the last remark of 3.5(ii). For an exact sequence $0 \to M \to L \to A^H \to 0$ in $\mathrm{MHS}(A)$, we have a splitting $s_B: B \to W_0 L_B$, and it induces a direct sum decomposition

$$L_B = M_B \oplus B \quad \text{compatible with } W. \tag{3.5.9}$$

Then we get $e_A \in M_B$, $e_{\mathbb{C}} \in W_0 M_{\mathbb{C}}$ such that $(e_A, 1) \in L_A$, $(e_{\mathbb{C}}, 1) \in F^0 W_0 L_{\mathbb{C}}$ by (3.5.9), using the bi-strictness of the morphisms of mixed Hodge structures, cf. [D2], where e_A and $e_{\mathbb{C}}$ are unique modulo M_A and $F^0 W_0 M_{\mathbb{C}}$. The ambiguity of s_B is given by $W_0 M_B$ and the dependence of $e_A, e_{\mathbb{C}}$ on the change of s_B is given just as in (3.5.4) (up to sign). Here note that the right exactness of $J(M)$ in (3.5.7) is equivalent to the vanishing of $\mathrm{Ext}^i(A_X, M)$ for $i > 1$ by effaceability, as remarked by Janssen.

Now we consider the polarizable case. For $M \in \mathrm{MHS}(A)^p$ we define $J'(M)$ by replacing $W_0 M_{\mathbb{C}}$ in (3.5.4) with

$$\mathrm{Ker}(W_0 M_{\mathbb{C}} \to Gr_0^W M_{\mathbb{C}}/(Gr_0^W M_B + F^0 Gr_0^W M_{\mathbb{C}})), \text{ cf. [B1].}$$

Then for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathrm{MHS}(A)^p$ we have

$$\begin{aligned} 0 &\to \mathrm{Hom}(A^H, M') \to \mathrm{Hom}(A^H, M) \to \mathrm{Hom}(A^H, M'') \\ &\longrightarrow J'(M') \longrightarrow J'(M) \longrightarrow J'(M'') \longrightarrow 0 \end{aligned} \tag{3.5.10}$$

because Gr_0^W of the short exact sequence splits after the scalar extension $\otimes_A B$ by polarization. If M has weights < 0 (resp. > 0), (3.5.5) (resp. (3.5.6)) holds with $J(M)$ replaced by $J'(M)$, and we have $J'(M) = J(M)$. If M is pure of weight 0 we have

$$J'(M) = \mathrm{Im}(M_B \to J(M)) \tag{3.5.11}$$

i.e. $J'(M)$ is the A-torsions of $J(M)$. Then we get

$$\mathrm{Ext}^1(A^H, M) = J'(M) \quad \text{for } M \in \mathrm{MHS}(A)^p \tag{3.5.12}$$

cf. [loc. cit], where Ext^1 is taken in $\mathrm{MHS}(A)^p$. In fact if $M, M' \in \mathrm{MHS}(A)^p$ are pure of same weight, an extension of M by M' in MHS(A) is polarizable iff the extension splits after scalar extension $\otimes_A B$.

3.6. THEOREM. *With the notation of* 3.1, *let* $M, N \in \mathrm{VMHS}(X, A)_{\mathrm{ad}}$, *and* $K = M_A, L = N_A$. *If* N *is torsion-free, we have a canonical short exact sequence*

$$\begin{aligned} 0 \to \mathrm{Ext}^1(A^H, H^0(X, N^* \otimes M)) &\xrightarrow{\alpha} \mathrm{Ext}^1(N, M) \\ &\xrightarrow{\beta} \mathrm{Hom}(A^H, H^1(X, N^* \otimes M)) \to 0 \end{aligned} \tag{3.6.1}$$

such that the morphism β *is identified with* $\mathrm{Ext}^1(N, M) \to \mathrm{Ext}^1(L, K) = H^1(X, L^* \otimes K)$ *induced by the forgetful functor. Here the middle* Ext *is taken in* $\mathrm{VMHS}(X, A)_{\mathrm{ad}}$, *and the first* Ext *and the last* Hom *in* $\mathrm{MHS}(A)^p$.

3.7. REMARKS. (i) If A is a field, 2.10 holds with $\mathbb{Q}$ replaced by A, and implies 3.6, because $\mathrm{VMHS}(X, A)_{\mathrm{ad}}$ is a full subcategory of MHM(X, A) (up to shift) stable by extensions, cf. (3.2.1).

(ii) If $N = A_X^H$, the morphisms α, β in (3.6.1) can be defined as follows. Let $0 \to M \to M' \to A_X^H \to 0$ be the short exact sequence corresponding to $e \in \mathrm{Ext}^1(A_X^H, M)$, and

$$0 \to H^0(X, M) \to H^0(X, M') \to A^H \xrightarrow{\partial} H^1(X, M) \to \tag{3.7.1}$$

the associated long exact sequence. We define $\beta(e)$ to be the image of 1 by ∂. Then this definition is compatible with the morphism induced by the forgetful functor in 3.6, because the adjunction for A-complexes is induced by $A \to (a_X)_* A_X$. The morphism α is defined by the composition

$$\mathrm{Ext}^1(A^H, H^0(X, M)) \xrightarrow{a_X^*} \mathrm{Ext}^1(A_X^H, a_X^*(H^0(X, M))) \to \mathrm{Ext}^1(A_X^H, M) \tag{3.7.2}$$

Then the exactness of (3.6.1) is clear except for the surjectivity of β. In fact, we have a section $s\colon \mathrm{Ker}\, \beta \to \mathrm{Ext}^1(A^H, H^0(X, M))$ by the short exact sequence deduced from (3.7.1), which is compatible with the restriction to the fiber at each point of X, i.e. we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, M) & \longrightarrow & H^0(X, M') & \longrightarrow & A^H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M_P & \longrightarrow & M'_P & \longrightarrow & A^H & \longrightarrow & 0 \end{array} \tag{3.7.3}$$

where $M_P, M'_P \in \mathrm{MHM}(A)^p$ are the fibers of M, M' at $P \in X$. This shows $\alpha s = \mathrm{id}$, where $\beta\alpha = 0$ is clear by the compatibility of (3.7.2) with the forgetful functor and $\mathrm{Ext}^1(A, H^0(X, L)) = 0$. For the proof of $s\alpha = \mathrm{id}$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, M) & \longrightarrow & M'' & \longrightarrow & A^H & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^0(X, M) & \longrightarrow & H^0(X, M') & \longrightarrow & A^H & \longrightarrow & 0 \end{array} \tag{3.7.4}$$

because X is connected, where e' and $e = \alpha(e')$ correspond to $0 \to H^0(X, M) \to M'' \to A^H \to 0$ and $0 \to M \to M' \to A^H_X \to 0$.

3.8. *Proof of* 3.6. By 3.4 we may assume $N = A_X$, and it is enough to show the surjectivity of β by 3.7(ii), because the diagram (3.5.1) is compatible with the forgetful functor. Here we may assume also $A = B$ by definition of A-structure on the extensions of admissible variations, and the property of β in 3.6.

For $P \in X$, let $i_P \colon X = P \times X \to X \times X$ the natural embedding, and $\delta \colon X \to X \times X$ the diagonal. Let $U = X \backslash P$, $Y = X \times U$, and $i'_P \colon U \to Y$, $\delta' \colon U \to Y$ the restrictions of i_P, δ to Y. Put $Y' = Y \backslash (\mathrm{Im}\, i_P \cup \mathrm{Im}\, \delta)$ with $j \colon Y' \to Y$ the natural inclusion. We have a triangle

$$\to j_! j^* M' \to M' \to (i'_P)_*(i'_P)^* M' \oplus \delta'_* \delta'^* M' \to \tag{3.8.1}$$

for $M' = pr_1^* M \in \mathrm{VMHS}(Y, A)_{\mathrm{ad}}$, where $pr_1 \colon Y \to X$, $pr_2 \colon Y \to U$ are the natural projections. We denote by $M_P \in \mathrm{MHS}(A)^p$ the fiber of M at P, and for $N \in \mathrm{MHS}(A)^p$, $N_U \in \mathrm{VMHS}(U, A)_{\mathrm{ad}}$ denotes the constant variation with fiber N. Taking the direct image of (3.8.1) by pr_2, we get a long exact sequence in $\mathrm{VMHS}(U, A)_{\mathrm{ad}}$:

$$\begin{aligned} 0 &\to H^0(X, M)_U \xrightarrow{\iota} (M_P)_U \oplus M|_U \to H^1(pr_2)_* j_! j^* M' \\ &\to H^1(X, M)_U \to 0 \end{aligned} \tag{3.8.2}$$

and a short exact sequence

$$0 \to \mathrm{Coker}\, \iota \to H^1(pr_2)_* j_! j^* M' \to H^1(X, M)_U \to 0 \tag{3.8.3}$$

so that $e \in \mathrm{Hom}(A^H, H^1(X, M))$ induces $e' \in \mathrm{Ext}^1(A^H_U, \mathrm{Coker}\, \iota)$. Consider the diagram of nine lemma:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & M|_U & \longrightarrow & M|_U & \longrightarrow & 0 \\
 & & \downarrow & & \cap\downarrow & & \cap\downarrow & & \\
0 & \longrightarrow & H^0(X,M)_U & \xrightarrow{\iota} & (M_P)_U \oplus M|_U & \longrightarrow & \operatorname{Coker}\iota & \longrightarrow & 0 \\
 & & \| & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
0 & \longrightarrow & H^0(X,M)_U & \xrightarrow{\iota'} & (M_P)_U & \longrightarrow & \operatorname{Coker}\iota' & \longrightarrow & 0
\end{array} \tag{3.8.4}$$

Let e'' be the image of e' in $\mathrm{Ext}^1(A_U^H, \operatorname{Coker}\iota')$. Then e'' comes from $\mathrm{Ext}^1(A^H, M_P/H^0(X,M))$ by the pull-back a_U^*, because we have a natural morphism of the triangle (3.8.1) to the pull-back of the triangle $\to j'_! j'^* M \to M \to i'_* M_P \to$ by pr_1, which induces the morphism of the associated long exact sequences and $\operatorname{Coker}\iota \to \operatorname{Coker}\iota'$, where $j'\colon U \to X, i'\colon P \to X$ are the natural inclusions. By the right exactness of Ext^1 on $\mathrm{MHS}(A)^P$, cf. 3.5 (iv), e'' is the image of $e_1 \in \mathrm{Ext}^1(A_U^H, (M_P)_U)$. Let e'_1 be the image of e_1 in $\mathrm{Ext}^1(A_U^H, \operatorname{Coker}\iota)$ by the natural morphism. Then the image of $e' - e'_1$ in $\mathrm{Ext}^1(A_U^H, \operatorname{Coker}\iota')$ is zero, and $e' - e'_1$ is the image of $e_2 \in \mathrm{Ext}^1(A_U^H, M_U)$. Here e_2 is extended to X, if it holds for the underlying A-local systems by the property of admissible variation.

Therefore by the property of β in 3.6 the proof of the surjectivity of β is reduced to the coincidence of the underlying extension class $\bar{e}_2$ of e_2 with the restriction to U of the underlying class $\bar{e}$ of e under the natural isomorphisms

$$\mathrm{Hom}(A, H^1(X,K)) = \mathrm{Hom}(A, (a_X)_* K[1]) = \mathrm{Ext}^1(A_X, K) \tag{3.8.5}$$

where K, K' are the underlying A-local systems of M, M'. Let

$$u = u_1 + u_2\colon (pr_2)_* K' = a_U^*(a_X)_* K \to a_U^* K_P \oplus K|_U$$

the first morphism in the direct image of the underlying triangle of (3.8.1). By $\mathrm{Ext}^1(A, K_P) = 0$ the underlying classes $\bar{e}''$, $\bar{e}_1$ of e'', e_1 are zero, and the composition of u_1 with $a_U^* \bar{e} \in \mathrm{Ext}^1(A_U, (pr_2)_* K')$ is zero, where the first isomorphism of (3.8.5) is used. Moreover the underlying class $\bar{e}'$ of e' is the image of $u \circ a_U^* \bar{e} \in \mathrm{Ext}^1(A_U, (K_P)_U \oplus K|_U)$, because $u \circ a_U^* \bar{e}$ is factorized by

$$\tau_{\leqslant 1} u\colon \tau_{\leqslant 1}(pr_2)_* K' \to (K_P)_U \oplus K|_U$$

and the quotient triangle of

$$\to \tau_{\leqslant 1}(pr_2)_* j_! j^* K' \to \tau_{\leqslant 1}(pr_2)_* K' \to (K_P)_U \oplus K|_U \to$$

by $\to 0 \to H^0(X, K)_U \to \operatorname{Im} \iota \to$ is equivalent to the underlying exact sequence of (3.8.3). On the other hand the composition of u_2 with $a_U^* \bar{e}$ is the restriction to U of the extension class in $\operatorname{Ext}^1(A_X, K)$ corresponding to $\bar{e}$ by definition of the second isomorphism of (3.8.5). Therefore by the long exact sequence associated with the (underlying) middle exact sequence of (3.8.4) the assertion is reduced to:

$$(0, e) \in \operatorname{Ext}^1(A_U, (K_P)_U \oplus K|_U) \quad \text{is zero if } (0, e) \in \operatorname{Im} \iota_* \tag{3.8.6}$$

where $\iota_*: \operatorname{Ext}^1(A_U, H^0(X, K)_U) \to \operatorname{Ext}^1(A_U, (K_P)_U \oplus K|_U)$. But this follows from the splitting of the injection $H^0(X, K) \to K_P$.

3.9. REMARK. The above proof of 3.6 can be applied to the analytic case if we use [KK]. In fact it is enough to show the long exact sequence (3.8.2) in $\mathrm{VMHS}(X, A)_{\mathrm{ad}}$, where we assume X has a Kähler compactification $\bar{X}$ (the condition of admissible variation and the mixed Hodge structure on the cohomology might depend on the meromorphic equivalent class of compactification). To show that $\mathscr{H}^1(pr_2)_* j_! j^* M'$ is admissible relative to $\bar{X}$, we embed pr_2 to $\overline{pr}_2: \bar{X} \times \bar{X} \to \bar{X}$, and for any curve C in $\bar{X}$ such that $C \cap U$ is dense in C, we take a desingularization of $(\overline{pr}_2^{-1} C, \overline{pr}_2^{-1} C \backslash Y')$, and apply the arguments in [S2][S4]. To show that the morphisms are compatible with mixed Hodge structures, we may restrict to each point Q of U and its fiber $pr_2^{-1}(Q)$. Let $j_Q: X \backslash (P \cup Q) \to X$ be the natural inclusion. Then the cohomology $H^{\cdot}(X, (j_Q)_! j_Q^* M)$ is calculated by taking $\pi: \bar{X}' \to \bar{X}$ the blow-up along P and Q, and the restriction of (3.8.2) to Q follows from the (perverse) Leray spectral sequence.

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