COMPOSITIO MATHEMATICA

JEFFREY LANG

Generic Zariski surfaces

Compositio Mathematica, tome 73, nº 3 (1990), p. 345-361

<http://www.numdam.org/item?id=CM_1990__73_3_345_0>

Compositio Mathematica **73**: 345–361, 1990.

Generic Zariski surfaces*

JEFFREY LANG

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

Received 30 June 1988; accepted 19 July 1989

Introduction

The simplest type of purely inseparable cover of a variety X with coordinate ring A in characteristic $p \neq 0$ is obtained by taking $Y = \mathrm{Spec}(A[\sqrt[p]{g}])$ for some $g \in A$. Efforts to relate the codimension one cocycles of X and Y ([2], [10]) have led to the ring-theoretic question, "If A is a UFD of characteristic $p \neq 0$, for what $g \in A$ is $A[\sqrt[p]{g}]$ a UFD?" A natural place to begin such investigations is with the case where A is a polynomial ring. Then we may ask, "For what $g \in k[x, y]$ is $k[x^p, y^p, g]$ a UFD?" Note that if g_x and g_y have no common factor in $k[x, y]$ then the coordinate ring of the surface $z^p = g$ is isomorphic to A([10], pg. 393).

The main result of this paper is motivated by the classical result of Max Noether, that a generic surface in $\mathbb{P}^3$ has Pic $\cong \mathbb{Z}$[7]. This result was extended to all characteristics by Deligne [5].

Let G be of degree n and a_{ij} its coefficients: $G = \sum a_{ij}x^i y^j \in k[x, y]$, with k an algebraically closed field of characteristic $p \neq 0$. We say that a property P is true *in general* for the surface $z^p = G(x, y)$ if there exists a non-zero $Q \in k[A_{ij}]$ such that P is true whenever $Q(a_{ij}) \neq 0$. We say that P is *generically* true, if it is true when the a_{ij} are algebraically independent over Fp.

This article completes the project of determining the group of Weil divisors of the surface $z^p = G(x, y)$ for a general choice of G. Consider the following theorem.

*THEOREM (Blass–Deligne–J. Lang). *The group of Weil divisors of the surface $z^p = G(x, y)$ is 0 (i.e., $k[x^p, y^p, G]$ is a UFD) if $n = \deg G \geqslant 4$ and $p > 2$, and is $\mathbb{Z}/2\mathbb{Z}$ if $n \geqslant 5$ and $p = 2$ in general.*

In [11] Lang shows that is enough to prove (*) for a generic G. Blass in [1] calculates the divisor class group of $z^p = G$ for a generic G in the case where $n = 0$ (mod p) and $p \geqslant 5$. Grant and Lang prove (*) for the remaining $p = 2$ and $p = 3$ cases.

---

* Partially supported by University of Kansas GRF Fund # 3761-20-0038

Blass [1] uses the fundamental group to study the curves on a disingularization of $z^p = G$ to arrive at his result, where the argument depends on a result of W. Lang [12] and the fact that there are no singularities at infinity. If $\deg G$ is not divisible by p, then this approach does not work, as the singularities at infinity present difficulties.

In [6] and this paper, this problem is overcome by combining the fundamental group methods with purely inseparable descent [16]. All three articles use techniques of Grothendieck [7] to study coverings of one curve by another, but in this paper obstacles such as singular points and wild ramification arise. Because of this, weaker results concerning the action of $\mathrm{Gal}(k\colon \mathbb{F}_p(a_{ij}))$ on the singular points are obtained (compare [2] page 273 and I. (5.7).), so that the arguments involving logarithmic derivatives II. (2.2) needed to be changed considerably.

Chapter I is quite long although the ideas are not difficult. If one is willing to accept the principal result in this chapter, Theorem 5.7, which intuitively seems true, then Chapter II provides a fairly brief and simple proof of the main theorem, II.(2.2).

A preliminary announcement of this article, coauthored by P. Blass, appeared in [4].

0. Notation and definitions

0.1 $k = \bar{k}$ is an algebraically closed field of characteristic $p \neq 0$. T_{ij} are indeterminates algebraically independent over $k, 0 \leqslant i + j \leqslant n$, where $n \geqslant 4$ is a fixed positive integer.

$$F(x, y) = \sum_{0 \leqslant i+j \leqslant n} T_{ij} x^i y^j.$$

$\sum$ stands for $\sum_{0 \leqslant i+j \leqslant n}$ unless stated otherwise.
F_x, F_y means $\partial F/\partial x, \partial F/\partial y$, etc.
$H(F) = F_{xx}F_{yy} - F_{xy}^2 =$ hessian of F.
$L = \overline{k(T_{ij})}$, the algebraic closure of $k(T_{ij})$.
$G = \mathrm{Gal}(L\colon k(T_{ij}))$.
$A = \mathrm{Spec}(k[T_{ij}])$.
$E = \mathrm{Spec}(k[T_{ij}]/(F_x, F_y))$.
There is a natural morphism

$$E \xrightarrow{\pi} A.$$

If $X \to A$ is a morphism, E_X will denote the scheme $E \times A$ and $\pi_X\colon E_X \to X$ the projection. If $U \subset A$ is open or closed, $\pi_U\colon E_U \to U$ has the foregoing meaning with respect to the inclusion map $U \to A$. Also the same conventions are applied to the map $X \to E$.

0.2 Closed points of A will be identified with polynomials of degree n in $k[x, y]$. Define a subset $V \subseteq A$ as follows: If $n \neq 0 \pmod p$, then a polynomial $g \in k[x, y]$ belongs to V if and only if g_x and g_y do not meet at infinity. If $n = 0 \pmod p$ then $g \in V$ if and only if the surface $z^p = g$ has no singularities at infinity. In both of these cases V is open and dense in A (see [2] page 267, no. (0.2) and [6] no. (0.2)). Now define a subset $U \subset V$ as follows: $g \in U$ if and only if $g \in V$ and $z^p = g$ has only non degenerate singularities (i.e., $g_x = g_y = 0$ implies hessian of $g \neq 0$). It turns out that U is a non empty open subset of V (see I(3.2) below).

0.3 With F as above, let $R = L[x, y, z]/(z^p - F(x, y))$ and $S = \operatorname{Spec} R$. Then all of the singularities of S are rational double points and there are $(n-1)^2$ of them if $n \neq 0 \pmod p$ and $n^2 - 3n + 3$ otherwise (see I(3.5)). When their coordinates need to be written, we will write $Q = (a_1, a_2, a_3)$. Thus we define $H(Q) = (F_{xx} - F_{xy}^2)\,(a_1, a_2)$.

0.4 Let X be a noetherian scheme, $Et(X)$ the category of finite étale coverings of X. Let Ω be an algebraically closed field. b: $\operatorname{Spec}\Omega \to X$, a geometric point of X. Let $Y \in Et(X)$. $F_b^X(Y)$ is the set of liftings

$$\begin{tikzcd} & Y \\ b\colon \operatorname{Spec}\Omega \arrow[ur, dashed] \arrow[r] & X \arrow[u] \end{tikzcd}$$

If $W \to X$ is a morphism, we then obtain a base change functor $Et(X) \to Et(W)$, which will be denoted by R_W or simply R. If X and Y are schemes, $X \cup Y$ denotes the disjoint union of X and Y.

0.5 In the following definition the ground field is assumed to be algebraically closed of characteristic $p \neq 0$. π: $A \to B$ is a finite separable morphism of curves with B irreducible and smooth.

0.6 Definition: π: $A \to B$ is called *r-simple* over a point $q \in B$ if there exists a point $p \in \pi^{-1}(q)$ such that for all $p' \neq p$ in $\pi^{-1}(q)$, p' is a nonsingular point of A, π is unramified at p', and such that the cardinality of $\pi^{-1}(q)$ is $\deg \pi - r + 1$.

0.7 If A is a Krull ring, $\mathrm{Cl}(A)$ will denote the divisor class group of A (see [15], pg. 4 for the definition). By a *surface*, we mean an irreducible, reduced, two-dimensional quasi-projective variety over an algebraically closed field. If E is a normal surface, $\mathrm{Cl}(E)$ will denote the divisor class group of the coordinate ring of E.

0.8 A_k^n stands for affine n-space over k. k^n is the set of all n-tuples of elements of k. For $g \in k[x,y]$, $Sg = \{(\alpha,\beta) \in k^2 \colon g_x(\alpha,\beta) = g_y(\alpha,\beta) = 0\}$.

I. THE GALOIS ACTION ON SINGULARITIES

1. Preliminaries

The proofs of the results in this section can be found in ([2], pgs. 275–276) or in [1]. They are based on the techniques described in Grothendieck's, SGAI, Chapter VII.

Let $i\colon Y \to X$ be a morphism of locally noetherian connected (regular) schemes and $b\colon \operatorname{Spec}\Omega \to Y$ be a geometric point of Y, where Ω is an algebraically closed field. We will abuse notation and let b also denote the corresponding geometric point of X.

1.1 The reader is reminded of the definition (see [7], pgs. 140–142) of the induced homomorphism

$$i\colon \pi_1(Y,b) \to \pi_1(X,b).$$

Consider the diagram of functors,

$$\begin{tikzcd} Et(Y) \arrow[d, "F_b^Y"] & Et(X) \arrow[l] \arrow[d, "F_b^X"] \\ \mathrm{ENS} & \mathrm{ENS} \end{tikzcd}$$

(ENS is the category of finite sets. See [7], pg. 146.) We have that $\pi_1(Y,b) = \operatorname{Aut}(F_b^Y)$ and $\pi_1(X,b) = \operatorname{Aut}(F_b^X)$. By SGAI (see [7], pg. 142) there is an isomorphism of functors:

$$F_b^Y \circ R \underset{\tau}{\overset{\mu}{\rightleftarrows}} F_b^X \quad \text{where } \mu\tau = \mathrm{id}(F_b^X) \quad \text{and} \quad \tau\mu = \mathrm{id}(F_b^Y \circ R).$$

If $\sigma \in \pi_1(Y,b) = \operatorname{Aut}(F_b^Y)$, we define $\bar{\sigma} = i_*(\sigma)$ by the diagram

$$\begin{tikzcd} F_b^X(W) \arrow[r, "\tau"] \arrow[d, "\bar{\sigma}_W"] & F_b^Y(W_Y) \arrow[d, "\sigma_{W_Y}"] \\ F_b^X(W) & F_b^Y(W_Y) \arrow[l, "\mu"] \end{tikzcd}$$

1.2. PROPOSITION. *If $W \in Et(X)$ is irreducible, then $\pi_1(X,b)$ acts transitively on $F_b^X(W)$ for any base point b in X.*

1.3. PROPOSITION. *Let $W \in Et(X)$ be irreducible and assume that $R(W) = W_Y$ decomposes into $W_Y = S(Y) \sqcup T$ where $s\colon Y \to W_Y$ is a section and T is irreducible. Then for any base point $b \in X$ the action of $\pi_1(X,b)$ on $F_b^X(W)$ is transitive and twice transitive.*

1.4. PROPOSITION. *Let $W \in Et(X)$, $R_W(S) = W_Y \in Et(Y)$ and let b be a base point in Y. Suppose that the action of $\pi_1(Y,b)$ on $F_b^Y(W_Y)$ includes a nontrivial permutation of r elements, then the action of $\pi_1(X,b)$ on $F_b^X(W)$ includes a nontrivial permutation of r elements. Also, if b' is any other base point in X not necessarily in Y, the action of $\pi_1(X,b')$ on $F_{b'}^X(W)$ also includes a non trivial permutation of r elements.*

2. Some results on curves

In this section $\pi\colon A \to B$ is a finite separable morphism of curves with B irreducible and smooth where the ground field k is assumed to be algebraically closed of characteristic $\neq 0$.

2.1. LEMMA. *Assume that π is r-simple over $q \in B$. Let p be the only point of $\pi^{-1}(q)$ where A may be singular. Let $\mathcal{O}_q^*$ be the henselization of $\mathcal{O}_q$ and K^* its quotient field. Then* $\mathrm{Spec}(K^*) \times_B A = S \sqcup \mathrm{Spec}(L_i) \sqcup \cdots \sqcup \mathrm{Spec}(L_s)$ *where S is a disjoint union of sections over* $\mathrm{Spec}(K^*)$, *L_j is a finite separable field extension of K^* for each $j, s \leqslant$ multiplicity of p on A, and $\Sigma[L_i\colon K^*] = r$.*

Proof. Let $W = \mathrm{Spec}\, T$ be an affine open neighborhood of q in B. Then $\pi^{-1}(W) = \mathrm{Spec}(R)$ is an affine open neighborhood of A containing the fibre $\pi^{-1}(q)$ of q in A since π is finite. By hypothesis $\pi^{-1}(q)$ consists of p and a finite number of remaining points $p_2, \ldots, p_n$ where A is smooth and unramified over q and where $n = \deg \pi - r$. Then $\mathrm{Spec}(\mathcal{O}_q^*) \times_B A = \mathrm{Spec}(\mathcal{O}_q^* \otimes_T R)$. $\mathcal{O}_q^* \otimes_T R$ is a finite integral extension of $\mathcal{O}_q^*$ and is therefore a direct sum $\mathcal{O}_q^* \otimes_T R = \oplus R_i$ where $R_1 = \mathcal{O}_q^* \otimes_T \mathcal{O}_p$ and $R_i = \mathcal{O}_q^* \otimes_T \mathcal{O}_{p_i}$ for $i = 2, \ldots, n$ (see [13], Theorem (43.15), pg. 185). The local rings $\mathcal{O}_q$ and $\mathcal{O}_{p_i}$ are discrete valuation rings with residue field k, so that $\mathcal{O}_q^*$ and R_i with $i \geqslant 2$ are as well ([14], Theorem (5.11.1), p. 193). Also, for $i \geqslant 2$, R_i is a finite integral unramified extension of $\mathcal{O}_q^*$. By Nakayama's lemma $R_i = \mathcal{O}_q^*$ for $i > 1$, so that $K^* \otimes_T R_i = K^*$ for $i > 1$.

Now let $\tilde{\mathcal{O}}_p$ be the integral closure of $\mathcal{O}_p$ in its total quotient field, and let $\mathcal{O}_p^* = \mathcal{O}_p \otimes_T \mathcal{O}_q^*$ and $\tilde{\mathcal{O}}_p^* = \tilde{\mathcal{O}}_p \otimes_T \mathcal{O}_q^*$. Then $\tilde{\mathcal{O}}_p^*$ is the integral closure of $\mathcal{O}_p^*$ in its total quotient ring ([15], page 101, Proposition 2) and we have that $\mathcal{O}_q^* \subseteq \mathcal{O}_p^* \subseteq \tilde{\mathcal{O}}_p^*$ are integral extensions. Let $m_1, \ldots, m_s$ be the maximal ideals of $\tilde{\mathcal{O}}_p$. By ([16], page 299, Corollary 1) $s \leqslant$ multiplicity of p on A. Again we have that $\tilde{\mathcal{O}}_p^* = \bigoplus_{j=1}^s R'_j$ where $R'_j = \mathcal{O}_{m_j} \otimes \mathcal{O}_q^*$ are discrete valuation rings with residue field k and whose valuation agrees with the valuation on $\mathcal{O}_{m_j}$([14], page 193). If we let t be a parameter for the maximal ideal of $\mathcal{O}_q$, it then follows that $\sum_{j=1}^s v_j(t) = r$ where v_j is the valuation on R'_j. Thus we have that $L_i = K^* \otimes_T R'_j$ are separable field extensions of K^* with $\sum_{j=1}^s [L_i: K^*] = r$, so that we are done if $K^* \otimes_T \mathcal{O}_p^* = K^* \otimes_T \tilde{\mathcal{O}}_p^*$. This is not difficult to see. Since they both have the same total quotient ring and $K^* \otimes_T \tilde{\mathcal{O}}_p^*$ is integral over $K^* \otimes_T \mathcal{O}_p^*$ we have that the conductor of $K^* \otimes \mathcal{O}_p^*$ in $K^* \otimes \tilde{\mathcal{O}}_p^*$ contains a nonzero divisor x. Then x is integral over K^* so that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in K^*$ with $a_n \neq 0$. Then a_n is in the conductor and is a unit in $K^* \otimes \mathcal{O}_p^*$.

2.2. THEOREM. *Let $\pi: A \to B, p \in A, q \in B$ be as in* (2.1). *Let $B^0 = B - \{q\}$ and $A^0 = \pi^{-1}(B^0)$. Assume that the induced morphism $A^0 \to B^0$ is étale. If $F_{b_0}^{B^0}(A^0)$ has $\deg \pi$ elements for some geometric point $b_0 \in B^0$, then each L_j in* (2.1) *is a Galois field extension of K^*.*

Proof. We have morphisms $\operatorname{Spec}(\overline{k(B^0)}) \to \operatorname{Spec}(K^*) \to \operatorname{Spec}(k(B^0)) \to B^0$. Thus we obtain a geometric point b of $\operatorname{Spec}(K^*)$ and the corresponding geometric point b_1 of B^0. By Grothendieck ([7], page 38), $F_b^{\operatorname{Spec}(K^*)}$ $(S \sqcup \operatorname{Spec}(L_1) \sqcup \cdots \sqcup \operatorname{Spec}(L_s)) \cong F_{b_1}^{B^0}(A^0) \cong F_{b_0}^{B^0}(A^0)$, which has $\deg \pi$ elements. Therefore card $F_b^{\operatorname{Spec}(K^*)}(S) + \sum_{j=1}^s$ card$(\operatorname{Aut}_{K^*}(L_j)) = \deg \pi$, so that $\sum_{j=1}^s$ card$(\operatorname{Aut}_{K^*}(L_j)) = r$. Since card$(\operatorname{Aut}_{K^*}(L_j)) \leqslant [L_j: K^*]$, it follows by (2.1) that $[L_j: K^*] = $ card$(\operatorname{Aut}_{K^*}(L_j))$ and hence L_j is Galois over K^* for each $j = 1, \ldots, s$.

2.3. COROLLARY. *If the multiplicity of p on A is less than r in* (2.2), *then for any base point b_1 of B^0, the action of $\pi_1(B^0, b_1)$ on $F_{b_1}(A^0)$ contains a nontrivial permutation of r-elements, holding all others fixed.*

Proof. Again let b and b_1 be the geometric points of $\operatorname{Spec}(K^*)$ and B^0 obtained from the morphisms $\operatorname{Spec}(\overline{k(B^0)}) \to \operatorname{Spec}(K^*) \to \operatorname{Spec}(k(B^0)) \to \operatorname{Spec}(B^0)$. By (2.1) and (2.2), we may assume that L_1 is a nontrivial Galois extension of K^*. Let σ be a nontrivial element of $\operatorname{Gal}(L_1, K^*)$. Then σ extends to an automorphism $\bar{\sigma}$ of $\overline{k(B^0)}^{\text{sep}}$ over K^*.

By Grothendieck ([7], p. 143, Proposition 8.1) we have $\pi_1(\operatorname{Spec}(K^*), b) = \operatorname{Gal}(\overline{k(B^0)}^{\text{sep}}, K^*)$. The element $\bar{\sigma} \in \pi_1(\operatorname{Spec}(K^*), b)$ induces a nontrivial permutation of the r elements of $F_b(\operatorname{Spec}(L_1) \sqcup \cdots \sqcup \operatorname{Spec}(L_s))$ and holds the $\deg \pi - r$ elements of $F_b(S)$ fixed. By (1.4), $\pi_1(B^0, b_1)$ induces a permutation of less than or equal to r elements of $F_{b_1}^{B^0}(A^0)$, holding all others fixed. The independence of base point also follows by (1.4).

3. The geometry of the map E → A

In this section we need to collect some facts about the geometry of the map $E \to A$. Many of the proofs are omitted because they could be found in [1] or [2].

3.1. PROPOSITION. *E is smooth, irreducible, and isomorphic to an affine space over k of dimension equal to the dimension of A* ([2], *p.* 281, (3.1.1)).

3.2. PROPOSITION. (a) *$U \subset V$ is open and dense,* (b) *$\pi_V: E_V \to V$ is a finite map,* (c) *$\pi_U: E_U \to U$ is étale* (*see* [2], *pages* 281–282 *and* [6], *Chapter I,* (3.5), (3.6), *and* (3.8)).

3.3. PROPOSITION. *For any base point $b \in U$, the action of $\pi_1(U, b)$ on $F_b(E_U)$ is transitive.*

Proof. As E_U is a dense open subscheme of E, it is irreducible and therefore connected. The result follows by (1.2).

3.4. THEOREM. *There exists a point $g \in V$ such that $\pi^{-1}(g)$ consists of $(n-1)^2$ unramified points (at which π is étale) if $n \neq 0 \pmod p$, $n^2 - 3n + 3$ unramified points otherwise. $k(E)$ is a field extension of $k(A)$ of degree $(n-1)^2$ if $n \neq 0 \pmod p$, $n^2 - 3n + 3$ otherwise.*

Proof. We will prove this for the case $n \neq 0 \pmod p$. The remaining case uses the same argument and is left as an exercise.

Assume first that $n \neq 2 \pmod p$. Let $g = xy = 1/n(x^n - y^n)$. Then $g_x = y + x^{n-1}, g_y = x - y^{n-1}$ and the hessian of g is $H = -(1 + (n-1)^2 x^{n-2} y^{n-2})$. Then $g \in V$ since $x_n - y_n$ has distinct factors.

$\pi^{-1}(g)$ is the set of points in k^2 where g_x and g_y meet. We have that g_x, g_y and H are never simultaneously 0. For if $(a, b) \in k^2$ is a point where $g_x = g_y = H = 0$, then $(n-1)^2 a^{n-2} b^{n-2} + 1 = 0$ which implies that $(n-1)^2 a^{n-1} b^{n-1} + ab = 0$, which gives $((n-1)^2 - 1)ab = 0$ since $a^{n-1} = -b$ and $b^{n-1} = a$. Therefore $n(n-2)ab = 0$ and hence $a = b = 0$. But then $H(a, b) = -1$.

Therefore, in fact $g \in U$ by (0.2). Thus $\pi^{-1}(g)$ consists of $(n-1)^2$ unramified points by (3.2). By (3.2) π is separable. It follows that $[k(E): k(A)] = (n-1)^2$. If $n = 2 \pmod p$, the same argument works with $g = x + xy + 1/n(x^n - y^n)$.

The proofs of the next two corollaries to (3.4) are proved by Blass ([2], page 287) for the case $n = 0 \pmod p$, but the arguments are independent of this assumption.

3.5. COROLLARY. *The surface S has $(n-1)^2$ singularities at finite distances if $n \neq 0 \pmod p$, $n^2 - 3n + 3$ otherwise.*

3.6. COROLLARY. *All of the singularities of S are nondegenerate.*

4. Some r-simple morphisms

This section begins by producing an example of a $g \in V - U$, such that $\pi^{-1}(g)$ is a set of $(\deg \pi) - 2$ distinct elements when $p = 3$ and $\deg(\pi) - 3$ distinct elements when $p = 2$.

4.1. EXAMPLE. In (a), (b), (c), $p = 3$ and $g \in V - U$ is such that the curves $g_x = 0$ and $g_y = 0$ meet at $\deg \pi - 3$ points of k^2 transversally with intersection multiplicity 1 and exactly one point with intersection multiplicity 3. In (d), (e), $p = 2$ and $g \in V - U$ is such that $g_x = 0$ and $g_y = 0$ meet at $\det(\pi) - 4$ points transversally with intersection multiplicity 1 and exactly one point Q with intersection multiplicity 4.

(a) If $n = 2 \pmod 3$, with $n \geqslant 5$, let $g = x^n + y^n + x^4 + xy - x$ if $n - 2 = 3s$ with $s = 1 \pmod 3$, otherwise let $g = x^n + y^n + xy^3 + xy - x$. In both cases $Q = (1, 1)$.

(b) If $n = 1 \pmod 3$, with $n \geqslant 7$, let $g = x^n + y^n + x^5 + x^4 + (y + (-1)^n)^3 x + y^2$. $Q = ((-1)^{n-1}, 0)$.

(c) If $n = 0 \pmod 3$ with $n \geqslant 6$, let $g = xy^{n-1} - x^{n-1} + x^{n-2}y + x^{n-4} + x$ if $n = 3s$ with $s = 1 \pmod 3$, otherwise let $g = xy^{n-1} + x^{n-2}y + x^{n-4}y^3 - x^{n-1} + x$. In both cases $Q = (1, 1)$.

(d) If $n = 2s + 3$ with $s \geqslant 1$, let $c \in k$ be such that $c \neq 0, 1$ and $g = (x + y)^2 (x + y + 1)^{2s} x + y^3(y + c)^{2s} + x^3 y$. $Q = (0, 0)$.

(e) If $n = 2s + 6$ with $s \geqslant 0$, let c be as in (d) and $g = xy(f(y) + x)^2 + (x + y)^2 (x + y + 1)^{2s} x + y^3(y + c)^2$ where $f(y)$ has degree $s + 2$, $f(y) + y$ has 0 as a root of multiplicity 1 and is such that $f(c) + c \neq 0, 1$. $Q = (0, 0)$.

The idea is now to construct using (4.1) a line L in A containing g so that the curve E_L lying above it in E is 3 simple over g if $p = 3$ and 4 simple over g if $p = 2$. When $p = 3$ we also want E_L to be nonsingular. The approach is to find $h(x, y) \in k[x, y]$ so that the line L defined by $g(x, y) + \lambda h(x, y)$: $\lambda \in k$, has the desired properties. We will do this explicitly for case (4.1(a)) and (4.1(d)) above, leaving the details for the remaining cases in (4.1) to the reader.

4.2. THEOREM. *Let $p = 2$ or 3, $n \geqslant 4$. Then there is a line L in $A = \operatorname{Spec} k[T_{ij}]$ containing g such that* (a) *E_L is irreducible* (b) *$\pi_L: E_L \to L$ is 3 simple over g if $p = 3$, 4 simple over g if $p = 2$,* (c) *E_L is nonsingular if $p = 3$. If $p = 2$, $\pi_L^{-1}(g)$ contains exactly one singular point of multiplicity 2.*

Proof. Case: $p = 3$, $n = 2 \pmod 3$, $g = x^n + y^n + x^4 + xy - x, n \geqslant 5$, $n - 2 = 3s$ *with* $s = 1 \pmod 3$. Let L be the line in $\operatorname{Spec} k[T_{ij}]$ corresponding to polynomials of the form $\lambda y + g, \lambda \in k$. Let $\pi_L: E_L \to L$ be the induced morphism. E_L is isomorphic to $\operatorname{Spec} k[x, y, \lambda]/(-x^{n-1} + x^3 + y, -y^{n-1} + x + \lambda) \cong k[x, y]/(-x^{n-1} + x^3 + y) \cong k[x]$, which is a line. This proves (a) and (c).

$E_L \to L$ is isomorphic to the projection to the $\operatorname{Spec}(k[\lambda])$-axis of the space curve in $\operatorname{Spec} k[x, y, \lambda]$ defined by $g_x = \lambda + g_y = 0$.

The matrix of partials with respect to x, y, and λ is

$$\begin{bmatrix} g_{xx} & g_{xy} & 0 \\ g_{xy} & g_{yy} & 1 \end{bmatrix}.$$

From (4.1) we have that if $\lambda = 0$, then $\det\begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \neq 0$ for every point of the space curve except the point Q defined by $\lambda = 0, x = 1, y = 1$. It then follows that there are $(n-1)^2 - 3$ points of $\pi^{-1}(g)$ where π is unramified and exactly one remaining point $Q \in \pi^{-1}(g)$. Therefore π is 3 simple over Q.

Case: $p = 3, n = 2 \pmod 3$, $n - 2 = 3s$ with $s \neq 1 \pmod 3$, $g = x^n + y^n + xy^3 + xy - x$. Apply the same argument with L defined by the space curve $g + \lambda y, \lambda \in k$.

Case: $p = 2, n = 2s + 3$ with $s \geqslant 1$ with $s \geqslant 1, g = (x+y)^2(x+y+1)^{2s}x + y^3(y+c)^{2s} + x^3y$, where $c \neq 0, 1 \in k$. Let L be the line in $\operatorname{Spec} k[T_{ij}]$ defined by polynomials of the form $g + \lambda x, \lambda \in k$. E_L is isomorphic to $\operatorname{Spec} k[x\,y, \lambda]/(g_x + \lambda, g_y) = k[x,y]/(y^2(y+c)^{2s} + x^3)$, hence E_L is irreducible. This proves (*a*).
This proves (*a*).

The matrix of partials with respect to x, y and λ for the ideal $(g_x + \lambda, g_y)$ $k[x, y, \lambda]$ is

$$\begin{bmatrix} 0 & x^2 & 1 \\ x^2 & 0 & 0 \end{bmatrix}.$$

From (4.1) we have that $\begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix} \neq 0$ at every point of E_L with $\lambda = 0$ except at the point Q given by $\lambda = 0$, $x = 0$, $y = 0$. Thus there are $(n-1)^2 - 4$ points of $\pi_L^{-1}(g)$ that are unramified over g and there is exactly one additional point of $\pi^{-1}(g)$ where E_L has a singularity of multiplicity 2. Therefore π_L is 4-simple over g.

The next theorem summarizes what has been shown in Sections 3 and 4.

4.3. THEOREM. *There exists a point $g \in V - U$ and a line L, closed in V, such that $g \in L$ and $L_U = L \cap U$ is open and dense in L and closed in U. Let L_1 be the open subset of L defined by $L_U \cup \{g\}$. Then we have induced coverings*

$$\begin{array}{ccccc} E_{L_U} & \subset & E_{L_1} & \subset & E_L \\ \downarrow & & \downarrow & & \downarrow \\ L_U & \subset & L_1 & \subset & L \end{array}$$

with $E_{L_1} \xrightarrow{\phi} L_1$ 3-simple over g if $p = 3$, 4-simple over g if $p = 2$. In the case $p = 2$, the fibre over g in E_{L_1} contains exactly one singular point of multiplicity 2.

4.4. REMARK. $E_U \to U$ is étale. Therefore by base change $E_{L_U} \to L_U$ is étale.

5. The action of G on sing(S)

5.1. PROPOSITION. *Let $b\colon \operatorname{Spec}\Omega \to L_U$ be any geometric base point; then the* *action of* $\pi_1(L_U, b)$ *on* $F_b^{L_U}(E_{L_U})$ *includes a permutation of r elements, holding all other elements fixed, with* $r = 3$ *if* $p = 3$, $r = 4$ *if* $p = 2$.

Proof. Consider the case $p = 3, n = 2 \pmod 3)$, L defined by $g + \lambda y$ in the proof of (4.2). Let $S_0 = \{Q \in \overline{k[\lambda]}^2 : g_x(Q) = g_y(Q) + \lambda = 0\}$. It is left as an exercise to verify (apply the same argument as (3.4) and (3.5)) that S_0 has $(n-1)^2 = \deg(\pi_{L_U})$ elements. Let $b_0 : \operatorname{Spec}(\overline{k(\lambda)}) \to \operatorname{Spec}(k(\lambda))$ be the base point of L_U such that $F_{b_0}(E_{L_U}) \cong S_0$. The proposition then follows by (1.4), (2.3), (4.3) and (4.4). The remaining cases are similar.

5.2. PROPOSITION. *For any geometric point b in U, the action of* $\pi_1(U, B)$ *on* $F_b(E_U)$ *includes a permutation of r elements, holding all other elements fixed, with* $r = 3$ *if* $p = 3$, $r = 4$ *if* $p = 2$.

Proof. Use (1.4) and (5.1).

5.3. Let $Z = \operatorname{Spec} k[T_{00}, T_{20}, T_{11}, T_{02}, \ldots]$. Z corresponds to polynomials g such that $z^p = g(x, y)$ has a singularity at the origin. Z_U then corresponds to g in U that have a singularity at the origin.

5.4. THEOREM. *For each base point b in* Z_U, *there exists an* $A \in F_b^{Z_U}(E_{Z_U})$ *whose stabilizer in* $\pi_1(Z_U, b)$ *acts transitively on* $F_b(E_{Z_U}) - \{A\}$. (*For the proof see* ([2], *page* 295, (3.3.1).)

5.5. COROLLARY. $\pi_1(U, b)$ *acts on* $F_b(E_U)$ *transitively and twice transitively for any base point b in U* ([2], *page* 295, (3.3.2)).

5.6. THEOREM. *If* $p = 3$, *then for any geometric point* $b : \operatorname{Spec} \Omega \to U$, *the action of* $\pi_1(U, b)$ *includes the alternating group on* $F_b(E_U)$. *If* $p = 2$ *then for each pair* $A, B \in F_b(E_U)$ *there is a pair* $C, D \in F_b(E_U) - \{A, B\}$ *such that* $\pi_1(U, b)$ *acts as the identity on* $F_b(E_U) - \{A, B, C, D\}$ *and permutes the elements of* $\{A, B, C, D\}$ *nontrivially.*

Proof. Assume $p = 3$. Let $b : \operatorname{Spec} \Omega \to U$ be a base point. By (5.2) and (5.5) we have for each pair $A, B \in F_b(E_U)$, there is a $C \in F_b(E_U)$ such that $\pi_1(U, b)$ includes a nontrivial permutation of $\{A, B, C\}$ which acts as the identity on $F_b(E_U) - \{A, B, C\}$. If this permutation is a transposition then by (5.5) we are done. If not then by (5.5) we have that for each pair A, B there is a $C \in F_b(E_U)$ such that the 3-cycle $(A, B, C) \in \pi_1(U, b)$. Then choose a 3-cycle $(C, D, E) \in \pi_1(U, b)$ with $D \neq A$ or B. If $E \neq A, B$, then $(C, D, E)^2(A, B, C)(C, D, E) = (A, B, D) \in \pi_1(U, b)$. If $E = B$, then $(C, D, B)^2(A, B, C) = (A, B, D) \in \pi_1(U, b)$. This shows that the action of $\pi_1(U, b)$ on $F_b(E_U)$ contains all 3-cycles of elements of $F_b(E_U)$. The statement for $p = 2$ follows immediately from (5.2) and (5.5). The independence of base point is by Grothendieck ([7], pg. 141).

Recall that $F = \sum_{0 \leqslant i+j \leqslant n} T_{ij} x^i y^j$, $L = \overline{k(T_{ij})}$, $S = \operatorname{Spec}(L[x, y, z]/(z^p - F))$, $G = \operatorname{Gal}(L : k(T_{ij}))$ and $\operatorname{Sing}(S) = S_F = \{Q \in L^2 : F_x(Q) = F_y(Q) = 0\}$.

5.7. THEOREM. $G = \operatorname{Gal}(\overline{k(T_{ij})} : k(T_{ij}))$ *acts on* $\operatorname{Sing}(S)$ *as the full symmetric* *group if* $p = 3$. *If* $p = 2$, *then for each pair* $Q_1, Q_2 \in S_F$, *there exists a pair* Q_3, Q_4 *in* $S_F - \{Q, Q_2\}$ *and a* $\sigma \in G$ *such that* $\sigma(Q_1) = Q_2$, $\sigma(Q_2) = Q_1$, $\sigma(Q_3) = Q_4, \sigma(Q_4) = Q_3$ *and such that* σ *acts as the identity on* $S_F - \{Q_1, Q_2, Q_3, Q_4\}$.

Proof. Let $b\colon \mathrm{Spec}(\overline{k(T_{ij})}) \to \mathrm{Spec}(k(T_{ij}))$ be the base point of U such $\mathrm{F}_b(E_U) \simeq \mathrm{Sing}(S)$. We have by Grothendieck ([7], pg. 143) a surjective homomorphism $G \to \pi_1(U, b)$. The identification $\mathrm{Sing}(S) \simeq F_b(E_U)$ is G-equivariant, where G acts on $F_b(E_U)$ via $G \to \pi_1(U, b)$. Thus by (5.6) if $p = 2$ the action of G on Sing(S) contains a 2-, 3- or 4-cycle or a disjoint product of 2-cycles. If this action contains a 4-cycle then its square is a disjoint product of 2-cycles. Then (5.5) gives us the desired result. If this action contains a 2-cycle then G acts as the full symmetric group on Sing(S) by (5.5), while if it contains a 3-cycle then this action includes the alternating group by the same argument used in the $p = 3$ case of (5.6). In each of these cases the result still holds.

If $p = 3$, then by (5.6) the action of G on Sing(S) contains the alternating group. Thus it is enough to show that this action includes at least one odd permutation. This can be accomplished by showing ([14], page 81) that $\delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ is not the square of an element in $F_3[T_{ij}]$ where the α_i are x-coordinates of the points in Sing(S), and F_3 is the prime subfield in L(δ is the discriminant of $\prod_i (x - \alpha_i)$). We are done then if we prove the corresponding result for a specialization of F, that is, for some choice $T_{ij} = \alpha_{ij} \in k$.

Several cases must be considered. We will consider only a few, admittedly simpler ones, leaving the remaining cases as an exercise.

If $n = 1 \pmod 3$ with n even let $g = x^n + y^n + xy$. Then the x coordinates of S_g are the roots in k of $f(x) = x^{(n-1)^2} - x$. δ will then equal the determinant of $-I$, where I is the $(n-1)^2$ identity matrix. Thus $\delta = -1$ which is not the square of an element in F_3.

If $n \equiv 2 \pmod 3$, with n odd, let $g = x^n + y^n + xy + y$. Then the x coordinates of S_g are the roots in k of $f(x) = x^{(n-1)^2} - x - 1$. δ is equal to the determinant of the matrix

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & \cdots & -1 & -1 & & & & \\
 & 1 & 0 & 0 & \cdots & 0 & -1 & -1 & & & \\
 & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & 1 & 0 & 0 & \cdots & \cdots & 0 & -1 & -1 \\
1 & 0 & 0 & \cdots & \cdots & -1 & & & & & \\
 & 1 & 0 & 0 & \cdots & \cdots & -1 & & & & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\
 & & & 1 & 0 & 0 & \cdots & \cdots & \cdots & -1 & \\
 & & & & 1 & 0 & 0 & \cdots & \cdots & 0 & -1
\end{bmatrix}$$

Therefore $\delta = (-1)^{n^2-2n} = -1$. If $n = 0 \pmod 3$ with n even, let $g = x^{n-1}y + xy^{n-2} + y^{n-1} + y$. The x-coordinates of S_g are the roots of $f(x) = (x^{n-1}+1)[(x^{n-1}+1)^{n-3} - x^{(n-2)^2}]$. Again $\delta = -1$ by a similar computation.

II. THE GROUP OF WEIL DIVISORS OF S

1. Techniques of purely inseparable descent.

If R is a noetherian integrally closed domain then R is a Krull ring ([16], pp. 1–4 for definition) and $X = \mathrm{Spec}(R)$ will be regular in codimension one and the group of Weil divisors of X ([9], pg. 130) and the divisor class group of R as defined in Samuel's notes ([16], pg. 18) are isomorphic.

Let k be an algebraically closed field of characteristic $p \neq 0$. Let $g \in k[x, y]$ be such that g_x and g_y have no common factors in $k[x, y]$. Define a derivation D on $k(x, y)$ by $D = g_y(\partial/\partial x) - g_x(\partial/\partial_y)$. For each non negative integer m, let $A_m = k[x^{p^m}, y^{p^m}, g]$ and let $X_m \subset A_k$ be the surface defined by $z^{p^m} = g$. Then $A_0 = k[x, y]$. Denote the quotient field of A_m by E_m. Each A_m is isomorphic to the coordinate ring of X_m ([10], pg. 404) and is thus noetherian integrally closed and hence a Krull ring. Since $A_m^p \subseteq A_{m+1} \subseteq A_m$ we have that A_m is integral over A_{m+1}. By Samuel ([16], pgs. 19–20) there is a well defined homomorphism $\phi_m\colon \mathrm{Cl}(A_{m+1}) \to \mathrm{Cl}(A_m)$. Define $D_m\colon E_m \to E_m$ as follows.

Given $\alpha \in E_m$, it can be written as $\alpha = \sum_{i=0}^{p^m-1} \alpha_i^{p^m} g^i$ for unique $\alpha_i \in k(x, y)$. Then define

$$D_m(\alpha) = \textstyle\sum_{i=0}^{p^m-1} (D\alpha_i)^{p^m} g^i.$$

D_m is a derivative on E_m ([10], pg. 404). For each $m \geqslant 1$, let $\mathscr{L}_m$ be the additive group of logarithmic derivatives of D_m in A_m. Thus $\mathscr{L}_m = \{f^{-1}D_m(f) \in A_m\colon f \in A_m\}$.

1.1. THEOREM. (a) *There exists $a \in k[x, y]$ such that $D^p = aD$*, (b) $\ker D_m \cap A_m = A_{m+1}$. (c) $\ker(\phi_m) \cong \mathscr{L}_m$, (d) $D_m^p = a^{p^m} D_m$, (e) *the order of $\mathscr{L}_m$ is p^M for some $M \leqslant \deg(g)\,(\deg(g) - 1)/2$.* ([2] pgs. 393, 394, 404.)

1.2. THEOREM. *Let $D\colon K \to K$ be a derivation of a field K of characteristic $p \neq 0$. Let $K' = \ker(D)$ and $[K\colon K'] = p$. An element $t \in K$ is a logarithmic derivative (i.e., there exists an $x \in K$ such that $t = Dx/x$) if and only if $D^{p-1}(t) - at + t^p = 0$* where $D^p = aD$ ([16], pg. 64, (3.2)).

1.3. THEOREM. *Let $D = g_y(\partial/\partial x) - g_x(\partial/\partial y)$ and $\beta \in k[x, y]$ be such that $D^p = \beta D$. If $(a, b) \in k^2$ is such that $g_x(a, b) = g_y(a, b) = 0$, then $\beta(a, b) = (\bar{H}(a, b))^{p-1/2}$ where $\bar{H} = g_{xy}^2 - g_{xx}g_{yy}$* ([3], Theorem 3.4).

1.4. LEMMA. *Let* $t = \sum_{j=0}^{p^m-1} \alpha_i^{p^m} g^j \in A_m$. If $t \in \mathscr{L}_m$ *then the degree of each* α_j *is less than or equal to* $\deg(g) - 2$ ([3], *Cor.* 3.6).

Consider Zariski surfaces $X: z^p = g$ *such that* g_x *and* g_y *meet transversally and in the maximum number of points of* k^2. *This number is* $(n-1)^2$ *if* $n \neq 0 \pmod p$, $n^2 - 3n + 3$ *otherwise, where* $n = \deg(g)$. *Such a g we will say satisfies condition* (*). *This is equivalent to saying that* $g \in U$ (*see* [2], *pg.* 268 *and* [6]). *In both of these cases, polynomials* $g \in k[x, y]$ *satisfy* (*) *for a general choice* ([2], *pg.* 282).

1.5. THEOREM. *Let g satisfy* (*). *Then for each* $m \geqslant 0$, $\mathscr{L}_m \simeq \mathscr{L}_0$, *the group of logarithmic derivatives of* $D = g_y(\partial/\partial x) - g_x(\partial/\partial y)$ *in* $k[x, y]$. ([16], *II* (2.1)).

1.6. LEMMA. *Let g satisfy* (*). *If* $0 \neq t \in \mathscr{L}_0$ *then* $t(Q) \neq 0$ *for at least one point of* $S_g = \{Q \in k^2 : g_x(Q) = g_y(Q) = 0.\}$ *Furthermore, if* $n = \deg(g) \neq 0 (\mathrm{mod}\, p)$ *then* $t(Q) \neq 0$ *for at least* $n - 2$ *points of* S_g([11], *pg.* 278,(2.9)).

For each $Q \in S_g$ let $\sqrt{\bar{H}(Q)}$ denote a root of the polynomial $\omega^2 = \bar{H}(Q)$ in k (if $p = 2$. $\sqrt{\bar{H}(Q)}$ is just $g_{xy}(Q)$.) Let $\mathbb{Z}/p\mathbb{Z}. \sqrt{\bar{H}(Q)}$ be the additive cyclic subgroup of k generated by $\sqrt{\bar{H}(Q)}$. If $t \in \mathscr{L}_0$, then $D^{p-1}t - at = -t^p$ by (1.2). By (1.3) this implies that $(t(Q))^p = (\sqrt{\bar{H}(Q)})^{p-1} t(Q)$. Thus $t(Q) \in \mathbb{Z}/p\mathbb{Z}.\sqrt{\bar{H}(Q)}$ for each $Q \in S_g$. We obtain a homomorphism $\Phi: \mathscr{L}_0 \to \bigoplus_{Q \in S_g} \mathbb{Z}/p\mathbb{Z}. \sqrt{\bar{H}(Q)}$ defined by $\Phi(t) = (t(Q)_{Q \in S_g})$. From (1.6) we have

1.7. LEMMA. *Let g satisfy* (*). *Then* Φ *is an injection.*

2. The generic class group

Let $(\mathbb{Z}/p\mathbb{Z})^s$ be a direct sum of s copies of $\mathbb{Z}/p\mathbb{Z}$, $p \neq 0, s \geqslant 3$. Let $C(S)$ be the group of permutations of elements of $(\mathbb{Z}/p\mathbb{Z})^s$ and T be the group of automorphisms of $(\mathbb{Z}/p\mathbb{Z})^s$ corresponding to sign changes of coordinates (if $p = 2$, $T = \{\mathrm{id}\}$). Let $p_1: C(S) \times T \to C(S)$ be the projection map. Let H be a subgroup of $C(S)$ that contains for each pair of coordinates of elements of $(\mathbb{Z}/p\mathbb{Z})^s$, an element $\sigma \in C(S)$ that permutes the given coordinates, permutes two others and acts as the identity on all other coordinates. Thus σ will be a product of two disjoint transpositions.

2.1. LEMMA. *Let* $G \subseteq C(S) \times T$ *be such that* $p_1(G)$ *contains* $\{H$ if $p = 2$, $C(S)$ if $p \geqslant 3\}$. *If W is an invariant subgroup of* $(\mathbb{Z}/p\mathbb{Z})^s$ *under the action of G, then* $W = 0$, $\mathbb{Z}/p\mathbb{Z}$, *or has a nonzero element which has at most four nonzero coordinates if* $p = 2, 3$ *nonzero coordinates if* $p \geqslant 3$.

Proof. Assume $p = 2$ and that $W \not\supseteq 0$ or $\mathbb{Z}/p\mathbb{Z}$. Then W contains an element of the form $(0, n_2, \ldots, n_s) = \boldsymbol{x}$ where at least one $n_j \neq 0$. We may assume without loss of generality that $n_2 \neq 0$. Let $\sigma \in H$ be a product of two disjoint 2-cycles, one of which permutes the first and second coordinates of elements of $(\mathbb{Z}/p\mathbb{Z})^s$. Then $\boldsymbol{x} - \sigma\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x} - \sigma\boldsymbol{x}$ has at most four nonzero coordinates.

Assume $p \geqslant 3$. Again if $W \not\cong 0$ or $\mathbb{Z}/p\mathbb{Z}$ then W contains an element of the form $\boldsymbol{x} = (0, n_2, \ldots, n_s)$ with $n_2 \neq 0$. Choose such a $\boldsymbol{x} \in W$ with the minimum number of nonzero coordinates. If this number is larger than 3, there is a $\sigma \in G$ that permutes the first two coordinates and holds all others fixed except for possible sign changes. Then either $\boldsymbol{x} + \sigma\boldsymbol{x}$ or $\boldsymbol{x} - \sigma\boldsymbol{x}$ has fewer non-zero coordinates than $\boldsymbol{x}$.

2.2. MAIN THEOREM. (Blass–Deligne–J. Lang). *Let k be an algebraically closed field of characteristic $p \neq 0$, let $n \geqslant 4$ if $p \geqslant 3$ and $n \geqslant 5$ if $p = 2$. Let $\{T_{ij}: 0 \leqslant i + j \leqslant n\}$ be a set of algebraically independent variables over k, $L = \overline{k(T_{ij})}$, $F = \sum_{0 \leqslant i+j \leqslant n} T_{ij}x^i y^j$ and $A = L[x^p, y^p, F]$. If $p > 2$, then* $\mathrm{Cl}(A) = 0$, *if* $p = 2$, *then* $\mathrm{Cl}(A) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The case $p \geqslant 5$ is proved in [6]. So assume $p = 2$ or 3. Let $F' = F - T_{10}x - T_{01}y - T_{00}$. Then by a change of coordinates we have that $z^p = F$ is isomorphic to $z^p = F'$ so that we may assume $F = T_{20}x^2 + T_{11}xy + T_{02}y^2 + \cdots$.

By (1.1), $\mathrm{Cl}(A) \cong \mathscr{L}_0$, the group of logarithmic derivatives of $D = F_y(\partial/\partial x) - F_x(\partial/\partial y)$ in $L[x, y]$. By (1.4) each element of $\mathscr{L}_0$ has degree at most $n - 2$. We proceed in a series of steps.

Step 1. Assume $n = 0 \pmod p$. Then $t \in \mathscr{L}_0$. Then the degree $(n-2)$ form of t is an integral multiple of $(\bar{F})y/x$, where $\bar{F}$ denotes the highest degree form of F. (Note that $x(\bar{F})_x + y(\bar{F})_y = 0$ by Euler's formula and that $(\bar{F})_y/x = -(\bar{F})_x/y \in L[x, y]$.)

$t = Dh/h$ for some $h \in L[x, y]$. Let $\bar{h}$ represent the highest degree form of h and $\bar{t}$ the degree $(n-2)$-form of t. Then $\overline{th} = (\bar{h})_x(\bar{F})_y - (\bar{h})_y(\bar{F})_x$. Thus $\overline{xth} = x((\bar{h})_x(\bar{F})_y - (\bar{h})_y(\bar{F}_x)) + y((\bar{h}_y(\bar{F})_y - (\bar{h})_y(\bar{F})_y) = \deg(h)\cdot\bar{h}(\bar{F})_y - \deg(F)\cdot\bar{F}\cdot(\bar{h})_y = \deg(h)\cdot\bar{h}\cdot\bar{F}_y$ by Euler's formula. Therefore $\bar{t} = \deg(h)\cdot(\bar{F})_y/x$.

Step 2. Assume $p = 2$ or 3 and $r \leqslant n - 2$. Let $V_r = \{t \in \mathscr{L}_0 : \deg(t) \leqslant r\}$. Then V_r is *not* isomorphic to $\mathbb{Z}/p\mathbb{Z}$ if $p = 3$ or if $p = 2$ and $r < n - 2$.

Suppose that $p = 3$ and $V_r \cong \mathbb{Z}/3\mathbb{Z}$. Let $t \neq 0 \in V_r$. Then by (1.6), $t(Q) \neq 0$ for some $Q \in S_F$. If $Q \neq Q' \in S_F$ then $t(Q') \neq 0$, for otherwise by (I.5.7) there exists a $\sigma \in \mathrm{Gal}(L: k(T_{ij}))$ that transposes Q and Q' and acts as the identity on $S_F - \{Q, Q'\}$. Then $\sigma(t) \in \mathscr{L}_0$ and by (1.7), t and $\sigma(t)$ are $\mathbb{Z}/p\mathbb{Z}$-independent. Thus $t(Q) \neq 0$ for all $Q \in S_F$.

Clearly $Q = (0, 0) \in S_F$. This implies that $t(0, 0) = s\sqrt{T_{11}^2 - T_{20}T_{02}}$ by (1.6) with $s = \pm 1$. We may assume $s = 1$.

For all $\sigma \in \mathrm{Gal}(L/k(T_{ij}))$, $\sigma(t) \in \mathscr{L}$. This clearly implies that all coefficients of t belong to $\sqrt{T_{11}^2 - T_{20}T_{02}}\cdot k(T_{ij})$.

By (1.2), $D^2 t - at = -t^3$ with $a = F_{xy}^2 - F_{xx}F_{yy}$ (a can be calculated as D^3x/Dx).

After comparing coefficients of t on both sides of this differential equation we see that in fact all coefficients of t must belong to $\sqrt{T_{11}^2 - T_{20}T_{02}}\cdot k[T_{ij}]$.

If we now set $T_{ij} = 0$ for $i + j \geqslant 7$, in the equality $D^2t - at = -t^3$, then the image of t will be a nonzero element of $\mathscr{L}_0$ for the case $n = 6$ by (1.2). By (1.1) this would imply that for $n = 6$, $\mathrm{Cl}(A) \neq 0$ which contradicts the explicit computation for this example I obtained in ([2], pg. 184). A similar argument works for the case $p = 2$, again using the computation of $\mathrm{Cl}(A)$ for $n = 5$ and 6 in ([2], page 181).

Step 3. Assume that $p = 2$. The cases $n = 5$ and 6 are proved in ([2], Chapter 3). Therefore we assume that $n \geqslant 7$. Then $D(F_y)/F_y = F_{xy} \in \mathscr{L}_0$. Therefore $\mathscr{L}_0 \neq 0$. If $n \neq 0 \pmod 2$ then by (1.6) each nonzero $t \in \mathscr{L}_0$ is such that $t(Q) \neq 0$ for at least 5 points $Q \in S_F$. By (I.5.7), (1.1) and (2.1), $\mathrm{Cl}(A) \cong \mathscr{L}_0 \cong \mathbb{Z}/2\mathbb{Z}$. If $n = 0 \pmod 2$ and $t \in \mathscr{L}_0$ then the degree $(n-2)$ form of t is equal to $s(\bar{F})_{xy}$ where $s = 0$ or 1 and $\bar{F}$ is the degree n form of F by step 1. Then $t - sF_{xy} \in \mathscr{L}_0$ and has degree at most $n - 3$. By step 2, $V_{n-3} \ncong \mathbb{Z}/2\mathbb{Z}$. Therefore by (I.5.7), (1.6), and (2.1) $V_{n-3} = 0$. Thus $t = sF_{xy}$ and by (1.1), $\mathrm{Cl}(A) \cong \mathbb{Z}/2\mathbb{Z}$.

Step 4. Assume $p = 3$. The case $n \leqslant 6$ is proved in ([2], Chapter3). Assume then that $n \geqslant 7$. If $n \neq 0 \pmod 3$, use (I.5.7), (1.6), (2.1) and step 2 to conclude that $\mathrm{Cl}(A) \cong \mathscr{L}_0 = 0$. If $n = 0 \pmod 3$, we have by the same argument that $V_{n-3} = 0$. Then by step 1, this implies $\mathscr{L}_0 \cong 0$ or $\mathbb{Z}/3\mathbb{Z}$. By step 2, $\mathscr{L}_0 = 0$.

2.3. COROLLARY. *Let $p \geqslant 3$, $n \geqslant 5$. Then for each $m \geqslant 0$, $\mathrm{Cl}(X_m) = 0$ where X_m is defined by $z^{p^m} = F$ over L.*

Proof. By (1.1) the kernel of the homomorphism $\phi_m\colon \mathrm{Cl}(A_{m+1}) \to \mathrm{Cl}(A_m)$ is isomorphic to $\mathscr{L}_m$ for each $m \geqslant 0$. By (1.5) and (2.2) ϕ_m is an injection for each m. Since $\mathrm{Cl}(A_0) = 0$ and the coordinate ring of X_m is isomorphic (but not in general k-isomorphic ([6], II.3.4)) to A_m, the result follows.

2.4. COROLLARY. *Let $p = 2$, $n \geqslant 5$. For all $m > 0$, $\mathrm{Cl}(X_m) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. By (2.2) we have that if $m = 2$, $\mathrm{Cl}(A_1) \cong \mathscr{L}_0 \cong \mathbb{Z}/2\mathbb{Z}$ and that $\mathscr{L}_0$ is generated by $F_{xy} = D(F_x)/F_x$. It follows from Samuel ([16], pg. 62) that $\mathrm{Cl}(A_1)$ is generated by $F_xL[x, y] \cap A_1$. Write $F = a^2x + b^2y + c^2xy$ where $a, b, c \in L[x, y]$. Then $\mathrm{Cl}(A_1)$ is generated by the height one prime $P_1 = (a^4 + c^4y^2, (a^2 + c^2y)(b^2 + c^2x))$ in $A_1 = L[x^2, y^2, F]$. Proceed now by induction to show that $\mathrm{Cl}(A_m) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $P_m = ((a^2 + c^2y)^{2^m}, (a^2 + c^2y)^{2^{m-1}}(b^2 + c^2x)^{2^{m-1}})$. It is not difficult to verify that $P_m = F_xL[x, y] \cap A_m$. The inclusions $A_m^2 \subseteq A_{m+1} \subseteq A_m$ induce homomorphisms $\mathrm{Cl}(A_m^2) \to \mathrm{Cl}(A_{m+1}) \to \mathrm{Cl}(A_m)$ by Samuel ([16], pg. 10, Theorem 6.2). By induction we obtain homomorphisms $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\alpha} \mathrm{Cl}(A_{m+1}) \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z}$. In [10], Lang showed that α is injective and since each height one prime in A_m ramifies over A_m^2, the composition $\beta\alpha$ is just multiplication by 2 ([9], pg. 403). We conclude that $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\alpha} \mathrm{Cl}(A_{m+1}) \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z}$ is exact. It is not difficult to see that $\mathrm{Cl}(A_{m+1})$ is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. Since α is an injection, $(P_{m+1}) \neq 0$ in $\mathrm{Cl}(A_{m+1})$. Since the ramification index of P_m over P_{m+1} is 2, it must be that α is multiplication by 1 and β is multiplication by 2. There β is the 0-map and α is an isomorphism.

In [11], Lang showed that if the divisor class group of $z^p = F$ is $(\mathbb{Z}/p\mathbb{Z})^s$ for some s as in (2.2) then the class group of $z^p = g$ is $(\mathbb{Z}/p\mathbb{Z})^s$ for all g in a dense open subset of A. Then by (1.1), (1.5) and (2.2) we obtain

2.5. COROLLARY. *There exists a dense open subset* W of A *such that for all* $g \in W$, $\mathrm{Cl}(z^{p^m} = g) = \{0 \quad \text{if} \quad p > 2, \ \mathbb{Z}/2\mathbb{Z} \quad \textit{if} \quad p = 2\}$.

The proof of the next two results are the same as those given in ([6], II(4.4) and II(4.5)) for the case $p \geqslant 5$.

2.6. COROLLARY. *The hypersurface* $z^{p^m} = F(x_1, \ldots, x_r)$ *has* 0 *divisor class group for a generic* F *of degree* $n \geqslant 4$ *if* $p > 2$. *If* $p = 2$, $n \geqslant 5$ *and* $r \geqslant 3$ *then* $\mathrm{Cl}(z^{2^m} = F(x_1, \ldots, x_r)) \cong 0$ *for a generic* F.

2.7. COROLLARY. (2.6) *holds for a general choice of F as well (see introduction).*

For each $m \geqslant 0$, *let* A_m *be the set of* $g \in A$ *of degree* n *for which the order of* $\mathrm{Cl}(z^p = g)$ *is* p^m.

2.8. CONJECTURE (M. Artin). If the surface $z^p = G$ has geometric genus p_g for a generic polynomial G of degree n and if the order of $\mathrm{Cl}(z^p = G)$ is p^s, then the codimension of $A_{m+s} \leqslant mp_g$ for all $m \geqslant 0$.

The author thanks P. Blass and W. Heinzer for useful suggestions and comments.

References

1. P. Blass, Picard groups of Zariski surfaces. *I. Comp. Math.* 54 (1985) 3–86.
2. P. Blass and J. Lang, Zariski Surfaces and Differential Equations in Characteristic $p > 0$. *Monographs and Textbooks in Pure and Applied Mathematics*, No. 106, Marcel Decker, New York (1987).
3. P. Blass and J. Lang, A method for computing the kernel of a map of divisor classes of local rings in characteristic $p \neq 0$, *Mich. Math. J.* 35 (1988).
4. P. Blass and J. Lang, Surfaces de Zariski factorielles, *C. R. Acad. Sc.* Paris, T. 306, Série I, (1988).
5. P. Deligne and N. Katz, Groupes de Monodromie en Geometrie Algébrique, *Lecture Notes in Mathematics 340*, Springer-Verlag, 1973.
6. A. Grant and J. Lang, Applications of the fundamental group and purely inseparable descent to the study of curves on Zariski surfaces, to appear in the Journal of Algebra.
7. A. Grothendieck, SGAI, *Lecture Notes in Mathematics*, No. 224, Springer-Verlag, New York, 1971.
8. R. Hartshorne, Equivalence relations on algebraic cycles and subvarieties of small codimension, *Proc. Symp. Pure Math.*, 29 (Arcata), American Mathematical Society (1975).
9. R. Hartshorne, *Algebraic Geometry*. Springer-Verlag, New York (1977).
10. J. Lang, The divisor class group of the surface $z^{p^m} = G(x, y)$ over fields of characteristic $p > 0$. *J. Alg.* 84, 2 (1983).
11. J. Lang, The factoriality of Zarisksi rings. *Compositio Mathematica* (1987).

12. W.E. Lang, Remarks on p-torsion of algebraic surfaces, *Compositio Math.*, 52 (2), (1984) 197–202.
13. M. Nagata, Local Rings. John Wiley & Sons, Inc. (1962).
14. M. Nagata, Field Theory. Marcel Dekker, Inc. (1977).
15. M. Raynaud, Anneaux Locaux Henséliens, *Lecture Notes in Mathematics*, Springer-Verlag, 1970.
16. P. Samuel, Lectures on Unique Factorization Domains. *Tata Lecture Notes* (1964).
17. O. Zariski and P. Samuel, *Commutative Algebra*, D. Van Nostrand Co., Inc. (1960).