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Completion regularity and τ-additivity of measures on product spaces

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0. Introduction

The first result on completion regularity is the following classical theorem of Kakutani [8]: If $(X_i)_{i\in I}$ is a family of compact metric spaces and for every $i \in$ I, μ_i is a Radon probability measure on X_i with full support, then the Radon product measure $\hat{\bigotimes}_{i\in I}\mu_i$ on $X = \Pi_{i\in I}X_i$ is completion regular. Since then, most of the work on completion regularity is concerned with Radon measures on compact spaces. In this paper we study completion regularity in products of completely regular (Hausdorff) spaces. Moreover, as it turns out in our investigation, the more general class of τ-additive measures (rather than the class of Radon measures) is relevant.

If one seeks an extension of Kakutani's theorem when every X_i is completely regular and every μ_i is τ-additive and completion regular (note that Borel measures on metric spaces are trivially completion regular), then the following difficulties arise: (a) the existence of a (unique) τ-additive extension $\hat{\bigotimes}_{i\in I}\mu_i$ of the simple product measure $\bigotimes_{i\in I}\mu_i$ is known only when I is countable (Ressel [13]); and (b) the product of two completion regular even Radon measures is not in general completion regular (Fremlin [4]).

Concerning (a) we prove in Section 2 that Ressel's theorem is valid for arbitrary I (Theorem 2.1). Then we show that the τ-additive product measure $\hat{\bigotimes}_{i\in I}\mu_i$ is completion regular if every μ_i has full support and $\hat{\bigotimes}_{i\in F}\mu_i$ is completion regular for every finite $F \subset I$ (Theorem 2.9). In view of this result, we prove in Section 3 that if μ is a completion regular measure on $X = \Pi_{i\in I}X_i$, where all X_i are separable metric spaces, and ν is a completion regular τ-additive measure on a completely regular space, then μ is τ-additive and $\mu \hat{\otimes} \nu$ is completion regular (Theorem 3.1). Thus, in spite of Fremlin's counter-example in (b), we have a positive result in an important special case.

1. Preliminaries and notations

All measure spaces $(X, \mathscr{A}, \mu)$ (simply denoted by (X, μ)) are assumed to be finite (i.e. $\mu(X) < \infty$). In fact, we usually assume for simplicity that μ is a probability measure (i.e. $\mu(X) = 1$). The inner measure and the outer measure (defined on all subsets of X) are denoted by μ_* and μ^*, respectively. If $\mathscr{V} = (V_\alpha)_{\alpha\in\Lambda}$ is a family of measurable sets (i.e. sets in $\mathscr{A}$), it will be convenient to use the following notation

$$\mu[\mathscr{V}] = \mu[(V_\alpha)_{\alpha\in\Lambda}] = \sup\left\{\mu\left(\bigcup_{\alpha\in\Gamma} V_\alpha\right) : \Gamma \text{ countable}, \Gamma \subset \Lambda\right\}.$$

It is clear that $\mu[\mathscr{V}] \leqslant \mu_*(\cup_{\alpha\in\Lambda} V_\alpha)$ and that $\mu[\mathscr{V}] = \mu(\cup_{\alpha\in\Gamma} V_\alpha)$ for some countable $\Gamma \subset \Lambda$.

We are primarily concerned with measures on topological spaces. Throughout, let X be a completely regular (Hausdorff) space. A *zero* set Z in X is a set of the form $Z = f^{-1}(\{0\})$, where $f: X \to \mathbb{R}$ is continuous. A *cozero* set in X is a complement of a zero set in X. It follows from the complete regularity of X that the family of cozero sets is a base for the topology of X. The family of *Baire* (resp. *Borel*) sets in X is the σ-algebra $\mathscr{B}(X)$ (resp. $\mathscr{B}_0(X)$) generated by the zero (resp. closed) sets in X. A Baire (resp. Borel) measure on X is a finite measure defined on the σ-algebra of Baire (resp. Borel) sets in X.

A Baire measure μ on X is called *completion regular* if every open (or, equivalently, every Borel) set G in X is μ-measurable, i.e. there exist Baire sets B_0 and B_1 in X such that $B_0 \subset G \subset B_1$ and $\mu(B_1 \backslash B_0) = 0$. A Borel measure is called completion regular if its restriction to the Baire sets is completion regular.

As open sets are precisely unions of cozero sets, completion regularity is a property of measurability of uncountable unions and it is not surprising that this property is related to the following notion of τ-additivity.

A Baire (resp. Borel) measure μ on X is called *τ-additive* if for every family $\mathscr{V} = (V_\alpha)_{\alpha\in\Lambda}$ of cozero (resp. open) sets in X,

$$\mu[\mathscr{V}] = \mu_*\left(\bigcup_{\alpha\in\Lambda} V_\alpha\right)\left(\text{resp. } \mu[\mathscr{V}] = \mu\left(\bigcup_{\alpha\in\Lambda} V_\alpha\right)\right), \tag{$*$}$$

i.e. there is a countable $\Gamma \subset \Lambda$ such that $\mu(\cup_{\alpha\in\Gamma} V_\alpha) = \mu_*(\cup_{\alpha\in\Lambda} V_\alpha)$ (resp. $\mu(\cup_{\alpha\in\Gamma} V_\alpha) = \mu(\cup_{\alpha\in\Lambda} V_\alpha)$). For information on τ-additive Baire and Borel measures we refer to [15] and [5].

The *support* S of a τ-additive Baire (resp. Borel) measure μ is the intersection of all zero (resp. closed) sets Z with $\mu(Z) = \mu(X)$. It follows that S is the least closed subset of X with $\mu^*(S) = \mu(X)$ (resp. $\mu(S) = \mu(X)$). If $S = X$, we say that μ has full support.

If $\mathscr{B}$ is a base for the topology of X consisting of open Baire (resp. open) sets, then it is easy to see that a Baire (resp. Borel) measure μ on X is τ-additive if and only if (*) holds for every subfamily $\mathscr{V} = (V_\alpha)_{\alpha\in\Lambda}$ of $\mathscr{B}$. A similar remark holds for the definition of the support of μ.

We shall use the fact that every τ-additive Baire measure μ has a unique extension to a τ-additive Borel measure, which we denote by $\tilde{\mu}$ (see [10] or [9]). Also, the restriction of a τ-additive Borel measure to the Baire sets is easily seen to be a τ-additive Baire measure.

Finally, we mention the following notion in case that X is a product space, $X = \Pi_{i\in I} X_i$ where every X_i is completely regular. A subset A of X is said to be *determined* by a nonempty $J \subset I$ if there exists a subset C of $\Pi_{i\in J} X_i$ such that $A = pr_J^{-1}(C)$, where pr_J denotes the canonical projection from X onto $\Pi_{i\in J} X_i$. In addition, X is determined by $\emptyset$. We shall use the result that if X satisfies the countable chain condition, i.e. every pairwise disjoint family of open sets in X is countable (which happens if all X_i are separable), then every Baire set in X is determined by countably many coordinates; in fact, the Baire sets in X have the form $pr_J^{-1}(C)$ where C is a Baire set in $\Pi_{i\in J} X_i$ and J is countable (see [14]).

2. Product measures

Throughout this section $(X_i)_{i\in I}$ is a family of completely regular spaces and $X = \Pi_{i\in I} X_i$. It is clear that $\mathscr{B}(X)$ (resp. $\mathscr{B}_0(X)$) contains the product σ-algebra $\otimes_{i\in I}\mathscr{B}(X_i)$ (resp. $\otimes_{i\in I}\mathscr{B}_0(X_i)$), but in general the inclusion may be strict. The basic result of this section is the following.

THEOREM 2.1. (a) *If μ_i is a τ-additive probability Baire measure on X_i for every $i \in I$, then there exists a unique τ-additive Baire measure μ on X extending the product measure $\otimes_{i\in I}\mu_i$.*

(b) *If ν_i is a τ-additive probability Borel measure on X_i for every $i \in I$, then there exists a unique τ-additive Borel measure ν on X extending the product measure $\otimes_{i\in I}\nu_i$.*

We call μ (resp. ν) the *τ-additive Baire* (resp. *Borel*) *product measure* of $(\mu_i)_{i\in I}$ (resp.$(\nu_i)_{i\in I}$) and write $\mu = \hat{\otimes}_{i\in I}\mu_i$ and $\nu = \hat{\otimes}_{i\in I}\nu_i$.

For the proof of this theorem we need some lemmas and the following definition. An *elementary open* (resp. *elementary open Baire*) set in X is a set of the form $pr_F^{-1}(\Pi_{i\in F} V_i)$, where F is a finite subset of I and V_i is open (resp. open Baire) in X_i for every $i \in F$.

In our first lemma we show the uniqueness of τ-additive product measures.

LEMMA 2.2. *If μ_1 and μ_2 are τ-additive Baire (or Borel) measures on X and coincide on $\otimes_{i\in I}\mathscr{B}(X_i)$, then $\mu_1 = \mu_2$.*

Proof. Assume that μ_1 and μ_2 are Baire measures and let U be a cozero set in X. Since U is a union of elementary open Baire sets, by the τ-additivity of μ_j, $j = 1, 2$, there exist sets $B_j, j = 1, 2$, such that B_j is a countable union of elementary open Baire sets, $B_j \subset U$ and $\mu_j(U) = \mu_j(B_j)$ for $j = 1, 2$. But $B_j \in \otimes_{i \in I} \mathscr{B}(X_i)$ and so $\mu_1(U) = \mu_1(B_1 \cup B_2) = \mu_2(B_1 \cup B_2) = \mu_2(U)$. Therefore $\mu_1 = \mu_2$.

If μ_1 and μ_2 are Borel measures, then by the above $\mu_1 | \mathscr{B}(X) = \mu_2 | \mathscr{B}(X)$ and so $\mu_1 = \mu_2$.

LEMMA 2.3. *Let μ_i be a τ-additive probability Baire measure on X_i for every $i \in I$ and ν a τ-additive probability Borel measure on X. If ν extends $\otimes_{i \in I} \mu_i$, then it also extends $\otimes_{i \in I} \tilde{\mu}_i$, where $\tilde{\mu}_i$ is the unique τ-additive Borel extension of μ_i (see* Section 1).

Proof. It suffices to prove that the measures $\nu_0 = \otimes_{i \in I} \tilde{\mu}_i$ and ν coincide on the family of elementary open sets (because $\otimes_{i \in I} \mathscr{B}_0(X_i)$ is the σ-algebra generated by this family which is closed under finite intersections; see [3], Corollary 1.6.2). Let G be an elementary open set, $G = pr_F^{-1}(\Pi_{i \in F} V_i)$, where $F \subset I$ is finite and V_i is open in X_i for every $i \in F$. Since every V_i is a union of cozero sets, by the τ-additivity of $\tilde{\mu}_i$ (and the fact that a countable union of cozero sets is cozero), there exist cozero sets U_i in $X_i, i \in F$, such that $U_i \subset V_i$ and $\tilde{\mu}_i(V_i) = \mu_i(U_i)$. We set $B_0 = pr_F^{-1}(\Pi_{i \in F} U_i)$. Then $B_0 \in \otimes_{i \in I} \mathscr{B}(X_i)$ and $\nu_0(G) = \Pi_{i \in F} \mu_i(U_i) = (\otimes_{i \in I} \mu_i)$ (B_0). On the other hand, using the τ-additivity of ν as in Lemma 2.2, we find $B \in \otimes_{i \in I} \mathscr{B}(X_i)$ such that $B \subset G$ and $\nu(G) = \nu(B) = (\otimes_{i \in I} \mu_i)(B)$. We then have $\nu_0(G) = (\otimes_{i \in I} \mu_i)$ $(B_0 \cup B) = \nu(G)$, which completes the proof.

LEMMA 2.4. *For every family $(X_i)_{i \in I}$ of completely regular spaces,* (a) *and* (b) *of Theorem* 2.1 *are equivalent.*

Proof. (a) $\Rightarrow$ (b). Let $(\nu_i)_{i \in I}$ be as in (b). Setting $\mu_i = \nu_i | \mathscr{B}(X_i)$ for every $i \in I$, by (a) the τ-additive Baire product measure $\mu = \hat{\otimes}_{i \in I} \mu_i$ exists. Then the Borel measure $\nu = \tilde{\mu}$ is τ-additive and extends $\otimes_{i \in I} \nu_i$ (by Lemma 2.3). The uniqueness of ν follows from Lemma 2.2.

(b) $\Rightarrow$ (a). Let $(\mu_i)_{i \in I}$ be as in (a). By (b) the τ-additive product measure $\hat{\otimes}_{i \in I} \tilde{\mu}_i$ exists. It is clear that the measure $\mu = (\hat{\otimes}_{i \in I} \tilde{\mu}_i) | \mathscr{B}(X)$ has the desired properties.

As mentioned in the introduction the following lemma is due to Ressel [13].

LEMMA 2.5. ([13], Theorem 2). *Part* (b) *of Theorem* 2.1 *holds if I is countable.*

The next lemma is based on an idea of the proof in [12] of Kakutani's theorem mentioned in the introduction.

LEMMA 2.6. *Let μ_i be a probability Baire measure on X_i with full support for every $i \in I$ and $\mu = \otimes_{i \in I} \mu_i$. Then for every family $\mathscr{V}$ of elementary open Baire sets, there exists a countable $J \subset I$ such that*

$$\mu[\mathscr{V}] = \mu[\{U^J : U \in \mathscr{V}\}],$$

where for every $A \subset X$ and $J \subset I$ we set $A^J = pr_J^{-1}(pr_J(A))$.

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{V}$ such that $\mu[\mathscr{V}] = \mu(\bigcup_{n=1}^{\infty} V_n)$. Choose a countable $J \subset I$ such that the set $V = \bigcup_{n=1}^{\infty} V_n$ is determined by J. It suffices to show that

$$\mu(V) \geqslant \mu[\{U^J : U \in \mathscr{V}\}] \tag{$*$}$$

because the other inequality is obvious.

Let U be a nonempty elementary open Baire set in $\mathscr{V}$. Then there exists an elementary open Baire set W determined by $I\backslash J$ such that $U = U^J \cap W$. We have

$$0 = \mu(U\backslash V) = \mu((U^J\backslash V) \cap W) = \mu(U^J\backslash V)\mu(W),$$

where $\mu(W) > 0$ because every μ_i has full support. Therefore $\mu(U^J\backslash V) = 0$ and $(*)$ follows.

REMARK. Lemma 2.6 remains valid if $\mu_i, i \in I$, are Borel measures and we replace "elementary open Baire" by "elementary open".

Proof of Theorem 2.1. By Lemma 2.4 it suffices to prove only part (b) of the theorem. This is done in step III below. However, for this purpose, we shall use special cases of (a) and (b), which we prove in steps I and II. Notice that in all cases we need to prove only the existence of the τ-additive product meaure because of Lemma 2.2.

Step I. Part (a) holds if μ_i has full support for every $i \in I$.

First we observe that X satisfies the countable chain condition because the product measure $\otimes_{i\in I}\mu_i$ is strictly positive on every elementary open Baire set in X. Thus, if B is a Baire set in X, there exists a countable $J \subset I$ and a Baire set C in $\Pi_{i\in J} X_i$ such that $B = pr_J^{-1}(C)$ (see Section 1). Using Lemmas 2.5 and 2.4 we set

$$\mu(B) = \left(\hat{\bigotimes_{i\in J}} \mu_i\right)(C).$$

It is easy to verify that μ is a well defined probability Baire measure on X extending the product measure $\otimes_{i\in I}\mu_i$. It remains to prove the τ-additivity of μ.

Let $\mathscr{V} = (V_\alpha)_{\alpha\in\Lambda}$ be a family of elementary open Baire sets in X such that $\bigcup_{\alpha\in\Lambda} V_\alpha = X$. It suffices to show that $\mu[\mathscr{V}] = 1$ (cf. [15], Part I, Theorems 24 and 25). By Lemma 2.6 there exists a countable $J \subset I$ such that

$$\mu[\mathscr{V}] = \mu[(\mathrm{V}_\alpha^J)_{\alpha\in\Lambda}].$$

Since the measure $\mu_J = \hat{\bigotimes}_{i\in J}\mu_i$ is τ-additive and the family $(pr_J(V_\alpha))_{\alpha\in\Lambda}$ is a covering of $\Pi_{i\in j}X_i$ by elementary open Baire sets, there exists a countable $\Gamma \subset \Lambda$ such that $\mu_J(\bigcup_{\alpha\in\Gamma} pr_J(V_\alpha)) = 1$. But

$$\mu_J\left(\bigcup_{\alpha\in\Gamma} pr_J(V_\alpha)\right) = \mu_J\left(pr_J\left(\bigcup_{\alpha\in\Gamma} V_\alpha\right)\right) = \mu\left(\left(\bigcup_{\alpha\in\Gamma} V_\alpha\right)^J\right) = \mu\left(\bigcup_{\alpha\in\Gamma} V_\alpha^J\right) \leqslant \mu[\mathscr{V}].$$

Therefore $\mu[\mathscr{V}] = 1$ and μ is τ-additive.

Step II. Part (b) holds if v_i has full support for every $i \in I$.

Using the fact that the restriction of a τ-additive Borel measure with full support to the Baire sets has also full support, this step follows from step I as in Lemma 2.4, (a) $\Rightarrow$ (b).

Step III. In this step we prove part (b), completing the proof of the theorem.

Let S_i be the support of v_i and $\lambda_i = v_i|\mathscr{B}_0(S_i)$ for every $i \in I$. Then λ_i is a τ-additive Borel probability measure on S_i with full support and, by step II, the τ-additive Borel product measure $\lambda = \hat{\bigotimes}_{i\in I}\lambda_i$ on $\Pi_{i\in I}S_i$ exists. We define $v(B) = \lambda(B \cap \Pi_{i\in I}S_i)$ for every $B \in \mathscr{B}_0(X)$. It is clear that v is a τ-additive Borel measure on X. Moreover, if $B \in \mathscr{B}_0(X)$ is of the form $pr_F^{-1}(\Pi_{i\in F}C_i)$, where $F \subset I$ is finite and $C_i \in \mathscr{B}_0(X_i)$ for every $i \in F$, then

$$v(B) = \lambda\left(pr_F^{-1}\left(\prod_{i\in F}(C_i \cap S_i)\right)\right) = \prod_{i\in F}\lambda_i(C_i \cap S_i) = \prod_{i\in F} v_i(C_i) = \left(\bigotimes_{i\in I} v_i\right)(B).$$

Therefore v extends $\bigotimes_{i\in I} v_i$.

COROLLARY 2.7. *Let μ_i be a τ-additive Baire probability measure on X_i for every $i \in I$ and $\mu = \hat{\bigotimes}_{i\in I}\mu_i$. Then $\tilde{\mu} = \hat{\bigotimes}_{i\in I}\tilde{\mu}_i$.*

Proof. Immediate from Theorem 2.1 and Lemma 2.3.

REMARK. Let v_i be a Radon probability measure on X_i for every $i \in I$. (We recall that a Radon measure is a Borel measure inner regular with respect to compact sets.) Is there a unique Radon measure on X extending the product measure $\bigotimes_{i\in I} v_i$? It is known that this happens if either every X_i is compact or I is countable (see e.g. [12] and [13]). Clearly the answer to the above question is "yes" if and only if the τ-additive product measure $v = \hat{\bigotimes}_{i\in I} v_i$ is Radon. We also observe that if v is Radon and we choose a compact set $K \subset X$ with $v(K) > 0$, then $v_i(pr_i(K)) = 1$ for all but a countable number of $i \in I$. It now follows easily from the above that there exists a unique Radon measure on X extending $\bigotimes_{i\in I} v_i$ if and only if v_i has compact support for all but a countable number of $i \in I$.

In the next theorem we show that completion regularity of a τ-additive product measure can be reduced to its finite subproducts. First we prove a lemma.

LEMMA 2.8. *Let v_i be a τ-additive Borel probability measure on X_i with full* *support for every $i \in I$ and $\nu = \hat{\bigotimes}_{i\in I}\nu_i$. Then for every open set G in X there exists a countable $J \subset I$ such that $\nu(G) = \nu(G^J)$, where $G^J = pr_J^{-1}(pr_J(G))$.*

Proof. Let $\mathscr{V}$ be the family of all elementary open sets included in G. By Lemma 2.6 (see also the remark following it), there exists a countable $J \subset I$ such that $\nu[\mathscr{V}] = \nu[\{V^J: V \in \mathscr{V}\}]$. By the τ-additivity of ν, $\nu(G) = \nu[\mathscr{V}]$ and $\nu(G^J) = \nu[\{V^J: V \in \mathscr{V}\}]$. Therefore $\nu(G) = \nu(G^J)$.

THEOREM 2.9. *Let μ_i be a τ-additive probability Baire (or Borel) measure on X_i for every $i \in I$. We assume that either I is countable or every μ_i has full support. Then, if $\hat{\bigotimes}_{i\in F}\mu_i$ is completion regular for every finite $F \subset I$, the measure $\mu = \hat{\bigotimes}_{i\in I}\mu_i$ is completion regular.*

Proof. It suffices to prove the theorem for Baire measures (see Corollary 2.7).

Assume that I is countable, $I = \mathbb{N}$, and let G be an open subset of X. Then $G = \bigcup_{n=1}^{\infty} G_n$, where each G_n has the form $G_n = V_n \times \Pi_{i=n+1}^{\infty} X_i$ for some open set V_n in $\Pi_{i=1}^{n} X_i$. Since $\hat{\bigotimes}_{i\leqslant n}\mu_i$ is completion regular, there are Baire sets B_n and C_n in $\Pi_{i=1}^{n} X_i$ such that $B_n \subset V_n \subset C_n$ and $(\hat{\bigotimes}_{i\leqslant n}\mu_i)(C_n \backslash B_n) = 0$. Then the sets $B = \bigcup_{n=1}^{\infty}(B_n \times \Pi_{i=n+1}^{\infty} X_i)$ and $C = \bigcup_{n=1}^{\infty}(C_n \times \Pi_{i=n+1}^{\infty} X_i)$ are Baire sets in X, $B \subset G \subset C$ and $\mu(C\backslash B) = 0$. Therefore μ is completion regular.

Now assume that every μ_i has full support and let G be an open subset of X. Because $\tilde{\mu} = \hat{\bigotimes}_{i\in I}\tilde{\mu}_i$ (Corollary 2.7), it follows from Lemma 2.8 that there exists a countable $J \subset I$ such that $\tilde{\mu}(G) = \tilde{\mu}(G^J)$. By the above, the measure $\hat{\bigotimes}_{i\in J}\mu_i$ is completion regular. So there are Baire sets C and C_1 in $\Pi_{i\in J} X_i$ such that $C \subset pr_J(G) \subset C_1$ and $(\hat{\bigotimes}_{i\in J}\mu_i)(C_1 \backslash C) = 0$. Then the sets $B = pr_J^{-1}(C)$ and $B_1 = pr_J^{-1}(C_1)$ are Baire sets in X, $B \subset G^J \subset B_1$ and $\mu(B_1 \backslash B) = 0$. It follows that $\tilde{\mu}(G^J) = \mu(B_1)$. On the other hand, by the τ-additivity of $\tilde{\mu}$ there is a Baire set B_0 in X such that $B_0 \subset G$ and $\tilde{\mu}(G) = \mu(B_0)$. Since $\tilde{\mu}(G) = \tilde{\mu}(G^J)$, it follows that $\mu(B_1 \backslash B_0) = 0$, completing the proof.

Let M be a subset of X. We say that M is invariant under countable changes if for every $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ in X such that $\{i \in I: x_i \neq y_i\}$ is countable we have $x \in M$ if and only if $y \in M$. The last result of this section (Corollary 2.10) provides, in a more general setting, an affirmative answer to a question of Mauldin and Mycielski in [12] concerning the measurability of sets invariant under countable changes. We notice that no assumption of full support is made in their formulation of this question, but such an assumption cannot be avoided (see Remark 1 below).

COROLLARY 2.10. *Let ν_i be a τ-additive probability Borel measure on X_i with full support for every $i \in I$ and $\nu = \hat{\bigotimes}_{i\in I}\nu_i$.* If $\emptyset \neq M \subsetneqq X$ *and M is invariant under countable changes (so I is uncountable), then $\nu_*(M) = 0$ and $\nu^*(M) = 1$.*

Proof. First notice that ν, as a τ-additive Borel measure, is inner regular with respect to closed sets (see [5], Theorem 5.4). Thus, in order to prove that $\nu^*(M) = 1$ it suffices to show that for every open set G in X with $G \supset M$, $\nu(G) = 1$. Indeed, by Lemma 2.8 there exists a countable $J \subset I$ such that $\nu(G^J) = \nu(G)$. If we assume that $X\backslash G^J$ is nonempty, then since $X\backslash G^J$ is determined by the countable set J and M is nonempty and invariant under countable changes it follows that $M \cap (X\backslash G^J) \neq \emptyset$, contradiction (because $G^J \supset M$). Therefore $G^J = X$ and so $\nu(G) = \nu(X) = 1$.

Similarly, replacing M by $X\backslash M$, we have $\nu^*(X\backslash M) = 1$ and so $\nu_*(M) = 0$.

REMARKS. 1. In Theorem 2.9 and Corollary 2.10 the assumption that the measures have full support cannot be dropped. Indeed, assume that I is uncountable, X_i contains at least two points and μ_i is the Dirac measure at some $x_i \in X_i$ (considered as a Baire or Borel measure) for every $i \in I$. Then $\mu = \hat{\bigotimes}_{i\in I}\mu_i$ is the Dirac measure at $x = (x_i)_{i\in I}$ and since $\{x\}$ is not a Baire set, μ is not completion regular. However, the finite products $\hat{\bigotimes}_{i\in F}\mu_i$ can be completion regular. Also, the conclusion of Corollary 2.10 fails because if every μ_i is considered as a Borel measure, then μ is a Borel measure and every subset of X is μ-measurable.

2. Under the assumptions of Theorem 2.9 (no assumption of completion regularity is made), every Baire set in X is $(\hat{\bigotimes}_{i\in I}\mu_i)$-measurable.

It suffices to prove the case of Baire measures. If I is finite this is proved in [1], Theorem 4.1. Assume that I is countable, $I = \mathbb{N}$, and let U be a cozero set in X. Then there is a continuous function $f: X \to \mathbb{R}$, $f \geqslant 0$, such that $U = \{x \in X: f(x) > 0\}$. Fix some $y = (y_n)_{n\in\mathbb{N}}$ in X and define

$$f_n: X \to \mathbb{R} \quad \text{with } f_n(x_1, x_2, \ldots) = f(x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots)$$

and $U_{n,k} = \{x \in X: f_n(x) \geqslant 1/k\}$ for every $n, k \in \mathbb{N}$. Then $\lim_n f_n(x) = f(x)$ and so $U = \bigcup_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}U_{n,k}$. Since $U_{n,k}$ is determined by the first n coordinates, it follows from the above that every $U_{n,k}$, hence also U, is $(\hat{\bigotimes}_{i\in I}\mu_i)$-measurable. Finally, if every μ_i has full support then X satisfies the countable chain condition and so every Baire set in X is determined by countably many coordinates. Now, as in the second part of the proof of Theorem 2.9, the result reduces to case where I is countable.

3. Measures on products of separable metric spaces

Fremlin [4] has proved that there exists a compact space X and a completion regular Radon measure μ on X such that (a) $\mu \hat{\otimes} \mu$ is not completion regular and (b) there exists an open subset of $X \times X$ which is not $\mu \otimes \mu$-measurable. In this section we show that none of the above holds for the product of two completion regular τ-additive measure spaces (X, μ) and (Y, ν), provided that X is homeomorphic to a product of separable metric spaces.

For the proof of the above result, which contains the main theorem of [6] as a special case, we use or refine several of the techniques of that paper. In the course of this investigation we obtain some characterizations of completion regularity in products of separable metric spaces and prove that this property implies τ-additivity. More precisely, the main result is the following.

THEOREM 3.1. *Let $(X_i)_{i\in I}$ be a family of separable metric spaces, $X = \Pi_{i\in I} X_i$ and μ a probability Baire (resp. Borel) measure on X.*

(a) *If μ is completion regular then μ is τ-additive.*

(b) *The following conditions are equivalent:*

(i) *μ is completion regular;*

(ii) *for every uncountable family $(U_\alpha)_{\alpha\in\Lambda}$ of nonempty cozero sets in X, if there exists a pairwise disjoint family $(I_\alpha)_{\alpha\in\Lambda}$ of countable subsets of I such that each U_α is determined by I_α, then there is a countable $\Gamma \subset \Lambda$ such that $\mu(\bigcup_{\alpha\in\Gamma} U_\alpha) = 1$ (i.e. $\mu[(U_\alpha)_{\alpha\in\Lambda}] = 1$);*

(iii) *for every open set G in X there exist cozero sets U and V such that $U \subset G \subset V$ and $\mu(V\backslash U) = 0$;*

(iv) *for every τ-additive probability Borel measure ν on a completely regular space Y, the Borel sets in $X \times Y$* are *$\mu \otimes \nu$-measurable; and*

(v) *for every completion regular τ-additive probability Baire (resp. Borel) measure ν on a completely regular space Y, the τ-additive product measure $\mu \hat{\otimes} \nu$ is completion regular.*

The proof of this theorem is essentially contained in the following five lemmas.

LEMMA 3.2. *If (X, μ) is as in Theorem* 3.1, *then* (i) $\Rightarrow$ (ii).

Proof. We assume that μ is completion regular and let $(U_\alpha)_{\alpha\in\Lambda}$ and $(I_\alpha)_{\alpha\in\Lambda}$ be as in (ii). It suffices to prove that for every Baire set B in X,

(1) $B \supset \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow B = X$; and

(2) $B \subset \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow B$ is covered by countably many U_α's.

Indeed, by (i) there are Baire sets B_0 and B_1 in X such that $B_0 \subset \bigcup_{\alpha\in\Lambda} U_\alpha \subset B_1$ and $\mu(B_1\backslash B_0) = 0$. Now, by (1) and (2), $B_1 = X$ and there is a countable $\Gamma \subset \Lambda$ such that $B_0 \subset \bigcup_{\alpha\in\Gamma} U_\alpha$. It is clear that $\mu(\bigcup_{\alpha\in\Gamma} U_\alpha) = 1$.

To prove (1) assume that $B \supset \bigcup_{\alpha\in\Lambda} U_\alpha$ and let J be a countable subset of I such that B is determined by J. Since $(I_\alpha)_{\alpha\in\Lambda}$ is pairwise disjoint there is some α such that $I_\alpha \cap J = \emptyset$. But $B \supset U_\alpha$ and so $B = X$.

To prove (2), assume that $B \subset \bigcup_{\alpha\in\Lambda} U_\alpha$. For every $\alpha \in \Lambda$, there is an open set V_α in $\Pi_{i\in I_\alpha} X_i$ such that $U_\alpha = pr_{I_\alpha}^{-1}(V_\alpha)$. If for some α $V_\alpha = \Pi_{i\in I_\alpha} X_i$ then $B \subset U_\alpha = X$. So we can assume that $V_\alpha \neq \Pi_{i\in I_\alpha} X_i$ for every $\alpha \in \Lambda$. Let J be a countable subset of I such that B is determined by J. It suffices to prove that $B \subset \cup\{U_\alpha : I_\alpha \cap J \neq \emptyset\}$.

Let $x = (x_i)_{i\in I} \in B$. We choose $y = (y_i)_{i\in I} \in X$ such that $y_i = x_i$ for every $i \in J \cup (\cup\{I_\alpha : I_\alpha \cap J \neq \emptyset\})$ and $(y_i)_{i\in I_\alpha} \in \Pi_{i\in I_\alpha} X_i \backslash V_\alpha$ for every α with $I_\alpha \cap J = \emptyset$. Then $y \in B\backslash \cup\{U_\alpha : I_\alpha \cap J = \emptyset\}$ and so there is some α such that $y \in U_\alpha$ and $I_\alpha \cap J \neq \emptyset$. Certainly $x \in U_\alpha$ for this α.

LEMMA 3.3. ([6], Lemma of §3). *Let (X, μ) and (Y, ν) be probability measure spaces and $Q \subset X \times Y$ a countable union of measurable rectangles (i.e. $Q = \bigcup_{m=1}^{\infty}(U_m \times V_m)$ where each U_m and V_m is measurable in X and Y, respectively. Then $(\mu \otimes \nu)(Q) = 1$ if (and only if) for every sequence $(A_n \times B)_{n\in\mathbb{N}}$ of measurable rectangles with $\mu(A_n)\, \nu(B) > 0$ for every n, there exists a sequence $(A'_n \times B')_{n\in\mathbb{N}}$ of measurable rectangles with $\mu(A'_n)\, \nu(B') > 0$, $A'_n \times B' \subset A_n \times B$ and $(\mu \otimes \nu)((A'_n \times B')\backslash Q) = 0$ for infinitely many n.*

LEMMA 3.4. *Let (X, μ) and (Y, ν) be probability measure spaces and $(U_\alpha)_{\alpha\in\Lambda}$ and $(V_\alpha)_{\alpha\in\Lambda}$ uncountable families of measurable sets in X and Y, respectively.*

(a) *If for every uncountable $\Lambda' \subset \Lambda$ $\mu[(U_\alpha)_{\alpha\in\Lambda'}] = 1$ and for every $\Lambda' \subset \Lambda$ with $\Lambda\backslash\Lambda'$ countable $\nu[(V_\alpha)_{\alpha\in\Lambda'}] = 1$, then $(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda}] = 1$.*

(b) *If for every uncountable $\Lambda' \subset \Lambda$ $\mu[(U_\alpha)_{\alpha\in\Lambda'}] = 1$, then there exists an uncountable $\Lambda_0 \subset \Lambda$ such that*

$$(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda_0}] = (\mu \otimes \nu)[(X \times V_\alpha)_{\alpha\in\Lambda_0}](= \nu[(V_\alpha)_{\alpha\in\Lambda_0}]).$$

Proof. (a) Let Γ be a countable subset of Λ with

$$(\mu \otimes \nu)\left(\bigcup_{\alpha\in\Gamma}(U_\alpha \times V_\alpha)\right) = (\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda}]$$

and set $Q = \bigcup_{\alpha\in\Gamma}(U_\alpha \times V_\alpha)$. Using Lemma 3.3 we prove that $(\mu \otimes \nu)(Q) = 1$.

Let $(A_n \times B)_{n\in\mathbb{N}}$ be a sequence of measurable rectangles such that $\mu(A_n)\nu(B) > 0$ for every n. We set

$$\Gamma_0 = \{\alpha \in \Lambda\colon \nu(V_\alpha \cap B) = 0\} \text{ and } \Gamma_n = \{\alpha \in \Lambda\colon \mu(U_\alpha \cap A_n) = 0\}, \quad n = 1, 2, \ldots$$

Since

$$\nu[(V_\alpha)_{\alpha\in\Gamma_0}] \leqslant \nu(Y\backslash B) = 1 - \nu(B) < 1,$$

by our assumption $\Lambda\backslash\Gamma_0$ is uncountable. Similarly Γ_n is countable for $n = 1, 2, \ldots$ We now choose $\alpha_0 \in \Lambda\backslash\bigcup_{n=0}^{\infty}\Gamma_n$ and set $B' = B \cap V_{\alpha_0}$ and $A'_n = A_n \cap U_{\alpha_0}$ for every $n \in \mathbb{N}$. Then $\mu(A'_n)\nu(B') > 0$, $A'_n \times B' \subset A_n \times B$ and $A'_n \times B' \subset U_{\alpha_0} \times V_{\alpha_0}$ for every n. Thus $(\mu \otimes \nu)((A'_n \times B')\backslash Q) = 0$ for every n and Lemma 3.3 implies that $(\mu \otimes \nu)(Q) = 1$.

(b) *Claim.* There exists an uncountable $\Lambda_0 \subset \Lambda$ such that

$$\nu[(V_\alpha)_{\alpha\in\Lambda_0}] = \nu[(V_\alpha)_{\alpha\in\Lambda'}] \text{ for every } \Lambda' \subset \Lambda_0 \text{ with } \Lambda_0\backslash\Lambda' \text{ countable.}$$

Choose $\alpha_\xi \in \Lambda$ for every $\xi < \omega_1$ with $\alpha_\xi \neq \alpha_{\xi'}$ for $\xi \neq \xi'$. Since ν is finite there exists $\xi_0 < \omega_1$ such that $\nu[(V_{\alpha_\xi})_{\xi \geqslant \xi_0}] = \nu[(V_{\alpha_\xi})_{\xi \geqslant \zeta}]$ for every $\zeta \geqslant \xi_0$. It is clear that the set $\Lambda_0 = \{\alpha_\xi : \xi \geqslant \xi_0\}$ satisfies the claim.

If $\nu[(V_\alpha)_{\alpha \in \Lambda_0}] = 0$, (b) is obvious. So we assume that $\nu[(V_\alpha)_{\alpha \in \Lambda_0}] > 0$ and choose a countable $\Gamma \subset \Lambda_0$ such that $\nu[(V_\alpha)_{\alpha \in \Lambda_0}] = \nu(\cup_{\alpha \in \Gamma} V_\alpha)$. Let ν' be the probability measure on Y given by

$$\nu'(C) = \frac{\nu((\bigcup_{\alpha \in \Gamma} V_\alpha) \cap C)}{\nu(\bigcup_{\alpha \in \Gamma} V_\alpha)}.$$

Then $\nu'[(V_\alpha)_{\alpha \in \Lambda'}] = 1$ for every $\Lambda' \subset \Lambda_0$ with $\Lambda_0 \backslash \Lambda'$ countable. Thus, applying (a) for the measures μ and ν' and the uncountable families $(U_\alpha)_{\alpha \in \Lambda_0}$ and $(V_\alpha)_{\alpha \in \Lambda_0}$, (b) follows.

Let $X = \Pi_{i \in I} X_i$ be as in Theorem 3.1. For every $i \in I$ we fix a countable base $\mathscr{B}_i$ for the topology of X_i such that $X_i \in \mathscr{B}_i$. In the next two lemmas, when we say that a subset of X is *basic elementary open* we mean that it is of the form $pr_F^{-1}(\Pi_{i \in I} V_i)$, where F is a finite subset of I and $V_i \in \mathscr{B}_i$ for every $i \in F$.

LEMMA 3.5. *Let (X, μ) be as in Theorem* 3.1 *with μ satisfying condition* (ii) *and (Y, ν) a probability measure space. If $(U_\alpha)_{\alpha < \rho}$ and $(V_\alpha)_{\alpha < \rho}$ are uncountable families and $k \in \mathbb{N}$ such that each U_α is a basic elementary open set in X determined by $\leqslant k$ coordinates, $U_\alpha \neq U_{\alpha'}$ for $\alpha \neq \alpha'$ and each V_α is measurable in Y, then there exist families $(\Lambda_\alpha)_{\alpha < \rho_1}$ and $(W_\alpha)_{\alpha < \rho_1}$ with the following properties*:

(a) *$(\Lambda_\alpha)_{\alpha < \rho_1}$ is a pairwise disjoint family of uncountable subsets of ρ and $\rho \backslash \bigcup_{\alpha < \rho_1} \Lambda_\alpha$ is countable; and*

(b) *$(W_\alpha)_{\alpha < \rho_1}$ is a family of basic elementary open sets in X determined by $\leqslant k - 1$ coordinates, $W_\alpha \supset \bigcup_{\beta \in \Lambda_\alpha} U_\beta$ and*

$$(\mu \otimes \nu)[(U_\beta \times V_\beta)_{\beta \in \Lambda_\alpha}] = (\mu \otimes \nu)[(W_\alpha \times V_\beta)_{\beta \in \Lambda_\alpha}] \textit{ for every } \alpha < \rho_1.$$

Moreover, the family $(W_\alpha)_{\alpha < \rho_1}$ can be chosen so that $W_\alpha \neq W_{\alpha'}$ for $\alpha \neq \alpha'$.

Proof. First we prove the following

Claim. For every uncountable $\Lambda \subset \rho$ there are an uncountable $\Lambda' \subset \Lambda$ and a basic elementary open set W in X determined by $\leqslant k - 1$ coordinates such that $W \supset \bigcup_{\alpha \in \Lambda'} U_\alpha$ and

$$(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha \in \Lambda'}] = (\mu \otimes \nu)[(W \times V_\alpha)_{\alpha \in \Lambda'}].$$

For every $\alpha \in \Lambda$ choose a subset I_α of I with k elements such that U_α is determined by I_α. Because the family of all basic elementary open sets determined by a fixed finite subset of I is countable, the set $\{I_\alpha : \alpha \in \Lambda\}$ is uncountable. Thus, by the Δ-lemma of Erdös and Rado (see [7], Lemma 22.6), there is an uncountable $\Lambda_0 \subset \Lambda$ and $J \subset I$ such that $I_\alpha \neq I_{\alpha'}$ and $I_\alpha \cap I_{\alpha'} = J$ for every $\alpha, \alpha' \in \Lambda_0$, $\alpha \neq \alpha'$. Clearly J has $\leqslant k - 1$ elements.

If $J = \emptyset$, then by condition (ii) of Theorem 3.1 $\mu[(U_\alpha)_{\alpha\in\Lambda'}] = 1$ for every uncountable $\Lambda' \subset \Lambda_0$. Thus, by Lemma 3.4, there exists an uncontable $\Lambda' \subset \Lambda_0$ such that

$$(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda'}] = (\mu \otimes \nu)[(X \times V_\alpha)_{\alpha\in\Lambda'}],$$

i.e. the claim for $W = X$.

We now assume that $J \neq \emptyset$ and choose an uncountable $\Lambda_1 \subset \Lambda_0$ and $S \subset \Pi_{i\in J} X_i$ such that $pr_J(U_\alpha) = S$ for every $\alpha \in \Lambda_1$. We set $W = pr_J^{-1}(S)$. Clearly, W is an elementary open set in X determined by $\leqslant k - 1$ coordinates and $W \supset \bigcup_{\alpha\in\Lambda_1} U_\alpha$. For every $\alpha \in \Lambda_1$, let W_α be an elementary open set determined by $I_\alpha \backslash J$ such that $W_\alpha \cap W = U_\alpha$. Since the family $(I_\alpha \backslash J)_{\alpha\in\Lambda_1}$ is pairwise disjoint, condition (ii) of Theorem 3.1 implies that $\mu[(W_\alpha)_{\alpha\in\Lambda'}] = 1$ for every uncountable $\Lambda' \subset \Lambda_1$. Thus, by Lemma 3.4, there exists an uncountable $\Lambda' \subset \Lambda_1$ such that

$$(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda'}] = (\mu \otimes \nu)[(X \times V_\alpha)_{\alpha\in\Lambda'}].$$

We then have

$$\begin{aligned}(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha\in\Lambda'}] &= (\mu \otimes \nu)[((W \cap W_\alpha) \times V_\alpha)_{\alpha\in\Lambda'}] \\ &= (\mu \otimes \nu)[(W \times V_\alpha)_{\alpha\in\Lambda'}],\end{aligned}$$

completing the proof of the claim.

Now, using the claim, it is easy to construct families $(\Lambda_\alpha)_{\alpha<\rho_1}$ and $(W_\alpha)_{\alpha<\rho_1}$ with properties (a) and (b) by transfinite induction on the ordinal α.

If there are $\alpha, \beta < \rho_1, \alpha \neq \beta$, with $W_\alpha = W_\beta$, we replace $(W_\alpha)_{\alpha<\rho_1}$ and $(\Lambda_\alpha)_{\alpha<\rho_1}$ by $(W'_\alpha)_{\alpha<\rho'_1}$ and $(\Lambda'_\alpha)_{\alpha<\rho'_1}$ such that $\{W'_\alpha : \alpha < \rho'_1\} = \{W_\alpha : \alpha < \rho_1\}$, $W'_\alpha \neq W'_\beta$ for $\alpha, \beta < \rho'_1, \alpha \neq \beta$, and $\Lambda'_\alpha = \cup\{\Lambda_\beta : \beta < \rho_1$ and $W_\beta = W'_\alpha\}$ for every $\alpha < \rho'_1$. It is easy to see that the new families also satisfy (a) and (b).

LEMMA 3.6. *Let (X, μ) be as in Theorem* 3.1 *with μ satisfying condition* (ii) *and ν a τ-additive probability Borel measure on a completely regular space Y. Then*

(a) *μ is τ-additive and for every open set G in X there are cozero sets U and V such that $U \subset G \subset V$ and $\mu(V \backslash U) = 0$; and*

(b) *for every open set G in $X \times Y$ there exist $H_0, H_1 \subset X \times Y$ of the form $H_i = \bigcup_{n=1}^{\infty} (S_n^i \times T_n^i)$, where S_n^i is cozero in X and T_n^i is open in Y for $n = 1, 2, \ldots$ and $i = 0, 1$, such that $H_0 \subset G \subset H_1$ and $(\mu \otimes \nu)(H_1 \backslash H_0) = 0$.*

Proof. In order to avoid repetitions, we first prove (b) under the assumption that μ is τ-additive. Then we show that this proof can be adapted to yield (a).

(b) Assume that μ is τ-additive. We write G in the form

$$G = \bigcup_{\alpha<\rho} (U_\alpha \times V_\alpha)$$

where $U_\alpha \neq U_{\alpha'}$ for $\alpha \neq \alpha'$, each U_α is basic elementary open in X and each V_α is open in Y. If we set $G_k = \bigcup\{U_\alpha \times V_\alpha : U_\alpha$ is determined by $\leqslant k$ coordinates$\}$, then $G = \bigcup_{k=1}^{\infty} G_k$. Thus, without loss of generality, we assume that for some $k \in \mathbb{N}$ each U_α is determined by $\leqslant k$ coordinates.

Let C be a countable subset of ρ such that $(\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha<\rho}] = (\mu \otimes \nu)(\bigcup_{\alpha\in C}(U_\alpha \times V_\alpha))$ and set $H_0 = \bigcup_{\alpha\in C}(U_\alpha \times V_\alpha)$. Then H_0 has the desired form and $H_0 \subset G$.

We shall construct by induction on i families

$$(U^i_\alpha)_{\alpha<\rho_i} \quad \text{and} \quad (V^i_\alpha)_{\alpha<\rho_i} \quad \text{for} \quad i = 0, 1, \ldots, m,$$

where $m \leqslant k$, $(U^i_\alpha)_{\alpha<\rho_i}$ is a family of basic elementary open sets in X determined by $\leqslant k - i$ coordinates with $U^i_\alpha \neq U^i_{\alpha'}$, for $\alpha \neq \alpha'$, and $(V^i_\alpha)_{\alpha<\rho_i}$ is a family of open sets in Y, as follows: We set $\rho_0 = \rho$, $U^0_\alpha = U_\alpha$ and $V^0_\alpha = V_\alpha$ for $\alpha < \rho_0$. Assume that $(U^i_\alpha)_{\alpha<\rho_i}$ and $(V^i_\alpha)_{\alpha<\rho_i}$ have been constructed for $i = 0, 1, \ldots, n-1$. If ρ_{n-1} is countable, we set $m = n - 1$ and the process stops. Otherwise we apply Lemma 3.5 for the families $(U^{n-1}_\alpha)_{\alpha<\rho_{n-1}}$ and $(V^{n-1}_\alpha)_{\alpha<\rho_{n-1}}$ and find families $(\Lambda^n_\alpha)_{\alpha<\rho_n}$ and $(U^n_\alpha)_{\alpha<\rho_n}$ with the following properties:

(a)$_n$ $(\Lambda^n_\alpha)_{\alpha<\rho_n}$ is a pairwise disjoint family of uncountable subsets of ρ_{n-1} and $C_n = \rho_{n-1} \setminus \bigcup_{\alpha<\rho_n} \Lambda^n_\alpha$ is countable; and

(b)$_n$ $(U^n_\alpha)_{\alpha<\rho_n}$ is a family of basic elementary open sets in X determined by $\leqslant k - n$ coordinates with $U^n_\alpha \neq U^n_{\alpha'}$ for $\alpha \neq \alpha'$, $U^n_\alpha \supset \bigcup_{\beta\in\Lambda^n_\alpha} U^{n-1}_\beta$ and

$$(\mu \otimes \nu)[(U^{n-1}_\beta \times V^{n-1}_\beta)_{\beta\in\Lambda^n_\alpha}] = (\mu \otimes \nu)[(U^n_\alpha \times V^{n-1}_\beta)_{\beta\in\Lambda^n_\alpha}]$$

for every $\alpha < \rho_n$.

Next we set $V^n_\alpha = \bigcup_{\beta\in\Lambda^n_\alpha} V^{n-1}_\beta$ for every $\alpha < \rho_n$ and the construction is completed. Finally, we have $m \leqslant k$. Indeed, if ρ_{k-1} is defined and is uncountable, then $U^k_\alpha = X$ for every $\alpha < \rho_k$ and so $\rho_k = 1$ is countable (because $U^k_\alpha \neq U^k_{\alpha'}$ for $\alpha \neq \alpha'$).

We set

$$H_1 = \left[\bigcup_{\alpha\in C_1} (U^0_\alpha \times V^0_\alpha)\right] \cup \cdots \cup \left[\bigcup_{\alpha\in C_m} (U^{m-1}_\alpha \times V^{m-1}_\alpha)\right] \cup \left[\bigcup_{\alpha<\rho_m} (U^m_\alpha \times V^m_\alpha)\right].$$

Since $C_1, \ldots, C_m$ and ρ_m are countable, H_1 has the desired form and, by (a)$_n$ and (b)$_n$ ($n = 1, \ldots, m$), $H_1 \supset G$.

Claim. For every $i = 0, 1, \ldots, m$ and every $\alpha < \rho_i$, there exists a countable subset Γ^i_α of $\rho = \rho_0$ such that

$$(\mu \otimes \nu)(U^i_\alpha \times V^i_\alpha) = (\mu \otimes \nu)\left(\bigcup_{\beta\in\Gamma^i_\alpha} (U_\beta \times V_\beta)\right).$$

The claim is trivial if $i = 0$. Assume that the claim is true for $i = n - 1$ (where $n = 1, \ldots, m$) and set $\lambda = \tilde{\mu} \hat{\otimes} \nu$ if μ is a Baire measure and $\lambda = \mu \hat{\otimes} \nu$ if μ is a Borel measure. Then using the τ-additivity of λ and $(\mathrm{b})_n$ we have

$$(\mu \otimes \nu)(U_\alpha^n \times V_\alpha^n) = \lambda\left(U_\alpha^n \times \bigcup_{\beta \in \Lambda_\alpha^n} V_\beta^{n-1}\right) = \lambda[(U_\alpha^n \times V_\beta^{n-1})_{\beta \in \Lambda_\alpha^n}]$$

$$= (\mu \otimes \nu)[(U_\alpha^n \times V_\beta^{n-1})_{\beta \in \Lambda_\alpha^n}] = (\mu \otimes \nu)[(U_\beta^{n-1} \times V_\beta^{n-1})_{\beta \in \Lambda_\alpha^n}]$$

and so the claim is true for $i = n$.

We now set

$$\Gamma = \left[\bigcup_{i=1}^{m} \bigcup_{\alpha \in C_i} \Gamma_\alpha^i\right] \cup \left[\bigcup_{\alpha < \rho_m} \Gamma_\alpha^m\right].$$

Clearly Γ is countable and it is easy to see that

$$(\mu \otimes \nu)(H_1) = (\mu \otimes \nu)\left(\bigcup_{\alpha \in \Gamma} (U_\alpha \times V_\alpha)\right)$$

$$\leqslant (\mu \otimes \nu)[(U_\alpha \times V_\alpha)_{\alpha < \rho}] = (\mu \otimes \nu)(H_0).$$

Therefore $(\mu \otimes \nu)(H_1 \backslash H_0) = 0$ and the proof of (b) is completed when μ is τ-additive.

(a) Let $(U_\alpha)_{\alpha < \rho}$ be a family of basic elementary open sets in X with $U_\alpha \neq U_{\alpha'}$ for $\alpha \neq \alpha'$, and set $G = \bigcup_{\alpha < \rho} U_\alpha$. It suffices to prove that there is a countable $C \subset \rho$ and a cozero set V in X such that $V \supset G$ and $\mu(V \backslash U) = 0$, where $U = \bigcup_{\alpha \in C} U_\alpha$. As in the proof of (b) we can assume that for some $k \in \mathbb{N}$ each U_α is determined by $\leqslant k$ coordinates.

Let C be a countable subset of ρ such that $\mu(\bigcup_{\alpha \in C} U_\alpha) = \mu[(U_\alpha)_{\alpha < \rho}]$ and set $U = \bigcup_{\alpha \in C} U_\alpha$.

We now proceed as in the proof of (b) ignoring the space Y (more precisely we assume that Y is a singleton) and construct families

$$(U_\alpha^i)_{\alpha < \rho_i} \quad \text{for} \quad i = 0, 1, \ldots, m.$$

Here, for example, the last relation in $(\mathrm{b})_n$ takes the form $\mu[(U_\beta^{n-1})_{\beta \in \Lambda_\alpha^n}] = \mu(U_\alpha^n)$ for every $\alpha < \rho_n$, from which we have immediately the following

Claim. For every $i = 0, 1, \ldots, m$ and every $\alpha < \rho_i$ there exists a countable subset Γ_α^i of $\rho = \rho_0$ such that $\mu(U_\alpha^i) = \mu(\bigcup_{\beta \in \Gamma_\alpha^i} U_\beta)$.
(Notice that in the proof of the corresponding claim in (b) the τ-additivity of μ was needed.)

Finally, we set

$$V = \left[\bigcup_{\alpha \in C_1} U_\alpha^0\right] \cup \cdots \cup \left[\bigcup_{\alpha \in C_m} U_\alpha^{m-1}\right] \cup \left[\bigcup_{\alpha < \rho_m} U_\alpha^m\right].$$

Then V is a cozero set, $V \supset G$ and, using the above claim as in (b), we see that $\mu(V \backslash U) = 0$.

Proof of Theorem 3.1. (i) ⇒ (ii) and (ii) ⇒ (iii) are proved in Lemmas 3.2 and 3.6(a), respectively. Since (iii) ⇒ (i) is trivial, (i), (ii) and (iii) are equivalent and so part (a) of the theorem follows from Lemma 3.6(a).

(ii) ⇒ (iv) follows from Lemma 3.6(b).

(ii) ⇒ (v) First notice that by part (a) $\mu \hat{\otimes} \nu$ is defined. Assume that μ and ν are Borel measures. Let G be an open subset of $X \times Y$ and let H_0 and H_1 be as in Lemma 3.6(b). Since ν is completion regular there are Baire sets B_n^i, $n = 1,2,\ldots, i = 0,1$, in Y such that $B_n^0 \subset T_n^0, \nu(B_n^0) = \nu(T_n^0)$ and $B_n^1 \supset T_n^1$, $\nu(B_n^1) = \nu(T_n^1)$. Then the sets $B_i = \bigcup_{n=1}^{\infty} S_n^i \times B_n^i, i = 0, 1$, are Baire sets in $X \times Y, B_0 \subset G \subset B_1$ and $(\mu \hat{\otimes} \nu)(B_1 \backslash B_0) = 0$. Therefore $\mu \hat{\otimes} \nu$ is completion regular.

Now assume that μ and ν are Baire measures. Then, by the above, $\tilde{\mu} \hat{\otimes} \tilde{\nu}$ is completion regular. Since $\mu \hat{\otimes} \nu$ is the restriction of $\tilde{\mu} \hat{\otimes} \tilde{\nu}$ to the Baire sets in $X \times Y$, if follows that $\mu \hat{\otimes} \nu$ is also completion regular.

Finally, (iv) ⇒ (i) and (v) ⇒ (i) are obvious (take Y to be a singleton). This completes the proof of the theorem.

REMARK. There are Baire measures on products of separable metric spaces (even on $\mathbb{R}^c$, where c is the continuum), that are not τ-additive; see [11] and the references given there. It follows from Theorem 3.1(a) that these measures are not completion regular.

COROLLARY 3.7. *Let $X = \Pi_{i \in I} X_i$, where all X_i are separable metric spaces and μ a completion regular probability Baire (or Borel) measure on X.*

(a) *If $A \subset X$ with $\mu^*(A) > 0$ (resp. $\mu_*(A) > 0$), then $pr_i(A)$ is dense in X_i (resp. $pr_i(A_i) = X_i$) for all but a countable number of $i \in I$.*

(b) *The support of μ is a zero set.*

(c) *If $\mu = \hat{\bigotimes}_{i \in I} \mu_i$, where μ_i is a probability Baire measure on X_i for every $i \in I$, then μ_i has full support for all but a countable number of $i \in I$.*

(d) *If μ is a Radon measure, then X_i is compact for all but a countable number of $i \in I$.*

Proof. (a) Assume that $\mu^*(A) > 0$ and set $J = \{i \in I : pr_i(A) \text{ is not dense in } X_i\}$. For every $i \in J$, we choose a nonempty cozero subset V_i of X_i such that $pr_i(A) \cap V_i = \emptyset$. If J were uncountable, then by Theorem 3.1, (i) ⇒ (ii), we should have $\mu(\bigcup_{i \in J_0} pr_i^{-1}(V_i)) = 1$ for some countable $J_0 \subset J$, a contradiction because $(\bigcup_{i \in J_0} pr_i^{-1}(V_i)) \cap A = \emptyset$ and $\mu^*(A) > 0$. Thus, J is countable.

If $\mu_*(A) > 0$ then, by the completion regularity of μ, A contains a nonempty Baire set B. The result now follows from the fact that B is determined by countably many coordinates.

(b) By Theorem 3.1(a), the support S of μ is defined. Since μ is completion regular there exists a Baire set $B \subset S$ such that $\mu(B) = 1$. Let J be a countable subset of I and C a Baire set in $\Pi_{i \in J} X_i$ such that $B = pr_J^{-1}(C)$. If F is the closure of

C in $\Pi_{i \in J} X_i$, then $pr_J^{-1}(F)$ is a zero set in X and $B \subset pr_J^{-1}(F) \subset S$. Therefore $\mu(pr_J^{-1}(F)) = 1$ and $S = pr_J^{-1}(F)$ is a zero set.

(c) By (b) the support of μ is determined by countably many coordinates and so (c) follows easily.

(d) follows from (a) when A is compact.

Our last corollary is an extension of Kakutani's theorem mentioned in the introduction, which also contains Theorem 3 in [2].

COROLLARY 3.8. *Let $(X_i)_{i \in I}$ be a family of spaces such that each X_i is homeomorphic to a product of separable metric spaces and Y a completely regular space. Let μ_i and ν be completion regular τ-additive Baire (or Borel) measures on X_i and Y, respectively, such that each μ_i has full support. Then the τ-additive product measure $(\hat{\bigotimes}_{i \in I} \mu_i) \hat{\otimes} \nu$ is completion regular.*

Proof. By Theorem 3.1, (i) $\Rightarrow$ (v), $\hat{\bigotimes}_{i \in F} \mu_i$ is completion regular for every finite $F \subset I$. Thus, by Theorem 2.9, $\hat{\bigotimes}_{i \in I} \mu_i$ is completion regular. The result now follows using once more Theorem 3.1.

Note added in proof

After this paper has been accepted, the authors learned that part (b) of Theorem 2.1 was obtained by D.H. Fremlin, *Quasi-Radon measure spaces*, unpublished notes of 10.8.76 and 2.6.82.

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