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De Rham cohomology of affinoid spaces

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The singular cohomology groups $H^{\cdot}(X, \mathbb{C})$ of a non-singular algebraic variety X over $\mathbb{C}$ can be obtained from the algebraic de Rham complex ([8]). For a non-singular variety Y over a finite field k this de Rham complex does not give the "correct" groups. A construction to remedy this has been proposed and carried out by Dwork, Monsky, Washnitzer, Berthelot and others. It can be described as follows. The affine non-singular variety Y is lifted to an affinoid space X over $K=$ the field of fractions of $W(k)$. Let A denote the ring of holomorphic functions on X. The de Rham complex $\Omega^{\cdot}$ of the holomorphic differential forms on X still does not give the correct cohomology groups. One refines the construction by introducing a subring $A^{\dagger} \subset A$ of overconvergent holomorphic functions. The de Rham complex $\Omega^{\cdot}(A^{\dagger})$ of overconvergent differential forms has in many cases the correct cohomology.

In this paper one studies the de Rham cohomology for general affinoid spaces $X = \text{spm}(A)$ over a field K of characteristic o. Such a space can be seen as a lift of an affine space $Y = \text{spec}(\bar{A})$ over the residue field of K. In general Y has singularities and one can no longer apply the Monsky-Washnitzer theory. In particular an affinoid algebra A need not have an overconvergent presentation.

In section 1 one uses Artin-approximation in order to show that the de Rham-cohomology groups of an affinoid space $X = \text{spm}(A)$ where A has an overconvergent presentation φ, do not depend on the choice of φ. Further it is shown that any non-singular X (the affine space Y can have arbitrary singularities) has at least locally for the Grothendieck topology on X an overconvergent presentation. This enables us to define de Rham cohomology sheaves on X.

In the special case that X is non-singular, connected and $\dim X = 1$, one uses an embedding of X in a non-singular projective curve over K to obtain an overconvergent presentation. In section 2 the same embedding is used for an explicit formula of $\dim \mathrm{H}^1_{DR}(X)$. For certain families of one-dimensional affinoid spaces $X \to S$ the method above gives the rank of the $\mathcal{O}(S)$-module $H^1_{DR}(X S)$. This resembles a result of Adolphson [1] and recent work of Baldassarri [3].

For affinoid spaces X with $\dim X > 1$ we have only some results in the case "X is the complement of a hypersurface $t = o$". In case $\bar{t} = 0$ defines a non-singular hypersurface over the residue field of K, the Monsky–Washnitzer theory has a residue map and a Gysin exact sequence ([11]) and one knows that $\dim H^{\bullet}_{DR}(X) < \infty$. In section 3 we allow $\bar{t} = 0$ (and also $t = 0$) to have singularities. A residue map is constructed and a Gysin-exact sequence. This leads to some results on $H^{\bullet}_{DR}(X)$. For the cohomology theory of Dwork, Monsky and Washnitzer we refer to [10, 11, 14, 15] and for affinoid spaces to [5, 6, 7].

Section 1. Overconvergence

In this section K is complete with respect to a non-archimedean valuation. Let $K\langle X_1, \ldots, X_n \rangle$ denote the free affinoid algebra over K in the explicitly given variables $X_1, \ldots, X_n$. An element $f = \sum a_\alpha X^\alpha \in K\langle X_1, \ldots, X_n \rangle$ is called *overconvergent* if for some $\lambda > 1$ one has $\lim |a_\alpha| \lambda^{|\alpha|} = 0$. The subring of overconvergent elements is denoted by $K\langle X_1, \ldots, X_n \rangle^\dagger$.

LEMMA 1.1. *Weierstrass preparation and division is valid for* $K\langle X_1, \ldots, X_n \rangle^\dagger$.

Proof. We follow the by now classical method (see [6] p. 55, 56). Let $F \in K\langle X_1, \ldots, X_n \rangle^\dagger$ have norm 1. A linear substitution $X_i \to \sum \lambda_{ij} X_j$ with $(\lambda_{ij}) \in \mathrm{Gl}(n, K^\circ)$ (where K° is the valuationring of K) or a substitution of the form $X_i \to X_i + X_n^{e_i} (i = 1, \ldots, n-1)$ and $X_n \to X_n$ makes F regular in X_n of some degree d. The substitution leaves $K\langle X_1, \ldots, X_n \rangle^\dagger$ invariant. For a suitable integer $N \geqslant 1$ and all $\lambda > 1, \lambda \in \sqrt{|K^*|}$ and λ close enough to 1, the element F remains regular in X_n of degree d on the polydisk $\{(X_1, \ldots, X_n) \in K^n \,\|\, |X_i| \leqslant \lambda$ for $i = 1, \ldots, n-1$ and $|X_n| \leqslant \lambda^N\}$. An element $G \in K\langle X_1, \ldots, X_n \rangle^\dagger$ extends to such a polydisk and hence in the usual Weierstrass division $G = QF + R$, the elements Q and R belong to $K\langle X_1, \ldots, X_n \rangle^\dagger$.

Among other properties of $K\langle X_1, \ldots, X_n \rangle^\dagger$ one can derive from (1.1) the following:

COROLLARY 1.2 (Monsky and Washnitzer [10]). $K\langle X_1, \ldots, X_n \rangle^\dagger$ *is noetherian.*

DEFINITION. Let A be an affinoid algebra over K. An overconvergent presentation φ of A is a surjective K-algebra homomorphism $\varphi: K\langle X_1, \ldots, X_n \rangle \to A$ such that the kernel of φ is generated by overconvergent elements. We define $(\varphi, A)^\dagger$ as $K\langle X_1, \ldots, X_n \rangle^\dagger / (\ker \varphi) \cap K\langle X_1, \ldots, X_n \rangle^\dagger$.

LEMMA 1.3.

(1) $K\langle X_1, \ldots, X_n \rangle^\dagger \hookrightarrow K\langle X_1, \ldots, X_n \rangle$ *is faithfully flat.*

(2) *Let* φ *be an overconvergent presentation of the affinoid algebra* A. *If* $(f_1, \ldots, f_m) = \ker \varphi$ *with* $f_1, \ldots, f_m \in K\langle X_1, \ldots, X_n \rangle^\dagger$ *then* $(\varphi, A)^\dagger = K\langle X_1, \ldots, X_n \rangle^\dagger / (f_1, \ldots, f_m)$. *Further* $(\varphi, A)^\dagger \to A$ *is faithfully flat.*

Proof. (1) For any maximal ideal $\underline{m}$ of $K\langle X_1, \ldots, X_n \rangle^\dagger$ one shows with the aid of (1.1) that $K\langle X_1,\ldots,X_n\rangle^\dagger/\underline{m}$ is a finite extension of K. (see [4] (II. 3.5)). Hence there exists a unique maximal ideal M of $K\langle X_1,\ldots,X_n\rangle$ with $M \cap K\langle X_1,\ldots,X_n\rangle^\dagger = \underline{m}$. The completions of $K\langle X_1,\ldots,X_n\rangle^\dagger$ localized at $\underline{m}$ and $K\langle X_1,\ldots,X_n\rangle$ localized at M are isomorphic and so $K\langle X_1,\ldots,X_n\rangle^\dagger \to K\langle X_1,\ldots,X_n\rangle$ is faithfully flat.

(2) is an immediate consequence of (1).

PROPOSITION (1.4) (S. Bosch [4]). *Let the complete field K have either characteristic 0 or satisfy $[K:K^p] < \infty$ with $0 \neq p = \text{char}\, K$, then $K\langle X_1,\ldots,X_n\rangle^\dagger$ has the Artin approximation property.*

Commentary. The statement of the Artin approximation in this case reads: "Let $f_1,\ldots,f_m$ belong to $K\langle X_1,\ldots,X_a,Y_1,\ldots,Y_b\rangle^\dagger$, let $\varepsilon > 0$ and let $\bar{y}_1,\ldots,\bar{y}_b$ in $K\langle X_1,\ldots,X_a\rangle$ have norms $\leqslant 1$ and satisfy $f_i(X_1,\ldots,X_a,\bar{y}_1,\ldots,\bar{y}_b) = 0$ $(i = 1,\ldots,m)$ Then there are $y_1,\ldots,y_b \in K\langle X_1,\ldots,X_a\rangle^\dagger$ with $\|y_i - \bar{y}_i\| \leqslant \varepsilon$ and $f_i(X_1,\ldots,X_a,y_1,\ldots,y_b) = 0$ for $i = 1,\ldots,m$".

Artin's proof in [2] can be adapted to the above case without any surprises. In case char $K = p \neq 0$ one has to add a verification of Lemma (2.2) [2] page 283. In the local analytic case such a verification is provided in [12] Section 8. This proof in the local case carries over to the case of overconvergent power series.

As in [2] Theorem (1.5a) we have the following consequences.

COROLLARY 1.5. *Suppose that* char $K = 0$ *or* char $K = p \neq 0$ *and* $[K:K^p] < \infty$. *Let A and B denote affinoid algebra's with overconvergent presentations φ and ψ. Let $u: A \to B$ be a morphism and let $\varepsilon > 0$. Then there exists a morphism $u': A \to B$ with $\|u - u'\| \leqslant \varepsilon$ and such that u' maps $(\varphi, A)^\dagger$ into $(\psi, B)^\dagger$. In particular if φ_1 and φ_2 are two overconvergent presentations of A and for any $\varepsilon > 0$ there exists an automorphism u of A with $\|u - 1\| \leqslant \varepsilon$ and $u((\varphi_1, A)^\dagger) = (\varphi_2, A)^\dagger$.*

COROLLARY 1.6. *$A = K\langle X_1,\ldots,X_n\rangle/I$ has an overconvergent presentation if and only if there exists an automorphism σ of $K\langle X_1,\ldots,X_n\rangle$ such that $\sigma(I)$ is generated by elements of $K\langle X_1,\ldots,X_n\rangle^\dagger$.*

Proof (1.5) follows from (1.4) along the lines of [2]. The only new thing one uses is: if $u: A \to A$ satisfies $\|u - 1\| < 1$ then u is an isomorphism. (1.6) The "if" parts as obvious. Suppose that A has an overconvergent presentation φ. Then there exists $u': K\langle X_1,\ldots,X_n\rangle \to A$ with $\|u' - u\| \leqslant \varepsilon < 1$ and $u'(K\langle X_1,\ldots,X_n\rangle^\dagger) \subseteq (\varphi, A)^\dagger$. With the help of the Weierstrass-Theorem 1.1 one shows that $u': K\langle X_1,\ldots,X_n\rangle^\dagger \to (\varphi, A)^\dagger$ is actually surjective. The faithful flatness implies that $\ker(u')$ is generated by overconvergent elements. So u' is also an overconvergent presentation. Further $u' = u\sigma$ for some automorphism of $K\langle X_1,\ldots,X_n\rangle$. This proves (1.6).

PROPOSITION 1.7. *Every reduced one-dimensional affinoid algebra has an overconvergent presentation.*

Proof. Let A denote the normalisation of this algebra. It suffices to show that every connected component of A has an overconvergent presentation. According to [13] a regular, connected, one-dimensional affinoid space can be embedded into a complete non-singular curve. This means a presentation by polynomial equations and hence an overconvergent presentation.

PROPOSITION 1.8. *Suppose that the affinoid algebra A is smooth over K. Then $X = \mathrm{Sp}(A)$ has a finite covering by rational subspaces $X_i = \mathrm{Sp}(A_i)$ such that each A_i carries an overconvergent presentation.*

Proof. Let B have an overconvergent presentation φ then every rational subspace U of $\mathrm{Sp}(B)$ carries an induced overconvergent presentation, since

$$\mathcal{O}(U) = B\langle T_1, \ldots, T_m\rangle/(f_1 - f_0 T_1, \ldots, f_m - f_0 T_m)$$

where $f_0, \ldots, f_m \in B$ can be chosen in the dense subring $(\varphi, B)^\dagger$ of B. According to Kiehl ([9] Folgerung (1.14)) a smooth X has locally the form $K\langle X_1, \ldots, X_n, 1/t\rangle[Y]/(P) = B$ where $t \in K\langle X_1, \ldots, X_n\rangle$ is an element with norm 1 and P is a monic polynomial in Y with coefficients in $K\langle X_1, \ldots, X_n\rangle$ such that $\mathrm{d}P/\mathrm{d}Y$ is invertible in B. Of course we may truncate t without changing B. Newton's method on approximation of roots shows that a monic polynomial $Q \in K\langle X_1, \ldots, X_n\rangle[Y]$ which is close enough to P defines an affinoid algebra isomorphic to B. So we are allowed to truncate the coefficients of P and we obtain that B can be defined by polynomial equations. The proposition follows.

In the sequel of this paper we assume that K has characteristic 0. Let A/K be a connected, non-singular, affinoid algebra of Krull-dimension n, which has an overconvergent presentation φ. By $\Omega^1(\varphi, A)^\dagger$ or $\Omega^1(A^\dagger)$ we denote the module of continuous differentials of $(\varphi, A)^\dagger$. If φ induces the isomorphism $(\varphi, A)^+ \cong K\langle X_1, \ldots, X_a\rangle^\dagger/(f_1, \ldots, f_b)$ then $\Omega^1(\varphi, A)^\dagger$ is an $(\varphi, A)^\dagger$-module generated by $\mathrm{d}x_1, \ldots, \mathrm{d}x_a$ and the relations between the generators are given by

$$\frac{\partial f_i}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f_i}{\partial x_a}\mathrm{d}x_a = 0 \ (i = 1, \ldots, b).$$

Clearly $\Omega^1(\varphi, A)^\dagger \otimes A$ is isomorphic to the usual module of continuous differentials of A/K. Further $\Omega^1(A, \varphi)^\dagger$ is a projective module of rank n. Put $\Omega^p(\varphi, A)^\dagger = \Lambda^p\Omega^1(\varphi, A)^\dagger$, then we have a De Rham complex $\Omega^\cdot(\varphi, A)^\dagger$. This complex depends on the choice of φ. The cohomology groups however do not depend on φ according to (1.5).

We will write $H^\cdot_{DR}(\varphi, A)$ or $H^\cdot_{DR}(X)$ or $H^\cdot_{DR}(\varphi, X)$, where $X = \mathrm{Spm}(A)$, for the cohomology of the de Rham complex $\Omega^\cdot(\varphi, A)^\dagger$. We will need a stronger version of the independence of φ. Let ε denote $p^{1/(p-1)}$ if the residue characteristic of K is $p \neq 0$ and 1 otherwise.

PROPOSITION 1.9. *Let A/K be as above and let φ and ψ denote two overconvergent presentations of A. Let u and v be automorphisms of A with $\|u-1\|, \|v-1\| < \varepsilon$ such that u and v are bijections $(\varphi, A)^\dagger \to (\psi, A)^\dagger$. Then u and v induce the same bijections $H^{\cdot}_{DR}(\varphi, A) \to H^{\cdot}_{DR}(\psi, A)$.*

Proof. It suffices to show that an automorphism u of A with $u(\varphi, A)^\dagger = (\varphi, A)^\dagger$ and $\|u-1\| < \varepsilon$ induces the identity on $H^{\cdot}_{DR}(\varphi, A)$.

One defines $D = \log(1 + (u-1)) = \sum(-1)^n(u-1)^{n+1}/n+1$ as endomorphism of A. Then D is a derivation of A over K and $\|D\| < \varepsilon$ and $u = \exp(D) = \sum_{n\geqslant 0}(D^n/n!)$. Consider the morphism of affinoid algebra $F: A \to A\langle T\rangle$ given by the formula

$$F(a) = \sum_{N\geqslant 0} \frac{D^n(a)}{n!} T^n.$$

Let $\alpha_0, \alpha_1: (\varphi, A)^\dagger\langle T\rangle^\dagger \to (\varphi, A)^\dagger$ denote the $(\varphi, A)^\dagger$-algebra homomorphism given by $\alpha_0(T) = 0$ and $\alpha_1(T) = l$. One easily verifies that F maps $(\varphi, A)^\dagger$ into $(\varphi, A)^\dagger\langle T\rangle^\dagger$ and that $\alpha_0.F = \mathrm{id}$ and $\alpha_1.F = u$.

It suffices now to show that α_0 and α_1 induce the same maps in the de Rham cohomology. We will show that α_0 and α_1 are homotopic. The space $\Omega^q(\varphi, A)^\dagger\langle T\rangle^\dagger)$ is the direct sum of $(\varphi, A)^\dagger\langle T\rangle^\dagger \otimes_{(\varphi, A)^\dagger} \Omega^q(\varphi, A)^\dagger$ and $(\varphi, A)^\dagger\langle T\rangle^\dagger \mathrm{d}T \otimes_{(\varphi, A)^\dagger} \Omega^{q-1}(\varphi, A)^\dagger$. The homotopy $\{\delta_q\}$ between $(\alpha_0)^*$ and $(\alpha_1)^*$ is given by: δ_q is zero on the first vectorspace and δ_q is integration from 0 to 1 with respect to T on the second vectorspace. This proves 1.9.

1.10. Sheaves of de Rham cohomology

Let again A/K denote a non-singular, affinoid algebra of Krull-dimension n. Let $X = \mathrm{Spm}(A)$ denote the associated affinoid space and let $U \subset X$ be a rational subset. Then there are $f_0, f_1, \ldots, f_m \in A$ generating the unit ideal such that

$$U = \{x \in X \,\|\, f_0(x)\| \geqslant |f_i(x)| \quad \text{for all } i\}.$$

Moreover

$$\mathcal{O}_X(U) = \mathcal{O}(U) = A\langle T_1, \ldots, T_m\rangle/(f_1 - f_0T_1, \ldots, f_m - f_0T).$$

For an overconvergent presentation φ of A one can choose $f_0, \ldots, f_m \in (\varphi, A)^\dagger$ and one finds an overconvergent presentation of $\mathcal{O}(U)$ not depending on the choices of $f_0, \ldots, f_m \in (\varphi, A)^\dagger$ but only depending on φ. We write $(\varphi, \mathcal{O}(U)^\dagger)$ for the corresponding subring of $\mathcal{O}(U)$ and $\Omega^q(\varphi, \mathcal{O}(U))^\dagger$ for the corresponding differential forms on U. The complex of sheaves $U \to \Omega^{\cdot}(\varphi, \mathcal{O}(U)^{\dagger})$ on X has sheaves of cohomology $U \to \mathscr{H}^{\cdot}(\varphi)(U)$ associated with the pre-sheave $U \to H^{\cdot}_{DR}(\varphi, U)$. We will consider the dependence on φ.

Let ψ be another overconvergent presentation of A and let $U \subset X = \mathrm{Spm}(A)$ be a rational subset. There is an $\varepsilon > 0$ (depending on U) such that for any automorphism u of A with $\|u - 1\| < \varepsilon$ the identity $u(U) = U$ holds. Choose u such that $\|u - 1\| < \varepsilon$ and $u(\varphi, A)^{\dagger} = (\psi, A)^{\dagger}$. Then $u(\varphi, \mathcal{O}(U))^{\dagger} = (\psi, \mathcal{O}(U))^{\dagger}$ and u induces a bijection $l^{\cdot}(U)\colon H^{\cdot}_{DR}(\varphi, U) \to H^{\cdot}_{DR}(\psi, U)$ depending only on φ, ψ, U. The resulting isomorphisms of sheaves $l^{\cdot}\colon \mathscr{H}^{\cdot}(\varphi) \to \mathscr{H}^{\cdot}(\psi)$ depend only on φ and ψ. Further $H^{\cdot}_{DR}(\varphi, X)$ can be recovered from $\mathscr{H}^{\cdot}(\varphi)$ with the spectral sequence $\{H^k(X, \mathscr{H}^l(\varphi)\}$.

The above enables us to define *the sheaves of the de Rham conomology for any rigid analytic space X over K which is non-singular and pure of dimension n.*

Indeed by (1.8), X has an admissible covering $\{X_i\}$ by affinoid spaces having overconvergent presentations $\{\varphi_i\} = \varphi$. The sheaves $\mathscr{H}^{\cdot}(\varphi_i)$ on X_i have canonical isomorphisms $\mathscr{H}^{\cdot}(\varphi_i)|X_i \cap X_j \to \mathscr{H}^{\cdot}(\varphi_j)|X_i \cap X_j$. So we find sheaves $(\mathscr{H}^{\cdot}, \varphi)$ on X. For another admissible covering of X and another family of overconvergent presentations ψ one finds a canonical isomorphism $(\mathscr{H}^{\cdot}, \varphi) \to (\mathscr{H}^{\cdot}, \psi)$.

The hypercohomology of the usual de Rham complex $\Omega^{\cdot}$ on X gives rise to a spectral sequence $E_r \Rightarrow \mathbb{H}^{\cdot}(\Omega^{\cdot})$ with $E_2^{p,q} = H^p(X, \mathscr{H}^q)$. It is possible to construct an overconvergent version $(E, \varphi)_r, r \geqslant 1$, of this spectral sequence with $(E, \varphi)_2^{p,q} = H^p(X, (\mathscr{H}^q, \varphi))$. This might lead to a definition of overconvergent de Rham cohomology on X as above.

In many cases, e.g. X is proper or X is an algebraic variety or $\dim X = 1$, the overconvergent presentations $\varphi = \{\varphi_i\}$ can be chosen such that φ_i and φ_j coincide on $X_i \cap X_j$ for all i, j. In such a case there is an overconvergent de Rham complex $(\Omega^{\cdot}, \varphi)$ and the overconvergent de Rham cohomology is defined as the hypercohomology of $(\Omega^{\cdot}, \varphi)$. (and does not depend on φ).

(1.11) AN EXAMPLE. Let $Z = \mathrm{Spm}(K\langle X, Y\rangle/(Y^2 - X(X - \pi)(X - 1)))$ where $0 < |\pi| < 1$. We take the obvious overconvergent presentation. The spectral sequence implies the exactness of

$$0 \to H^1(Z, (\mathscr{H}^0, \varphi)) \to H^1_{DR}(Z) \to H^0(Z, (\mathscr{H}^1, \varphi)) \to 0.$$

$(\mathscr{H}^0, \varphi)$ is the constant sheaf with stalk K and the bad reduction of Z implies $H^1(Z, K) = K$. Using Section 2 one can calculate $\dim H^1_{DR}(Z) = 2$ and so $\dim H^0(Z, (\mathscr{H}^1, \varphi)) = 1$.

Section 2. Dimension one

The field K is supposed to have characteristic 0 and to be algebraically closed.

THEOREM 2.1. (Compare [1]). *Let X be a connected, non-singular, one-dimensional affinoid space. Then X can be embedded in a complete non-singular curve* $\hat{X}$ *of genus g such that* $X = \hat{X} - (B_1 \cup \cdots \cup B_n)$ *where the* B_i *are distinct open subspaces of* $\hat{X}$ *isomorphic to* $\{z \in K \mid |z| < 1\}$. *The de Rham cohomology groups of X are:*

$$H^0_{DR}(X) = K; H^1_{DR}(X) = K^{2g+(n-1)}; H^i_{DR}(X) = 0 \quad \textit{for } i > 1.$$

Proof. The embedding $X \hookrightarrow \hat{X}$ is constructed in [13]. We consider first the case $n = 1$. Let $\tau\colon \{z \in K \mid |z| < 1\} \xrightarrow{\sim} B_1$ be an analytic isomorphism. Choose a sequence $\rho_1 < \rho_2 < \ldots$ in $|K^*|$ with $\lim \rho_m = 1$. Put $X_m = \hat{X} - \tau\{z \in K \mid |z| < \rho_m\}$ and $\partial X_m = \tau\{z \in K \mid |z| = \rho_m\}$. Then $\mathcal{O}(X)^\dagger = \varinjlim \mathcal{O}(X_m)$ and $\Omega^1(X_m)^\dagger = \varinjlim \Omega^1(X_m)$ are provided with the direct limit topology. The kernel of the continuous map $d\colon \mathcal{O}(X)^\dagger \to \Omega^1(X)^\dagger$ consists of the constant functions on X, (i.e. K) and we have only to show that coker(d) has dimension $2g$.

To any $f \in \mathcal{O}(X_m)$ we associate $f \circ \tau$ defined on $\{z \in K \mid \rho_m \leqslant |z| < 1\}$ and its expansion $f \circ \tau = \sum_{n=-\infty}^{\infty} a_n z^n$.

LEMMA 2.2. $\|f\|_m :=$ *the supremum-norm of f on* X_m *is equal to* $\max_{n \leqslant 0} |a_n| \rho_m^n$.

Proof. In the canonical reduction $X_m \to \bar{X}_m$ the subset $X_m - \partial X_m$ is mapped to one point. So for every $f \in \mathcal{O}(X_m)$ we have that $\|f\|_m$ equals the supremum norm on $\partial X_m = \max_{n \in \mathbb{Z}}(|a_n| \rho_m^n)$.

This expression decreases when ρ_m increases. It follows that $|a_k| \rho_m^k < \max_{n \in \mathbb{Z}} |a_n| \rho_m^n$ for every $k > 0$. This proves (2.2).

LEMMA 2.3. *The image of* $d\colon \mathcal{O}(X)^\dagger \to \Omega^1(X)^\dagger$ *is closed.*

Proof. Let E denote the image. We have to show that $E \cap \Omega^1(X_m)$ is closed for every m. Choose a converging sequence $\omega_i \in E \cap \Omega^1(X_m)$. Let $f_i \in \mathcal{O}(X)^\dagger$ satisfy $df_i = \omega_i$. The expansion of $f_i \circ \tau$ is $\sum_{n=-\infty}^{\infty} a_n(i) z^n$ where we have chosen $a_0(i) = 0$. It is convergent on $\rho_m < |z| < 1$ since $\omega_i \circ \tau = d(f_i \circ \tau) = \sum n a_n(i) z^{n-1}\, dz$ converges on $\rho_m \leqslant |z| < 1$.

Further $f_i \circ \tau$ is a Cauchy sequence for the supremum norm on $\{z \in K \mid |z| = \rho_{m+1}\}$. Indeed, according to (2.2)

$$\begin{aligned} &\|f_i \circ \tau - f_{i+1} \circ \tau\|_{|z| = \rho_{m+1}} \\ &\quad = \max_{n<0} |a_n(i) - a_n(i+1)| \rho_{m+1}^n \\ &\quad = \max_{n<0} (|n a_n(i) - n a_n(i+1)| \rho_m^{n-1} (|n|^{-1} \rho_m^{1-n} \rho_{m+1}^n)); \end{aligned}$$

let the constant c satisfy $c \geqslant |n|^{-1} \rho_m^{1-n} \rho_{m+1}^n$ for all $n < 0$ then one finds

$$\|f_i \circ \tau - f_{i+1} \circ \tau\|_{|z| = \rho_{m+1}} \leqslant c \|\omega_i - \omega_{i+1}\|_m.$$

So $\{f_i\}$ is according to (2.2) a Cauchy sequence in $\mathcal{O}(X_{m+1})$. Then $f_\infty = \lim f_i \in \mathcal{O}(X_{m+1})$ satisfies $d(f_\infty) = \lim \omega_i$. Hence $\lim \omega_i \in E \cap \Omega^1(X_m)$.

We continue now the proof of (2.1). Let $\mathcal{O}_a(\hat{X} - \tau(0))$ and $\Omega^1_a(\hat{X} - \tau(0))$ denote the meromorphic (or rational) functions and differential forms on $\hat{X}$ with only a pole in $\tau(0)$.

The differentiation $d_1: \mathcal{O}_a(\hat{X} - \tau(0)) \to \Omega^1_a(\hat{X} - \tau(0))$ has a cokernel H of dimension $2g$ as one easily computes with the help of Riemann–Roch. This yields an injective map $H \to H^1_{DR}(X) = \text{coker}(d: \mathcal{O}(X)^\dagger \to \Omega^1(X)^\dagger)$.

The vectorspace $H^1_{DR}(X)$ provided with the topology induced by $\Omega^1(X)^\dagger$ is a locally convex Hausdorff space. It induces on H the usual topology since $\dim H < \infty$ and the topology is Hausdorff. So H is complete as a subspace of $H^1_{DR}(X)$. Since $\Omega^1_a(\hat{X} - \tau(0))$ is dense in $\Omega^1(X)^\dagger$ one finds that H is dense in $H^1_{DR}(X)$. This implies $H = H^1_{DR}(X)$ and it proves the case $n = 1$.

The exact and commutative diagram

$$\begin{array}{ccccccccc} 0 \to \mathcal{O}(\hat{X}) & \to & \mathcal{O}(\hat{X} - B_1)^\dagger \oplus \mathcal{O}(\hat{X} - B_2)^\dagger & \to & \mathcal{O}(\hat{X} - B_1 \cup B_2)^\dagger & \to & H^1(\hat{X}, \mathcal{O}) \to 0 \\ \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 \to \Omega^1(\hat{X}) & \to & \Omega^1(\hat{X} - B_1)^\dagger \oplus \Omega^1(\hat{X} - B_2)^\dagger & \to & \Omega^1(\hat{X} - B_1 \cup B_2)^\dagger & \to & H^1(\hat{X}, \Omega^1) \to 0 \end{array}$$

and $H^1(\hat{X}, \Omega^1) \cong K; H^1(\hat{X}, \mathcal{O}) \cong K^g$ implies the case $n = 2$ of the theorem. By induction, with a similar proof of the induction step, one obtains the general statement.

EXAMPLE 2.4. Let X be the affinoid subspace of $\mathbb{P}^1$ of the form $X = \{z \in K \mid |z| \leqslant 1\} - B_1 \cup B_2 \ldots B_n$, where the B_i's are disjoint open discs of radii $|\pi_i|$ and with centers a_i.

Every element f of $\mathcal{O}(X)$ has a unique expression

$$f = \sum_{m \geqslant 0} a_m(0) z^m + \sum_{i=1}^{n} \sum_{m>0} a_m(i) \left(\frac{\pi_i}{z - a_i} \right)^m$$

in which each $\sum_{m \geqslant 0} a_m(i) T^m (i = 0, \ldots, n)$ is a power series with $\lim a_m(i) = 0$. The element f is overconvergent if and only if each $\sum a_m(i) T^m$ is overconvergent. An easy calculation shows that the images of the differential forms $(\pi_i/z - a_i)\, dz (i - 1, \ldots, n)$ form a basis of $H^1_{DR}(X)$.

EXAMPLE 2.5. Let $\mathcal{L}$ be a compact subset of $\mathbb{P}^1$ not containing ∞ and let X denote the open subspace $\mathbb{P}^1 - \mathcal{L}$ of $\mathbb{P}^1$. The kernel of $d: \mathcal{O}(X) \to \Omega^1(X)$ is of course K. The cokernel of d can be identified with the finite additive K-valued measures μ on $\mathcal{L}$ with total measure $\mu(\mathcal{L}) = 0$. Equivalently one can describe the cokernel of d as the K-vectorspace of K-valued currents on the tree of the reduction of X. Let $\omega \in \Omega^1(X)$. The measure μ corresponding to ω can be described as follows. Let $U \subset \mathcal{L}$ be a compact open subset. There exists a connected affinoid $Y \subset X$ containing ∞, such that $\mathbb{P}^1 - Y = B_1 \cup \cdots \cup B_n$, the

$B_1, \ldots, B_n$ are open discs and the corresponding closed discs are still disjoint. Further it can be arranged such that $(B_1 \cup \cdots \cup B_s) \cap \mathscr{L} = U$. Then

$$\mu(U) := \sum_{i=1}^{s} \text{res}_{\partial B_i}(\omega).$$

The Example 2.4 shows that $\mu = 0$ is equivalent to ω is exact. On the other hand, a construction analogous to [6] I.8.9. shows that every such measure μ is the image of a differential form ω.

REMARK 2.6. The condition "K algebraically closed" in Theorem 2.1 is superfluous. In general for a finite extension L of the field K one sees that $H^i_{DR}(X) \otimes_K L \cong H^i_{DR}(X \otimes_K L)$. If X is absolutely non-singular and connected of dimension 1 then there exists a finite extension L of K such that $X \otimes_K L$ can be embedded in a complete, non-singular curve Y over L such that $Y - X \otimes_K L$ is the disjoint union of n subspaces isomorphic to $\{z \in L \mid |z| < 1\}$. The proof of (2.1) yields $H^1_{DR}(X \otimes_K L) = L^{2g+(n-1)}$ and this determines $H^1_{DR}(X)$.

COROLLARY 2.7. *Let* $X, \hat{X}, B_1, \ldots, B_n$ *be as in* (2.1). *Choose* $a_i \in B_i$ *for* $i = 1, \ldots, n$. *Then the natural maps of the algebraic De Rham cohomology groups* $H^i_{DR}(\hat{X} - \{a_1, \ldots, a_n\})$ *into the analytic De Rham cohomology groups* $H^i_{DR}(X)$ *are isomorphisms.*

Proof. For $i = 0$, this is obvious. For $i = 1$, both spaces have dimension $2g + (n - 1)$ and one has to show that the map is injective. Let ω be an algebraic differential form on $\hat{X} - \{a_1, \ldots, a_n\}$ and suppose that $\omega = df$ for some $f \in \mathcal{O}(X)^\dagger$. There are open discs $B'_i \subsetneqq B_i (i = 1, \ldots n)$ such that f is holomorphic on $X' = \hat{X} - (B'_1 \cup \cdots \cup B'_n)$ and $\omega = df$ holds on X'. Using isomorphisms $\tau_i\colon B_i \xrightarrow{\sim} \{z \in K \mid |z| < 1\}$ such that $\tau_i(a_i) = 0$ and $\tau_i(B'_i) = \{z \in K \mid |z| < \rho_i\}$ for some $\rho_i < 1$ we find that $\omega|_{B_i}$ has the form $\sum_{n \gg -\infty} a_{n,i} \tau_i^n d_i$. The terms $a_{-1,i}$ are zero since $\omega = df$ holds on $\rho_i < |\tau_i| < 1$ and $f = \text{constant} + \sum_{n \neq -1} (a_{n,i}/n + 1)\tau_i^{n+1}$ extends to a meromorphic function on B_i with possibly a pole at a_i. So f is a rational function on $\hat{X}$ with poles $\subseteq \{a_1, \ldots, a_n\}$. This shows that the map between the H^1_{DR}-groups is injective.

2.8. A generalization of theorem 2.1.

For certain families $\rho\colon X \to S$ of one-dimensional affinoid spaces we will generalize (2.1). Here X and S are connected affinoid spaces, ρ is smooth, the fibres of ρ have dimension one and ρ has an overconvergent presentation. The last statement means that $\mathcal{O}(X)$ can be written in the form $\mathcal{O}(S)\langle T_1, \ldots, T_n\rangle / (f_1, \ldots, f_m)$ where each f_i is overconvergent w.r.t. $T_1, \ldots, T_n$. The problem is to determine $\ker(d)$ and $\text{coker}(d) = H^1_{DR}(X/S)$ for

$$d\colon \mathcal{O}(X)^\dagger \to \Omega^\dagger_{X/S}.$$

We suppose that X/S is obtained from a curve $\tilde{X}/S$ by deleting open, disjoint discs $B_1, \ldots, B_n$. This means the following:

$\tilde{X}/S$ is a connected and smooth curve of genus g. The open disc B_i is the image of an open immersion $u_i\colon S \times \{t \in K \mid |t| < 1\} \to \tilde{X}$ such that $\rho \circ u_i$ is the projection onto S. Further $X = \tilde{X} - B_i \cup \cdots \cup B_n$. The embedding $X \subset \tilde{X}$ induces an obvious overconvergent presentation for $X \to S$. One easily verifies that the proof of (2.1) extends to the new situation. One finds as result: $\ker(d) = \mathcal{O}(S)$ and $\operatorname{coker}(d)$ is a projective $\mathcal{O}(S)$-module of rank $2g + (n - 1)$.

EXAMPLES 2.9.

(1) $S = \operatorname{Spm}(K\langle \lambda, (1/\lambda(1-\lambda))\rangle)$ and

$$X = \operatorname{Spm}(\mathcal{O}(S)\langle X, Y\rangle/(Y^2 - X(X-1)(X-\lambda))).$$

Then $H^1_{DR}(X/S)$ is a free $\mathcal{O}(S)$-module of rank 2

(2) S as above and X the affinoid space with algebra:

$$\mathcal{O}(S)\langle X, (1/X(X-1)(X-\lambda)), Y\rangle/(Y^N - X^A(X-1)^B(X-\lambda)^C), \text{ where}$$

$$(N, A, B, C) = 1 \text{ and } 0 < A, B, C < N.$$

Then $H^1_{DR}(X/S)$ is a free $\mathcal{O}(S)$-module of rank $2N + 1$.

In the examples above the Gauss–Manin connection can be defined on $H^1_{DR}(X/S)$. This differential equation is a direct sum of hypergeometric equations.

3. The complement of a hypersurface

In this section we do some calculations on the de Rham cohomology of the affinoid space $X = \operatorname{Sp}(A)$ in which A has the form $A = K\langle X_1, \ldots, X_n, t^{-1}\rangle$ and $t \in K\langle X_1, \ldots, X_n\rangle$ is an element with norm 1. The algebra A depends only on the zero-set of the residue class $\bar{t} \in \bar{K}[X_1, \ldots, X_n]$ of t. So we may suppose that t and $\bar{t}$ are polynomials of degree d and that $\bar{t}$ has no multiple factors. Further $X = \operatorname{Sp}(A)$ is an affinoid subset of $\{X \in K^n \mid t(X) \neq 0\}$ = the complement of the hypersurface.

In the special case where $\bar{t} = 0$ is a non-singular variety in $\bar{K}^n$ one can apply the Gysin exact sequence of [11] II p. 231.

$$\to H^{i-2}_{DR}(B) \to H^i_{DR}(A') \to H^i_{DR}(A) \to H^{i-1}_{DR}(B) \to \cdots$$

with $A' = K\langle X_1, \ldots, X_n\rangle$; $A = K\langle X_1, \ldots, X_n, t^{-1}\rangle$; $B = K\langle X_1, \ldots, X_n\rangle/(t)$.

Indeed, $\bar{A}'$ and $\bar{A}'/(\bar{t})$ are non-singular complete intersections. This gives a reduction in the dimenson for the calculation of the H^i_{DR}. In the special case $n = 2$ one finds the following:

Let $(\bar{t} = 0) \subset \bar{K}^2$ have s components $Y_1, \ldots, Y_s$. Let g_i denote the genus of $\hat{Y}_i$ and $n_i = \#(\hat{Y}_i - Y_i)$. Then:

$$\dim H^i_{DR}(K\langle X_1, X_2, t^{-1}\rangle) = \begin{cases} 1 & \text{for } i = 0 \\ s & \text{for } i = 1 \\ \sum_{i=1}^{s} (2g_i + (n_i - 1)) & \text{for } i = 2 \end{cases}$$

This follows from the Gysin sequence and Theorem (2.1).

In this section we try to give calculations for the dimensions if "$\bar{t} = 0$" and even "$t = 0$" have singularities. In order to do this we give a detailed description of the residue map.

A general linear transformation of the coordinates (for convenience we suppose that $\bar{K}$ is infinite) brings $\bar{t}$ in the form: $\bar{t}$ is a monic polynomial in X_n of degree d and the gcd. of $\bar{t}$ and $\partial\bar{t}/\partial X_n$ is 1. Lifting $\bar{t}$ to t we may suppose that t is a polynomial of total degree d; t is monic in X_n of degree d and the discriminant $\Delta \in K[X_1, \ldots, X_{n-1}]$ of t wrt. X_n has norm 1 as an element of $K\langle X_1, \ldots, X_{n-1}\rangle$.

We want to define a residue map $\mathrm{Res}\colon A^\dagger \to B^\dagger$, where B denotes $K\langle X_1, \ldots, X_n, \Delta^{-1}\rangle/(t)$, which generalizes the usual residues of meromorphic differential forms in one variable. First we make a formal computation.

LEMMA 3.1. *Let $t_0, \ldots, t_{d-1}$ denote indeterminates; let $t \in \mathbb{Z}[t_0, \ldots, t_{d-1}][X]$ denote the polynomial $X^d + t_{d-1}X^{d-1} + \cdots + t_0$; let Δ denote the discriminant of t and let $[n]$ denote the least common multiple of $1, 2, \ldots, n$.*

Then every rational expression $(\sum_{i<md} a_i X^i)t^{-m}$ with $a_i \in \mathbb{Z}[t_0, \ldots, t_{d-1}]$ can uniquely be written as

$$\left(\sum_{i=0}^{d-1} b_i X^i\right)t^{-1} + \frac{d}{\mathrm{d}x}\left(\left(\sum_{i<(m-1)d} c_i X^i\right)t^{-m+1}\right).$$

The coefficients satisfy $\Delta^{dm} b_i$ and $[m-1]\Delta^{dm} c_j$ belong to $\mathbb{Z}[t_0, \ldots, t_{d-1}]$.

Proof. Introduce indeterminates $\lambda_1, \ldots, \lambda_d$ such that $t = (X - \lambda_1)\ldots(X - \lambda_d)$ and $\mathbb{Z}[t_0, \ldots, t_{d-1}]$ is seen as a subring of $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$. The given expression can uniquely be written in the form

$$\sum_{i=1}^{d} \sum_{n=1}^{m} \frac{c(i, n)}{(X - \lambda_i)^n}$$

where $\Delta^{dm}c(i,n)$ belong to $\mathbb{Z}[\lambda_1,\dots,\lambda_d]$. Indeed Δ belongs to the ideal $(t/(X-\lambda_1),\dots,t/(X-\lambda_d))$ of $\mathbb{Z}[\lambda_1,\dots,\lambda_d]$ and so

$$\Delta^{dm}\in\left(\frac{t}{X-\lambda_1},\dots,\frac{t}{X-\lambda_d}\right)^{dm}\subseteq\left(\left(\frac{t}{X-\lambda_1}\right)^{m},\dots,\left(\frac{t}{X-\lambda_d}\right)^{m}\right).$$

This implies that t^{-m} has the required form and the same holds for Pt^{-m} where P is a polynomial of degree $<md$. There is a unique decomposition as

$$\sum_{i=1}^{d}\frac{c(i,1)}{X-\lambda_i}+\frac{d}{\mathrm{d}x}\left(\sum_{i=1}^{d}\sum_{n=2}^{m}\frac{c(i,n)}{(1-n)(X-\lambda_i)^{n-1}}\right).$$

Rewriting this in the form

$$\left(\sum_{i=0}^{d-1}b_iX^i\right)t^{-1}+\frac{d}{\mathrm{d}x}\left(\left(\sum_{i<(m-1)d}c_iX^i\right)t^{-m+1}\right)$$

one finds that $\Delta^{dm}b_i$ and $[m-1]\Delta^{dm}c_j$ belong to $\mathbb{Z}[\lambda_1,\dots,\lambda_d]$. Those elements are invariant under the permutations of $\{\lambda_1,\dots,\lambda_d\}$ and so $\Delta^{md}b_i,\Delta^{md}[m-1]c_j\in\mathbb{Z}[t_0,\dots,t_{d-1}]$.

DEFINITION OF RES 3.2.
Res: $K[X_1,\dots,X_n,t^{-1}]\to K[X_1,\dots,X_n,\Delta^{-1}]/(t)$ is given by using (3.1) with $\mathbb{Z}[t_0,\dots,t_{d-1}]$ replaced by $K[X_1,\dots,X_{n-1}]$ and X by X_n and

$$\mathrm{Res}\left(\left(\sum_{i<md}a_iX_n^i\right)t^{-m}\right)=\sum_{i=0}^{d-1}b_iX_n^i\ \mathrm{modulo}(t).$$

In order to extend this to: $K\langle X_1,\dots,X_n,t^{-1}\rangle^{\dagger}=A^{\dagger}\to K\langle X_1,\dots,X_n,\Delta^{-1}\rangle^{\dagger}/(t)=B^{\dagger}$, we introduce some norms.

For any $\lambda>1$, $\|\ \|_\lambda$ on $K[X_1,\dots,X_n,t^{-1}]$ is the supremumnorm on the set

$$\{(x_1,\dots,x_n)\in K^n\,|\,|x_1|\leqslant\lambda,\dots,|x_n|\leqslant\lambda,|t^{-1}|\leqslant\lambda\}.$$

(For notational convenience we assume K algebraically closed). For any $\rho>1$ we define $\|\ \|_\rho$ on $K[X_1,\dots,X_n,\Delta^{-1}]/(t)$ as the norm induced by the supremumnorm on $K[X_1,\dots,X_n,\Delta^{-1}]$ with respect to the set

$$\{(X_1,\dots,X_n)\in K^n\,|\,|X_1|\leqslant\rho,\dots,|X_n|\leqslant\rho,|\Delta^{-1}|\leqslant\rho\}.$$

Similar for $K[X_1,\dots,X_n,\Delta^{-1},t^{-1}]$.

We apply 3.1 to $X_n^i t^{-m}(0 \leqslant i < d$ and $m \geqslant 1)$. One calculates then that there exists a constant c such that $\Delta^{md} b_i$ and $[m-1]\Delta^{md} c_j$ are polynomials in $\mathbb{Z}[t_0, \dots, t_d]$ of total degree $\leqslant m.c$. It follows that $\| b_i \|_\rho \leqslant \rho^{mc'}$ and

$$\left\| \left(\sum_{i<(m-1)d} c_i X^i \right) t^{-m+1} \right\|_\rho \leqslant m\rho^{mc''}$$

for some constants $c', c'' > 0$ since $|[m-1]|^{-1} < m$. Let $L\colon K[X_1, \dots, X_n, t^{-1}] \to K[X_1, \dots, X_n, t^{-1}, \Delta^{-1}]$ be the $K[X_1, \dots, X_{n-1}]$-linear map given by the formula:

$$a = \left(\sum_{i=0}^{d-1} b_i X_n^i \right) t^{-1} + \frac{\partial}{\partial X_n}(L(a)) \quad \text{and} \quad \operatorname{Res}(a) = \sum_{i=0}^{d-1} b_i X_n^i \operatorname{mod}(t).$$

If $\rho > 1$ is chosen small enough with respect to $\lambda > 1$ then one calculates from the estimates above that $\| \operatorname{Res}(a) \|_\rho \leqslant C \| a \|_\lambda$ and $\| L(a) \|_\rho \leqslant C \| a \|_\lambda$ for some constant $C > 0$. So Res and L can be extended by continuity to maps on the completions with respect to $\| \ \|_\rho$ and $\| \ \|_\lambda$. Taking the direct limit over all $\lambda > 1$ and $\rho = \rho(\lambda) > 1$ one finds (continuous) maps

$$\operatorname{Res}\colon K\langle X_1, \dots, X_n, t^{-1} \rangle^\dagger \to (K\langle X_1, \dots, X_n, \Delta^{-1} \rangle / (t))^\dagger$$
$$L\colon K\langle X_1, \dots, X_n, t^{-1} \rangle^\dagger \to K\langle X_1, \dots, X_n, t^{-1}, \Delta^{-1} \rangle^\dagger.$$

The domain of definition of Res and L can also be extended to $K\langle X_1, \dots, X_n, \Delta^{-1}, t^{-1} \rangle^\dagger$. We note that for any $a \in K\langle X_1, \dots, X_n, \Delta^{-1}, t^{-1} \rangle^\dagger$ one has again

$$a = \left(\sum_{i=0}^{d-1} b_i X_n^i) t^{-1} + \frac{\partial}{\partial X_n}(L(a) \right) \quad \text{where} \quad \operatorname{Res}(a) = \sum_{i=0}^{d-1} b_i X_n^i \operatorname{mod}(t). \qquad (*)$$

PROPOSITION (3.3). *The following sequences are exact.*

$$0 \to K\langle X_1, \dots, X_{n-1} \rangle^\dagger \to K\langle X_1, \dots, X_n, t^{-1} \rangle^\dagger \xrightarrow{\partial/\partial X_n}$$
$$K\langle X_1, \dots, X_n, t^{-1} \rangle^\dagger \xrightarrow{\operatorname{Res}} K\langle X_1, \dots, X_n^{-1} \rangle^\dagger/(t),$$
$$0 \to K\langle X_1, \dots, X_{n-1}, \Delta^{-1} \rangle^\dagger \to K\langle X_1, \dots, X_n, t^{-1}, \Delta^{-1} \rangle^\dagger \xrightarrow{\partial/\partial X_n}$$
$$K\langle X_1, \dots, X_n, t^{-1}, \Delta^{-1} \rangle^\dagger \xrightarrow{\operatorname{Res}} K\langle X_1, \dots, X_n, \Delta^{-1} \rangle^\dagger/(t) \to 0.$$

Proof. The exactness of the second sequence follows easily from $(*)$, with the exception of the calculation of kernel $\partial/\partial X_n$. Write $b \in K\langle X_1, \dots, X_n, t^{-1}, \Delta^{-1} \rangle^\dagger$

in the form

$$b = b_0 + \sum_{i=0}^{d-1} \sum_{m \geqslant 1} b(i,m) X_n^i t^{-m}$$

with $b_0 \in K\langle X_1, \ldots, X_n, \Delta^{-1}\rangle^\dagger$ and all $b(i,m) \in K\langle X_1, \ldots, X_{n-1}, \Delta^{-1}\rangle^\dagger$. Put $X_n^i(\partial/\partial X_n)t = A_i t + B_i$ with A_i, B_i polynomials of degree $<d$ wrt. X_n. Then $(\partial/\partial X_n)(b)$ has the form

$$\frac{\partial}{\partial X_n}(b_0) + \sum_{m \geqslant 1} t^{-m}\left(\sum_{i=1}^{d-1} b(i,m)(iX_n^{i-1} - mA_i) + \sum_{i=0}^{d-1} (1-m) b(i, m-1) B_i\right).$$

We note that $A_i (i \neq 0)$ has the form $\mathrm{d}X_n^{i-1}$ + lower degree and that $B_0 = \mathrm{d}X_n^{d-1}$ + lower degree terms.

If $(\partial/\partial X_n)(b) = 0$ then $(\partial/\partial X_n)(b_0) = 0$ and every coefficient of t^{-m} is zero. Hence $b_0 \in K\langle X_1, \ldots, X_{n-1}, \Delta^{-1}\rangle^\dagger$. For $m = 1$ one finds $b(i,1) = 0$ for $i = 1, \ldots, d-1$. For $m = 2$ one finds $b(i,2) = 0$ for $i = 1, \ldots, d-1$ and $b(0,1) = 0$ etc. So all $b(i,m) = 0$ and $b = b_0$ has the required form. A similar argument yields: if $(\partial/\partial X_n)(b)$ lies in $K\langle X_1, \ldots, X_n, t^{-1}\rangle^\dagger$ and $b_0 = 0$ then $b \in K\langle X_1, \ldots, X_n, t^{-1}\rangle^\dagger$.

This shows that in the first exact sequence one has ker(Res) = im$(\partial/\partial X_n)$. The remaining verifications are easy.

NOTATIONS AND DEFINITIONS 3.4. $A = K\langle X_1, \ldots, X_n, t^{-1}\rangle^\dagger$; $B = K\langle X_1, \ldots, X_n, \Delta^{-1}\rangle^\dagger/(t)$; $C = K\langle X_1, \ldots, X_{n-1}\rangle^\dagger$ and $A' = A\langle \Delta^{-1}\rangle^\dagger$; $C' = C\langle \Delta^{-1}\rangle^\dagger$. For every $p \geqslant 0$ one defines a residue map $\mathrm{Res}_p\colon \Omega^p(A') \to B \otimes_C \Omega^{p-1}(C)$ by the formule

$$\mathrm{Res}_p\left(\sum a_\alpha \,\mathrm{d}X_{\alpha_1} \wedge \cdots \wedge \mathrm{d}X_{\alpha_p}\right) = \sum_{\alpha_1 < \cdots < \alpha_p = n} \mathrm{Res}(a_\alpha) \mathrm{d}X_{\alpha_1} \wedge \cdots \wedge \mathrm{d}X_{\alpha_{p-1}}.$$

Put $M = \mathrm{Res}(A)$, this is a C-submodule of B and for each p one has $\mathrm{Res}(\Omega^p(A)) = M \otimes \Omega^{p-1}(C)$.

We note that $B \otimes_C \Omega^{p-1}(C)$ equals $B \otimes_{C'} \Omega^{p-1}(C')$. Define $\nabla\colon B \to B \otimes \Omega^1(C')$ such that $\nabla \circ \mathrm{Res}_1 = \mathrm{Res}_2 \circ d^1$. One easily verifies that ∇ exists and is unique, and that ∇ is a connection. Using ∇ one defines maps $\nabla^q\colon B \otimes \Omega^q(C') \to B \otimes \Omega^{q+1}(C')$ by

$$\nabla^q\left(\sum b_\alpha \,\mathrm{d}X_{\alpha_1} \wedge \cdots \wedge \mathrm{d}X_{\alpha_q}\right) = \sum \nabla(b_\alpha) \wedge \mathrm{d}X_{\alpha_1} \wedge \cdots \wedge \mathrm{d}X_{\alpha_q}.$$

A straight-forward verification shows that: $\nabla^{q-1} \circ \mathrm{Res}_q = \mathrm{Res}_{q+1} \circ d^q$ for all $q \geqslant 1$. In particular it follows that ∇ is an integrable connection and that

$\{B \otimes \Omega^q(C'), \nabla^q\}$ is the de Rham-complex associated to ∇. Also $\{M \otimes \Omega^q(C), \nabla^q\}$ is the De Rham-complex associated to $\nabla: M \to M \otimes \Omega^1(C)$.

COROLLARIES. 3.5.

(i) *The canonical morphisms* $\Omega^\cdot(C) \to \ker(\Omega^\cdot(A) \overset{\alpha}{\twoheadrightarrow} M \otimes \Omega^\cdot(C))$ *and* $\Omega^\cdot(C') \to \ker(\Omega^\cdot(A') \overset{\alpha'}{\twoheadrightarrow} B \otimes \Omega^\cdot(C'))$ *are quasi-isomorphisms.*
(ii) $H^i_{DR}(A) \cong H^{i-1}(M \otimes \Omega^\cdot(C))$.
(iii) $\dim H^1_{DR}(A) \leqslant d$.
(iv) *The complex* $\{B \otimes \Omega^\cdot(C'), \nabla^\cdot\}$ *is quasi-isomorphic to* $\{\Omega^\cdot(B), d^\cdot\}$.
(v) *For* $n = 2$ *the dimensions of* $H^i_{DR}(A)$ *are finite.*

Proof. (i) Let $\omega \in \Omega^p(A)$ have $\alpha(\omega) = \mathrm{Res}(\omega) = 0$ and $d(\omega) = 0$. Then ω has the form

$$\sum \frac{\partial}{\partial x_n}(b_\alpha)\, dX_{\alpha_1} \wedge \cdots \wedge dX_{\alpha_{p-1}} \wedge dX_n + \sum_{\beta p < n} a_\beta\, dX_{\beta_1} \wedge \cdots \wedge dX_{\beta_p}$$

with all $b_\alpha \in A$ (follows from (3.3))

With $\eta = (-1)^p \sum b_\alpha\, dX_{\alpha_1} \wedge \cdots \wedge dX_{\alpha_{p-1}}$ one has

$$\omega - d\eta = \sum_{\alpha_p < n} c_\alpha\, dX_{\alpha_1} \wedge \cdots \wedge dX_{\alpha_p}.$$

Each $\partial c_\alpha / \partial x_n = 0$ and according to (3.3) each $c_\alpha \in K\langle X_1, \ldots, X_{n-1}\rangle^\dagger$.

So $\omega = d\eta^+$ an element of $\Omega^p(C)$. The same argument proves the second statement.

(ii) is obvious from (i).

(iii) $H^1_{DR}(A) \cong H^0(M \otimes \Omega^\cdot(C)) = \ker \nabla \subseteq \ker(\nabla: B \to B \otimes \Omega'(C'))$. The last ∇ is a differential equation of order d over C' and the vectorspace of solutions has dimension $\leqslant d$.

(iv) From (i) it follows that $D = \Omega^\cdot(A')/\Omega^\cdot(K\langle X_1, \ldots, X_n, \Delta^{-1}\rangle^\dagger) \to B \otimes \Omega^\cdot(C')$ is a quasi-isomorphism. The well known morphism $\Omega^\cdot(B) \to D$ given by $\omega \mapsto \tilde{\omega} \wedge dt/t$ where $\tilde{\omega} \in \Omega^\cdot(A')$ is a lift of ω, is also a quasi-isomorphism. This follows of course from [11] II. But in this case it follows easily from (3.3). Indeed $\omega \in \Omega^p(A)$ can be written as

$$d(\eta) + \sum_{\alpha_p < n} a_\alpha\, dX_{\alpha_1} \wedge \cdots \wedge dX_{\alpha_p} + \sum_{\beta_{p-1} < n} b_\beta\, dX_{\beta_1} \wedge \cdots \wedge dX_{\beta_{p-1}} \wedge \frac{dt}{t}$$

in which $b_\beta \in K\langle X_1, \ldots, X_{n-1}, \Delta^{-1}\rangle^\dagger[X_n]$ has degree $< d$ in X_n. If $d(\omega) = 0$ then all $\partial(a_\alpha)/\partial X_n = 0$ and the a_α lie in $K\langle X_1, \ldots, X_{n-1}, \Delta^{-1}\rangle^\dagger$. This shows that the map $H^{p-1}(\Omega^\cdot(B)) \to H^p(D)$ is surjective. The injectivity follows also from (3.3).

(v) Using (i) and (iv) one finds the Gysin-exact sequence for the cohomologies of the rings $K\langle X_1, \Delta^{-1}\rangle^\dagger, K\langle X_1, X_2, \Delta^{-1}, t^{-1}\rangle^\dagger$, $K\langle X_1, X_2, \Delta^{-1}\rangle^\dagger/(t)$. The first and the last ring have dimension 1 and hence their DR-cohomology is finite dimensional. So the DR-cohomology of $K\langle X_1, X_2, \Delta^{-1}, t^{-1}\rangle^\dagger$ is finite dimensional. The next step is to construct a Gysin sequence for the rings $K\langle X_1, X_2, t^{-1}\rangle^\dagger, K\langle X_1, X_2, t^{-1}, \Delta^{-1}\rangle^\dagger$ and $K\langle X_1, X_2, t^{-1}\rangle^\dagger/(\Delta)$. From this (v) follows.

Of course we can replace Δ by an element δ of the form $(X_1 - \lambda_1)\dots(X_1 - \lambda_s)$ where $\bar{\lambda}_1, \dots, \bar{\lambda}_s \in \bar{K}$ are distinct. Applying the method of (3.1), (3.2) and (3.3) to δ and X_1 (the discriminant is 1 in this case) one obtains the required exact sequence.

3.6. In a rather special case we can calculate M = the image of Res. It is the case where t has the form $X_n^d - a$ with $|d| = 1$ and $a \in K\langle X_1, \dots, X_{n-1}\rangle^\dagger$ with norm 1. The discriminant Δ equals d.a. An easy calculation shows that $\mathrm{Res}_1(X_n^{d-1} t^{-m}\, \mathrm{d}X_n) = 0$ for $m > 1$ and that

$$\mathrm{Res}_1(X_n^i t^{-m}\, \mathrm{d}X_n) = (-1)^{m-1} \binom{m - 1 - \frac{i+1}{d}}{m-1} a^{1-m} X_n^i$$

for $m \geqslant 1$ and $i \leqslant d - 2$. It follows that

$$M = \sum_{i=0}^{d-2} K\langle X_1, \dots, X_{n-1}, a^{-1}\rangle^\dagger X_n^i + K\langle X_1, \dots, X_{n-1}\rangle^\dagger X_n^{d-1} \,\mathrm{mod}(t).$$

We continue this example for the case $n = 2$; write X, Y for the two variables and $t = Y^d - a$. After identifying M and MdX, the operator $\nabla\colon M \to M$ has the form

$$\nabla(m_0, \dots, m_{d-1}) = (m_0', \dots, m_{d-1}') +$$

$$+ \left(\frac{1-d}{d}\frac{a'}{a} m_0, \frac{2-d}{d}\frac{a'}{a} m_1, \dots, \frac{-1}{d}\frac{a'}{a} m_{d-2}, 0\right)$$

where $m_0, \dots, m_{d-2} \in K\langle X, a^{-1}\rangle^\dagger$ and $m_{d-1} \in K\langle X\rangle^\dagger$.

For this operator ∇ we have to calculate ker and coker. We note that a and $b \in K\langle X\rangle^\dagger$ will give the same answer if $\bar{a} = \bar{b}$. This means that a can be supposed to have the form $\lambda(X - \lambda_1)^{n_1}\dots(X - \lambda_s)^{n_s}$ with $|\lambda| = 1, |\lambda_i| \leqslant 1$ and $|\lambda_i - \lambda_j| = 1$ for $i \neq j$.

LEMMA 3.6.1. *Let a be as above. The differential operator L on* $K\langle X, a^{-1}\rangle^\dagger$

given by $L(m) = m' - (i/d)(a'/a)m$ *and* $0 < i < d$ *satisfies*:

(i) $\dim(\ker L) = 1$ *if* d *divides all* $in_1, \ldots, in_s$. *Otherwise* $\ker L = 0$
(ii) $\dim(\ker L) - \dim(\operatorname{coker} L) = -s + 1$.

Proof (i) is rather obvious.

(ii) If $\ker L \neq 0$ then $m \mapsto b^{-1}L(bm)$, where $b \neq 0$ satisfies $L(b) = 0$, is the ordinary differentiation on $K\langle X, a^{-1}\rangle^\dagger$ and we have already shown (ii) in that case. If $\ker L = 0$ then one can show that the image of L is closed in $K\langle X, a^{-1}\rangle^\dagger$. The cokernel of $L: K[X, a^{-1}] \to K[X, a^{-1}]$ has dimension $s - 1$ and is represented by a basis $1/X - \lambda_1, \ldots, 1/X - \lambda_{s-1}$. The cokernel of L on $K\langle X, a^{-1}\rangle^\dagger$ has the same dimension.

COROLLARY (3.6.2.). *The de Rham cohomology groups of* $K\langle X, Y, t^{-1}\rangle^\dagger$ *with* $t = Y^d - \lambda(X - \lambda_1)^{n_1} \ldots (X - \lambda_s)^{n_s}$ *have the following dimensions*:

$\dim H^0_{DR} = 1$, $\dim H'_{DR} = \gcd(d, n_1, \ldots, n_s)$ *and* $\dim H^2_{DR}$ *equals* $1 + (d-1)(s-1) - \gcd(d, n_1, \ldots, n_s)$.

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