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The irreducible unitary GL (n − 1, ℝ)-spherical representations of SL(n, ℝ)

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0. Introduction

In 1969 Kostant published a fundamental paper [Ko] on the existence and irreducibility of K-spherical representations of a connected non-compact semisimple Lie group G with a maximal compact subgroup K. G is assumed to have finite centre. In general it is still an open question which spherical representations are unitary. For split-rank one groups however the unitarizability was completely solved by Kostant and later, in a different context, by Flensted-Jensen and Koornwinder [F-K]. The problem can be reformulated by posing the question: which spherical functions (eventually bounded spherical functions) are positive-definite. It is this problem which is solved by Flensted-Jensen and Koornwinder in the split-rank one case. Recently Bang-Jensen [Ba] made some progress on the unitarizability in the higher rank case.

Since 1980 harmonic analysis on general pseudo-Riemannian symmetric spaces G/H has attracted the interest of a lot of people. We especially have to mention the fundamental work of Faraut, Flensted-Jensen, Molcanov, Oshima and Sekiguchi. In relation to the problem mentioned above, a lot of effort has been put into the determination of the H-spherical representations. The unitarizability problem is much more difficult here, since H may not be compact. A reformulation into H-spherical distributions is helpful, but in general a positive-definite spherical distribution need not to be extremal (in contrast to the case $H = K$). For rank one pseudo-Riemannian symmetric spaces nevertheless some progress has been made: The unitary spherical dual is

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completely determined for the *isotropic* rank one spaces by Faraut [Fa] and M.T. Kosters [MKo]. The greater part of the irreducible spherical unitary representations are induced from a suitable parabolic subgroup $P = MAN$. The lucky fact that in this case the pair $(K, M \cap K)$ is Gelfand-pair helps very much in solving the unitarizability problem.

In this paper we give a complete classification of the irreducible unitary H-spherical representations of $G = SL(n, \mathbb{R})$, where $H = S(GL(1, \mathbb{R}) \times GL(n-1, \mathbb{R}))$ which is isomorphic to $GL(n-1, \mathbb{R})$. We take $n > 2$. The space $X = G/H$ is pseudo-Riemannian, not isotropic, of rank one. It can be seen as the next item in the list of rank one pseudo-Riemannian symmetric spaces. Here $K = SO(n, \mathbb{R})$ and $M \cap K = SO(n-2, \mathbb{R})$. Clearly $(K, M \cap K)$ is not a Gelfand-pair in this case.

$GL(n-1, \mathbb{R})$-spherical representations of $SL(n, \mathbb{R})$ were extensively studied by M.T. Kosters and Van Dijk [MKo-D], M.T. Kosters [MKo], W.A. Kosters, [WKo], and by Van Dijk and Poel [D-P]. A complete classification was known only for $n = 3$ and is due to Molcanov [M] and Poel [P]. The methods used for $n > 3$ in this paper are quite different of those used in the case $n = 3$. Our method is much more in the spirit of the work of Vogan [Vol]: The role of K-types is decisive at several stages.

The classification method we use works also for other rank one spaces, especially for the isotropic ones, and so one might treat all rank one spaces on an equal footing. We have decided to stick to the space $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$. We think this to be most convenient for the reader (and the authors). This is of course a question of taste.

The outline of the paper is as follows. In Sections 1 and 2 we will deduce, among other things, irreducibility results for the H-spherical principal series. These are certain series of induced representations from a parabolic subgroup of G which is closely related to the symmetric space X. In Section 3 we study which of these representations are unitarizable. The related spherical distributions are determined in Section 4. The main result is in Section 5 where the spherical dual is given.

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1. The $GL(n-1, \mathbb{R})$-spherical representations

Let $G = SL(n, \mathbb{R})$ and $\mathfrak{g}$ its real Lie algebra, the complexification of $\mathfrak{g}$ is denoted by $\mathfrak{g}_c$. Similar notation will be used for other Lie groups, Lie algebras and linear spaces. In particular the subscript c will always stand for the complexification of a $\mathbb{R}$-linear vector space.

Let $P_0 = M_0 A_0 N_0$ be the standard minimal parabolic subgroup of G, where

$$A_0 = \left\{ \begin{pmatrix} e^{t_1} & & 0 \\ & \ddots & \\ 0 & & e^{t_n} \end{pmatrix} \middle| t_j \in \mathbb{R},\ \ j = 1,\dots,n;\ \ \sum_1^n t_j = 0 \right\},$$

$$M_0 = \left\{ \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \middle| \varepsilon_j = \pm 1,\ \ j = 1,\dots,n;\ \ \sum_1^n \varepsilon_j = 1 \right\},$$

$$N_0 = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.$$

N_0 is the full group of unipotent upper triangular matrices. Let $\mathfrak{a}_0$ be the Lie algebra of A_0,

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \middle| t_j \in \mathbb{R},\ \ j = 1,\dots,n;\ \ \sum_1^n t_j = 0 \right\},$$

the roots of $\mathfrak{a}_0$ on the Lie algebra $\mathfrak{g}$ of G are given by

$$\alpha_{i,j} \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} = t_i - t_j, \quad i,j = 1,\dots,n \quad \text{and} \quad i \neq j.$$

A root $\alpha_{i,j}$ is called positive if $i < j$. Let Δ be the set of simple roots,

$$\Delta = \{\alpha_{1,2}, \alpha_{2,3}, \dots, \alpha_{n-1,n}\}.$$

For any subset $F \subset \Delta$ let $P_F = M_F A_F N_F$ be the Langlands decomposition of the parabolic subgroup P_F associated to F, then $P_0 \subset P_F, M_0 \subset M_F, A_F \subset A_0$ and $N_F \subset N_0$. For details on parabolic subgroups see Varadarajan [Val]. Taking $F = \{\alpha_{2,3}, \alpha_{3,4}, \dots, \alpha_{n-2,n-1}\}$ we get

$$A_F = \left\{ \begin{pmatrix} e^{t_1} & & & & 0 \\ & e^u & & & \\ & & \ddots & & \\ & & & e^u & \\ 0 & & & & e^{t_n} \end{pmatrix} \middle| t_1, t_n, u \in \mathbb{R};\ \ t_1 + t_n + (n-2)u = 0 \right\},$$

$$M_F = \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \varepsilon_n \end{pmatrix} \middle| \varepsilon_1 = \pm 1,\ \varepsilon_n = \pm 1;\ h \in GL(n-2, \mathbb{R});\ \varepsilon_1 \varepsilon_n \det h = 1 \right\},$$

$$N_F = \left\{ \begin{pmatrix} 1 & x_1 \dots x_{n-2} & & z \\ & 1 & & y_1 \\ & & \ddots & \vdots \\ & & 1 & y_{n-2} \\ & & & 1 \end{pmatrix} \middle| x_i, y_j, z \in \mathbb{R},\quad i,j = 1, \dots, n-2 \right\}. \quad (1.1)$$

P_F is a parabolic subgroup of parabolic rank 2.
Let

$$A = \left\{ a_t = \begin{pmatrix} e^t & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & e^{-t} \end{pmatrix} \middle| t \in \mathbb{R} \right\}, \quad \text{and}$$

$$M_0 = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \alpha\varepsilon \end{pmatrix} \middle| \alpha \in \mathbb{R},\quad h \in GL(n-2, \mathbb{R}),\quad \varepsilon = \pm 1; \alpha^2 \varepsilon \det h = 1 \right\}, \quad (1.2)$$

Then $M_F A_F = MA$. Put $N = N_F, P = P_F$ then $P = MAN$. This is of course not the Langlands decomposition of P, but this decomposition is used to define the H-spherical principal series representations of G, cf. [MKo-D]. Let χ be the character of M defined by

$$\chi(m) = \varepsilon \quad \text{if } m = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \alpha\varepsilon \end{pmatrix}. \quad (1.3)$$

(Observe that χ is trivial on $M \cap g_0 H g_0^{-1}$, where g_0 is defined in (4.1)). Put $\rho = n - 1$ and define for $s \in \mathbb{C}$

$$E_{0,s} = \{ f \in C^\infty(G) \mid f(gma_t n) = e^{-(s+\rho)t} f(g),$$
$$\text{for all } (g, m, a_t, n) \in G \times M \times A \times N \},$$
$$E_{l,s} = \{ f \in C^\infty(G) \mid f(gma_t n) = \chi(m^{-1}) e^{-(s+\rho)t} f(g),$$
$$\text{for all } (g, m, a_t, n) \in G \times M \times A \times N \}.$$

G acts on $E_{i,s} (i = 1, 2)$ by left translations. The associated representations are denoted by $\pi_{i,s}(i = 1,2$ and $s \in \mathbb{C})$ and constitute the so-called $GL(n-1,\mathbb{R})$-spherical principal series of $SL(n,\mathbb{R})$, cf. loc. cit.

REMARK. The representations $E_{i,s}$ are in [MKo-D] denoted by $E_{i,-s}$.

The first step is to study the irreducibility of these spherical representations.

2. Irreducibility

First we will study the irreducibility of $E_{0,s}$, by determining sufficient conditions under which the K-fixed vector is cyclic in $E_{0,s}$. Put $\mu_s = (s/2)\alpha_{1,n}$ and $\rho_1 = ((n-1)/2)\alpha_{1,n}$, then it is easily seen that

$$\begin{aligned} E_{0,s} &= \{f \in C^\infty(G) \mid f(gman) = \mathrm{e}^{-(\mu_s+\rho_1)(\log a)}f(g), \\ &\qquad \text{for all } (g,m,a,n) \in G \times M_F \times A_F \times N_F\}, \\ E_{1,s} &= \{f \in C^\infty(G) \mid f(gman) = \chi(m^{-1})\mathrm{e}^{-(\mu_s+\rho_1)(\log a)}f(g), \\ &\qquad \text{for all } (g,m,a,n) \in G_F \times M_F \times A_F \times N_F\}, \end{aligned}$$

where $P_F = M_F A_F N_F$ is the parabolic subgroup defined in (1.1). The above construction of $E_{i,s}$ is the standard parabolic induction procedure. In general if $\tilde{P} = \tilde{M}\tilde{A}\tilde{N}$ is a parabolic subgroup of G, δ a unitary representation of $\tilde{M}$ on a Hilbert space $\mathscr{H}$ and $\lambda \in \tilde{\mathfrak{a}}_c^*$ one can define the induced representation by

$$\begin{aligned} I(G,\tilde{P},\delta,\lambda) = \{f \in C^\infty(G,\mathscr{H}) \mid f(gman) = \delta(m^{-1})\mathrm{e}^{-(\lambda+\tilde{\rho})\log a}f(g), \\ \text{for all } (g,m,a,n) \in G \times \tilde{M} \times \tilde{A} \times \tilde{N}\}, \end{aligned} \tag{2.1}$$

where $\tilde{\rho} \in \tilde{\mathfrak{a}}_c^*$ is defined by

$$\tilde{\rho}(H) = \mathrm{Tr}(\mathrm{ad}\, H\,|_{\tilde{\mathfrak{n}}}), \mathrm{H} \in \mathfrak{a}.$$

$I(G,\tilde{P},\delta,\lambda)$ is invariant under left translations by elements of G, the corresponding representation is denoted by $\pi(G,\tilde{P},\delta,\lambda)$. When it's clear from the context the parameter $\tilde{P}$ and/or G is most of the time deleted. Moreover $I(\delta,\lambda)$ can be endowed with a pre-Hilbert space structure with norm

$$\| f \|^2 = \int_K \| f(k) \|_{\mathscr{H}}^2 \,\mathrm{d}k, \tag{2.2}$$

where $K = SO(n,\mathbb{R})$ and $\mathrm{d}k$ the normalized Haar measure on K. Furthermore the sesqui-linear form

$$(f,h) = \int_K (f(k),h(k))_{\mathscr{H}} \,\mathrm{d}k \quad f,h \in C^\infty(G,\mathscr{H}) \tag{2.3}$$

defines a non-degenerate G-invariant pairing between $I(\delta, \lambda)$ and $I(\delta, -\bar{\lambda})$. Thus this also holds for $E_{0,s}$ and $E_{1,s}$, in particular (2.3) defines a G-invariant pairing between $E_{i,s}$ and $E_{i,-\bar{s}}$, $i = 1, 2$ and $s \in \mathbb{C}$, in this case $\mathscr{H} = \mathbb{C}$, cf. Knapp [Kn], Chapter VII.

Now define for $\lambda \in \mathfrak{a}_{0c}^*$ arbitrary

$$E_0(G/P_0, \lambda) = \{f \in C^\infty(G) \mid f(gman) = \mathrm{e}^{-(\lambda+\rho_0)(\log a)} f(g),\ \text{for all } (g, m, a, n) \in G \times M_0 \times A_0 \times N_0\}.$$

Put

$$\chi_1 \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} = \varepsilon_1 \varepsilon_n,$$

χ_1 is a character if M_0, and define

$$E_1(G/P_0, \lambda) = \{f \in C^\infty(G) \mid f(gman) = \chi_1(m^{-1}) \mathrm{e}^{-(\lambda+\rho_0)(\log a)} f(g),\ \text{for all } (g, m, a, n) \in G \times M_0 \times A_0 \times N_0\}.$$

Thus $E_0(G/P_0, \lambda) = I(P_0, 1, \lambda)$ and $E_1(G/P_0, \lambda) = I(P_0, \chi_1, \lambda)$, cf. (2.1).

Define the following special elements of $\mathfrak{a}_{0c}^*$:

$$\rho_0 = \frac{1}{2} \sum_{i<j} \alpha_{i,j},$$

$$\lambda(s) = -\rho_0 + \left(\frac{s}{n-1} + 1\right) \rho_1 \quad (s \in \mathbb{C}).$$

The restriction of the Killing form $\mathscr{B}$ to $\mathfrak{a}_{0c}$ is non-degenerate and therefore we can identify $\mathfrak{a}_{0c}$ with $\mathfrak{a}_{0c}^*$ via $\mathscr{B}$. Hence $\mathscr{B}$ defines a $\mathbb{C}$-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}_{0c}^*$. If we make usual identification $\mathfrak{a}_{0c} = \mathfrak{a}_c^* = \{z \in \mathbb{C}^n \mid z_1 + z_2 + \cdots + z_n = 0\}$ then $\langle \cdot, \cdot \rangle$ is given by $\langle \lambda, \mu \rangle = \lambda_1 \mu_1 + \lambda_2 \mu_2 + \cdots + \lambda_n \mu_n$. Under this identification the $\alpha_{i,j}$'s and $\lambda(s)$ are given by

$$\begin{aligned} &\alpha_{i,j} = e_i - e_j, \\ &\lambda(s) = \left(\frac{s}{2} - \frac{n-3}{2} - \frac{n-5}{2}, \ldots, \frac{n-5}{2}, \frac{n-3}{2} - \frac{s}{2}\right), \end{aligned} \tag{2.4}$$

where $\{e_i \mid i = 1, \ldots, n\}$ is the standard basis of $\mathbb{C}^n$.

LEMMA 2.1. *Let $s\in\mathbb{C}$, then*

$$E_{i,s}\subset E_i(G/P_0,\lambda(s)),\, i=0,1.$$

The proof is left to the reader. By the duality (2.3) we deduce

COROLLARY 2.2. *Let $s\in\mathbb{C}$, then $E_{i,-s}$ is a quotient of $E_i(G/P_0,-\lambda(s)), i=0,1$.*

For $\lambda\in\mathfrak{a}_{0c}^*$ define

$$e(\lambda)^{-1}=\prod_{i<j}\Gamma\left(\frac{1}{2}\left(\frac{3}{2}+\frac{\langle\lambda,\alpha_{i,j}\rangle}{\langle\alpha_{i,j},\alpha_{i,j}\rangle}\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}+\frac{\langle\lambda,\alpha_{i,j}\rangle}{\langle\alpha_{i,j},\alpha_{i,j}\rangle}\right)\right),$$

$e(\lambda)$ is the denumerator of Harish–Chandra's c-function, cf. Helgason [H].

The Iwasawa decomposition of G is given by $G=KA_0N_0$. Define

$$1_\lambda(kan)=\mathrm{e}^{-(\lambda+\rho_0)\log a}\quad(k\in K, a\in A_0, n\in N_0),$$

then $1_\lambda\in E_0(G/P_0,\lambda)$.

THEOREM 2.3. (Helgason [H]) *1_λ is a cyclic vector in $E_0(G/P_0,\lambda)$ if and only if $e(\lambda)\neq 0$.*

LEMMA 2.4. *$e(-\lambda(-s))\neq 0$ for $s\notin\{-1,-3,\ldots\}\cup\{n-5,n-7,\ldots\}$.*

Proof. Using the identification (2.4) one easily gets

$$-4\frac{\langle\lambda(-s),\alpha_{i,j}\rangle}{\langle\alpha_{i,j},\alpha_{i,j}\rangle}=\begin{cases}2(j-i) & i\neq 1, j\neq n\\ 2s & i=1, j=n\\ s-n-1+2j & i=1, j\neq n\\ s+n+1-2i & i\neq 1, j=n\end{cases}$$

and some manupilations with gamma functions gives the result.

Since also $G=KMAN$, the function $1_s(kma_tn)=\mathrm{e}^{-(s+\rho)t}(k\in K, m\in M, t\in\mathbb{R},$ and $n\in N)$ is well-defined and is in $E_{0,s}$. Corollary 2.2 combined with the above results gives

LEMMA 2.5. *1_s is cyclic in $E_{0,s}$ as soon as $e(-\lambda(-s))\neq 0$.*

COROLLARY 2.6. *$E_{0,s}$ is (topologically) irreducible for all $s\in\mathbb{C}$ with $s\notin\mathbb{Z}$, if n is odd, and for all $s\in\mathbb{C}$ with $s\notin 2\mathbb{Z}+1$ if n is even.*

Proof. This follows easily from Lemma 2.5 and the observation that $\pi_{0,s}$ is irreducible if and only if both 1_s is cyclic in $E_{0,s}$ and 1_{-s} is cyclic in $E_{0,-s}$.

By using a more sophisticated argument which is explained in the Appendix one can also prove

LEMMA 2.7. *Let n be odd and s an even integer with* $|s| \leqslant n-3$. *Then* $E_{0,s}$ *is topologically irreducible. In particular is* 1_s *cyclic in* $E_{0,s}$ *in this case.*

The proof of this lemma is postponed to the Appendix.

Next we are going to study the irreducibility of $\pi_{1,s}$ this is done by applying the method of "translation of parameters" introduced by N. Wallach [W]. His method of translation of parameters was the first step to the translation functors of G. Zuckermann [Z], cf. J.N. Bernstein & S.I. Gelfand [B – G].

Consider the adjoint action of $G = SL(n, \mathbb{R})$ on $V = sl(n, \mathbb{C})$, $V = \mathfrak{g}_c$ the complexified Lie algebra of G. This action is irreducible since $\mathfrak{g}$ is simple. Put

$$v_0 = \begin{Bmatrix} 0 & 0 & 1 \\ \vdots & & 0 \\ 0 & \cdots & 0 \end{Bmatrix}, w_0 = \begin{Bmatrix} 0 & \cdots & 0 \\ 0 & & \vdots \\ 1 & 0 & 0 \end{Bmatrix}$$

v_0 is the highest weight vector for the adjoint action and w_0 is the lowest weight vector.

$$\begin{aligned} &a_t \cdot v_0 = e^{2t} v_0 \\ &a_t \cdot w_0 = e^{-2t} w_0 \quad t \in \mathbb{R} \\ &m \cdot v_0 = \chi(m) v_0 \\ &m \cdot w_0 = \chi(m) w_0 \quad m \in M. \\ &n \cdot v_0 = v_0 \\ &{}^t n \cdot w_0 = w_0 \qquad n \in N, \text{ and } {}^t n \text{ it's transposed.} \end{aligned} \tag{2.5}$$

Define

$$F_0(g) = \mathscr{B}(g \cdot v_0, w_0), \quad (g \in G)$$

recall that $\mathscr{B}$ is the Killing form on V. For any $v \in V$ define the matrix coefficient

$$c_v(g) = \mathscr{B}(g \cdot v_0, v), \quad (g \in G).$$

The map $v \mapsto c_v$ is a G-equivariant injective linear map from V into $E_{1,-\rho-2}$. Since $G = KMAN$ we may conclude that w_0 is a cyclic vector for $\tau_0 = \mathrm{Ad}|_K$ on V, because w_0 is cyclic for G and satisfies (2.5).

LEMMA 2.8. (Wallach [W], Ch. 8.13.9) $F_0 \cdot 1_s$ *is cyclic in* $E_{1,s-2}$ *as soon as* 1_s *is cyclic in* $E_{0,s}$.

Proof. Observe that $G = \bar{N}AMK$, where $\bar{N} = \{{}^t n \mid n \in N\}$, therefore the G-invariant subspace generated by $F_0 \cdot 1_s$ clearly contains $[\pi_{0,s}(G)1_s] \cdot F_0$. Hence if $f_1 \in E_{1,-(s-2)}$ is K-finite and orthogonal to $\pi_{1,s-2}(G)(F_0 \cdot 1_s)$ then $f_1 \cdot F_0 = 0$, so $f_1 = 0$.

COROLLARY 2.9. *Let $s \in \mathbb{C}$.*

(i) *For n even is $E_{1,s}$ topologically irreducible if $s \notin 2\mathbb{Z} + 1$.*

(ii) *For n odd is $E_{1,s}$ topologically irreducible if*

(a) *$s \notin \mathbb{Z}$, or*

(b) *s an even integer with $|s| \leqslant n - 3$.*

Proof. The statement follows from an observation used in the proof of Corollary 2.6. but now applied to $F_0 \cdot 1_s$, namely that $E_{1,s}$ is irreducible if and only if both $F_0 \cdot 1_{s+2}$ is cyclic in $E_{1,s}$ and $F_0 \cdot 1_{-s+2}$ is cyclic in $E_{1,-s}$.

3. Unitarizability

In the study of unitarizability of the $\pi_{i,s}$ it's well-known that certain intertwining operators play an important role. In this case the operator of interest is

$$A_{i,s}\colon E_{i,s} \rightarrowtail E_{i,-s}$$

$$A_{i,s}(f)(g) = \int_{\bar{N}} f(gw\bar{n})\,\mathrm{d}\bar{n} \quad (g \in G, f \in E_{i,s})$$

where w is given by

$$w = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & & & 0 \\ 0 & & I_{n-2} & \vdots \\ -1 & 0 & \dots & 0 \end{pmatrix}, \tag{3.1}$$

and $\bar{N} = {}^tN = wNw^{-1}$. This integral is absolutely convergent for $\operatorname{Re} s > \rho - 2$ ($\rho = n - 1$), cf Knapp [Kn] and [MKo–D]. The first step is to decompose these intertwining operators in simpler parts. Let $\tilde{n}_y \in \tilde{N}$ be defined by

$$\tilde{n}_y = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & y & & 1 \end{pmatrix} \quad y = (y_1, \dots, y_{n-2}) \in \mathbb{R}^{n-2}.$$

Define the intertwining operator $B_{i,s}\colon E_{i,s} \rightarrowtail \tilde{E}_{i,s}$ by the integral

$$(B_{i,s}f)(g) = \int_{\mathbb{R}^{n-2}} f(g\tilde{n}_y)\,\mathrm{d}y.$$

This integral is absolutely convergent for $\operatorname{Re} s > \rho - 2$. The adjoint operator $B^*_{i,s}\colon \tilde{E}_{i,-s} \rightarrowtail E_{i,-s}$ is given by

$$(B^*_{i,s}f)(g) = \int_{\mathbb{R}^{n-2}} f(g({}^t\tilde{n}_y))\,\mathrm{d}y,$$

this integral is also absolutely convergent for $\operatorname{Re} s > \rho - 2$ and both have a meromorphic extension to all $s \in \mathbb{C}$ with at most poles for s in the set $\{\rho - 2 - 2k \mid k \in \mathbb{Z}_{\geqslant 0}\}$. The intertwining operator $\tilde{A}_{i,s}$ is given by

$$[\tilde{A}_{i,s}f](g) = \int_{-\infty}^{\infty} f(gwv_x)\,\mathrm{d}x, \quad (g \in G,\ f \in \tilde{E}_{i,s},\ K\text{-finite}) \tag{3.2}$$

where

$$v_x = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \cdot & & 0 \\ 0 & & \cdot & 0 \\ x & 0 & 0 & 1 \end{pmatrix}.$$

This integral is absolutely convergent for $\operatorname{Re} s > 0$. Moreover one can check the following useful relation, cf. Knapp [Kn] Ch.XIV, Section 4.

PROPOSITION 3.1. *Let* $i = 0, 1$ *and* $\operatorname{Re} s > \rho - 2$,

$$A_{i,s} = c \cdot B^*_{i,s}\tilde{A}_{i,s}B_{i,s}$$

where c *is a positive constant, independent of* s. *By analytic continuation this identity holds for all* $s \in \mathbb{C}$ *for which both sides can be defined by analytic continuation.*

It follows for $f, h \in E_{i,s}$ and s real that

$$(f, A_{i,s}h) = c(B_{i,s}f, \tilde{A}_{i,s}B_{i,s}h). \tag{3.3}$$

(See (2.3) for the definition of $(.,.)$.)

LEMMA 3.2. *If* $\tilde{A}_{i,s}$ *defines a G-invariant unitary structure on* $\operatorname{Im} B_{i,s}$ *then* $A_{i,s}$ *does the same on* $E_{i,s}$, *provided all operators are well-defined.*

The image of $B_{i,s}$ can be described as follows. Let

$$\tilde{N} = \left\{ \begin{pmatrix} 1 & q_1,\dots,q_{n-2} & x \\ 0 & 1 & 0 \\ & 1 & \\ 0 & p_1,\dots,p_{n-2} & 1 \end{pmatrix} \middle| \, p_i, q_i, x \in \mathbb{R} \right\}, \text{ and}$$

$$\tilde{P} = MA\tilde{N}.$$

Observe that $\tilde{P}$ has the same Levi factor as P (cf. Section 1). $\tilde{P}$ is standard for the choice $\tilde{\Delta} = \{\alpha_{1,n}, -\alpha_{3,n}, \alpha_{3,4}, \dots, \alpha_{n-2,n-1}, -\alpha_{2,n-1}\}$ of simple roots, and corresponds to the set $\tilde{F} = \tilde{\Delta}\backslash\{\alpha_{1,n}, -\alpha_{3,n}\} \subset \tilde{\Delta}$. Define

$$\tilde{E}_{0,s} = I(\tilde{P}, 1, \mu_s),$$

$$\tilde{E}_{1,s} = I(\tilde{P}, \chi, \mu_s). \text{ (cf. (1.3) and Section 2.)}$$

Again both $\tilde{E}_{0,s}$ and $\tilde{E}_{1,s}$ are G-spaces, G acting by left translations. The associated representations are denoted by $\tilde{\pi}_{i,s}$, $i = 1, 2$.

LEMMA 3.3. (Knapp [Kn], Ch.XIV)

$$\operatorname{Im} B_{i,s} \subset \tilde{E}_{i,s},$$

for all $s \in \mathbb{C}$ for which $B_{i,s}$ is defined by meromorphic continuation, $i = 0, 1$.

Moreover

LEMMA 3.4. (Vogan [Vol]: Proposition 4.1.20.) *Let $i = 0, 1$ and $s \in \mathbb{C}$. Then $\tilde{\pi}_{i,s}$ and $\pi_{i,s}$ have equivalent composition series.*

The next step is to apply induction by stages to $(\tilde{\pi}_{i,s}, \tilde{E}_{i,s})$. Define $F_1 = \tilde{F} \cup \{\alpha_{1n}\}$ and put $P_1 = M_1 A_1 N_1$ for Langlands decomposition of the parabolic subgroup P_1 associated to F_1. Then

$$A_1 = \left\{ \begin{pmatrix} e^t & 0 & & & 0 \\ & e^u & & & \\ 0 & & e^u & & \\ & & & \cdot & 0 \\ & & & e^u & \\ 0 & & 0 & & e^t \end{pmatrix} \middle| \, t, u \in \mathbb{R}; \quad 2t + (n-2)u = 0 \right\},$$

$$M_1 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & h & 0 \\ c & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL^{\pm}(2,\mathbb{R}), \quad h \in SL^{\pm}(n-2,\mathbb{R}), \quad (\mathrm{ad}-\mathrm{bc})\det h = 1 \right\},$$

$$N_1 = \left\{ \begin{pmatrix} 1 & q_1, \dots, q_{n-2} & 0 \\ 0 & 1 & 0 \\ & 1 & \\ 0 & p_1, \dots, p_{n-2} & 1 \end{pmatrix} \middle| p_i, q_i, x \in \mathbb{R} \right\}.$$

Let $Q = M'A'N'$ denote the parabolic subgroup of M_1 given by $Q = M_1 \cap \tilde{P}$, then

$$M' = M_F = \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \varepsilon_n \end{pmatrix} \middle| \varepsilon_1 = \pm 1, \quad \varepsilon_n = \pm 1; \quad h \in GL(n-2, \mathbb{R}); \varepsilon_1 \varepsilon_n \det h = 1 \right\}, \tag{3.4}$$

$$A' = M_1 \cap A_F = \{a_t | t \in \mathbb{R}\},$$

where a_t is defined in (1.2) and

$$N' = M_1 \cap \tilde{N} = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & & 0 \\ 0 & & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}.$$

Observe that $M_1 \cong S(SL^{\pm}(2, \mathbb{R}) \times SL^{\pm}(n-2, \mathbb{R}))$ and $A_F = A' A_1$. Now we want to induce the characters $(m, a, n) \mapsto \delta(m) e^{\mu_s \log a}$, $(\delta = 1, \chi)$, of $M'A'N'$ to a representation of M_1 and afterwards induce this representation of M_1, to a representation of G. This procedure is called induction by stages, cf Vogan [Vol].

Before going on let us put forward a few facts, which will be used heavily, about the principal series of the group $SL^{\pm}(2, \mathbb{R})$ with respect to its standard minimal parabolic subgroup. Let us write $G_2 = SL^{\pm}(2, \mathbb{R})$, all subgroups and representations of G_2 will be provided with the subscript 2. Define

$$K_2 = O(2, \mathbb{R}),$$

$$A_2 = \left\{ a_\tau = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} \middle| \tau \in \mathbb{R} \right\},$$

$$M_2 = \left\{ \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \middle| \varepsilon_1, \varepsilon_2 = \pm 1 \right\},$$

$$N_2 = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}.$$

Let δ_1 be the character of M_2 given by

$$\delta_1 : \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \mapsto \varepsilon_1 \varepsilon_2,$$

and let δ_0 be the trivial character of M_2. For any $s \in \mathbb{C}$ and $i = 1, 2$ define

$$E(i,s)_2 = \{f \in C^\infty(G_2) \mid f(gma_\tau n) = \delta_i(m^{-1}) e^{-(s+1)\tau} f(g) \\ \text{for all } (g, m, a_\tau, n) \in G_2 \times M_2 \times A_2 \times N_2\}.$$

Thus $E(i,s)_2 = \mathrm{I}(P_2, \delta_i, s)_2$, $P_2 = M_2 A_2 N_2$, and G_2 acts on $E(i,s)_2$ by left translations. Call the corresponding representations $\pi(i,s)_2$, $i = 1, 2$ and $s \in \mathbb{C}$.

PROPOSITION 3.5. *Let $i = 0, 1$. The representations $\pi(i,s)_2$ are topologically irreducible for $s \notin 2\mathbb{Z} + 1$.*

Let γ_p be the two-dimensional representation of K_2 defined by

$$\gamma_p \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} e^{ip\theta} & 0 \\ 0 & e^{-ip\theta} \end{pmatrix},$$

$$\gamma_p \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

γ_p is irreducible for $p \neq 0$. Let γ_0^+ denote the trivial representation of K_2 and γ_0^- the one-dimensional determinant–representation which is trivial on $SO(2)$.

LEMMA 3.6. *The K_2-types of $\pi(0,s)_2$ are $\{\gamma_0^+, \gamma_2, \gamma_4, \ldots\}$. The K_2-types of $\pi(1,s)_2$ are $\{\gamma_0^-, \gamma_2, \gamma_4, \ldots\}$. Each K_2-type occurs with multiplicity one.*

For $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$ define the intertwining operator

$$B(i,s)_2 : E(i,s)_2 \mapsto E(i,-s)_2$$

by the absolutely convergent integral

$$[B(i,s)_2 f](g) = \int_{-\infty}^{\infty} f(gwv_x)\, dx (f \in E(i,s)_2 \text{ and } g \in G_2)$$

where $v_x = {}^t n_x$, $x \in \mathbb{R}$, and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular $B(i,s)_2$ intertwines the K_2-action and since every K_2-type occurs at most with multiplicity one, we have

$$B(i,s)_2|_{\gamma_p} = c_i(s,p) \mathrm{Id}_{\gamma_p}$$

on the space of γ_p.

PROPOSITION 3.7. (i) *For* $s \in \mathbb{C}$, $\operatorname{Re} s > 0$, *we have*

$$c_0(s,p) = c_1(s,p) = \pi(-1)^{p/2} 2^{1-s} \frac{\Gamma(s)}{\Gamma(\frac{1}{2}(s+p+1))\Gamma(\frac{1}{2}(s-p+1))}$$

with $p \in 2\mathbb{Z}_{\geqslant 0}$.

(ii) *For* $k \in \mathbb{Z}_{\geqslant 0}$ *the* K_2*-types of* $\operatorname{Ker} B(i, 2k+1)_2$ *are*

$$\{\gamma_p \mid p > 2k,\ p \in 2\mathbb{Z}_{\geqslant 0}\}.$$

For $h_1 \in E(i,s)_2$ and $h_2 \in E(i, -\bar{s})_2$ define

$$(h_1, h_2) = \int_{K_2} h_1(k)\overline{h_2(k)}\, dk, \tag{3.5}$$

this induces a G-invariant sesquilinear pairing between $E(i,s)_2$ and $E(i,-\bar{s})_2$, $i = 0, 1$ and $s \in \mathbb{C}$.

PROPOSITION 3.8. *Let* $i = 0, 1$.

(i) $\pi(i,s)_2$ *is unitary for* $s \in \mathbb{C}$ *with* $\operatorname{Re} s = 0$. *The* G_2*-invariant scalar product is given by* (3.5).

(ii) $\pi(i,s)_2$ *is unitarizable for* $0 < s < 1$. *The* G_2*-invariant scalar product is given by*

$$\langle f, h \rangle = (f, B(i,s)_2 h) \quad (f, h \in E(i,s)_2).$$

(iii) $\pi(i,s)$ *is* not *unitarizable for all other* $s \in \mathbb{C}$ *with* $\operatorname{Re} s \geqslant 0$.

PROPOSITION 3.9. *Let* $k \in \mathbb{Z}_{\geqslant 0}$ *and* $i = 0, 1$.

(i) $\operatorname{Ker} B(i, 2k+1)_2$ *is a* G_2*-invariant irreducible subspace of* $E(i, 2k+1)_2$. *The corresponding representations* $\pi^d(0, 2k+1)_2$ *and* $\pi^d(1, 2k+1)_2$ *are equivalent and will be denoted by* $\pi^d(2k+1)_2$.

(ii) *For* $f, h \in \operatorname{Ker} B(i, 2k+1)_2$ *define*

$$\langle f, h \rangle = \lim_{s \to 2k+1} \frac{(-1)^{k+1}}{s - (2k+1)} (f, B(i,s)_2 h),$$

then $\langle .,. \rangle$ *defines a* G_2*-invariant scalar product on* $\operatorname{Ker} B(i, 2k+1)_2$. *Thus* $\pi^d(2k+1)_2$ *is an irreducible unitary representation of* G_2. ($\pi^d(2k+1)_2$ *is a discrete series representation of* G_2.)

This concludes for this moment the treatment of some series of representations of $SL^{\pm}(2, \mathbb{R})$ which will be used in the sequel.

By abuse of notation define

$$\delta_0(m)=1,\text{ and }\delta_1(m)=\varepsilon_1\varepsilon_n, m=\begin{pmatrix}\varepsilon_1 & 0 & 0\\ 0 & h & 0\\ 0 & 0 & \varepsilon_n\end{pmatrix}\in M'.$$

Observe that $\delta_1(m)=\operatorname{sgn}(\det h)$. Let $\pi(\delta_1,s)_1$ be the induced representation of M_1 on the space $I(M_1,Q,\delta_i,\mu_s)$, $i=0,1$, where we consider μ_s as a character on A' by restriction, cf. (3.4). Recall that $M_1\cong S(SL^{\pm}(2,\mathbb{R})\times SL^{\pm}(n-2,\mathbb{R}))$ and that Q is the parabolic subgroup of M_1 defined in (3.4). Extend δ_i in a natural way to M_1; if $a\times h\in M_1$, $a\in SL^{\pm}(2,\mathbb{R})$ and $h\in SL^{\pm}(n-2,\mathbb{R})$, put

$$\delta_i(a\times h)=(\operatorname{sgn}(\det h))^i,\quad i=0,1.$$

Then clearly

$$\pi(\delta_i,s)_1=\pi(i,s)_2\otimes\delta_i,\quad i=0,1.$$

The next step is to induce the representations

$$(m,a,n)\to\pi(\delta_i,s)_1(m)\quad(m\in M_1,a\in A_1,n\in N_1)$$

of P_1 to a representation of G.

THEOREM 3.10. *Let $s\in\mathbb{C}$ and $i=0,1$. The representations $\tilde{\pi}_{i,s}$ and $I(P_1,\pi(\delta_i,s)_1,0)$ are equivalent.*

The proof follows from the observation that the restriction of μ_s to $\mathfrak{a}_1$ is zero and the procedure of induction by stages, cf. Vogan [Vol]: Proposition 4.1.18. Since $I(P_1,\pi(\delta_i,s)_1,0)$ is unitarizable for $s\in\sqrt{-1}\mathbb{R}$ and $0<s<1$ we deduce from Lemma 3.2, Corollary 2.9, and the above theorem

COROLLARY 3.11. *Let $i=0,1$. Then $\pi_{0,s}$ and $\pi_{1,s}$ are irreducible and unitarizable for $s\in\sqrt{-1}\mathbb{R}$ and $0<s<1$.*

LEMMA 3.12. *Let $s\in\mathbb{R}$ with $s>0$. If $\pi_{0,s}$ or $\pi_{1,s}$ is irreducible and unitarizable then $0<s<1$.*

Proof. If $\pi_{i,s}$ is irreducible then also $\tilde{\pi}_{i,s}$ (Lemma 3.4) and thus $I(P_1,\pi(\delta_i,s)_1,0)$, hence consider $I(P_1,\pi(\delta_i,s)_1,0)$. $B(i,s)_2$ induces by induction in a natural manner an intertwining operator

$$\operatorname{Ind}B(i,s)_2\colon \tilde{E}_{i,s}\mapsto\tilde{E}_{i,-s}$$

Ind $B(i, s)_2$ is given by essentially the same formula as $B(i, s)_2$, if we take into account the inbedding of $SL^{\pm}(2, \mathbb{R})$ into M_1:

$$[\text{Ind } B(i, s)_2(f)](g) = \int_{-\infty}^{\infty} f(gwv_x)\, dx,$$

$g \in G$, $f \in \tilde{E}_{i,s}$, K-finite, where v_x is given by (3.2). Hence

$$\text{Ind } B(i, s)_2 = \tilde{A}_{i,s}, \quad s \in \mathbb{C}, \text{ Re } s > 0.$$

Now suppose under the conditions of the lemma that $\pi_{i,s}$ (and thus $\tilde{\pi}_{i,s}$) is irreducible and unitarizable. Since Ker $\tilde{A}_{i,s} = I(P_1, \text{Ker } B(\delta_1, s)_2, 0)$ we see that Ker $\tilde{A}_{i,s}$ is G-invariant and non-trivial for $s > 0$, $s \in 2\mathbb{Z} + 1$. Since $\tilde{\pi}_{i,s}$ is assumed to be irreducible we have furthermore that $s \notin 2\mathbb{Z} + 1$. But then the G-invariant scalar product on $\tilde{E}_{i,s}$ must be a real scalar multiple of

$$(f, h) \rightarrowtail (f, \tilde{A}_{i,s} h) \quad (f, h \in \tilde{E}_{i,s}, K\text{-finite})$$

where the right-hand side is the sequi-linear form (2.3), with $\mathscr{H} = \mathbb{C}$. Since

$$f \rightarrowtail (f, B(i, s)_2 f) \quad (f \in E(i, s)_2)$$

assumes strictly positive and strictly negative values for $s > 1$, $s \notin 2\mathbb{Z} + 1$, the same holds for

$$f \rightarrowtail (f, \tilde{A}_{i,s} f) \quad (f \in \tilde{E}_{i,s}),$$

because $\tilde{A}_{i,s}$ is the intertwining operator $B(\delta_i, s)_2$ induced to G. Hence s satisfies $0 < s < 1$.

REMARK 3.13. Let $s \in \mathbb{R}$, $s > 0$, and $i = 0,1$. If $\pi_{i,s}$ is irreducible then $\tilde{A}_{i,s}$ yields an equivalence between $\pi_{i,s}$ and $\pi_{i,-s}$. For $s \in \mathbb{C}$, Re $s = 0$, $\pi_{i,s}$ and $\pi_{i,-s}$ have the same character, so provided $\pi_{i,s}$ is irreducible, $\pi_{i,s}$ and $\pi_{i,-s}$ are equivalent.

Let $k = 0, 1, 2, \ldots$ and $i = 0, 1$. Ker $B(i, 2k + 1)_2$ gives rise, by induction, to a closed G-invariant subspace of $\tilde{E}_{i,2k+1}$. This subspace is nothing else but Ker $\tilde{A}_{i,2k+1}$. By Proposition 3.9 this space carries a G-invariant scalar product, given by

$$\langle f, h \rangle = \lim_{s \to 2k+1} \frac{(-1)^{k+1}}{s - (2k + 1)} (f, \tilde{A}_{i,s} h) \ (f, h \in \text{Ker } \tilde{A}_{i,2k+1}).$$

Call $\tilde{\pi}^d_{i,2k+1}$ the corresponding representation on Ker $\tilde{A}_{i,2k+1}$. We will come back to these representations in the next section.

4. The spherical distributions

We start with a review of the construction of H-invariant distribution vectors of $\pi_{i,s}$. For details we refer to [MKo-D]. Consider the map

$$(h, m, a, n) \rightarrowtail hg_0 man$$

of $H \times M \times A \times N$ into G, where

$$g_0 = \left(\begin{array}{cc|ccc|c} \tfrac{1}{2}\sqrt{2} & \tfrac{1}{2}\sqrt{2} & 0 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ \hline 0 & 0 & & & & 0 \\ & & & I_{n-3} & & \\ 0 & 0 & & & & 0 \\ \hline \tfrac{1}{2}\sqrt{2} & -\tfrac{1}{2}\sqrt{2} & 0 & & 0 & 0 \end{array}\right) \tag{4.1}$$

This is a C^∞-map and it has an open and dense image in G. The fibre above $x = hg_0man$ is equal to $\{(hl, g_0^{-1}lg_0m, a, n) | l \in g_0Mg_0^{-1} \cap H\}$. Now define for $\mathrm{Re}\, s > \rho$ the functions $\mathscr{P}_{0,s}$ and $\mathscr{P}_{1,s}$ by

$$\begin{cases} \mathscr{P}_{0,s}(hg_0ma_tn) = \dfrac{1}{\Gamma^2(\frac{1}{4}(\bar{s} - \rho + 2))}\, \mathrm{e}^{(\bar{s}-\rho)t} \\ \mathscr{P}_{0,s}(g) = 0 \quad \text{if} \quad g \notin Hg_0P \end{cases}$$

$$\begin{cases} \mathscr{P}_{1,s}(hg_0ma_tn) = \dfrac{1}{\Gamma^2(\frac{1}{4}(\bar{s} - \rho + 4))}\, \chi(m)\mathrm{e}^{(\bar{s}-\rho)t} \\ \mathscr{P}_{1,s}(g) = 0 \quad \text{if} \quad g \notin Hg_0P \end{cases}$$

$\mathscr{P}_{1,s}$ is well-defined since χ is trivial on $M \cap g_0^{-1}Hg_0$, see [MKo-D]. Moreover $\mathscr{P}_{0,s}$ and $\mathscr{P}_{1,s}$ have the following properties: Let $i = 0, 1$

(i) $\mathscr{P}_{i,s}$ defines an H-invariant element of $E_{i,s}^{-\infty}$, the anti-dual of $E_{i,s}$, for $\mathrm{Re}\, s > \rho$ by

$$f \rightarrowtail \int_K \overline{f(k)}\mathscr{P}_{i,s}(k)\, \mathrm{d}k \quad (f \in E_{i,s}).$$

(ii) The mapping $s \rightarrowtail \mathscr{P}_{i,s}(f)$ $(f \in E_{i,s})$ has an H-invariant anti-holomorphic extension, denoted by $\mathscr{P}_{i,s}$, to all of $\mathbb{C}$ and $\mathscr{P}_{i,s} \in E_{i,s}^{-\infty}$ for all $s \in \mathbb{C}$.

(iii) Let $\pi_{i,s}^{-\infty}$ be the transposed action of G on $E_{i,s}^{-\infty}$. For $s \in \mathbb{C}$ and $\varphi \in D(G)$

$$\pi_{i,s}^{-\infty}(\varphi)\mathscr{P}_{i,s} = \varphi * \mathscr{P}_{i,s} \in E_{i,-\bar{s}}.$$

(iv) Let Ω be the Casimir operator of G. Then

$$\pi_{i,s}^{-\infty}(\Omega\varphi)\mathscr{P}_{i,s} = \frac{\bar{s}^2 - \rho^2}{4n}\, \pi_{i,s}^{-\infty}(\varphi)\mathscr{P}_{i,s}, \varphi \in D(G) \quad \text{and} \quad s \in \mathbb{C}.$$

We now come to the definition of the Fourier transform

$$\mathscr{F}_{i,s}: D(G) \rightarrowtail E_{i,s},$$
$$\mathscr{F}_{i,s}(\varphi) = \pi_{i,-\bar{s}}^{-\infty}(\varphi)\mathscr{P}_{i,-\bar{s}}$$

$i = 0, 1$ and $s \in \mathbb{C}$. The Fourier transform is a right H-invariant and G-equivariant mapping from $D(G)$ into $E_{i,s}$:

$$\mathscr{F}_{i,s}(R_h\varphi) = \mathscr{F}_{i,s}(\varphi) \quad \text{and} \quad \mathscr{F}_{i,s}(L_g\varphi) = \pi_{i,s}(g)\mathscr{F}_{i,s}(\varphi) \tag{4.2}$$

for all $\varphi \in D(G)$, $h \in H$, $g \in G$, $i = 0, 1$, and $s \in \mathbb{C}$, where R, L stands for the right, left action of G on G respectively. Finally we can define the spherical distributions $\zeta_{i,s} \in D'(G)$ by

$$\langle \zeta_{i,s}, \varphi \rangle = \mathscr{P}_{i,-\bar{s}}(\pi_{i,s}^{-\infty}(\varphi)\mathscr{P}_{i,s}) = \mathscr{P}_{i,-\bar{s}}(\mathscr{F}_{i,-\bar{s}}(\varphi))$$

$\varphi \in D(G)$, $i = 0, 1$, and $s \in \mathbb{C}$. Notice that for $\varphi, \psi \in D(G)$

$$\langle \zeta_{i,s}, \tilde{\varphi} * \psi \rangle = \int_K \mathscr{F}_{i,s}(\varphi)(k)\overline{\mathscr{F}_{i,-\bar{s}}(\psi)(k)}\mathrm{d}\,k, \tag{4.3}$$

where $\tilde{\varphi}(g) = \bar{\varphi}(g^{-1})$, $g \in G$. ($D'(G)$ is considered as the *anti-dual* of $D(G)$.)

PROPOSITION 4.1. ([MKo-D]; Proposition 4.1 and Lemma 8.2) *Let $s \in \mathbb{C}$ and $i = 0, 1$.*

(i) *$\zeta_{i,s}$ is a spherical distribution with eigenvalue $(\bar{s}^2 - \rho^2)/4n$, i.e. $\Omega\zeta_{i,s} = ((\bar{s}^2 - \rho^2)/4n)\zeta_{i,s}$.*

(ii) *$\zeta_{i,s} = \zeta_{i,-s}$.*

For $\lambda \in \mathbb{C}$ put

$$D'_{\lambda,H}(X) = \left\{ T \in D'(X) \,|\, T \text{ is } H\text{-invariant and } \Omega T = \frac{\lambda}{4n} T \right\}$$

LEMMA 4.2. ([MKo-D]; Proposition 7.9) $\dim D'_{\lambda,H}(X) = 2$ *for all* $\lambda \in \mathbb{C}$.

If we consider the spherical distributions $\zeta_{i,s}$ as distributions on X, $i = 0, 1$ and $s \in \mathbb{C}$, we get the following list of basis elements, cf. loc. cit. Theorem 8.5.

PROPOSITION 4.3. *Put $\lambda = (\bar{s}^2 - \rho^2)/4n$. Then we have*
(i) *If $s \notin \mathbb{Z}$ then $\{\zeta_{0,s}, \zeta_{1,s}\}$ is a basis of $D'_{\lambda,H}(X)$*
(ii) If n is even *then for*
s even: $\{\zeta_{0,s}, \zeta_{1,s}\}$ is a basis of $D'_{\lambda,H}(X)$
s odd; put $s = s_r = \rho + 2r$, $r \in \mathbb{Z}$.

(1) $|s_r| \geqslant \rho$, *r even: basis* $\left\{\left(\dfrac{d}{ds}\right)\zeta_{0,s}|_{s=s_r}, \zeta_{1,s_r}\right\}$

(2) $|s_r| \geqslant \rho$, *r odd: basis* $\left\{\zeta_{0,s_r}, \left(\dfrac{d}{ds}\right)\zeta_{1,s}|_{s=s_r}\right\}$

(3) $|s_r| < \rho$. *In this case*

$$\zeta_{0,s_r} = \frac{\Gamma(\frac{1}{2}(-\rho - r + 2))^2\,\Gamma(\frac{1}{2}\rho + r + 2))^2}{\Gamma(-\frac{1}{2}r)^2\Gamma(\frac{1}{2}(r+1))^2}\zeta_{1,s_r} \quad (r \text{ even})$$

$$\zeta_{0,s_r} = \frac{\Gamma(\frac{1}{2}(1 - r))^2\,\Gamma(\frac{1}{2}(r+2))^2}{\Gamma(\frac{1}{2}(\rho + r))^2\,\Gamma(\frac{1}{2}(-\rho - r + 1))^2}\zeta_{1,s_r} \quad (r \text{ odd})$$

Write $\zeta_{0,s_r} = c_r\zeta_{1,s_r}$ $(c_r > 0)$ and define

$$\Theta_r = \frac{d}{ds}(\zeta_{0,s} - c_r\zeta_{1,s})|_{s=s_r}.$$

Then $\{\zeta_{0,s_r}, \Theta_r\}$ is a basis of $D'_{\lambda,H}(X)$. Furthermore Θ_r is not positive–definite.
(iii) If n is odd *then for*
s even: $\{\zeta_{0,s}, \zeta_{1,s}\}$ is a basis of $D'_{\lambda,H}(X)$.
s odd: put $s = s_r = \rho + 2r + 1$, $r \in \mathbb{Z}$. There is a constant $c_r > 0$ such that $\zeta_{0,s_r} = c_r\zeta_{1,s_r}$. Define

$$\Theta_r = \frac{d}{ds}(\zeta_{0,s} - c_r\zeta_{1,s})|_{s=s_r},$$

then $\{\zeta_{0,s_r}, \Theta_r\}$ is a basis of $D'_{\lambda,H}(X)$. Furthermore Θ_r is not positive–definite.

Let T be a bi-H-invariant, positive–definite, extremal distribution on G. (That is T corresponds to an irreducible unitary representation of G with a non-trivial H-invariant distribution vector.) Then T is spherical:

$$\Omega T = \frac{\lambda}{4n}T$$

for some $\lambda \in \mathbb{C}$. Putting $\lambda = \bar{s}^2 - \rho^2$, the positive–definite property of T implies that λ is real, since Ω acts as a real scalar on the Hilbert subspace reproduced by T, hence we may assume that $s \in \sqrt{-1}\mathbb{R}$ or $s \in \mathbb{R}$, $s > 0$. Let us look first for a family of positive–definite bi-H-invariant extremal distributions associated to the representations $\pi_{i,s}$, $i = 0, 1$ and $s \in \mathbb{C}$.

1. Let $s \in \sqrt{-1}\mathbb{R}$, then both $\pi_{0,s}$ and $\pi_{1,s}$ are irreducible and unitary. Moreover we deduce from (4.3) that in this case

$$\langle \zeta_{i,s}, \tilde{\varphi} * \psi \rangle = \int_K \mathscr{F}_{i,s}(\varphi)(k)\overline{\mathscr{F}_{i,s}(\psi)(k)}\mathrm{d}k, \quad \varphi, \psi \in D(G)$$

and $\zeta_{i,s} \neq 0$, $i = 0, 1$. So $\mathscr{P}_{i,s}$ is a non-zero H-invariant distribution vector of $\pi_{i,s}$ and the corresponding reproducing distribution is $\zeta_{i,s}$, which is extremal and positive–definite, $i = 0, 1$.

2. Let $s \in \mathbb{R}$ and $s > 0$. We recall the following fact from [MKo-D], Proposition 8.3.

LEMMA 4.4. *Let $s \in \mathbb{C}$, then provided both sides are defined we have*

$$A_{i,s} \circ \mathscr{F}_{i,s} = c_i(s)\mathscr{F}_{i,-s} \quad (i = 0, 1)$$

where

$$c_0(s) = \frac{2^{-\rho}\Gamma(\frac{1}{2}\rho)\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{4}(s - \rho + 2))^2}{\pi\Gamma(\frac{1}{4}(s + \rho))^2\Gamma(\frac{1}{2}(1 - s))},$$

$$c_1(s) = \frac{2^{-\rho+1}\Gamma(\frac{1}{2}(\rho + 2))\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{4}(s - \rho + 4))^2}{\pi(n - 1)\Gamma(\frac{1}{4}(s + \rho + 2))^2\Gamma(\frac{1}{2}(1 - s))}.$$

By Corollary 3.9 we know that $\pi_{i,s}$ is irreducible and unitarizable for $0 < s < 1$. The G-invariant scalar product is given by

$$\langle f, h \rangle = (f, A_{i,s}(h)) = \int_K f(k)\overline{A_{i,s}(h)(k)}\,\mathrm{d}k$$

$f, h \in E_{i,s}$ and $i = 0, 1$, cf. (3.3).

On the other hand (4.3) gives for $\varphi, \psi \in D(G)$

$$\langle \zeta_{i,s}, \tilde{\varphi} * \psi \rangle = (\mathscr{F}_{i,s}(\varphi), \mathscr{F}_{i,-s}(\psi))$$
$$= \frac{1}{c_i(s)}(\mathscr{F}_{i,s}(\varphi), A_{i,s} \circ \mathscr{F}_{i,s}(\psi)).$$

So $c_i(s)\zeta_{i,s}$ is positive–definite, it is the reproducing distribution of $\pi_{i,-s}$ and $\mathscr{P}_{i,-s}$ is a non-zero H-invariant distribution vector of $\pi_{i,-s}$, $i=0$, 1 and $0<s<1$.

3. Let $s=s_r=\rho+2r$ for *n even*, $s=s_r=\rho+2r+1$ for *n odd*, with $r\in\mathbb{Z}$ and $s_r>0$. Observe that in both cases s_r is an odd positive integer. From the Plancherel formula for the space G/H (cf. [D-P]) we obtain:

– *if r is odd then ζ_{0,s_r} is positive–definite and extremal.*

– *if r is even, then ζ_{1,s_r} is positive–definite and extremal.*

Let k be such that $2k+1=s_r$, i.e. $k=\frac{1}{2}(s_r-1)$, and let r be odd. (Recall that for r odd $\zeta_{1,s_r}=0$, n even, and $\zeta_{0,s_r}=c_r\zeta_{1,s_r}$, n odd, cf. Proposition 4.3.) Put

$$\mathscr{A}_{0,s}=\frac{1}{\Gamma(\frac{1}{2}(s-\rho+2))^2}A_{0,s}.$$

It follows from Proposition 3.1 and the fact that $B_{0,s}$ has only poles in the set $\{n-3-2k\,|\,k\in\mathbb{Z}_{\geqslant 0}\}$ that $\mathscr{A}_{0,s}$ depends analytically on s for $\operatorname{Re}s\geqslant 0$. By Lemma 4.4

$$\mathscr{A}_{0,s}\circ\mathscr{F}_{0,s}=c_0\cdot 2^{-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{4}(s+\rho))^2\Gamma(\frac{1}{2}(1-s))\Gamma(\frac{1}{2}(s-\rho+4))^2}\mathscr{F}_{0,-s}$$

where c_0 is a positive constant not depending on s, $\operatorname{Re}s>0$. Clearly $\mathscr{A}_{0,s_r}\circ\mathscr{F}_{0,s_r}=0$, so $\operatorname{Im}\mathscr{F}_{0,s_r}\subset\operatorname{Ker}\mathscr{A}_{0,s_r}$ and one has for $\varphi,\psi\in D(G)$

$$\lim_{s\to s_r}\frac{(-1)^{k+1}}{s-s_r}(\mathscr{F}_{0,s}(\varphi),\mathscr{A}_{0,s}\circ\mathscr{F}_{0,s}(\psi))$$

$$=\lim_{s\to s_r}\frac{(-1)^{k+1}}{s-s_r}((\mathscr{F}_{0,s_r}(\varphi),\mathscr{A}_{0,s}\circ\mathscr{F}_{0,s_r}(\psi))$$

$$d_{0,s_r}\langle\zeta_{0,s_r},\tilde{\varphi}*\psi\rangle,$$

where

$$\mathrm{d}_{0,s_r}=c_0\cdot 2^{-s_r}\frac{\Gamma(\frac{1}{2}s_r)\Gamma(\frac{1}{2}(s_r+1))}{\Gamma(\frac{1}{4}(\rho+s_r))^2\Gamma(\frac{1}{2}(s_r-\rho+4))^2},\quad r\text{ odd.}$$

Similarly if one defines

$$\mathscr{A}_{1,s}=\frac{1}{\Gamma(\frac{1}{4}(s-\rho+2))^2}A_{1,s}$$

then one gets by similar arguments that $\mathscr{A}_{1,s}$ depends analytically on s for $\operatorname{Re} s > 0$, $\operatorname{Im} \mathscr{F}_{1,-(2k+1)} \subset \operatorname{Ker} \mathscr{A}_{1,2k+1}$ for r even, $(2k + 1 = s_r)$, and

$$\lim_{s\to s_r} \frac{(-1)^{k+1}}{s - s_r} (\mathscr{F}_{1,s}(\varphi), \mathscr{A}_{1,s} \circ \mathscr{F}_{1,s}(\psi))$$

$$= d_{1,s_r}\langle \zeta_{1,s_r}, \tilde{\varphi} * \psi \rangle,$$

where

$$d_{1,s_r} = c_1 \cdot 2^{-s_r} \frac{\Gamma(\frac{1}{2}s_r)\Gamma(\frac{1}{2}(s_r + 1))}{\Gamma(\frac{1}{4}(\rho + s_r + 2))^2\Gamma(\frac{1}{2}(s_r - \rho + 2))^2}, \quad r \text{ even}$$

c_1 a positive constant independent of s_r. It follows from the observations after Remark 3.13 and relation (3.3) that $\operatorname{Ker} \mathscr{A}_{i,s_r}$ carries a G-invariant scalar product given by

$$\langle f, h \rangle = \lim_{s\to s_r} \frac{(-1)^{k+1}}{s - s_r} (f, \mathscr{A}_{i,s} h) \quad (f, h \in \operatorname{Ker} \mathscr{A}_{i,s_r}), \quad i = 0, 1.$$

Moreover the restriction of π_{0,s_r} to $\operatorname{Ker} \mathscr{A}_{0,s_r}$ is equivalent to the restriction of π_{1,s_r} to $\operatorname{Ker} \mathscr{A}_{1,s_r}$. (This follows from the analogous result for the group G_2, cf. Proposition 3.9) The corresponding representation is denoted by $\pi^d(s_r)$. So the representation associated to ζ_{0,s_r}, for r odd, and ζ_{1,s_r}, for r even, is the restriction of π_{0,s_r} to the closure of $\operatorname{Im} \mathscr{F}_{0,s_r}$, r odd, and the restriction of π_{1,s_r} to the closure of $\operatorname{Im} \mathscr{F}_{1,s_r}$, r even.

CONJECTURE: *$\pi^d(s_r)$ is an irreducible representation of G.*

4. For $s = \rho$, $E_{0,\rho}$ contains the trivial representation as a subquotient which has a reproducing distribution T_0 given by

$$\langle T_0, \varphi \rangle = \int_G \bar{\varphi}(g)\, \mathrm{d}g, \quad (\varphi \in D(G)).$$

For n odd T_0 is a scalar multiple of $\zeta_{0,\rho}$, and for n even T_0 is a scalar multiple of $(\mathrm{d}/\mathrm{d}s)(\zeta_{0,s})|_{s=\rho}$.

REMARK 4.5. For n odd one doesn't need to regularize the intertwining operator $A_{i,s}$ since in that case $A_{i,s}$ is well-defined in the odd integers and has only poles in the even integers.

5. The $GL(n-1\ \mathbb{R})$-spherical dual of $SL(n,\mathbb{R})$

With all the definitions and constructions out of the way we can state the main result.

THEOREM 5.1. *The $GL(n-1,\mathbb{R})$-spherical dual of $SL(n,\mathbb{R})$ consists of the following representations.*

(i) *The principal series representations:*
$\pi_{i,s}$, $i = 0, 1$, *and* $s \in \sqrt{-1}\mathbb{R}$, $s \geqslant 0$.
The positive–definite spherical distribution corresponding to $\pi_{i,s}$ *is* $\zeta_{i,s}$, $i = 0, 1$, *and* $s \in \sqrt{-1}\mathbb{R}$, $s \geqslant 0$.

(ii) *The complementary series:*
$\pi_{i,s}$, $i = 0, 1$, *and* $0 < s < 1$.
The positive–definite spherical distribution corresponding to $\pi_{i,s}$ *is* $c_i(s)\zeta_{i,s}$, $i = 0, 1$, *and* $0 < s < 1$.
Put $s_r = \rho + 2r$, *if n is even, and* $s_r = \rho + 2r + 1$, *if n is odd,* $r \in \mathbb{Z}$.

(iii) *The relative discrete series representations:*
  (a) π_{1,s_r} *restricted to the closure of* $\operatorname{Im}\mathscr{F}_{1,s_r}$, *r even and* $s_r > 0$. *The corresponding positive–definite spherical distribution is* $d_{1,s_r}\zeta_{1,s_r}$.
  (b) π_{0,s_r} *restricted to the closure of* $\operatorname{Im}\mathscr{F}_{0,s_r}$, *r odd and* $s_r > 0$. *The corresponding positive–definite spherical distribution is* $d_{0,s_r}\zeta_{0,s_r}$.

(iv) *The trivial representation. The corresponding spherical distribution is* 1. (*Which equals* $2^{-n}(-1)^{n/2}\pi^{-1}\Gamma^2(n/2)(\mathrm{d}/\mathrm{d}s)(\zeta_{0,s})|_{s=\rho}$ *for n even and* $(-1)^{(n-1)/2}2^{1-n}\pi^{-3}\Gamma^2(n/2)\zeta_{0,\rho}$ *for n odd.*)

COROLLARY 5.2. *Let the notation be as in the above theorem. T is a bi-$GL(n-1,\mathbb{R})$-invariant positive–definite spherical distribution on $SL(n,\mathbb{R})$ if and only if T is contained in one of the following sets.*

(i) $\{a\zeta_{0,s} + b\zeta_{1,s} \mid a, b \geqslant 0,\quad s \in \mathbb{C}; \operatorname{Re} s = 0$ *and* $\operatorname{Im} s \geqslant 0\}$
(ii) $\{a\zeta_{0,s} + b\zeta_{1,s} \mid a, b \geqslant 0$ *and* $0 < s < 1\}$
(iii) $\{a\zeta_{0,s_r} \mid a \geqslant 0,\ s_r > 0;\ r$ *odd*$\}$
(iv) $\{a\zeta_{1,s_r} \mid a \geqslant 0,\ s_r > 0;\ r$ *even*$\}$
(v) $\{a\cdot 1 \mid a \geqslant 0\}$

Before starting to prove the theorem let us recall some facts about some special K-types occuring in $\pi_{i,s}$. For π a representation of G and δ any irreducible unitary representation of K let $m(\delta,\pi)$ be the multiplicity of δ in π restricted to K. Let 1 be the trivial representation, and τ_0 the special representation of K constructed in [MKo-D], section 6. Then for $s \in \mathbb{C}$

$$\begin{aligned} m(1,\pi_{0,s}) &= m(\tau_0,\pi_{1,s}) = 1,\\ m(1,\pi_{1,s}) &= m(\tau_0,\pi_{0,s}) = 0. \end{aligned} \tag{5.1}$$

Also introduce the following terminology: let δ be as above, we say that $\varphi \in D(G)$ is of left K-type δ if the left-translations of φ by elements of K span a finite dimensional subspace of $D(G)$, and left translation on this space defines a representation of K, equivalent with a multiple of δ. Put

$$D(\delta; G) = \{\varphi \in D(G) \mid \varphi \text{ is of left-}K\text{-type } \delta\}.$$

It also follows, loc. cit., that if $s \notin 2\mathbb{Z} + 1$ then there exists a function $\varphi_0 \in D(1; G)$ such that $\mathscr{F}_{0,s}(\varphi_0) \neq 0$ and a $\psi_1 \in D(\tau_0; G)$ such that $\mathscr{F}_{1,s}(\psi_1) \neq 0$. Moreover $\mathscr{F}_{0,s}(\varphi) = 0$ for all $\varphi \in D(1; G)$ and $\mathscr{F}_{1,s}(\psi) = 0$ for all $\psi \in D(\tau_0; G)$, cf. (5.1) and (4.2).

The proof of Theorem 5.1. Let π be an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$ with a non-trivial H-invariant distribution vector. Let T be the corresponding reproducing distribution, then T is a positive–definite bi-H-invariant extremal distribution on G. (Recall that $D'(G)$ is the *anti-dual* of $D(G)$.) Moreover T is spherical, i.e.

$$\Omega T = \frac{\lambda}{4n} T$$

for certain λ. Put $\lambda = (\bar{s})^2 - \rho$, then $\operatorname{Re} s = 0$ or $s \in \mathrm{R}$, since λ must be real. The proof consists of 4 steps:

(1) Suppose $\operatorname{Re} s = 0$, we may assume $\operatorname{Im} s \geqslant 0$. $\zeta_{0,s}$ and $\zeta_{1,s}$ are both positive–definite and extremal. By Proposition 4.3 there are constants a and b such that $T = a\zeta_{0,s} + b\zeta_{1,s}$. Taking $\varphi \in D(1; G)$ we get $\varphi * T = a(\varphi * \zeta_{0,s})$, since the multiplicity of the trivial K-type in $\pi_{1,s}$ is 0. Therefore

$$\langle \varphi * T, \varphi \rangle = \langle T, \tilde{\varphi} * \varphi \rangle = a \langle \zeta_{0,s}, \tilde{\varphi} * \varphi \rangle = (\mathscr{F}_{0,s}(\varphi), \mathscr{F}_{0,s}(\varphi))$$

and $\mathscr{F}_{0,s}(\varphi)$ is non-zero for some left K-invariant φ, cf. [MKo-D], Section 6. Hence $a \geqslant 0$. Similarly by taking $\varphi \in D(\tau_0; G)$ we get $b \geqslant 0$. But then, since T is extremal, T is a positive multiple of either $\zeta_{0,s}$ or $\zeta_{1,s}$ and π is equivalent to either $\pi_{0,s}$ or $\pi_{1,s}$.

(2) Suppose $s \in \mathbb{R}$, $|s| \geqslant \rho$, we may assume $s \geqslant 0$. Put $s_r = \rho + 2r$, for n even, and $s = \rho + 2r + 1$, for n odd, with $r \in \mathbb{Z}_{\geqslant 0}$. Since every matrixcoefficient of π must vanish at infinity, unless π is the trivial representation, we assume that π is not the trivial representation. Then it follows from the asymptotic analysis for spherical distributions in [D-P], Section 4, that there exist constants $a, b \in \mathbb{C}$ such that

$T = a\zeta_{0,s_r}$ if r is odd and
$T = b\zeta_{1,s_r}$ if r is even.

Hence if $s\in\mathbb{R}$, $|s|\geqslant\rho$, then π is equivalent to the trivial representation or to one of the relative discrete series representations of G w.r.t. H.

3. Suppose $s\in\mathbb{R}$, $|s|<\rho$, and $s\notin 2\mathbb{Z}+1$. We may assume $s>0$. Recall from Corollary 2.9 and Proposition 3.6 that under the above assumptions $\pi_{i,s}$ is irreducible but not unitarizable for $s>1$. If $0<s<1$ then $\pi_{i,s}$ is irreducible and unitarizable, $i=0,1$. Again by Proposition 4.3 there are complex constants a and b such that $T=a\zeta_{0,s}+b\zeta_{1,s}$. Let $\varphi\in D(1;G)$. Then $\varphi*T=a(\varphi*\zeta_{0,s})$ since $m(1,\pi_{1,s})=0$. Let $\varphi_0\in D(1;G)$, be such that $\mathscr{F}_{0,s}(\varphi_0)\neq 0$, it follows from (4.3) that $\varphi_0*\zeta_{0,s}\neq 0$. Suppose $a\neq 0$, then the representation π associated to T has a non-trivial K-fixed vector namely the one corresponding to φ_0*T. The mapping

$$\varphi*T\to a(\varphi*\zeta_{0,s})\quad \varphi\in D(1;G),$$

induces a $(\mathfrak{U}(\mathfrak{g})^{\mathfrak{k}},\mathfrak{k})$ equivariant isomorphism from the K-fixed vectors in $\mathscr{H}$ onto the K-fixed vectors in $E_{0,s}$ (recall that $\mathscr{H}$ is the representation space corresponding to T). Applying [D], Theorem 9.1.12, yields the existence of a $(\mathfrak{g},K)$-equivariant isomorphism of $\mathscr{H}_K$ onto $(E_{0,s})_K$, the subscript K stands for the subspace of K-finite vectors. So $\pi_{0,s}$ is unitarizable, hence $0<s<1$ and π is equivalent to $\pi_{0,s}$.

If $a=0$ then $T=b\zeta_{1,s}$ which immediately implies $0<s<1$ and π is equivalent to $\pi_{1,s}$.

(4) Left to consider the case $s\in 2\mathbb{Z}+1$, $|s|<\rho$. Again we may assume $s>0$. Put $s_r=\rho+2r$, for n even, and $s=\rho+2r+1$, for n odd, with $r\in\mathbb{Z}_{\leqslant 0}$. According to Proposition 4.3: $\zeta_{0,s_r}=c_r\zeta_{1,s_r}$ and $\{\zeta_{0,s_r},\Theta_r\}$ is a basis of $D'_{\lambda,H}(X)$, where $\Theta_r=(\mathrm{d}/\mathrm{d}s)(\zeta_{0,s}-c_r\zeta_{1,s})|_{s=s_r}$. Moreover loc. cit., Θ_r is not positive–definite. There exist constants $a,b\in\mathbb{C}$ such that $T=a\zeta_{0,s}+b\Theta_r$. Since $\zeta_{0,s_r}=c_r\zeta_{1,s_r}$ one has $\varphi*\zeta_{0,s_r}=0$ for all $\varphi\in D(1;G)\cup D(\tau_0;G)$. Hence $\langle T,\tilde{\varphi}*\varphi\rangle=b\langle\Theta_r,\tilde{\varphi}*\varphi\rangle$ for such φ. Calculating $\mathscr{F}_{i,s_r}(\varphi)$, $\varphi\in D(1;G)\cup D(\tau_0;G)$, implies $\langle\Theta_r,\tilde{\varphi}*\varphi\rangle>0$ for some $\varphi\in D(1;G)$ and $\langle\Theta_r,\tilde{\psi}*\psi\rangle<0$ for some $\psi\in D(\tau_0;G)$ if r is even. Thus $b=0$ if r is even. If r is odd one should interchange φ and ψ to get the desired result. Hence $b=0$ for all r and π is equivalent to one of the relative discrete series representations of G w.r.t. H. This finishes the proof of Theorem 5.1.

Appendix

In this appendix we will give a proof of

LEMMA 2.7. *Let n be odd and s an even integer with $|s|\leqslant n-3$. Then $E_{0,s}$ is topologically irreducible. In particular is 1_s cyclic in $E_{0,s}$ in this case.*

The proof of this lemma is due to J. Bang-Jensen. Let the notation be as before. An outline of the proof is as follows. For each s satisfying the assumptions of the Lemma we will construct a $\lambda\in\mathfrak{a}^*_{0c}$ in the closure of the positive Weyl chamber such that

$$E_0(G/P_0, \lambda)_K \twoheadrightarrow (E_{0,s})_K \hookrightarrow (E_0(G/P_0, -\lambda)_K, \tag{a.1}$$

where the subscript K stands for the subspace of K-finite vectors, and $\twoheadrightarrow$ ($\hookrightarrow$) stands for a $(\mathfrak{g}, K)$-equivariant surjective (injective) map. We refer to Section 2 for the definition of $E_0(G/P_0, \lambda)$. Let $J(\lambda)$ be the irreducible spherical $(\mathfrak{g}, K)$-module with infinitesimal character λ. (Spherical means that the multiplicity of the trivial K-type in $J(\lambda)$ is 1, cf [Ba]). Observe that the modules in (a.1) are $(\mathfrak{g}, K)$-modules. Since λ is in the closure of the positive Weyl chamber we have the following facts, cf. [Ko] Theorem 2.10.3.:

(i) *$J(\lambda)$ is the unique irreducible quotient of $E_0(G/P_0, \lambda)_K$.*
(ii) *$J(\lambda)$ is the unique irreducible submodule of $E_0(G/P_0, -\lambda)_K$.*
Moreover
(iii) *The multiplicity of the trivial K-type in $(E_{0,s})_K$ is 1.*

From these 3 facts and (a.1) it follows that $(E_{0,s})_K = J(\lambda)$. Indeed if $V \subset (E_{0,s})_K$ is a non-trivial submodule then it follows from (i), (ii) and (a.1) that both V and $(E_{0,s})_K/V$ must contain $J(\lambda)$. But then the multiplicity of the trivial K-type in $(E_{0,s})_K$ is at least 2, contradicting (iii). Hence $(E_{0,s})_K$ is an irreducible $(\mathfrak{g}, K)$-module. Whence $E_{0,s}$ is an irreducible G-module.

In order to prove the Lemma we need some more notation. Let $G_1 = GL(n, \mathbb{R})$, $\mathfrak{g}_1$ it's Lie algebra and $\mathfrak{a}_1 \subset \mathfrak{g}_1$ the Cartan subalgebra consisting of diagonal matrices. Let $K_1 = O(n)$ the standard maximal compact subgroup of G_1. For $\nu \in \mathfrak{a}_{1c}^*$ let $J_1(\nu)$ the irreducible K_1-spherical $(\mathfrak{g}_1, K_1)$-module with infinitesimal character ν. Let W_1 be the Weyl group of the pair $(\mathfrak{g}_1, \mathfrak{a}_1)$, then $W_1 \cong S_n$. It's well-known that

PROPOSITION A.1. *$J_1(\nu)$ is equivalent to $J_1(\mu)$ if and only if $\mu \in W_1 \cdot \nu$.*

In the sequel we will identify

$$\mathfrak{a}_{1c}^* = \mathfrak{a}_c = \mathbb{C}^n$$
$$W_1 \cong S_n.$$

As before let $G = SL(n, \mathbb{R})$, $K = SO(n) \subset G$ a maximal compact subgroup, and $\mathfrak{a}_0 \subset \mathfrak{g}$ the Cartan subalgebra consisting of diagonal matrices with trace 0. Then we identify

$$\mathfrak{a}_{0c}^* = \mathfrak{a}_{0c} = \{z \in \mathbb{C}^n \mid z_1 + z_2 + \cdots + z_n = 0\} \subset \mathfrak{a}_{1c}.$$

In order to get a short notation write $P(p \times q^k \times r)$, $p + kq + r = n$, for the standard parabolic subgroup of G_1 (or G) with Levi component $L(p \times q^k \times r) =$

$GL(p,\mathbb{R}) \times GL(q,\mathbb{R}) \times \cdots \times GL(q,\mathbb{R}) \times GL(r,\mathbb{R})$ $(L(p \times q^k \times r) \cap G)$, with the factor $GL(q,\mathbb{R})$ occuring k-times. Also if $x \in \mathbb{C}$ then x can be seen as a character on $GL(1)$ given by

$$x\colon \lambda \rightarrowtail |\lambda|^x$$

This character is also denoted by $J_1(x)$.

LEMMA A.2. (cf. Vogan [Vo2], Lemma 13.5) *Let $n \in \mathbb{N}$, $n \geqslant 2$, be an integer and write $G_1 = GL(n,\mathbb{R})$. Let $m \in \mathbb{Z}_{\geqslant 0}$ and let $x \in \mathbb{R}$ be such that $m - n + 2 \leqslant x \leqslant m$. Then with the notation as above*

$$J_1(x, m, m-1, \ldots, m-n+2) \cong$$
$$\operatorname{ind}(P(1 \times (n-1)) \uparrow G_1)(J_1(x) \otimes J_1(m, m-1, \ldots, m-n+2)),$$

isomorphic as $(\mathfrak{g}_1, K_1)$-modules.

The proof is exactly the same as the proof of Lemma 13.5 in Vogen [Vo2]. It should be remarked that ind stands for induction in the category of $(\mathfrak{g}, K)$-modules, cf [Vo1], Chapter 6. By the same argument

$$J_1(m, m-1, \ldots, m-n+2, x) \cong$$
$$\operatorname{ind}(P((n-1) \times 1) \uparrow G_1)(J_1(m, m-1, \ldots, m-n+2) \otimes J_1(x)),$$

and we deduce from Proposition A.1:

COROLLARY A.3. *Under the assumptions of the previous Lemma*

$$\operatorname{ind}(P(1 \times (n-1)) \uparrow G_1)(J_1(x) \otimes J_1(m, m-1, \ldots, m-n+2)) \cong$$
$$\operatorname{ind}(P((n-1) \times 1) \uparrow G_1)(J_1(m, m-1, \ldots, m-n+2) \otimes J_1(x)).$$

Observe that the statement of Lemma 2.7 is true for $n = 3$. Thus we may assume that n is odd and $n \geqslant 5$. Let s be a positive even integer with $|s| \leqslant n-3$. Put $m = (n-1)/2$, $k = m - \frac{1}{2}s$, thus $s = 2(m-k)$ and $0 < k < m$. Also write

$$\lambda = (m-1, m-2, \ldots, m-k+1, m-k, m-k, m-k-1, \ldots,$$
$$k-m+1, k-m, k-m, k-m-1, \ldots, -(m-1))$$

Then we can define

$$I(\lambda) = \operatorname{ind}(P((1)^n) \uparrow G)(\lambda).$$

Observe that $I(\lambda)$ is isomorphic to $E_0(G/P_0, \lambda)_K$, cf. [Vo1] Proposition 6.3.5.

Since λ is in the closure of the positive Weyl chamber it's well-known that $J(\lambda)$ is the unique irreducible quotient of $I(\lambda)$, cf. [Ko].

Moreover there exists a $(\mathfrak{g}, K)$-equivariant surjective map of $I(\lambda)$ onto

$$\operatorname{ind}(P(k \times (1)^{n-2k} \times (k)) \uparrow G)(J_1(m-1, \ldots, m-k), (m-k, \ldots, k-m), J_1(k-m-1, \ldots, 1-m)).$$

This latter space is by Lemma A.2 and induction by stages isomorphic to

$$\mathcal{Y} = \mathit{ind}(P(1 \times k \times (1)^{n-2k-2} \times k \times 1) \uparrow G)(Y),$$

where

$$Y = (m-k, J_1 m-1, \ldots, m-k), m-k-1, \ldots, 1+k-m, J_1(k-m, \ldots, 1-m), k-m).$$

Again by induction by stages there is a surjective $(\mathfrak{g}, K)$-equivariant map

$$\mathcal{Y} \rightarrowtail\!\!\!\!\twoheadrightarrow \operatorname{ind}(P(1 \times (n-2) \times 1) \uparrow G)(m-k, J_1(m-1, \ldots, 1-m), k-m).$$

($\rightarrowtail\!\!\!\!\twoheadrightarrow$ stands for a $(\mathfrak{g}, K)$-equivariant surjective map.) The latter space is isomorphic to $(E_{0,s})_{K'}$ with $s = 2(m-k)$. The conclusion is that

$$I(\lambda) \rightarrowtail\!\!\!\!\twoheadrightarrow (E_{0,s})_{K'} \quad s = 2(m-k). \tag{a.2}$$

Also by Lemma A.2 and induction by stages we see that

$$\begin{aligned}(E_{0,s})_K &\cong \operatorname{ind}(P((n-2) \times 1 \times 1) \uparrow G)(J_1(m-1, \ldots 1-m), m-k, k-m) \\ &\cong \operatorname{ind}(P((n-2) \times 2) \uparrow G)(J_1(m-1, \ldots, 1-m) \otimes J_1(m-k, k-m)).\end{aligned}$$

But $J_1(m-k, k-m) \cong J_1(k-m, m-k)$ and $J_1(m-1, \ldots, 1-m) \cong J_1(1-m, \ldots, m-1)$, cf. Proposition A.1. The result is that

$$(E_{0,s})_K \cong \operatorname{ind}(P(1 \times (n-2) \times 1) \uparrow G)(k-m, J_1(1-m, \ldots, m-1), m-k).$$

This latter space can by induction by stages $(\mathfrak{g}, K)$-equivariantly embedded into

$$\tilde{\mathcal{Y}} = \operatorname{ind}(P(1 \times k \times (1)^{n-2k-2} \times k \times 1) \uparrow G)(\tilde{Y}),$$

where

$$\tilde{Y} = (k-m, J_1(1-m, \ldots, k-m), 1+k-m, \ldots, m-k-1, J_1(m-k, \ldots m-1), m-k).$$

By Lemma A.2 $\tilde{\mathscr{Y}}$ is isomorphic to

$$\operatorname{ind}(P(k\times(1)^{n-2k}\times k)\uparrow G)(J_1(1-m,\dots,k-m),k-m,\dots,m-k,\\ J_1(m-k,\dots,m-1)),$$

and this space can be embedded into $I(-\lambda)$. Thus

$$(E_{0,s})_K \hookrightarrow I(-\lambda), s=2(m-k) \tag{a.3}$$

($\hookrightarrow$ stands for a $(\mathfrak{g},K)$-equivariant injective map.) Combining (a.2) and (a.3) gives

$$I(\lambda) \twoheadrightarrow (E_{0,s})_K \hookrightarrow I(-\lambda), s=2(m-k), s\geqslant 0.$$

By duality we also have

$$I(\lambda) \twoheadrightarrow (E_{0,-s})_K \hookrightarrow I(-\lambda), s=2(m-k), s\geqslant 0.$$

Combining this we get

$$I(\lambda) \twoheadrightarrow (E_{0,s})_K \hookrightarrow I(-\lambda), s=2(m-k), 0\leqslant |s|\leqslant n-3, \tag{a.1}$$

and this, as outlined above, proves Lemma 2.7.

References

[Ba] J. Bang-Jensen: *The Multiplicities of Certain K-types in Irreducible Spherical Representations of Semisimple Lie Groups.* Thesis Massachusetts Institute of Technology, 1987.

[B-G] J.N. Bernstein and S.I. Gelfand: Tensor products of finite and infinite dimensional representations of semisimple Lie algebras. *Comp. Math.* 41 (1980), p. 245–285.

[D-P] G. van Dijk & M. Poel: The Plancherel formula for the pseudo-Riemannian space $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$. *Comp. Math.* 58 (1986) p. 371–397.

[D] *J. Dixmier: Enveloping Algebras.* North-Holland Publishing Company, Amsterdam/New York/Oxford, 1977.

[Fa] J. Faraut: Distributions sphériques sur les espaces hyperboliques. *J. Math. Pures et Appl.* 58 (1979), p. 369–444.

[F-K] M. Flensted-Jensen & T.H. Koornwinder: Positive definite spherical functions on a non-compact rank one symmetric space. In: *Lect. Notes in Math.* 739, Springer Verlag Berlin etc., 1979, p. 249–282.

[H] S. Helgason: A Duality for Symmetric Spaces with Applications to Group Representations I. *Adv. Math.* 5 (1970), p. 1–154.

[Kn] A.W. Knapp: *Representation Theory of Semisimple Groups. An Overview Based on Examples.* Princeton University Press, Princeton N.J., 1986.

[Ko] B. Kostant: On the existence and irreducibility of certain series of representations. In: I.M. Gelfand (ed.) *Lie groups and their representations*, Halsted Press, New-York, 1975, p. 231–329. (See also *Bull. A.M.S.* 75 (1969), p. 627–642.)

[MKo] M.T. Kosters: *Spherical distributions on rank one symmetric spaces.* Thesis University of Leiden, 1983.

[MKo-D] M.T. Kosters & G. van Dijk: Spherical Distributions on the Pseudo-Riemannian space $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$. *J. Funct. Anal.* 68 (1986), p. 168–213.

[WKo] W.A. Kosters: Eigenspaces of the Laplace-Beltrami operator on $SL(n, \mathbb{R})/S(GL(1) \times GL(n-1))$, I and II. *Ind. Math.* 47 (1985), p. 99–145.

[M] V.F. Molčanov: The Plancherel formula for the pseudo-Riemannian space $SL(3, \mathbb{R})/GL(2, \mathbb{R})$. *Sibirsk Math. J.* 23 (1982), p. 142–151.

[P] M. Poel: *Harmonic Analysis on* $SL(n, \mathbb{R})/GL(n-1, \mathbb{R})$. Thesis University of Utrecht, 1986.

[T] E.G.F. Thomas: The theorem of Bochner-Schwartz-Godement for generalized Gelfand pairs. In: K.D. Bierstedt and B. Fuchsteiner (eds.), *Functional Analysis: Surveys and recent results III*, Elseviers Science Publishers B.V. (North Holland) (1984).

[Va] V.S. Varadarajan: *Harmonic Analysis on Real Reductive Groups. Lecture Notes in Mathematics*, No. 576, Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[Vo1] D.A. Vogan: *Representations of Real Reductive Lie Groups.* Birkhäuser, Boston/Basel/Stuttgart, 1981.

[Vo2] D.A. Vogan: The unitary dual of $GL(n)$ over an archimedian field. *Invent. Math.* 83 (1985), p. 449–505.

[W] N.R. Wallach: *Harmonic Analysis on Homogeneous Spaces.* Dekker, New-York, 1977.

[Z] G. Zuckerman: Tensor products of finite and infinite dimensional representations of semisimple Lie groups. *Ann. Math.* 106 (1977), p. 295–308.