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A two-dimensional van Aardenne-Ehrenfest theorem in irregularities of distribution

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1. Introduction

In 1935 van der Corput [4] conjectured that no infinite sequence in the unit interval can have bounded discrepancy relative to the family of subintervals. This conjecture was proved in 1945 by van Aardenne-Ehrenfest [1]. In 1949 she [2] proved the following quantitative refinement. Let $A = (a_1, a_2, a_3, \dots)$ be an arbitrary infinite sequence in $U = [0, 1)$. Let

$$d(A, n, x) = \sum_{\substack{0 \leqslant a_i < x \\ 1 \leqslant i \leqslant n}} 1 - nx$$

and

$$d(A, n) = \sup_{0 < x \leqslant 1} |d(A, n, x)|.$$

Then

$$d(A, n) \gg \frac{\log \log n}{\log \log \log n} \tag{1.1}$$

for infinitely many n. (Following the common usage among number theorists, we use Vinogradov's notation $f(n) \gg g(n)$ to mean $f(n) > c \cdot g(n)$ for some positive absolute constant c depending at most on the dimension but independent of n.)

In 1954 Roth [9], by an analytic method, improved (1.1) to $(\log n)^{1/2}$. Finally, in 1972, Schmidt [10], by a combinatorial approach, obtained the long-standing conjecture $\log n$. This result is the best possible, since already in 1904 Lerch [7] proved that, if α is an irrational number which has bounded partial denominators in its continued fraction expansion, then the sequence (here $\{x\}$ stands for the fractional part of the real number x)

$$A = (\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots)$$

satisfies

$$d(A, n) \ll \log n.$$

We also refer to Hardy and Littlewood [6] and Ostrowski [8].

By introducing an ingenious variation of Roth's method, Halász [5] was able to give, among others, an alternative proof of Schmidt's theorem. We quote Halász: "Our atttempt has been motivated by the fact that Schmidt's clever elementary argument does not seem to generalize to higher dimensions, whereas Roth's more analytic method works efficiently there giving the best results known today. Unfortunately, some surprising (or natural) obstacle prevented us, too, from getting any improvement in higher dimension and to illustrate the applicability of our version of Roth's method we have to be contented with proving some new results for dimension 1."

In this paper we show how to modify Roth-Halász method to get an improvement in higher dimension. The object of this paper is to prove the two-dimensional analogue of van Aardenne-Ehrenfest's theorem.

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots)$ be an infinite sequence of points in the k-dimensional unit cube $U^k = [0, 1)^k (k \geqslant 2)$. For $\mathbf{x} = (x_1, \dots, x_k) \in U^k$, let $B(\mathbf{x})$ denote the box consisting of all $\mathbf{y} = (y_1, \dots, y_k) \in U^k$ such that $0 \leqslant y_i < x_i$ $(1 \leqslant i \leqslant k)$. Write

$$d(\mathbf{A}, n, \mathbf{x}) = \sum_{\substack{\mathbf{a}_i \in B(\mathbf{x}) \\ 1 \leqslant i \leqslant n}} 1 - n x_1 x_2 \dots x_k,$$

and

$$d(\mathbf{A}, n) = \sup_{\mathbf{x} \in U^k} |d(\mathbf{A}, n, \mathbf{x})|.$$

First consider the case $k = 2$. Note that for every $N(\geqslant 2)$, there is an N-element *set* $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N\} \subset U^2$ such that

$$\sup_{\mathbf{x} \in U^2} \left| \sum_{\mathbf{a}_i \in B(\mathbf{x})} 1 - N x_1 x_2 \right| \ll \log N. \tag{1.2}$$

Indeed, let α be an irrational number which has bounded partial denominators in its continued fraction expansion, and let

$$\mathbf{a}_i = \left(\{i\alpha\}, \frac{i-1}{N}\right) \in U^2, \quad i = 1, 2, \ldots, N.$$

Then Lerch's theorem implies (1.2). It is impossible to improve (1.2), since Schmidt's theorem mentioned above is equivalent to the following two-dimensional statement. Let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N\}$ be an arbitrary set of N points in the unit square U^2. Then there is a rectangle $B(\mathbf{x})$ with $0 < x_1 \leqslant 1, 0 < x_2 \leqslant 1$ such that

$$\left| \sum_{\mathbf{a}_i \in B(\mathbf{x})} 1 - N x_1 x_2 \right| \gg \log N.$$

Now the two-dimensional analogue of van der Corput's conjecture goes as follows. Does there exist an *infinite sequence* $\mathbf{A}$ of points in the unit square such that $d(\mathbf{A}, n)/\log n$ remains bounded? We shall prove that no such sequence exists, i.e.

$$\limsup_{n \to \infty} \frac{d(\mathbf{A}, n)}{\log n} = \infty.$$

THEOREM 1.1. *Let $\varepsilon > 0$. Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots)$ be an infinite sequence of points in the unit square $U^2 = [0, 1)^2$. Then*

$$d(\mathbf{A}, n) > (\log n)(\log \log n)^{(1/8) - \varepsilon} \tag{1.3}$$

for infinitely many n.

Note that (1.3) is probably very far from being best possible. Let $\mathbf{A}$ be an infinite sequence of points in U^k $(k \geqslant 2)$. It is conjectured that

$$d(\mathbf{A}, n) \gg (\log n)^k$$

for infinitely many n (the case $k = 1$ was settled by Schmidt). It is most doubtful that our approach could be modified to get this strong conjecture. An adaptation of our method to three-dimensional infinite sequences yields that the relation

$$\limsup_{n \to \infty} \frac{d(\mathbf{A}, n)}{(\log n)^{k/2}} = \infty \tag{1.4}$$

holds for $k = 3$. Due to the great technical difficulties of the proof we are not at present certain whether our method gives (1.4) for arbitrary $k \geqslant 4$. Note that (1.4) is only a slight improvement on the old result of Roth [9] that for every k-dimensional sequence $\mathbf{A}$,

$$d(\mathbf{A}, n) \gg (\log n)^{k/2}$$

for infinitely many n. We also note that the three-dimensional analogue of van der Corput's conjecture is still open.

We shall derive Theorem 1.1 from the following three-dimensional result.

THEOREM 1.2. *Let $\varepsilon > 0$. Let $\mathscr{P} = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_N\}$ be an arbitrary distribution of N points, not necessarily distinct, in the unit cube $U^3 = [0, 1)^3$. For every box*

$$B(\mathbf{x}) = \{\mathbf{y} \in U^3 : 0 \leqslant y_i < x_i, 1 \leqslant i \leqslant 3\}$$

with $\mathbf{x} \in U^3$, let us define the discrepancy function

$$D(\mathbf{x}) = D(\mathscr{P}; \mathbf{x}) = \sum_{\mathbf{P}_i \in B(\mathbf{x})} 1 - N x_1 x_2 x_3.$$

Then

$$\sup_{\mathbf{x} \in U^3} |D(\mathbf{x})| > (\log N)(\log\log N)^{(1/8)-\varepsilon}$$

provided $N > N_0(\varepsilon)$.

Finally, we derive Theorem 1.1 from Theorem 1.2. We figure the N points of a distribution $\mathscr{P}$ in U^3 as the points $\mathbf{P}_i = (\mathbf{a}_i, (i-1)/N)$, $i = 1, 2, \ldots, N$, and denote by $Z(\mathbf{x})$ the number of these points in the box $B(\mathbf{x})$. For every $\mathbf{x} = (x_1, x_2, x_3) \in U^3$, let $\mathbf{x}^* = (x_1, x_2)$. For $(m-1)/N < x_3 \leqslant m/N$ we have

$$\begin{aligned} |D(\mathbf{x})| &= |Z(\mathbf{x}) - N x_1 x_2 x_3| \\ &= |d(\mathbf{A}, m, \mathbf{x}^*) + x_1 x_2 (m - N x_3)| \leqslant |d(\mathbf{A}, m, \mathbf{x}^*)| + 1, \end{aligned}$$

and so

$$\sup_{\mathbf{x} \in U^3} |D(\mathbf{x})| \leqslant \max_{1 \leqslant n \leqslant N} d(\mathbf{A}, n) + 1.$$

It follows from Theorem 1.2 that there is an $n = n(N) \leqslant N$ such that

$$d(\mathbf{A}, n) \geqslant \sup_{\mathbf{x} \in U^3} |D(\mathbf{x})| - 1 \gg (\log N)(\log\log N)^{(1/8)-\varepsilon}.$$

Since $n \geqslant d(\mathbf{A}, n)$, we get that $n(N) \to \infty$ as $N \to \infty$. Hence (1.3) holds for the infinity of values $n(N)$, $N = 1, 2, 3, \ldots$, and Theorem 1.1 follows.

2. Proof of Theorem 1.2

Following Roth [9], we shall construct an auxiliary function $F(\mathbf{x}) = F(\mathscr{P}; \mathbf{x})$ in U^3 such that

$$\int_{U^3} F(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x} > (\log N)(\log\log N)^{(1/8)-\varepsilon} \tag{2.1}$$

and

$$\int_{U^3} |F(\mathbf{x})|\, d\mathbf{x} < 2 + \varepsilon, \tag{2.2}$$

provided $N > N_1(\varepsilon)$ (note that we shall actually make a "random construction"). These inequalities give

$$\sup_{\mathbf{x} \in U^3} |D(\mathbf{x})| \geqslant \frac{\int_{U^3} F(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}}{\int_{U^3} |F(\mathbf{x})|\, d\mathbf{x}}$$

$$> \frac{1}{2+\varepsilon} (\log N)\, (\log\log N)^{(1/8)-\varepsilon},$$

so that Theorem 1.2 follows easily.

Following the basic idea of Roth [9], we shall build up $F(\mathbf{x})$ from *modified* Rademacher functions. In fact, we shall follow Schmidt's variant of the method (see [11]).

Any $x \in [0, 1)$ can be written in the form

$$x = \sum_{j=0}^{\infty} \beta_j(x) \cdot 2^{-j-1}$$

where $\beta_j(x) = 0$ or 1 and such that the sequence $\beta_j(x)$, $j = 0, 1, 2, \ldots$ does not end with $1, 1, 1, \ldots$. For $r = 0, 1, 2, \ldots$ let

$$R_r(x) = (-1)^{\beta_j(x)}$$

(these are called the Rademacher functions).

By an r-interval, we mean a dyadic interval of the form

$$[m2^{-r}, (m+1)2^{-r})$$

where $0 \leqslant m < 2^r$.

By an r-function, we mean a function $f(x)$ defined in $U = [0, 1)$ such that in every r-interval, $f(x) = R_r(x)$ or $f(x) = -R_r(x)$.

Clearly, if $f(x)$ is an r-function, then

$$\int_U f(x)\,\mathrm{d}x = 0.$$

LEMMA 2.1. *Suppose that* $f_1, f_2, \ldots, f_t$ *are* $r_1-, r_2-, \ldots, r_t$*-functions, respectively.*

(a) *If an odd number among* $r_1, r_2, \ldots, r_t$ *are equal to* $r = \max\{r_1, r_2, \ldots, r_t\}$, *then the product* $f_1 \cdot f_2 \cdots f_t$ *is an* r*-function, and so*

$$\int_U f_1(x) \cdots f_t(x)\,\mathrm{d}x = 0.$$

(b) *If* $0 \leqslant r_1 < r_2 < \cdots < r_t \leqslant n$, *then*

$$\int_U (f_{r_1}(x) + \cdots + f_{r_t}(x))^2\,\mathrm{d}x = t \leqslant n+1,$$

and for $m \geqslant 2$,

$$\int_U (f_{r_1}(x) + \cdots + f_{r_t}(x))^{2m}\,\mathrm{d}x < ((2m)(n+1))^m.$$

Proof of Lemma 2.1. (a) is trivial.

(b) $m = 1$ is trivial from case (a).

(c) $m \geqslant 2$: Expanding the $2m$th power, we have

$$(f_{r_1} + \cdots + f_{r_t})^{2m} = \sum f_{r_{i(1)}} \cdot f_{r_{i(2)}} \cdots f_{r_{i(2m)}}.$$

The integral

$$\int_U f_{r_{i(1)}}(x) \cdots f_{r_{i(2m)}}(x)\,\mathrm{d}x$$

is 0 unless $r_{i(1)}, \ldots, r_{i(2m)}$ form m pairs of equal numbers. The number of divisions of $\{1, 2, \ldots, 2m\}$ into pairs is

$$(2m-1)(2m-3)\cdots 3\cdot 1 < (2m)^m,$$

and the number of possibilities for $r_{i(1)}, \ldots, r_{i(2m)}$ is $\leqslant (n+1)^m$. Hence

$$\int_U (f_{r_{i(1)}}(x) + \cdots + f_{r_{i(2m)}}(x))^{2m}\, dx < (2m)^m (n+1)^m,$$

which gives (b). □

Suppose that $\mathbf{r} = (r_1, \ldots, r_k)$ is a k-tuple of nonnegative integers. Let

$$|\mathbf{r}| = r_1 + \cdots + r_k;$$

and for any $\mathbf{x} = (x_1, \ldots, x_k) \in U^k$, let

$$R_{\mathbf{r}}(\mathbf{x}) = R_{r_1}(x_1)\cdot R_{r_2}(x_2)\cdots R_{r_k}(x_k). \tag{2.3}$$

By an $\mathbf{r}$-box, we mean the cartesian product $I_1 \times I_2 \times \cdots \times I_k \subset U^k$ of r_j-intervals I_j $(1 \leqslant j \leqslant k)$.

By an $\mathbf{r}$-function, we mean a function $f(\mathbf{x})$ defined in U^k such that in every $\mathbf{r}$-box, $f(\mathbf{x}) = R_{\mathbf{r}}(\mathbf{x})$ or $f(\mathbf{x}) = -R_{\mathbf{r}}(\mathbf{x})$. We sometimes call the $\mathbf{r}$-functions *modified Rademacher functions of order* $\mathbf{r}$.

LEMMA 2.2.

(a) *The square of an* $\mathbf{r}$*-function is identically* 1.

(b) *An* $\mathbf{r}$*-function is an* r_j*-function in the variable* x_j $(1 \leqslant j \leqslant k)$.

Let $f_1, f_2, \ldots, f_t$ *be* $\mathbf{r}_1-, \mathbf{r}_2-, \ldots, \mathbf{r}_t$*-functions, respectively. For every* $i = 1, 2, \ldots, t$, *write*

$$\mathbf{r}_i = (r_{i1}, r_{i2}, \ldots, r_{ik}).$$

(c) *Suppose that there is a* $j \in \{1, \ldots, k\}$ *such that an odd number among* $r_{1j}, r_{2j}, \ldots, r_{tj}$ *are equal to* $\max\{r_{1j}, r_{2j}, \ldots, r_{tj}\}$. *Then*

$$\int_{U^k} f_{\mathbf{r}_1}(\mathbf{x}) f_{\mathbf{r}_2}(\mathbf{x}) \ldots f_{\mathbf{r}_t}(\mathbf{x})\, d\mathbf{x} = 0.$$

(d) *Suppose that for every* $j \in \{1, \ldots, k\}$, *the* j*th coordinates* r_{ij} $(1 \leqslant i \leqslant t)$ *are* *all different. Let*

$$r_j^* = \max\{r_{1j}, r_{2j}, \ldots, r_{tj}\}, \quad j = 1, \ldots, k;$$

and let

$$\mathbf{r}^* = (r_1^*, r_2^*, \ldots, r_k^*).$$

Then the product $f_{\mathbf{r}_1} \cdot f_{\mathbf{r}_2} \cdots f_{\mathbf{r}_t}$ *is an* $\mathbf{r}^*$*-function.*

Proof of Lemma 2.2. (a) and (b) are trivial.

(c) follows from combining Lemma 2.1(a) and Lemma 2.2(a).

(d) follows from (2.3) and Lemma 2.1(a). □

The case $t = 2$ of Lemma 2.2(c) yields that modified Rademacher functions of distinct orders are pairwise orthogonal. In order to get some information on the "higher moments", we need the following lemma (see Schmidt [11]; see also Lemma 2.4 in the monograph [3]).

LEMMA 2.3. *Let*

$$\mathscr{X} = \{\mathbf{r} = (r_1, r_2, r_3) \colon r_i \geqslant 0 \ (1 \leqslant i \leqslant 3) \quad \text{and} \quad |\mathbf{r}| = r_1 + r_2 + r_3 = n\}.$$

Let $\mathscr{Y} \subset \mathscr{X}$ *be an arbitrary subset, and for every* $\mathbf{r} \in \mathscr{Y}$ *let* $f_{\mathbf{r}}$ *be an* $\mathbf{r}$*-function. Then for every* $m \geqslant 1$,

$$\int_{U^3} \left(\sum_{\mathbf{r} \in \mathscr{Y}} f_{\mathbf{r}}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} < (2m)^{3m} \cdot (n+1)^{2m}.$$

We recall that

$$D(\mathbf{x}) = \sum_{\mathbf{P}_i \in B(\mathbf{x})} 1 - N x_1 x_2 x_3$$

is the discrepancy function of the point distribution $\mathscr{P} = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_N\}$.

The following lemma is a variant of the key lemma of Roth [9] (see Schmidt [11]; see also Lemma 2.5 in the monograph [3]).

LEMMA 2.4. *Suppose that* $2^n \geqslant 2N$. *Then for every* $\mathbf{r} = (r_1, r_2, r_3)$ *satisfying* $r_i \geqslant 0, |\mathbf{r}| = r_1 + r_2 + r_3 = n$, *there is an* $\mathbf{r}$*-function* $f_{\mathbf{r}}$ *satisfying*

$$\int_{U^3} f_{\mathbf{r}}(\mathbf{x}) D(\mathbf{x}) \, \mathrm{d}\mathbf{x} \geqslant 2^{-n-7} \cdot N. \tag{2.4}$$

In what follows, let n be chosen to satisfy

$$4N > 2^n \geqslant 2N, \tag{2.5}$$

and for every $\mathbf{r}$ with $r_i \geqslant 0, |\mathbf{r}| = n$, let $g_{\mathbf{r}}$ stand for an $\mathbf{r}$-function satisfying (2.4).

REMARK. Roth [9] considered the following auxiliary function

$$G(\mathbf{x}) = \frac{1}{n+1} \sum_{|\mathbf{r}|=n} g_{\mathbf{r}}(\mathbf{x}).$$

By Lemma 2.4 and (2.5),

$$\begin{aligned}\int_{U^3} G(\mathbf{x})D(\mathbf{x})\,d\mathbf{x} &\geqslant 2^{-n-7}\cdot N\cdot\frac{1}{n+1}\sum_{|\mathbf{r}|=n} 1\\ &= 2^{-n-7}\cdot N\cdot\frac{1}{n+1}\cdot\frac{(n+1)(n+2)}{2} \gg \log N.\end{aligned}$$

On the other hand, by using the orthogonality of the functions $g_{\mathbf{r}}$,

$$\begin{aligned}\int_{U^3} |G(\mathbf{x})|\,d\mathbf{x} &\leqslant \left(\int_{U^3} (G(\mathbf{x}))^2\,d\mathbf{x}\right)^{1/2} = \left(\int_{U^3} \frac{1}{(n+1)^2}\sum_{|\mathbf{r}|=n} 1\right)^{1/2}\\ &= \left(\frac{n+2}{2(n+1)}\right)^{1/2} \leqslant 1.\end{aligned}$$

These inequalities give

$$\sup_{\mathbf{x}\in U^3} |D(\mathbf{x})| \gg \log N.$$

This means that Roth's auxiliary function $G(\mathbf{x})$ is only just not enough to prove the two-dimensional van der Corput conjecture. Our auxiliary function $F(\mathbf{x})$ will be more complicated. Motivated by the success of the "Riesz product" of Halász [5], our starting point will be a "short" product of the sums of certain $\mathbf{r}$-functions $g_{\mathbf{r}}$ rather than simply a sum of these functions (see (2.6) below).

Throughout this paper, let

$$\mathscr{X} = \{\mathbf{r} = (r_1, r_2, r_3): r_i \geqslant 0 (1 \leqslant i \leqslant 3) \quad \text{and} \quad |\mathbf{r}| = r_1 + r_2 + r_3 = n\}$$

where n is specified by (2.5). The cardinality $|\mathscr{X}|$ of $\mathscr{X}$ equals $(n+1)(n+2)/2$ (throughout $|H|$ denotes the number of elements of a finite set H).

Let $q \geqslant 1$ be an integer, and let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_q$ be *disjoint* subsets of $\mathscr{X}$, i.e. $\mathscr{A}_i \cap \mathscr{A}_j = \emptyset$ if $1 \leqslant i < j \leqslant q$.

Let

$$\varrho = \frac{q^{(1/4)-\varepsilon}}{n+1},$$

and write

$$P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) = \prod_{v=1}^{q} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{A}_v} g_{\mathbf{r}}(\mathbf{x})\right). \tag{2.6}$$

Expanding the product (2.6), we have

$$P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) = 1 + S_1(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) + S_2(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}), \tag{2.7}$$

where

$$S_1(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) = \sum_{l=1}^{q} \varrho^l \sum_{1 \leqslant v_1 < \cdots < v_l \leqslant q} \sum{}' g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \tag{2.8}$$

and the summation Σ' extends over all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ such that

$$\begin{aligned} &\forall j (1 \leqslant j \leqslant l)\colon \mathbf{r}_j = (r_{j1}, r_{j2}, r_{j3}) \in \mathscr{A}_{v_j}, \quad \text{and} \\ &\forall j\, \forall k (1 \leqslant j < k \leqslant l)\colon r_{j1} \neq r_{k1}, r_{j2} \neq r_{k2}, r_{j3} \neq r_{k3}; \end{aligned} \tag{2.9}$$

moreover,

$$S_2(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) = \sum_{l=2}^{q} \varrho^l \sum_{1 \leqslant v_1 < \cdots < v_l \leqslant q} \sum{}'' g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \tag{2.10}$$

where the summation Σ'' extends over all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ such that

$$\begin{aligned} &\forall j (1 \leqslant j \leqslant l)\colon \mathbf{r}_j = (r_{j1}, r_{j2}, r_{j3}) \in \mathscr{A}_{v_j}, \quad \text{and} \\ &\exists i \in \{1, 2, 3\}, \quad \exists j \in \{1, 2, \ldots, l\}, \quad \exists k \in \{1, 2, \ldots, l\} \end{aligned} \tag{2.11}$$

such that $j \neq k$ and $r_{ji} = r_{ki}$.

REMARK. $S_2(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})$ represents those terms of the expansion of $P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})$ where "index-coincidence" occurs.

The following three basic lemmas will be proved in Sections 3–5.

LEMMA 2.5. *Let $\mathscr{A}_1, \dots, \mathscr{A}_q$ be disjoint subsets of $\mathscr{X}$ such that*

$$\sum_{l=1}^{q} |\mathscr{A}_l| \geqslant \frac{|\mathscr{X}|}{2}.$$

Then

$$\int_{U^3} S_1(\mathscr{A}_1, \dots, \mathscr{A}_q; \mathbf{x}) D(\mathbf{x})\, d\mathbf{x} > 2^{-12} \cdot q^{(1/4)-\varepsilon} \cdot (n+1)$$

provided N is sufficiently large.

LEMMA 2.6. *Let $\mathscr{A}_1, \dots, \mathscr{A}_q$ be disjoint subsets of $\mathscr{X}$. Then*

$$\int_{U^3} |S_2(\mathscr{A}_1, \dots, \mathscr{A}_q; \mathbf{x})|\, d\mathbf{x} < 2^{4q^2}(n+1)^{-1/8}$$

provided $q > q_1$.

LEMMA 2.7. *Let $\mathscr{A}_1, \dots, \mathscr{A}_q$ be disjoint subsets of $\mathscr{X}$. Then*

$$\int_{U^3} (P(\mathscr{A}_1, \dots, \mathscr{A}_q; \mathbf{x}))^2\, d\mathbf{x} < 1 + 2^{16q^2}(n+1)^{-1/8} + e^{q^{(1/2)-\varepsilon}}$$

provided $q > q_2$.

In order to guarantee the relation

$$2^{16q^2}(n+1)^{-1/8} = o(1),$$

let (note that [ ] stands for *integral part*)

$$q = [(\log n)^{(1/2)-\varepsilon}]. \tag{2.12}$$

We shall specify the disjoint subsets $\mathscr{A}_1, \dots, \mathscr{A}_q \subset \mathscr{X}$ by a probabilistic argument.

For every $\mathbf{r} \in \mathscr{X}$ and for every $i = 1, 2, \dots, q$, let $\xi_{\mathbf{r}}^{(i)}$ be a random variable with common distribution

$$\Pr\left(\xi_{\mathbf{r}}^{(i)} = 1 - \frac{1}{q}\right) = \frac{1}{q}, \Pr\left(\xi_{\mathbf{r}}^{(i)} = -\frac{1}{q}\right) = 1 - \frac{1}{q}. \tag{2.13}$$

Clearly the common expectation $E(\xi_{\mathbf{r}}^{(i)})$ is 0. Suppose that the random variables

$\xi_{\mathbf{r}}^{(i)}, \mathbf{r} \in \mathscr{X}, 1 \leqslant i \leqslant q$ are *mutually independent*. Note that the *existence* of this sequence of independent random variables follows easily from a standard product measure construction or one can use Kolmogorov's extension theorem (see any textbook). Let $(\Omega, \mathscr{F}, \Pr)$ denote the underlying probability measure space. Consider the following sum

$$\sum_{\mathbf{r} \in \mathscr{X}} \left(\xi_{\mathbf{r}}^{(1)} + \frac{1}{q} \right) g_{\mathbf{r}}. \tag{2.14}$$

Observe that in (2.14) the coefficient of $g_{\mathbf{r}}$ is 0 or 1. In other words, (2.14) represents a "random 0–1-sum" of $g_{\mathbf{r}}$'s, that is,

$$\sum_{\mathbf{r} \in \mathscr{X}} \left(\xi_{\mathbf{r}}^{(1)} + \frac{1}{q} \right) g_{\mathbf{r}} = \sum_{\mathbf{r} \in \tilde{\mathscr{A}}_1} g_{\mathbf{r}}$$

where $\tilde{\mathscr{A}}_1$ is a "random subset" of $\mathscr{X}$. Similarly, we have

$$\sum_{\mathbf{r} \in \mathscr{X} \setminus \tilde{\mathscr{A}}_1} \xi_{\mathbf{r}}^{(2)} g_{\mathbf{r}} + \frac{1}{q} \sum_{\mathbf{r} \in \mathscr{X} \setminus \tilde{\mathscr{A}}_1} g_{\mathbf{r}} = \sum_{\mathbf{r} \in \tilde{\mathscr{A}}_2} g_{\mathbf{r}}$$

where $\tilde{\mathscr{A}}_2$ is a random subset of $\tilde{\mathscr{X}}_1 = \mathscr{X} \setminus \tilde{\mathscr{A}}_1$.

Repeating this argument q times, we have with

$$\tilde{\mathscr{X}}_{i-1} = \mathscr{X} \setminus (\tilde{\mathscr{A}}_1 \cup \tilde{\mathscr{A}}_2 \cup \ldots \cup \tilde{\mathscr{A}}_{i-1}),$$

$$\sum_{\mathbf{r} \in \tilde{\mathscr{X}}_{i-1}} \xi_{\mathbf{r}}^{(i)} g_{\mathbf{r}} + \frac{1}{q} \sum_{\mathbf{r} \in \tilde{\mathscr{X}}_{i-1}} g_{\mathbf{r}} = \sum_{\mathbf{r} \in \tilde{\mathscr{A}}_i} g_{\mathbf{r}}, \quad (i = 1, \ldots, q) \tag{2.15}$$

where $\tilde{\mathscr{A}}_i$ is a random subset of $\tilde{\mathscr{X}}_{i-1}$.

The following two probabilistic lemmas will be proved in Section 3.

LEMMA 2.8. *Assume that n is sufficiently large. Then*

$$\Pr\left(\sum_{i=1}^{q} |\tilde{\mathscr{A}}_i| \geqslant |\mathscr{X}|/2 \right) > 1/2.$$

LEMMA 2.9. *Let λ be a real number satisfying*

$$\log q \leqslant \lambda \leqslant 2(n+1)q^{-1/2}.$$

For every $1 \leqslant i \leqslant q$, let

$$\tilde{\mathscr{U}}_i(\lambda) = \left\{ \mathbf{x} \in U^3 : \sum_{\mathbf{r} \in \tilde{\mathscr{X}}_{i-1}} \xi_{\mathbf{r}}^{(i)} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\lambda(n+1)q^{-1/2} \right\} \tag{2.16}$$

(*note that* $\tilde{\mathscr{U}}_i(\lambda)$ *is a random subset of* U^3). *Then*

$$\Pr\{\mathrm{Vol}\,(\tilde{\mathscr{U}}_i(\lambda)) \leqslant e^{-\lambda^2/4} \quad \text{for every} \quad 1 \leqslant i \leqslant q\} > 1/2$$

provided n is sufficiently large. Here Vol *stands for the three-dimensional Lebesgue measure.*

By using Lemmas 2.2–2.9, one can complete the proof of Theorem 1.2 as follows. Let

$$\lambda = \tfrac{1}{2}q^{(1/4)+\varepsilon}. \tag{2.17}$$

From Lemmas 2.8–2.9, we have

$$\Pr\left\{\sum_{l=1}^{q} |\tilde{\mathscr{A}}_l| \geqslant |\mathscr{X}|/2 \quad \text{and} \quad \mathrm{Vol}(\tilde{\mathscr{U}}_i(\lambda)) \leqslant e^{-\lambda^2/4} \forall i \ (1 \leqslant i \leqslant q)\right\} > 0.$$

Thus there is a sequence $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_q$ of q disjoint deterministic subsets of $\mathscr{X}$ such that

$$\sum_{l=1}^{q} |\mathscr{A}_l| \geqslant |\mathscr{X}|/2, \tag{2.18}$$

and, by using (2.15)–(2.17) and the notation $\mathscr{X}_{i-1} = \mathscr{X} \backslash (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_{i-1})$,

$$\mathrm{Vol}\left\{\mathbf{x} \in U^3 \colon \sum_{\mathbf{r} \in \mathscr{A}_i} g_{\mathbf{r}}(\mathbf{x}) - \frac{1}{q} \sum_{\mathbf{r} \in \mathscr{X}_{i-1}} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\frac{n+1}{2(q^{(1/4)-\varepsilon})}\right\} \leqslant e^{-\lambda^2/4} \tag{2.19}$$

for every $1 \leqslant i \leqslant q$.

We can now define the desired auxiliary function $F(\mathbf{x})$: let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_q$ be a sequence of $q = [(\log n)^{(1/2)-\varepsilon}]$ disjoint subsets of $\mathscr{X}$ satisfying (2.18) and (2.19), and let

$$F(\mathbf{x}) = S_1(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}).$$

We have to check inequalities (2.1) and (2.2). Combining (2.5), (2.12), (2.18) and Lemma 2.5, inequality (2.1) follows easily. It remains to verify (2.2).

By (2.7), (2.12) and Lemma 2.6, we have

$$\begin{aligned}\int_{U^3} |F(\mathbf{x})|\,d\mathbf{x} &= \int_{U^3} |P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) - 1 - S_2(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})|\,d\mathbf{x} \\ &\leqslant \int_{U^3} |P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})|\,d\mathbf{x} + 1 + \int_{U^3} |S_2(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})|\,d\mathbf{x} \\ &\leqslant \int_{U^3} |P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})|\,d\mathbf{x} + 1 + o(1).\end{aligned} \tag{2.20}$$

Therefore, in order to prove (2.2), it suffices to prove

$$\int_{U^3} |P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})| \, d\mathbf{x} < 1 + \varepsilon. \tag{2.21}$$

For every $1 \leqslant i \leqslant q$, let

$$Z_i = \left\{ \mathbf{x} \in U^3 : \sum_{\mathbf{r} \in \mathscr{A}_i} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\frac{n+1}{q^{(1/4)-\varepsilon}} \right\} \tag{2.22}$$

For notational convenience, write

$$P(\mathbf{x}) = P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}),$$

and for $i = 1, 2$, write

$$S_i(\mathbf{x}) = S_i(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}).$$

Since

$$\varrho = \frac{q^{(1/4)-\varepsilon}}{n+1},$$

from (2.6) and (2.22), we have

$$\begin{aligned} \int_{U^3} |P(\mathbf{x})| \, d\mathbf{x} &= \int_{U^3} P(\mathbf{x}) \, d\mathbf{x} + 2 \int_{\{\mathbf{x}: P(\mathbf{x}) < 0\}} |P(\mathbf{x})| \, d\mathbf{x} \\ &\leqslant \int_{U^3} P(\mathbf{x}) \, d\mathbf{x} + 2 \sum_{i=1}^{q} \int_{Z_i} |P(\mathbf{x})| \, d\mathbf{x}. \end{aligned} \tag{2.23}$$

By (2.7) and Lemma 2.6,

$$\begin{aligned} \int_{U^3} P(\mathbf{x}) \, d\mathbf{x} &= \int_{U^3} (1 + S_1(\mathbf{x}) + S_2(\mathbf{x})) \, d\mathbf{x} \\ &\leqslant 1 + \int_{U^3} S_1(\mathbf{x}) \, d\mathbf{x} + \int_{U^3} |S_2(\mathbf{x})| \, d\mathbf{x} \\ &= 1 + \int_{U^3} S_1(\mathbf{x}) \, d\mathbf{x} + o(1). \end{aligned}$$

By (2.8), (2.9) and Lemma 2.2(c),

$$\int_{U^3} S_1(\mathbf{x})\,d\mathbf{x} = 0,$$

and so we have

$$\int_{U^3} P(\mathbf{x})\,d\mathbf{x} = 1 + o(1). \tag{2.24}$$

Let us return to (2.23). By Cauchy-Schwarz inequality, Lemma 2.7 and (2.12),

$$\begin{aligned}\int_{Z_i} |P(\mathbf{x})|\,d\mathbf{x} &\leqslant (\mathrm{Vol}(Z_i))^{1/2}\left(\int_{U^3} (P(\mathbf{x}))^2\,d\mathbf{x}\right)^{1/2}\\ &\leqslant (\mathrm{Vol}(Z_i))^{1/2}\,(2e^{q^{(1/2)-\varepsilon}})^{1/2}.\end{aligned} \tag{2.25}$$

We are going to estimate $\mathrm{Vol}(Z_i)$ from above. For every $1 \leqslant i \leqslant q$, let, with $\mathscr{X}_{i-1} = \mathscr{X}\backslash(\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1})$,

$$U_i = \left\{\mathbf{x} \in U^3\colon \sum_{\mathbf{r}\in\mathscr{A}_i} g_{\mathbf{r}}(\mathbf{x}) - \frac{1}{q}\sum_{\mathbf{r}\in\mathscr{X}_{i-1}} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\frac{n+1}{2(q^{(1/4)-\varepsilon})}\right\}$$

and

$$\begin{aligned}V_i &= \left\{\mathbf{x} \in U^3\colon \frac{1}{q}\sum_{\mathbf{r}\in\mathscr{X}_{i-1}} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\frac{n+1}{2(q^{(1/4)-\varepsilon})}\right\}\\ &= \left\{\mathbf{x} \in U^3\colon \sum_{\mathbf{r}\in\mathscr{X}_{i-1}} g_{\mathbf{r}}(\mathbf{x}) \leqslant -\frac{q^{(3/4)+\varepsilon}(n+1)}{2}\right\}.\end{aligned}$$

By (2.22),

$$Z_i \subset U_i \cup V_i,$$

and so

$$\mathrm{Vol}(Z_i) \leqslant \mathrm{Vol}(U_i) + \mathrm{Vol}(V_i) \quad (1 \leqslant i \leqslant q). \tag{2.26}$$

From (2.17) and (2.19), we have

$$\mathrm{Vol}(U_i) \leqslant e^{-\lambda^2/4} \leqslant e^{-q^{1/2}} \tag{2.27}$$

provided q is sufficiently large. Next, let $\mathscr{Y} = \mathscr{X}_{i-1} = \mathscr{X} \backslash (\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1})$. Applying Lemma 2.3 with

$$m = [q^{1/2}] \tag{2.28}$$

(integral part), we obtain

$$(2m)^{3m}(n+1)^{2m} > \int_{U^3} \left(\sum_{\mathbf{r} \in \mathscr{Y}} g_{\mathbf{r}}(\mathbf{x}) \right)^{2m} d\mathbf{x}$$
$$\geqslant \mathrm{Vol}(V_i) \cdot (q^{(3/4)+\varepsilon}(n+1)/2)^{2m}.$$

By (2.28) we have

$$\mathrm{Vol}(V_i) \leqslant \left(\frac{8m^3}{q^{(3/2)+2\varepsilon}/4} \right)^m < e^{-m} \tag{2.29}$$

provided q is sufficiently large. By (2.26)–(2.29),

$$\mathrm{Vol}(Z_i) \leqslant e^{-q^{1/2}} + e^{-m} \leqslant 2e^{-m} \tag{2.30}$$

provided q is sufficiently large. Thus, by using (2.25), (2.28) and (2.30),

$$\int_{Z_i} |P(\mathbf{x})| \, d\mathbf{x} \leqslant (2e^{-m} \cdot 2e^{q^{(1/2)-\varepsilon}})^{1/2} \leqslant q^{-2} \tag{2.31}$$

provided q is sufficiently large. Summarizing, by (2.23), (2.24) and (2.31),

$$\int_{U^3} |P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x})| \, d\mathbf{x} = \int_{U^3} |P(\mathbf{x})| \, d\mathbf{x} \leqslant 1 + o(1) + 2q \cdot q^{-2} < 1 + \varepsilon, \tag{2.32}$$

provided q is sufficiently large. From (2.5), (2.12) and (2.32), we conclude that (2.21) holds if N is sufficiently large. Finally, inequalities (2.20) and (2.21) give (2.2). This completes the deduction of Theorem 1.2 from Lemmas 2.2–2.9.

3. Proofs of Lemmas 2.5, 2.8 and 2.9

We begin with the proofs of the probabilistic lemmas.

Proof of Lemma 2.8. From (2.13) and (2.15), we have

$$\sum_{i=1}^{q} |\tilde{\mathscr{A}}_i| = \sum_{\mathbf{r} \in \mathscr{X}} \chi_{\mathbf{r}}$$

where

$$\chi_{\mathbf{r}} = 1 - \prod_{i=1}^{q} (1 - 1/q - \xi_{\mathbf{r}}^{(i)}).$$

We have

$$\Pr(\chi_{\mathbf{r}} = 0) = (1 - 1/q)^q, \quad \Pr(\chi_{\mathbf{r}} = 1) = 1 - (1 - 1/q)^q,$$

and these events are mutually independent for all $\mathbf{r} \in \mathscr{X}$. The expectation of

$$\zeta = \sum_{\mathbf{r} \in \mathscr{X}} \chi_{\mathbf{r}}$$

is

$$\mathrm{E}\zeta = |\mathscr{X}|(1 - (1 - 1/q)^q) \approx |\mathscr{X}|(1 - 1/e)$$

and the variance is ζ is

$$\mathrm{Var}(\zeta) = |\mathscr{X}|(1 - (1 - 1/q)^q)(1 - 1/q)^q \approx |\mathscr{X}|(1 - 1/e)1/e$$

(provided q is sufficiently large). We recall that $|\mathscr{X}| = (n + 1)(n + 2)/2$. By using Chebishev's inequality (i.e. the "second moment method")

$$\Pr(|\zeta - \mathrm{E}\zeta| \geqslant y(\mathrm{Var}(\zeta))^{1/2}) \leqslant y^{-2}$$

with

$$y = (-|\mathscr{X}|/2 + \mathrm{E}\zeta)(\mathrm{Var}(\zeta))^{-1/2},$$

we obtain

$$\Pr(\zeta \leqslant |\mathscr{X}|/2) \leqslant \Pr(|\zeta - \mathrm{E}\zeta| \geqslant y(\mathrm{Var}(\zeta))^{1/2}) \leqslant y^{-2} \leqslant 100n^{-2}$$

provided n and q are sufficiently large. Hence

$$\Pr\left\{\sum_{i=1}^{q} |\tilde{\mathscr{A}}_i| < \frac{|\mathscr{X}|}{2}\right\} = \Pr\left\{\sum_{\mathbf{r} \in \mathscr{X}} \chi_{\mathbf{r}} < \frac{|\mathscr{X}|}{2}\right\}$$
$$= \Pr\left\{\zeta < \frac{|\mathscr{X}|}{2}\right\} \leqslant 100n^{-2} < \tfrac{1}{2},$$

if n is sufficiently large. Since $q \approx (\log n)^{(1/2)-\varepsilon}$, Lemma 2.8 follows. □

Proof of Lemma 2.9. From (2.13) and (2.15) we have

$$\sum_{\mathbf{r}\in\mathcal{X}_{i-1}} \xi_{\mathbf{r}}^{(i)}\cdot g_{\mathbf{r}} = \sum_{\mathbf{r}\in\mathcal{X}} \xi_{\mathbf{r}}^{(i)}\cdot\chi_{\mathbf{r}}^{(i)}\cdot g_{\mathbf{r}}$$

where

$$\chi_{\mathbf{r}}^{(i)} = \prod_{j=1}^{i-1}\left(1-\frac{1}{q}-\xi_{\mathbf{r}}^{(j)}\right).$$

Let $i\in\{1,2,\ldots,q\}$ be fixed, and let

$$\eta_{\mathbf{r}} = \eta_{\mathbf{r}}^{(i)} = \xi_{\mathbf{r}}^{(i)}\cdot\chi_{\mathbf{r}}^{(i)} \quad (\mathbf{r}\in\mathcal{X},\ 1\leqslant i\leqslant q).$$

Since $\xi_{\mathbf{r}}^{(i)}$ and $\chi_{\mathbf{r}}^{(i)}$ are independent, the expectation of $\eta_{\mathbf{r}}$ is

$$\mathrm{E}\eta_{\mathbf{r}} = \mathrm{E}\xi_{\mathbf{r}}^{(i)}\cdot\mathrm{E}\chi_{\mathbf{r}}^{(i)} = 0.$$

Moreover, we have $|\eta_{\mathbf{r}}|\leqslant 1$, and the variance of $\eta_{\mathbf{r}}$ is

$$\mathrm{Var}(\eta_{\mathbf{r}}) = \mathrm{E}(\eta_{\mathbf{r}})^2 \leqslant \mathrm{E}(\xi_{\mathbf{r}}^{(i)})^2 = (1-1/q)1/q < 1/q.$$

Denote by $(\Omega,\mathcal{F},\Pr)$ the underlying probability measure space.

Let

$$\zeta(\omega,\mathbf{x}) = \sum_{\mathbf{r}\in\mathcal{X}} \eta_{\mathbf{r}}(\omega)g_{\mathbf{r}}(\mathbf{x}) = \sum_{\mathbf{r}\in\mathcal{X}_{i-1}} \xi_{\mathbf{r}}^{(i)}(\omega)g_{\mathbf{r}}(\mathbf{x}) \quad (\omega\in\Omega,\mathbf{x}\in U^3).$$

From the mutual independence of the random variables $\eta_{\mathbf{r}}, \mathbf{r}\in\mathcal{X}$, for every $\mathbf{x}\in U^3$, we have

$$\int_\Omega e^{t\zeta(\omega,\mathbf{x})}\,d\Pr(\omega) = \mathrm{E}(e^{t\zeta(\mathbf{x})}) = \prod_{\mathbf{r}\in\mathcal{X}} \mathrm{E}(e^{t\eta_{\mathbf{r}}g_{\mathbf{r}}(\mathbf{x})})$$

where the real parameter t will be specified later.

The linearity of the expectation yields

$$\begin{aligned}\mathrm{E}(e^{t\eta}) &= 1 + t\cdot\mathrm{E}\eta + \sum_{k=2}^{\infty}\frac{t^k\cdot\mathrm{E}(\eta^k)}{k!}\\ &\leqslant 1 + \sum_{k=2}^{\infty}\frac{|t|^k\cdot\mathrm{E}(|\eta|^k)}{k!}\\ &\leqslant 1 + \mathrm{E}(\eta^2)\sum_{k=2}^{\infty}\frac{|t|^k}{k!} \leqslant 1 + t^2\mathrm{E}(\eta^2)\end{aligned}$$

if $E\eta = 0$, $|\eta| \leqslant 1$, $|t| \leqslant 1$. Thus we have (note that $\exp(y) = e^y$)

$$\begin{aligned} E(\exp(t\eta_r g_r(\mathbf{x}))) &\leqslant 1 + t^2 E(\eta_r)^2 \\ &= 1 + t^2 \cdot \mathrm{Var}(\eta_r) < 1 + t^2/q, \end{aligned}$$

and so

$$E(\exp(t \cdot \zeta(\mathbf{x}))) < (1 + t^2/q)^{|\mathscr{X}|}$$

if $|t| \leqslant 1$.

We recall

$$\tilde{U}_i(\lambda) = \{\mathbf{x} \in U^3 \colon \zeta(\omega, \mathbf{x}) \leqslant -\lambda(n+1)q^{-1/2}\}.$$

Let

$$t = -\lambda(n+1)^{-1} q^{1/2}/2.$$

By the hypothesis of the lemma, $|t| \leqslant 1$. Therefore,

$$\begin{aligned} (1 + t^2/q)^{|\mathscr{X}|} &\geqslant \int_{U^3} E(\exp(t\zeta(\mathbf{x})))\, d\mathbf{x} \\ &= \int_{\Omega} \left(\int_{U^3} \exp(t\zeta(\omega, \mathbf{x}))\, d\mathbf{x} \right) d\Pr(\omega) \\ &\geqslant \int_{\Omega} \mathrm{Vol}\,(\tilde{U}_i(\lambda)) \exp(|t|\lambda(n+1)q^{-1/2})\, d\Pr(\omega), \end{aligned}$$

and by using $|\mathscr{X}| = (n+1)(n+2)/2$, we have for $n \geqslant 2$,

$$\begin{aligned} E(\mathrm{Vol}(\tilde{U}_i(\lambda))) &\leqslant (1 + t^2/q)^{|\mathscr{X}|} \exp(-|t|\lambda(n+1)q^{-1/2}) \\ &\leqslant \exp(t^2(n+1)(n+2)/(2q) - |t|\lambda(n+1)q^{-1/2}) \\ &\leqslant e^{-\lambda^2/3}. \end{aligned}$$

Hence

$$\Pr\{\mathrm{Vol}(\tilde{U}_i(\lambda)) \geqslant e^{-\lambda^2/4}\} \cdot e^{-\lambda^2/4} \leqslant E(\mathrm{Vol}(\tilde{U}_i(\lambda))) \leqslant e^{-\lambda^2/3},$$

and so we have

$$\Pr\{\mathrm{Vol}(\tilde{U}_i(\lambda)) \geqslant e^{-\lambda^2/4}\} \leqslant e^{-\lambda^2/3 + \lambda^2/4} = e^{-\lambda^2/12}.$$

It follows that

$$\Pr\{\mathrm{Vol}(\tilde{U}_i(\lambda)) < e^{-\lambda^2/4} \ \forall i (1 \leqslant i \leqslant q)\} \geqslant 1 - qe^{-\lambda^2/12} > \tfrac{1}{2},$$

since, by hypothesis, $\lambda \geqslant \log q$, and $q \approx (\log n)^{1/2-\varepsilon}$ is sufficiently large. Lemma 2.9 follows. □

Proof of Lemma 2.5. Note that

$$S_1(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) = \sum_{i=1}^{q} \sum_{\mathbf{r} \in \mathscr{A}_i} \varrho \cdot g_{\mathbf{r}}(\mathbf{x}) + \sum_{l=2}^{q} \varrho^l \cdot G_l(\mathbf{x}) \tag{3.1}$$

where

$$\varrho = \frac{q^{(1/4)-\varepsilon}}{n+1}$$

and for $l = 2, 3, \ldots, q$,

$$G_l(\mathbf{x}) = \sum_{1 \leqslant v_1 < \cdots < v_l \leqslant q} {\sum}' g_{\mathbf{r}_1}(\mathbf{x}) \cdots g_{\mathbf{r}_l}(\mathbf{x}) \tag{3.2}$$

where the summation Σ' is taken over all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ of index-vectors

$$\mathbf{r}_j = (r_{j1}, r_{j2}, r_{j3})$$

such that

$$\mathbf{r}_j \in \mathscr{A}_{v_j} \quad \forall j (1 \leqslant j \leqslant l), \tag{3.3}$$

and $r_{j1} \neq r_{k1}, r_{j2} \neq r_{k2}, r_{j3} \neq r_{k3} \quad \forall j \, \forall k (1 \leqslant j < k \leqslant l)$

(i.e. there is no coincidence among the corresponding coordinates of $\mathbf{r}_1, \ldots, \mathbf{r}_l$).

By Lemma 2.2(d), every product $g_{\mathbf{r}_1} \ldots g_{\mathbf{r}_l}$ in (3.2) forms an s-function $g_{\mathbf{s}}(\mathbf{x})$ where $\mathbf{s} = (s_1, s_2, s_3)$ and for every $i = 1, 2, 3$,

$$s_i = \max\{r_{1i}, r_{2i}, \ldots, r_{li}\}.$$

For notational convenience, write

$$\mathbf{s} = \max\{\mathbf{r}_1, \ldots, \mathbf{r}_l\}.$$

We decompose the integral

$$\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x} = \int_{U^3} g_{\mathbf{s}}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x} \tag{3.4}$$

into integrals over s-boxes. Let $B \subset U^3$ be an s-box given by

$$B = \{\mathbf{x} \in U^3 \colon m_i 2^{-s_i} \leqslant x_i < (m_i + 1)2^{-s_i}, i = 1, 2, 3\}.$$

Let B^* be the box

$$B^* = \{\mathbf{x} \in U^3 \colon m_i 2^{-s_i} \leqslant x_i < (m_i + \tfrac{1}{2})2^{-s_i},\ i = 1, 2, 3\}.$$

It follows from the definition of modified Rademacher functions that the integral

$$\int_B g_{\mathbf{s}}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}$$

is equal to

$$\int_{B^*} \sum_{\varepsilon_1=0}^{1} \sum_{\varepsilon_2=0}^{1} \sum_{\varepsilon_3=0}^{1} (-1)^{\varepsilon_1+\varepsilon_2+\varepsilon_3+1} D((y_1 + \varepsilon_1 2^{-s_1-1}, y_2 + \varepsilon_2 2^{-s_2-1}, y_3 + \varepsilon_3 2^{-s_3-1}))\, d\mathbf{y} \quad \text{or its negative.} \tag{3.5}$$

We recall that

$$D(\mathbf{x}) = \sum_{\mathbf{P}_i \in B(\mathbf{x})} 1 - N x_1 x_2 x_3 = Z(\mathbf{x}) - N x_1 x_2 x_3 \tag{3.6}$$

where $Z(\mathbf{x})$ denotes the number of points of the given N-element set $\mathscr{P} = \{\mathbf{P}_1, \ldots, \mathbf{P}_N\}$ in the box

$$B(\mathbf{x}) = \{\mathbf{z} \in U^3 \colon 0 \leqslant z_i < x_i,\ i = 1, 2, 3\}.$$

We claim

$$\left| \sum_{\varepsilon_1=0}^{1} \sum_{\varepsilon_2=0}^{1} \sum_{\varepsilon_3=0}^{1} (-1)^{\varepsilon_1+\varepsilon_2+\varepsilon_3+1} Z((y_1 + \varepsilon_1 2^{-s_1-1}, y_2 + \varepsilon_2 2^{-s_2-1}, y_3 + \varepsilon_3 2^{-s_3-1})) \right| \leqslant |\mathscr{P} \cap B|. \tag{3.7}$$

Indeed, the left-hand side of (3.7) is the number of points of $\mathscr{P}$ in the box

$$\prod_{i=1}^{3} [y_i, y_i + 2^{-s_i-1}).$$

This box is contained in B, and (3.7) follows.

Note also that

$$\sum_{\varepsilon_1=0}^{1} \sum_{\varepsilon_2=0}^{1} \sum_{\varepsilon_3=0}^{1} (-1)^{\varepsilon_1+\varepsilon_2+\varepsilon_3+1} \prod_{i=1}^{3} (y_i + \varepsilon_i 2^{-s_i-1})$$
$$= 2^{-s_1-s_2-s_3-3} = 2^{-|\mathbf{s}|-3}. \tag{3.8}$$

From (3.5)–(3.8) we have

$$\left|\int_{U^3} g_{\mathbf{s}}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}\right| \leqslant \sum_B{}' \left|\int_B g_{\mathbf{s}}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}\right|$$
$$\leqslant \sum_B{}' \operatorname{Vol}(B^*)(|\mathscr{P} \cap B| + N 2^{-|\mathbf{s}|-3}),$$

where the summation Σ' is taken over all s-boxes in U^3.

By using

$$\operatorname{Vol}(B^*) = 2^{-|\mathbf{s}|-3}$$

and

$$\sum_B{}' |\mathscr{P} \cap B| = |\mathscr{P}| = N,$$

we obtain

$$\left|\int_{U^3} g_{\mathbf{s}}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}\right| \leqslant 2^{-|\mathbf{s}|-3} N + 2^{|\mathbf{s}|} 2^{-|\mathbf{s}|-3} N 2^{-|\mathbf{s}|-3} < N 2^{-|\mathbf{s}|-2},$$

that is, by (3.4),

$$\left|\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_l}(\mathbf{x}) D(\mathbf{x})\, d\mathbf{x}\right| < N 2^{-|\mathbf{s}|-2} \tag{3.9}$$

where

$$\mathbf{s} = \max\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_l\}.$$

By (3.2) and (3.9), we have (we also use the disjointness of the sets $\mathscr{A}_1, \ldots, \mathscr{A}_q \subset \mathscr{X}$)

$$\left|\int_{U^3} G_l(\mathbf{x}) D(\mathbf{x})\,d\mathbf{x}\right|$$

$$\leqslant \sum_{1\leqslant v_1<\cdots<v_l\leqslant q} \sum_{\substack{(\mathbf{r}_1,\ldots,\mathbf{r}_l)\\(3.3)}}' \left|\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_l}(\mathbf{x}) D(\mathbf{x})\,d\mathbf{x}\right|$$

$$\leqslant \sum_{1\leqslant v_1<\cdots<v_l\leqslant q} \sum_{\substack{(\mathbf{r}_1,\ldots,\mathbf{r}_l)\\(3.3)}} N 2^{-|\max\{\mathbf{r}_1,\ldots,\mathbf{r}_l\}|-2}$$

$$\leqslant \sum_{\{\mathbf{r}_1,\ldots,\mathbf{r}_l\}\subset\mathscr{X}} N 2^{-|\max\{\mathbf{r}_1,\ldots,\mathbf{r}_l\}|-2}. \tag{3.10}$$

Let $\mathbf{r} = (r_1, r_2, r_3)$ with $|\mathbf{r}| = n$ and $\mathbf{s} = (s_1, s_2, s_3)$ with $|\mathbf{s}| = h$ such that $0 \leqslant r_i \leqslant s_i, i = 1, 2, 3$. Then

$$n - s_2 - s_3 \leqslant n - r_2 - r_3 = r_1 \leqslant s_1 = h - s_2 - s_3,$$

that is,

$$r_1 \in [n - s_2 - s_3, h - s_2 - s_3]. \tag{3.11}$$

Similarly,

$$r_2 \in [n - s_1 - s_3, h - s_1 - s_3]. \tag{3.12}$$

Since r_1 and r_2 determine the triplet $\mathbf{r} = (r_1, r_2, r_3)$, from (3.11) and (3.12) we get that the number of l-sets

$$\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} \subset \mathscr{X} = \{\mathbf{r} : r_i \geqslant 0, i = 1, 2, 3, |\mathbf{r}| = n\}$$

such that $\max\{\mathbf{r}_1, \ldots, \mathbf{r}_l\} = \mathbf{s}$ is fixed and $|\mathbf{s}| = h$, is less or equal than

$$((h - n + 1)^2)^l = (h - n + 1)^{2l}.$$

Note that the number of triplets $\mathbf{s} = (s_1, s_2, s_3), s_i \geqslant 0, i = 1, 2, 3$, with $|\mathbf{s}| = s_1 + s_2 + s_3 = h$ is $(h + 1)(h + 2)/2$. Returning now to (3.10), we obtain

$$\left|\int_{U^3} G_l(\mathbf{x}) D(\mathbf{x})\,d\mathbf{x}\right| \leqslant \sum_{h=n+l-1}^{3n} \frac{(h+1)(h+2)}{2}(h - n + 1)^{2l} N 2^{-h-2}$$

$$\leqslant n^2 \sum_{h=n+l-1}^{3n} (h - n + 1)^{2l} 2^{-(h-n+1)},$$

since $4N > 2^n \geqslant 2N$. Hence

$$\left|\int_{U^3}\left(\sum_{l=2}^{q}\rho^l G_l(\mathbf{x})\right)D(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|$$

$$\leqslant \sum_{l=2}^{q}\varrho^l\left|\int_{U^3} G_l(\mathbf{x})\,D(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|$$

$$\ll \sum_{l=2}^{q} n^2\varrho^l \sum_{h=n+l-1}^{3n}(h-n+1)^{2l}2^{-(h-n+1)}. \tag{3.13}$$

Since

$$q \leqslant (\log n)^{(1/2)-\varepsilon} \leqslant \log n \quad \text{and} \quad \varrho = \frac{q^{(1/4)-\varepsilon}}{n+1} \leqslant \frac{\log n}{n},$$

we have

$$\sum_{l=2}^{q} n^2\varrho^l \sum_{h=n+l-1}^{3n}(h-n+1)^{2l}2^{-(h-n+1)}$$

$$\leqslant \sum_{l=2}^{\log n}\left(\frac{\log n}{n}\right)^l \sum_{h=n+l-1}^{3n}(h-n+1)^{2l}2^{-(h-n+1)}$$

$$\leqslant \sum_{l=2}^{\log n}\frac{(\log n)^l}{n^{l-2}}\sum_{k=1}^{\infty}k^{2l}2^{-k}$$

$$= \sum_{l=2}^{\log n}\sum_{k=l}^{\infty}\frac{(k^2\log n)^l}{n^{l-2}}2^{-k}. \tag{3.14}$$

If n is sufficiently large, then

$$\sum_{l=2}^{\log n}\sum_{k=1}^{\infty}\frac{(k^2\log n)^l}{n^{l-2}}2^{-k}$$

$$= \sum_{k=2}^{\infty}(k^2\log n)^2 2^{-k} + \sum_{l=3}^{\log n}\sum_{k=1}^{\infty}\frac{(k^2\log n)^l}{n^{l-2}}2^{-k}$$

$$= O((\log n)^2) + \sum_{l=3}^{\log n}\sum_{k=1}^{\infty}\frac{k^{2l}}{n^{l/4}}2^{-k}$$

$$= O((\log n)^2) + \sum_{k=1}^{n^{1/8}/2} \sum_{l=3}^{\log n} \left(\frac{k}{n^{1/8}}\right)^{2l} 2^{-k} +$$

$$+ \sum_{k>n^{1/8}/2} \sum_{l=3}^{\log n} \frac{k^{2l}2^{-k}}{n^{l/4}}$$

$$= O((\log n)^2) + O(1) + O(2^{-n^{1/9}}) = O((\log n)^2). \tag{3.15}$$

Thus, by (3.13)–(3.15),

$$\left| \int_{U^3} \left(\sum_{l=2}^{q} \varrho^l G_l(\mathbf{x}) \right) D(\mathbf{x})\, d\mathbf{x} \right| = O((\log n)^2). \tag{3.16}$$

On the other hand, from Lemma 2.4 we have

$$\int_{U^3} \left(\sum_{i=1}^{q} \sum_{\mathbf{r}\in\mathscr{A}_i} \varrho g_{\mathbf{r}}(\mathbf{x}) \right) D(\mathbf{x})\, d\mathbf{x}$$

$$\geqslant \varrho(2^{-n-7}N) \sum_{i=1}^{q} |\mathscr{A}_i|$$

$$\geqslant \varrho 2^{-10}|\mathscr{X}| = \frac{q^{(1/4)-\varepsilon}}{n+1} 2^{-10} \frac{(n+1)(n+2)}{2}$$

$$> 2^{-11} q^{(1/4)-\varepsilon}(n+1), \tag{3.17}$$

since $4N > 2^n \geqslant 2N$ and $\Sigma_{i=1}^{q} |\mathscr{A}_i| \geqslant |\mathscr{X}|/2$. From (3.1), (3.16), (3.17), we conclude that

$$\int_{U^3} S_1(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}) D(\mathbf{x})\, d\mathbf{x} \geqslant \int_{U^3} \left(\sum_{i=1}^{q} \sum_{\mathbf{r}\in\mathscr{A}_i} g_{\mathbf{r}}(\mathbf{x}) \right) D(\mathbf{x})\, d\mathbf{x} -$$

$$- \left| \int_{U^3} \left(\sum_{l=2}^{q} \rho^l G_l(\mathbf{x}) \right) D(\mathbf{x})\, d\mathbf{x} \right|$$

$$> 2^{-11} q^{(1/4)-\varepsilon}(n+1) - O((\log n)^2) > 2^{-12} q^{(1/4)-\varepsilon}(n+1)$$

if n is sufficiently large. This proves Lemma 2.5. □

4. Proofs of Lemmas 2.6 and 2.7

Here and in the following section we shall utilize some elementary concepts and facts from graph theory. We shall use the following standard terminology.

A *graph* $G = (V, E)$ consists of a finite nonempty set $V = V(G)$ of *vertices* and a finite set $E = E(G)$ of *edges*. With every edge, an unordered pair of vertices, called its *endvertices*, is associated. We assume that the two endvertices of an edge are distinct. We denote an edge with endvertices u and v by $\{u, v\}$. Two edges are called *parallel* if they have the same endvertices. A graph without parallel edges is called *simple*. The *multiplicity* of an edge $e \in E$ of a graph G is the number of edges of G parallel to e (e is included).

A vertex that is not incident to any edge is called *isolated*. In this paper we exclusively deal with *graphs without isolated vertices*. For this graphs, the edge-set E uniquely determines the vertex-set $V =$ the set of endvertices of all edges in E. Therefore, it will not cause any misunderstanding to identify a graph without isolated vertices with its edge-set, i.e. $G = E(G)$.

Two vertices that are joined by an edge are called *neighbours*. The number of neighbours of a vertex $v \in V$ is the *degree* of the vertex. The *maximum degree* of a graph is the maximum of the degrees of its vertices.

A *matching* is a graph G such that no two edges of G have a common endvertex. A simple graph is called a *clique* if every two of its vertices are joined by an edge. A graph is called an *m-parallel clique* if every two of its vertices are joined by m parallel edges.

In a graph a *walk* is a finite sequence $v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$, in which vertices v_i and edges e_j appear alternatively such that for $i = 1, 2, \ldots, k$ the endvertices of every edge e_i are the vertices v_{i-1}, v_i. If $v_0, v_1, \ldots, v_k$ are distinct, the walk is called a *path of length k*. If $v_0, v_1, \ldots, v_{k-1}$ are distinct and $v_0 = v_k$, the walk is called a *circuit of length k*. A graph is *connected* if every two of its vertices are connected by a walk. The *components* of a graph are the maximal connected subgraphs of the graph.

In order to get a common generalization of Lemmas 2.6 and 2.7, we introduce the following notation. We recall that

$$\mathscr{X} = \{\mathbf{r} = (r_1, r_2, r_3)\colon r_i \geqslant 0, i = 1, 2, 3, \quad \text{and} \quad |\mathbf{r}| = r_1 + r_2 + r_3 = n\}.$$

Let p be an integer satisfying

$$p \geqslant q.$$

Let $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_p$ be arbitrary (not necessarily disjoint) subsets of $\mathscr{X}$. Let

$$\varrho = \frac{q^{(1/4)-\varepsilon}}{n+1},$$

and write

$$P(\mathscr{B}_1,\ldots,\mathscr{B}_p;\mathbf{x}) = \prod_{\nu=1}^{p}\left(1+\varrho\sum_{\mathbf{r}\in\mathscr{B}_\nu} g_{\mathbf{r}}(\mathbf{x})\right).$$

Expanding this product, we have

$$P(\mathscr{B}_1,\ldots,\mathscr{B}_p;\mathbf{x}) = 1 + \sum_{l=1}^{p}\ \sum_{1\leqslant\nu_1<\cdots<\nu_l\leqslant p}\ \sum_{\substack{\mathbf{r}_i\in\mathscr{B}_{\nu_i}\\ 1\leqslant i\leqslant l}} \varrho^l g_{\mathbf{r}_1}(\mathbf{x})\cdots g_{\mathbf{r}_l}(\mathbf{x}). \tag{4.1}$$

Every term $\varrho^l g_{\mathbf{r}_1}(\mathbf{x})\cdots g_{\mathbf{r}_l}(\mathbf{x})$ on the right-hand side of (4.1) can be uniquely represented by a sequence

$$(\mathbf{r}_1,\nu_1;\mathbf{r}_2,\nu_2;\ldots;\mathbf{r}_l,\nu_l) \quad \text{satisfying } \mathbf{r}_i\in\mathscr{B}_{\nu_i} \quad (1\leqslant i\leqslant l). \tag{4.2}$$

We shall associate with every sequence (4.2) a *graph*, called the "index-coincidence graph" of (4.2). For every $i = 1,2,3$, let

$$W_i = \{(k.i)\colon 1\leqslant k\leqslant p\} \quad \text{and} \quad W = W_1\cup W_2\cup W_3.$$

Let K_i denote the clique on the vertex-set W_i $(i = 1,2,3)$, and let $K = K_1\cup K_2\cup K_3$. Now the index-coincidence graph

$$G = G(\mathbf{r}_1,\nu_1;\mathbf{r}_2,\nu_2;\ldots;\mathbf{r}_l,\nu_l)$$

of (4.2) will be a subgraph of K as follows:

An edge $e = \{u,v\}\in K$ with $u = (k.i)$ and $v = (k'.i)$ belongs to G if and only if both k and k' occur among $\nu_1,\nu_2,\ldots,\nu_l$, let say $k = \nu_s$ and $k' = \nu_t$, and then $r_{si} = r_{ti}$ where $\mathbf{r}_s = (r_{s1},r_{s2},r_{s3})$ and $\mathbf{r}_t = (r_{t1},r_{t2},r_{t3})$.

Note that every index-coincidence graph G is the union of the vertex-disjoint subgraphs $G\cap K_i$ $(i = 1,2,3)$, and every subgraph $G\cap K_i$ is the union of vertex-disjoint cliques.

Let $W_0 = \{1,2,\ldots,p\}$, and let K_0 denote the 3-parallel clique on the vertex-set W_0. We shall associate with every subgraph $G\subseteq K$ another graph $G_0\subseteq K_0$ as follows. Let us consider the following 3 canonical bijections

$$\phi_i\colon W_i\to W_0 \quad \text{where} \quad \phi_i((k.i)) = k\ (1\leqslant k\leqslant p, 1\leqslant i\leqslant 3).$$

For every edge $e = \{u,v\}\in K$, let

$$\phi_i(e) = \{\phi_i(u),\phi_i(v)\}\in K_0.$$

Finally, for every $G \subseteq K$, let

$$G_0 = \phi_1(G \cap K_1) \cup \phi_2(G \cap K_2) \cup \phi_3(G \cap K_3) \subseteq K_0.$$

We call G_0 the "row-graph" of G. The 3 edges

$$e_i = \{(k.i), (k'.i)\} \in K \quad (1 \leqslant i \leqslant 3)$$

of K are called "row-parallel". Note that $\phi_1(e_1)$, $\phi_2(e_2)$, $\phi_3(e_3)$ are 3 parallel edges of K_0.

A subgraph $G \subseteq K$ is called "row-connected" if its row-graph G_0 is connected.

The fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$, implies that if $G \subseteq K$ is an index-coincidence graph of G contains two row-parallel edges, then it contains the third one as well. In other words, if $G \subseteq K$ is an index-coincidence graph, then every $e \in G_0$ has multiplicity 1 or 3.

A subgraph $G \subseteq K$ is called *special*, if it is the union of vertex-disjoint cliques and every $e \in G_0$ has multiplicity 1 or 3. Note that every index-coincidence graph $G \subseteq K$ is special.

A special graph $G \subseteq K$ is called *3-parallel*, if ever edge $e \in G_0$ has multiplicity 3 (note that the empty graph is 3-parallel).

A special graph $G \subseteq K$ is called *non-3-parallel*, if there is an edge $e \in G_0$ with multiplicity 1.

For every subgraph $G \subseteq K$ or $G \subseteq K_0$, let $V(G)$ denote the set of endvertices of all edges in G.

Let us return to (4.1). We have

$$\begin{aligned} P(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}) &= 1 + \sum_{l=1}^{p} \sum_{(\mathbf{r}_1, v_1; \ldots; \mathbf{r}_l, v_l)} \varrho^l g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \\ &= 1 + S_1(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}) + S_2(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}), \end{aligned} \tag{4.3}$$

where

$$S_1(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}) = \sum_{l=1}^{p} \sum{}' \varrho^l g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x})$$

and the summation Σ' extends over all sequences $(\mathbf{r}_1, v_1; \ldots; \mathbf{r}_l, v_l)$ such that the index-coincidence graph $G(\mathbf{r}_1, v_1; \ldots; \mathbf{r}_l, v_l)$ is 3-parallel, and

$$S_2(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}) = \sum_{l=2}^{p} \sum{}'' \varrho^l g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x})$$

where the summation Σ'' extends over all sequences $(\mathbf{r}_1, \nu_1; \ldots; \mathbf{r}_l, \nu_l)$ such that the index-coincidence graph $G(\mathbf{r}_1, \nu_1; \ldots; \mathbf{r}_l, \nu_l)$ is non-3-parallel.

The object of this section is to prove

LEMMA 4.1. *For arbitrary subsets* $\mathscr{B}_1, \ldots, \mathscr{B}_p \subset \mathscr{X}$, *we have*

$$\int_{U^3} |S_2(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x})| \, d\mathbf{x} < 2^{4p^2}(n+1)^{-1/8}$$

provided $p > p_1$ *(i.e. p is sufficiently large).*

First we derive Lemmas 2.6–2.7. from Lemma 4.1. Note that Lemma 2.6 follows easily by choosing

$$p = q, \quad \mathscr{B}_1 = \mathscr{A}_1, \quad \mathscr{B}_2 = \mathscr{A}_2, \ldots, \quad \mathscr{B}_q = \mathscr{A}_q.$$

The deduction of Lemma 2.7 is a slightly more difficult. Let

$$p = 2q, \quad \mathscr{B}_1 = \mathscr{B}_2 = \mathscr{A}_1, \quad \mathscr{B}_3 = \mathscr{B}_4 = \mathscr{A}_2, \ldots, \quad \mathscr{B}_{2q-1} = \mathscr{B}_{2q} = \mathscr{A}_q.$$

Then by (4.3) and Lemma 4.1, we have

$$\int_{U^3} (P(\mathscr{A}_1, \ldots, \mathscr{A}_q; \mathbf{x}))^2 \, d\mathbf{x} = \int_{U^3} P(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x}) \, d\mathbf{x}$$

$$= 1 + \int_{U^3} S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x}) \, d\mathbf{x} + \int_{U^3} S_2(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x}) \, d\mathbf{x}$$

$$\leqslant 1 + \int_{U^3} S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x}) \, d\mathbf{x} + \int_{U^3} |S_2(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x})| \, d\mathbf{x}$$

$$\leqslant 1 + \int_{U^3} S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x}) \, d\mathbf{x} + 2^{16q^2}(n+1)^{-1/8} \tag{4.4}$$

if q is sufficiently large.

Let $\varrho^l g_{\mathbf{r}_1} \ldots g_{\mathbf{r}_l}$ be an arbitrary term in $S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x})$. Since the index-coincidence graph is 3-parallel, we have

$$g_{\mathbf{r}_1} \ldots g_{\mathbf{r}_l} = (g_{\mathbf{r}_{i(1)}})^2 \ldots (g_{\mathbf{r}_{i(k)}})^2 \cdot g_{\mathbf{r}_{j(1)}} \ldots g_{\mathbf{r}_{j(l-2k)}}$$

where the index-coincidence graph of the term $g_{\mathbf{r}_{j(1)}} \ldots g_{\mathbf{r}_{j(l-2k)}}$ is the empty graph, i.e. there is *no* index-coincidence.

By Lemma 2.2(a),

$$g_{\mathbf{r}_1} \cdots g_{\mathbf{r}_l} = g_{\mathbf{r}_{j(1)}} \cdots g_{\mathbf{r}_{j(l-2k)}},$$

and if $l > 2k$, then by Lemma 2.2(c),

$$\int_{U^3} g_{\mathbf{r}_{j(1)}}(\mathbf{x}) \ldots g_{\mathbf{r}_{j(l-2k)}}(\mathbf{x})\, d\mathbf{x} = 0.$$

We conclude that the integral of the product $g_{\mathbf{r}_1} \ldots g_{\mathbf{r}_l}$ is 0 unless $\mathbf{r}_1, \ldots, \mathbf{r}_l$ form $l/2$ pairs of equal triplets $\mathbf{r}_i = (r_{i1}, r_{i2}, r_{i3})$. Hence

$$\int_{U^3} S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x})\, d\mathbf{x} = \int_{U^3} \left(\prod_{\nu=1}^{q} \left(1 + \varrho^2 \sum_{\mathbf{r} \in \mathscr{A}_\nu} (g_{\mathbf{r}}(\mathbf{x}))^2 \right) - 1 \right) d\mathbf{x}$$

$$= \prod_{\nu=1}^{q} (1 + \varrho^2 |\mathscr{A}_\nu|) - 1 < \exp\left(\varrho^2 \sum_{\nu=1}^{q} |\mathscr{A}_\nu| \right) \leqslant \exp(\varrho^2 |\mathscr{X}|).$$

Since

$$\varrho^2 |\mathscr{X}| = \left(\frac{q^{(1/4)-\varepsilon}}{n+1} \right)^2 (n+1)(n+2)/2 < q^{(1/2)-\varepsilon},$$

we have

$$\int_{U^3} S_1(\mathscr{B}_1, \ldots, \mathscr{B}_{2q}; \mathbf{x})\, d\mathbf{x} < \exp(q^{(1/2)-\varepsilon}). \tag{4.5}$$

Lemma 2.7 follows from (4.4) and (4.5).

Therefore, in order to complete the proof of Theorem 1.2, it suffices to prove Lemma 4.1.

Proof of Lemma 4.1. Let $H \subseteq K = K_1 \cup K_2 \cup K_3$ be an arbitrary subgraph. Let $|V(H_0)| = l$, i.e. the number of (non-isolated) vertices of the row-graph H_0 of H is l. Write

$$S(H; \mathbf{x}) = \sum_{G(\mathbf{r}_1, \nu_1; \ldots; \mathbf{r}_l, \nu_l) = H} \varrho^l g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}),$$

if H is an index-coincidence graph; otherwise let

$$S(H; \mathbf{x}) = 0.$$

Let

$$S(\supseteq H;\mathbf{x}) = \sum_{\substack{H\subseteq F\subseteq K\\ F\ i.c.g.}} S(F;\mathbf{x})$$

where *i.c.g.* stands for *index-coincidence graph.* Evidently

$$S(\supseteq H;\mathbf{x}) = \sum_{\substack{H\subseteq F\subseteq K\\ F\ \text{special}}} S(F;\mathbf{x}) = \sum_{H\subseteq F\subseteq K} S(F;\mathbf{x}).$$

For every $H \subseteq \tilde{K} \subseteq K$, let

$$S(\tilde{K} \supseteq, \supseteq H;\mathbf{x}) = \sum_{\substack{H\subseteq F\subseteq \tilde{K}\\ F\ i.c.g.}} S(F;\mathbf{x})$$

where *i.c.g.* stands for *index-coincidence graph.* Evidently

$$S(\tilde{K} \supseteq, \supseteq H;\mathbf{x}) = \sum_{\substack{H\subseteq F\subseteq \tilde{K}\\ F\ \text{special}}} S(F;\mathbf{x}) = \sum_{H\subseteq F\subseteq \tilde{K}} S(F;\mathbf{x}).$$

Let $\bar{H}$ be the minimal special graph such that

$$H \subseteq \bar{H} \subseteq K.$$

Note that $V(\bar{H}_0) = V(H_0)$. Since every index-coincidence graph is special, we have

$$S(\supseteq H;\mathbf{x}) = S(\supseteq \bar{H};\mathbf{x}) \tag{4.6}$$

and

$$S(\tilde{K} \supseteq, \supseteq H;\mathbf{x}) = S(\tilde{K} \supseteq, \supseteq \bar{H};\mathbf{x}). \tag{4.7}$$

Consider the decomposition of $\bar{H}$ into maximal row-connected subgraphs

$$\bar{H} = H^{(1)} \cup H^{(2)} \cup \cdots \cup H^{(h)}.$$

For every $i = 1, 2, \ldots, h$, let (note that $H_0^{(i)}$ is the row-graph of $H^{(i)}$, and $V(H_0^{(i)})$ is the set of (non-isolated) vertices of H_0)

$$V(H_0^{(i)}) = \{j_1^{(i)}, j_2^{(i)}, \ldots, j_{l(i)}^{(i)}\} \subset W_0 = \{1, 2, \ldots, p\}.$$

We have

$$S(\supseteq H;\mathbf{x}) = S(\supseteq \bar{H};\mathbf{x}) = \Pi_1 \cdot \Pi_2 \tag{4.8}$$

where

$$\Pi_1 = \prod_{i=1}^{h} \left(\varrho^{l_i} \sum_{G(\mathbf{r}_1^{(i)}, j_1^{(i)}; \ldots; \mathbf{r}_{l(i)}^{(i)}, j_{l(i)}^{(i)}) \supseteq H^{(i)}} g_{\mathbf{r}_1^{(i)}}(\mathbf{x}) \ldots g_{\mathbf{r}_{l(i)}^{(i)}}(\mathbf{x}) \right),$$

and

$$\Pi_2 = \tilde{\prod_{v}} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{B}_v} g_{\mathbf{r}}(\mathbf{x}) \right)$$

where the product $\tilde{\Pi}$ extends over all

$$v \in W_0 \backslash V(H_0) = \{1, 2, \ldots, p\} \backslash \bigcup_{i=1}^{h} \{j_1^{(i)}, \ldots, j_{l(i)}^{(i)}\}.$$

By definition, we have

$$S_2(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x}) = \sum_{H \subset K}^{*} S(H;\mathbf{x}) \tag{4.9}$$

where the summation Σ^* is extended over all non-3-parallel special subgraphs.

Let $H \subset K$ be an arbitrary non-3-parallel special subgraph. We distinguish two cases according as the row-graph H_0 of H has maximum degree 1 or $\geqslant 2$.

Case 1: H_0 has maximum degree 1.

Because H is non-3-parallel, there is an edge $e \in H_0$ with multiplicity 1. Let say

$$e_1 = \phi_1^{-1}(e) \in H, \quad e_2 = \phi_2^{-1}(e) \notin H, \quad e_3 = \phi_3^{-1}(e) \notin H.$$

We shall use the following version of the classical inclusion–exclusion principle.

Inclusion–exclusion formula: *Let $A_1, A_2, \ldots, A_t \subseteq \Omega$ where Ω is a finite set. Let $f\colon \Omega \to \mathbb{R}$ be a real-valued function defined on Ω. For each subset $T \subseteq \{1, 2, \ldots, t\}$, let*

$$A_T = \bigcap_{i \in T} A_i \quad \text{and} \quad A_{\emptyset} = \Omega.$$

Then

$$\sum_{\omega\in\Omega\setminus(\bigcup_{i=1}^{t}A_i)} f(\omega) = \sum_{T\subseteq\{1,\ldots,t\}} (-1)^{|T|} \sum_{\omega\in A_T} f(\omega).$$

Now let

$$\tilde{K} = K\setminus\{e_2, e_3\} \quad \text{and} \quad \tilde{K}\setminus H = \{e^{(1)}, e^{(2)}, \ldots, e^{(t)}\}.$$

Let

$$\Omega = A_{\emptyset} = \{F\colon H \subseteq F \subseteq \tilde{K}\},$$

and for $i = 1, \ldots, t$,

$$A_i = \{F\colon H \cup \{e^{(i)}\} \subseteq F \subseteq \tilde{K}\}.$$

Let $\mathbf{x} \in U^3$ be arbitrary but fixed. Let

$$f = f_{\mathbf{x}}\colon \Omega \to \mathbb{R}$$

be defined by

$$f(F) = S(F; \mathbf{x}), \quad F \in \Omega.$$

For every $T \subseteq \{1, \ldots, t\}$, we have

$$A_T = \bigcap_{i\in T} A_i = \{F\colon H \cup \{e^{(i)}\colon i \in T\} \subseteq F \subseteq \tilde{K}\}.$$

The inclusion–exclusion formula gives with $G = G_T = H \cup \{e^{(i)}\colon i \in T\}$,

$$S(H; \mathbf{x}) = \sum_{H\subseteq G\subseteq \tilde{K}} (-1)^{|G\setminus H|} S(\tilde{K} \supseteq, \supseteq G; \mathbf{x}). \tag{4.10}$$

Let G be a graph satisfying

$$H \subseteq G \subseteq \tilde{K} = K\setminus\{e_2, e_3\}$$

If the row-graph G_0 of G has maximum degree $\geqslant 2$, then we have

$$S(\tilde{K} \supseteq, \supseteq G; \mathbf{x}) = -S(\supseteq G \cup \{e_2, e_3\}; \mathbf{x}) + S(\supseteq G; \mathbf{x}). \tag{4.11}$$

Therefore, by (4.10) and (4.11),

$$S(H;\mathbf{x}) = \sum_G{}'(-1)^{|G\setminus H|}S(\tilde{K}\supseteq,\supseteq G;\mathbf{x}) - \\ - \sum_G{}''(-1)^{|G\setminus H|}S(\supseteq G;\mathbf{x}) + \sum_G{}'''(-1)^{|G\setminus H|}S(\supseteq G;\mathbf{x}) \tag{4.12}$$

where Σ' extends over all graphs G such that

$$H \subseteq G \subseteq \tilde{K} = K\setminus\{e_2, e_3\}$$

and the row-graph G_0 of G has maximum degree 1, Σ'' extends over all graphs G such that

$$H \cup \{e_2, e_3\} \subseteq G \subseteq K$$

and the row-graph G_0 has maximum degree $\geqslant 2$, Σ''' extends over all graphs G such that $H \subseteq G \subseteq \tilde{K}$ and G_0 has maximum degree $\geqslant 2$.

Case 2: H_0 has maximum degree $\geqslant 2$
Again from the inclusion–exclusion formula, we have

$$S(H;\mathbf{x}) = \sum_{H\subseteq G\subseteq K}(-1)^{|G\setminus H|}S(\supseteq G;\mathbf{x}). \tag{4.13}$$

The proof of Lemma 4.1 is based on the following two lemmas.

LEMMA 4.2. *Let $G \subset K$ be a non-3-parallel special subgraph. Let e_1, e_2, e_3 be three parallel edges in K such that*

$$e_1 \in G, \quad e_2 \notin G, \quad e_3 \notin G.$$

Suppose that the row-graph G_0 of G has maximum degree 1. Then, with

$$\tilde{K} = K\setminus\{e_2, e_3\},$$

$$\int_{U^3} |S(\tilde{K}\supseteq,\supseteq G;\mathbf{x})|\,d\mathbf{x} \leqslant (8p)^{4p}(n+1)^{-1/4}.$$

LEMMA 4.3. *Let $G \subseteq K$ be a special subgraph. Suppose that the row-graph G_0 of G has maximum degree $\geqslant 2$. Then*

$$\int_{U^3} |S(\supseteq G;\mathbf{x})|\,d\mathbf{x} < (100p)^{10p}(n+1)^{-1/8}.$$

First we derive Lemma 4.1 from Lemmas 4.2 and 4.3. Note that the total number of subgraphs $G \subseteq K = K_1 \cup K_2 \cup K_3$ is precisely

$$2^{\binom{p}{2}} \cdot 2^{\binom{p}{2}} \cdot 2^{\binom{p}{2}} = 2^{3\binom{p}{2}}.$$

Thus from (4.6), (4.7), (4.9), (4.12), (4.13) and Lemmas 4.2–4.3, we have

$$\int_{U^3} |S_2(\mathscr{B}_1, \ldots, \mathscr{B}_p; \mathbf{x})| \, d\mathbf{x}$$

$$\leqslant \sum_{\substack{H \subset K \\ \text{Case 1}}}^{*} \left(\sum_{G}' \int_{U^3} |S(\tilde{K} \supseteq, \supseteq \bar{G}; \mathbf{x})| \, d\mathbf{x} \right.$$

$$\left. + \sum_{G}'' \int_{U^3} |S(\supseteq \bar{G}; \mathbf{x})| \, d\mathbf{x} + \sum_{G}''' \int_{U^3} |S(\supseteq \bar{G}; \mathbf{x})| \, d\mathbf{x} \right)$$

$$+ \sum_{\substack{H \subset K \\ \text{Case 2}}}^{*} \sum_{H \subseteq G \subseteq K} \int_{U^3} |S(\supseteq \bar{G}; \mathbf{x})| \, d\mathbf{x}$$

$$\leqslant 2^{3\binom{p}{2}} \cdot 2^{3\binom{p}{2}} ((8p)^{4p}(n+1)^{-1/4} + (100p)^{10p}(n+1)^{-1/8}) < 2^{4p^2}(n+1)^{-1/8}$$

if $p > p_1$, i.e. p is sufficiently large. This proves Lemma 4.1.

The rest of this section is devoted to the proof of Lemma 4.2. Lemma 4.3 will be proved in the next section.

Proof of Lemma 4.2. Since the row-graph G_0 of G has maximum degree 1, G is a matching. That is, G is a union of vertex-disjoint edges

$$G = \{e_1, f^{(1)}, \ldots, f^{(t)}\} \cup \{f_1^{(t+1)}, f_2^{(t+1)}, f_3^{(t+1)}, \ldots, f_1^{(t+s)}, f_2^{(t+s)}, f_3^{(t+s)}\}$$

where $f_1^{(l)}, f_2^{(l)}, f_3^{(l)}$ $(t+1 \leqslant l \leqslant t+s)$ are row-parallel edges. Let

$$e_1 = \{(k.1), (k'.1)\},$$
$$f^{(j)} = \{(m_j.i_j), (m'_j.i_j)\} \quad (1 \leqslant j \leqslant t),$$
$$f_i^{(l)} = \{(m_l.i), (m'_l.i)\} \quad (t+1 \leqslant l \leqslant t+s, i = 1, 2, 3).$$

Analogously to (4.8), we have, with $\tilde{K} = K \backslash \{e_2, e_3\}$,

$$S(\tilde{K} \supseteq, \supseteq G; \mathbf{x}) = P_0(\mathbf{x}) \cdot P_1(\mathbf{x}) \cdot P_2(\mathbf{x}) \cdot P_3(\mathbf{x}) \tag{4.14}$$

where

$$P_0(\mathbf{x}) = \varrho^2 \sum_{\substack{\mathbf{r}_\alpha \in \mathscr{B}_k \\ \mathbf{r}_\alpha \neq \mathbf{r}_\beta}} \sum_{\substack{\mathbf{r}_\beta \in \mathscr{B}_{k'} \\ r_{\alpha 1} = r_{\beta 1}}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}), \tag{4.15}$$

$$P_1(\mathbf{x}) = \prod_{j=1}^{t} \left(\varrho^2 \sum_{\mathbf{r}_\alpha \in \mathscr{B}_{m_j}} \sum_{\substack{\mathbf{r}_\beta \in \mathscr{B}_{m'_j} \\ r_{\alpha i_j} = r_{\beta i_j}}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) \right),$$

$$P_2(\mathbf{x}) = \prod_{l=t+1}^{t+s} \left(\varrho^2 \sum_{\mathbf{r} \in \mathscr{B}_{m_l} \cap \mathscr{B}_{m'_l}} (g_\mathbf{r}(\mathbf{x}))^2 \right)$$

$$= \prod_{l=t+1}^{t+s} \left(\varrho^2 \sum_{\mathbf{r} \in \mathscr{B}_{m_l} \cap \mathscr{B}_{m'_l}} 1 \right),$$

$$P_3(\mathbf{x}) = \tilde{\prod_{v}} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{B}_v} g_\mathbf{r}(\mathbf{x}) \right)$$

where the product $\tilde{\prod}$ is extended over all

$$v \in W_0 \backslash V(G_0) = \{1, \ldots, p\} \backslash \{k, k', m_1, m'_1, \ldots, m_{t+s}, m'_{t+s}\}$$
and $\mathbf{r}_\alpha = (r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3})$, $\mathbf{r}_\beta = (r_{\beta 1}, r_{\beta 2}, r_{\beta 3})$.

We need the following iterated variants of the Cauchy-Schwarz inequality.

LEMMA 4.4. *For every $k \geqslant l \geqslant 1$, we have*

(a) $$\int_{U^3} |h_1(\mathbf{x}) \ldots h_l(\mathbf{x})| \, d\mathbf{x} \leqslant \prod_{i=1}^{l} \left(\int_{U^3} (h_i(\mathbf{x}))^{2k} \, d\mathbf{x} \right)^{1/2k},$$

(b) $$\int_{U^3} |h_0(\mathbf{x}) h_1(\mathbf{x}) \ldots h_l(\mathbf{x})| \, d\mathbf{x} \leqslant \left(\int_{U^3} (h_0(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2} \prod_{i=1}^{l} \left(\int_{U^3} (h_i(\mathbf{x}))^{4k} \, d\mathbf{x} \right)^{1/4k}.$$

Proof of Lemma 4.4. (a) Let $l \leqslant 2^m < 2l$, and let

$$h_{l+1}(\mathbf{x}) \equiv \ldots h_{2^m}(\mathbf{x}) \equiv 1.$$

By iterated application of Cauchy-Schwarz inequality,

$$\int_{U^3} |h_1(\mathbf{x}) \ldots h_{2^m}(\mathbf{x})| \, d\mathbf{x}$$
$$\leqslant \left(\int_{U^3} (h_1(\mathbf{x}) \ldots h_{2^{m-1}}(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{U^3} (h_{2^{m-1}+1}(\mathbf{x}) \ldots h_{2^m}(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2}$$

$$\leqslant \cdots \leqslant \prod_{i=1}^{2^m} \left(\int_{U^3} (h_i(\mathbf{x}))^{2^m} \, d\mathbf{x} \right)^{2^{-m}} = \prod_{i=1}^{l} \left(\int_{U^3} (h_i(\mathbf{x}))^{2^m} \, d\mathbf{x} \right)^{2^{-m}}.$$

Since $2^m < 2l \leqslant 2k$, we have

$$\left(\int_{U^3} (h(\mathbf{x}))^{2^m} \, d\mathbf{x} \right)^{2^{-m}} \leqslant \left(\int_{U^3} (h(\mathbf{x}))^{2k} \, d\mathbf{x} \right)^{1/2k},$$

and inequality (a) follows.

(b) Again by Cauchy-Schwarz inequality,

$$\int_{U^3} |h_0(\mathbf{x}) h_1(\mathbf{x}) \ldots h_l(\mathbf{x})| \, d\mathbf{x}$$

$$\leqslant \left(\int_{U^3} (h_0(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2} \left(\int_{U^3} \left(\prod_{i=1}^{l} h_i(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2}. \tag{4.16}$$

By using case (a),

$$\int_{U^3} \left(\prod_{i=1}^{l} h_i(\mathbf{x}) \right)^2 d\mathbf{x} \leqslant \prod_{i=1}^{l} \left(\int_{U^3} (h_i(\mathbf{x}))^{4k} \, d\mathbf{x} \right)^{1/2k}. \tag{4.17}$$

Combining (4.16) and (4.17), inequality (b) follows. □

Applying Lemma 4.4(b) with $k = p$ to (4.14), we have

$$\int_{U^3} |S(\tilde{K} \supseteq, \supseteq G; \mathbf{x})| \, d\mathbf{x} \leqslant Q_0 \cdot Q_1 \cdot Q_2 \cdot Q_3, \tag{4.18}$$

where

$$Q_0 = \left(\int_{U^3} (P_0(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2}, \tag{4.19}$$

$$Q_1 = \varrho^{2t} \prod_{j=1}^{t} \left(\int_{U^3} \left(\sum_{\mathbf{r}_\alpha \in \mathscr{B}_{m_j}} \sum_{\substack{\mathbf{r}_\beta \in \mathscr{B}_{m'_j} \\ r_{\beta i_j} = r_{\alpha i_j}}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) \right)^{4p} d\mathbf{x} \right)^{1/4p}, \tag{4.20}$$

$$Q_2 = \prod_{l=t+1}^{t+s} \left(\varrho^2 \sum_{\mathbf{r} \in \mathscr{B}_{m_l} \cap \mathscr{B}_{m'_l}} 1 \right),$$

$$Q_3 = \prod_{v}^{\sim} \left(\int_{U^3} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{B}_v} g_{\mathbf{r}}(\mathbf{x}) \right)^{4p} d\mathbf{x} \right)^{1/4p}. \tag{4.21}$$

First we have

$$\int_{U^3}\left(1+\varrho\sum_{\mathbf{r}\in\mathscr{B}_\nu} g_{\mathbf{r}}(\mathbf{x})\right)^{4p}d\mathbf{x}$$

$$\leqslant 2^{4p-1}\left(\int_{U^3} 1\,d\mathbf{x}+\varrho^{4p}\int_{U^3}\left(\sum_{\mathbf{r}\in\mathscr{B}_\nu} g_{\mathbf{r}}(\mathbf{x})\right)^{4p}d\mathbf{x}\right),$$

and so by Lemma 2.3,

$$\int_{U^3}\left(1+\varrho\sum_{\mathbf{r}\in\mathscr{B}_\nu} g_{\mathbf{r}}(\mathbf{x})\right)^{4p}d\mathbf{x}\leqslant 2^{4p-1}+2^{4p-1}\varrho^{4p}(4p)^{6p}(n+1)^{4p}<(4p)^{8p}$$

since $\varrho = q^{(1/4)-\varepsilon}/(n+1)$ and $p \geqslant q$. Thus by (4.21),

$$Q_3 < (4p)^{2p}. \tag{4.22}$$

Secondly, we have

$$Q_2=\prod_{l=t+1}^{t+s}\left(\varrho^2\sum_{\mathbf{r}\in\mathscr{B}_m\cap\mathscr{B}_{m'}} 1\right)\leqslant(\varrho^2|\mathscr{X}|)^s\leqslant q^{s/2}\leqslant p^{p/4} \tag{4.23}$$

since

$$\varrho = q^{(1/4)-\varepsilon}/(n+1),\ |\mathscr{X}|=(n+1)(n+2)/2,\ p\geqslant q \text{ and } p\geqslant 2s.$$

In order to estimate Q_0 and Q_1, we need

LEMMA 4.5. *We have*
(a) $Q_1 \leqslant q^{t/2}(16p)^t$
(b) $Q_0 \leqslant 4q^{1/2}(n+1)^{-1/4}$.

Proof of Lemma 4.5. (a) We have

$$\left(\sum_{\mathbf{r}_\alpha\in\mathscr{B}_m}\ \sum_{\substack{\mathbf{r}_\beta\in\mathscr{B}_{m'}\\ r_{\beta i}=r_{\alpha i}}} g_{\mathbf{r}_\alpha}\cdot g_{\mathbf{r}_\beta}\right)^{4p}$$

$$=\prod_{l=1}^{4p}\left(\sum_{\mathbf{r}_\alpha^{(l)}\in\mathscr{B}_m}\ \sum_{\substack{\mathbf{r}_\beta^{(l)}\in\mathscr{B}_{m'}\\ r_{\beta i}^{(l)}=r_{\alpha i}^{(l)}}} g_{\mathbf{r}_\alpha^{(l)}}\cdot g_{\mathbf{r}_\beta^{(l)}}\right)=\sum_{a_1=0}^{n}\sum_{a_2=0}^{n}\cdots\sum_{a_{4p}=0}^{n}\Pi \tag{4.24}$$

where

$$\Pi = \left(\prod_{l=1}^{4p} \left(\sum_{\substack{\mathbf{r}_\alpha^{(l)} \in \mathscr{B}_m \\ r_{\alpha i}^{(l)} = a_l}} g_{\mathbf{r}_\alpha^{(l)}} \right) \right) \cdot \left(\prod_{l=1}^{4p} \left(\sum_{\substack{\mathbf{r}_\beta^{(l)} \in \mathscr{B}_{m'} \\ r_{\beta i}^{(l)} = a_l}} g_{\mathbf{r}_\beta^{(l)}} \right) \right).$$

We need

LEMMA 4.6. *Let $\mathscr{B} \subset \mathscr{X}$, let $i \in \{1, 2, 3\}$ and let $a \in \{0, 1, \ldots, n\}$. Then*

$$\int_{U^3} \left(\sum_{\substack{\mathbf{r} \in \mathscr{B} \\ r_i = a}} g_\mathbf{r}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} \leqslant (2m(n+1))^m$$

if $m \geqslant 2$, and the integral is $\leqslant n + 1$ if $m = 1$.

Proof of Lemma 4.6. For notational convenience, suppose that $i = 1$. The restriction of the functions $g_\mathbf{r}$ ($\mathbf{r} \in \mathscr{B}, r_i = a$) to the line-segment

$$U(y_1, y_3) = \{\mathbf{x} \in U^3 : x_1 = y_1, x_3 = y_3, 0 \leqslant x_2 < 1\}$$

form 1-dimensional modified Rademacher functions of different orders (see Lemma 2.2(b)). Thus by Lemma 2.1(b),

$$\int_{U(y_1, y_3)} \left(\sum_{\substack{\mathbf{r} \in \mathscr{B} \\ r_1 = a}} g_\mathbf{r}(\mathbf{x}) \right)^{2m} \mathrm{d}x_2 \leqslant (2m(n+1))^m$$

if $m \geqslant 2$, and the integral is $\leqslant (n + 1)$ if $m = 1$. Since

$$\int_{U^3} \left(\sum_{\substack{\mathbf{r} \in \mathscr{B} \\ r_1 = a}} g_\mathbf{r}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} = \int_0^1 \int_0^1 \left(\int_{U(y_1, y_3)} \left(\sum_{\substack{\mathbf{r} \in \mathscr{B} \\ r_1 = a}} g_\mathbf{r}(\mathbf{x}) \right)^{2m} \mathrm{d}x_2 \right) \mathrm{d}y_1 \, \mathrm{d}y_3,$$

Lemma 4.6 follows. □

By Lemma 4.6 with $m = 8p$,

$$\int_{U^3} \left(\sum_{\substack{\mathbf{r}_\alpha^{(l)} \in \mathscr{B}_m \\ r_{\alpha i}^{(l)} = a_l}} g_{\mathbf{r}_\alpha^{(l)}}(\mathbf{x}) \right)^{16p} \mathrm{d}\mathbf{x} \leqslant (16p(n+1)^{8p}. \tag{4.25}$$

Thus, by (4.24), Lemma 4.4(a) with $k = l = 8p$ and (4.25),

$$\int_{U^3}\left(\sum_{\mathbf{r}_\alpha\in\mathscr{B}_m}\ \sum_{\substack{\mathbf{r}_\beta\in\mathscr{B}_{m'}\\ r_{\beta i}=r_{\alpha i}}} g_{\mathbf{r}_\alpha}(\mathbf{x})g_{\mathbf{r}_\beta}(\mathbf{x})\right)^{4p}\,d\mathbf{x}$$

$$\leqslant (n+1)^{4p}\cdot(16p(n+1))^{4p} = (16p)^{4p}(n+1)^{8p}.$$

Returning now to (4.20), we have

$$Q_1 \leqslant \varrho^{2t}(16p(n+1)^2)^t = \left(\frac{q^{(1/4)-\varepsilon}}{n+1}\right)^{2t}(16p(n+1)^2)^t \leqslant q^{t/2}(16p)^t,$$

and Lemma 4.5(a) follows.

Next we prove Lemma 4.5(b). By (4.15), (4.19) and Lemma 2.2(c),

$$(Q_0)^2 = \varrho^4\tilde{\sum}\int_{U^3} g_{\mathbf{r}_\alpha}(\mathbf{x})g_{\mathbf{r}_\beta}(\mathbf{x})g_{\mathbf{r}_\gamma}(\mathbf{x})g_{\mathbf{r}_\delta}(\mathbf{x})\,d\mathbf{x} \tag{4.26}$$

where the summation $\tilde{\Sigma}$ is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ such that

$$\begin{gathered}\{\mathbf{r}_\alpha, \mathbf{r}_\gamma\}\subset\mathscr{B}_k,\ \{\mathbf{r}_\beta, \mathbf{r}_\delta\}\subset\mathscr{B}_{k'},\\ r_{\alpha 1}=r_{\beta 1},\quad \mathbf{r}_\alpha\neq\mathbf{r}_\beta,\quad r_{\gamma 1}=r_{\delta 1},\quad \mathbf{r}_\gamma\neq\mathbf{r}_\delta,\\ |\{r_{\alpha 2}, r_{\beta 2}, r_{\gamma 2}, r_{\delta 2}\}|\leqslant 3,\quad |\{r_{\alpha 3}, r_{\beta 3}, r_{\gamma 3}, r_{\delta 3}\}|\leqslant 3.\end{gathered} \tag{4.27}$$

This means that

there are $\zeta, \eta, \vartheta, \mu \in \{\alpha, \beta, \gamma, \delta\}$ such that

$$\zeta\neq\eta,\quad \vartheta\neq\mu,\quad r_{\zeta 2}=r_{\eta 2},\quad r_{\vartheta 3}=r_{\mu 3}. \tag{4.28}$$

In this way, we can associate with every quadruplet $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (4.27) another quadruplet

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu)$$

satisfying (4.28) (if the mapping ψ is not uniquely determined, then we choose among the possible quadruplets $(\zeta, \eta, \vartheta, \mu)$ arbitrarily).

Since by (4.27),

$$r_{\alpha 1}=r_{\beta 1},\quad \mathbf{r}_\alpha\neq\mathbf{r}_\beta,\quad |\mathbf{r}_\alpha|=|\mathbf{r}_\beta|=n,$$

we have $r_{\alpha 2} \neq r_{\beta 2}, r_{\alpha 3} \neq r_{\beta 3}$. Similarly, $r_{\gamma 2} \neq r_{\delta 2}, r_{\gamma 3} \neq r_{\delta 3}$. Hence, by (4.28) we get, say,

$$\zeta \in \{\alpha, \beta\}, \quad \eta \in \{\gamma, \delta\}, \quad \vartheta \in \{\alpha, \beta\}, \quad \mu \in \{\gamma, \delta\}. \tag{4.29}$$

For notational convenience, let

$$g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}).$$

We have

$$\tilde{\sum_{(4.27)}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= (\tilde{\Sigma}_2 + \tilde{\Sigma}_3 + \tilde{\Sigma}_4) \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \tag{4.30}$$

where the summation $\tilde{\Sigma}_i (i = 2, 3, 4)$ extends over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (4.27) such that the quadruplet

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu)$$

has precisely i distinct coordinates, i.e. $|\{\zeta, \eta, \vartheta, \mu\}| = i$.

We can write

$$\tilde{\Sigma}_4 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= \sum_{(\zeta,\eta,\vartheta,\mu)}' \Sigma_4^{(\zeta,\eta)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \tag{4.31}$$

where Σ' is taken over the 4 permutations $(\zeta, \eta, \vartheta, \mu)$ of $\alpha, \beta, \gamma, \delta$ satisfying (4.29), and $\Sigma_4^{(\zeta,\eta)}$ is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (4.27) such that

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu).$$

Let $(\zeta, \eta, \vartheta, \mu)$ be one of these 4 permutations. We have

$$\Sigma_4^{(\zeta,\eta)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= \sum_{\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta,\eta,\vartheta,\mu)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \tag{4.32}$$

$$= \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \int_{U^3} \sum_{r_{\vartheta 3}=r_{\mu 3}}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

where Σ' is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ with

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu)$$

such that (see (4.27)–(4.28))

$$r_{\alpha 1} = r_{\beta 1} = a, \quad r_{\gamma 1} = r_{\delta 1} = b, \quad r_{\zeta 2} = r_{\eta 2} = c, \quad r_{\vartheta 3} = r_{\mu 3}. \tag{4.33}$$

Note that Σ' is actually extended over the coordinate $r_{\vartheta 3}$ only, since the quadruplet

$$(a, b, c, d) \text{ with } d = r_{\vartheta 3} = r_{\mu 3} \tag{4.34}$$

uniquely determines $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$.

We need

LEMMA 4.7. *Let* $\mathscr{B}_1 \subset \mathscr{X}, \mathscr{B}_2 \subset \mathscr{X}, i \in \{1,2,3\}, j \in \{1,2,3\}, i \neq j,$ *and let* $a \in \{0,1,\ldots,n\}, b \in \{0,1,\ldots,n\}, a \neq b.$

Then

$$\int_{U^3} \left(\sum_{\substack{\mathbf{r}_1 \in \mathscr{B}_1 \\ r_{1i}=a}} \sum_{\substack{\mathbf{r}_2 \in \mathscr{B}_2, r_{2j}=r_{1j} \\ r_{2i}=b}} g_{\mathbf{r}_1}(\mathbf{x}) g_{\mathbf{r}_2}(\mathbf{x}) \right)^2 d\mathbf{x} \leqslant n+1.$$

Proof of Lemma 4.7. Without loss of generality, we can assume that $a < b$, $i = 1, j = 3$. Since $|\mathbf{r}_1| = |\mathbf{r}_2| = n$, we have

$$r_{12} = n - a - r_{13} > r_{22} = n - b - r_{23} = n - b - r_{13}.$$

By Lemma 2.2(b), the restriction of the product $g_{\mathbf{r}_1} \cdot g_{\mathbf{r}_2}$ to the line segment

$$U(y_1, y_3) = \{\mathbf{x} \in U^3 : x_1 = y_1, x_3 = y_3, 0 \leqslant x_2 < 1\}$$

forms a product of 1-dimensional modified Rademacher functions of orders r_{12} and r_{22}. Since $r_{12} > r_{22}$, from Lemma 2.1(a) we have that this product is a one-dimensional modified Rademacher function of order r_{12}, i.e. an r_{12}-function $f_{r_{12}}$. By using the orthogonality of one-dimensional modified Rademacher functions of different orders, we obtain

$$\int_{U(y_1,y_3)} \Bigg(\sum_{\substack{\mathbf{r}_1 \in \mathscr{B}_1 \\ r_{11}=a}} \sum_{\substack{\mathbf{r}_2 \in \mathscr{B}_2, r_{23}=r_{13} \\ r_{21}=b}} g_{\mathbf{r}_1}(\mathbf{x}) g_{\mathbf{r}_2}(\mathbf{x}) \Bigg)^2 \mathrm{d}x_2$$

$$= \int_{U(y_1,y_3)} \Bigg({\sum_{r_{12}}}' f_{r_{12}}(x_2) \Bigg)^2 \mathrm{d}x_2 = {\sum_{r_{12}}}' 1 \leqslant n + 1. \tag{4.35}$$

Integrating (4.35) over $0 \leqslant y_1 \leqslant 1, 0 \leqslant y_3 \leqslant 1$, Lemma 4.7 follows. □

Let us return to (4.32). For every fixed triplet (a, b, c) where

$$a = r_{\alpha 1} = r_{\beta 1}, \quad b = r_{\gamma 1} = r_{\delta 1}, \quad a \neq b, \quad c = r_{\zeta 2} = r_{\eta 2},$$

by using Cauchy-Schwarz inequality and Lemma 4.7, we have

$$\left| \int_{U^3} \Bigg({\sum_{(4.33)}}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \Bigg) \mathrm{d}\mathbf{x} \right|$$

$$= \left| \int_{U^3} g_{\mathbf{r}_\zeta}(\mathbf{x}) g_{\mathbf{r}_\eta}(\mathbf{x}) \Bigg({\sum_{(4.33)}}' g_{\mathbf{r}_\vartheta}(\mathbf{x}) g_{\mathbf{r}_\mu}(\mathbf{x}) \Bigg) \mathrm{d}\mathbf{x} \right|$$

$$\leqslant \Bigg(\int_{U^3} \Bigg(\sum_{(4.33)} g_{\mathbf{r}_\vartheta}(\mathbf{x}) g_{\mathbf{r}_\mu}(\mathbf{x}) \Bigg)^2 \mathrm{d}\mathbf{x} \Bigg)^{1/2} \leqslant (n+1)^{1/2}.$$

Hence

$$\left| \sum_{a=0}^{n} \sum_{\substack{b=0 \\ b \neq a}}^{n} \sum_{c=0}^{n} \int_{U^3} {\sum_{r_{\vartheta 3} = r_{\mu 3}}}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, \mathrm{d}\mathbf{x} \right|$$

$$\leqslant \sum_{a=0}^{n} \sum_{\substack{b=0 \\ b \neq a}}^{n} \sum_{c=0}^{n} (n+1)^{1/2} < (n+1)^{7/2}. \tag{4.36}$$

If $a = b$, then by (4.34),

$$\left| \sum_{a=b=0}^{n} \sum_{c=0}^{n} \sum_{d=0}^{n} \int_{U^3} \Bigg({\sum_{r_{\vartheta 3} = r_{\mu 3}}}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \Bigg) \mathrm{d}\mathbf{x} \right|$$

$$\leqslant \sum_{a=b=0}^{n} \sum_{c=0}^{n} \sum_{d=0}^{n} 1 = (n+1)^3. \tag{4.37}$$

Thus by (4.31), (4.32), (4.36) and (4.37),

$$\tilde{\Sigma}_4 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \leqslant 4((n+1)^{7/2} + (n+1)^3) \leqslant 8(n+1)^{7/2}. \tag{4.38}$$

Let us return to (4.30). We have

$$\tilde{\Sigma}_3 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} = \sum_{(\zeta,\eta,\vartheta,\mu)}'' \Sigma_3^{(\zeta,\eta,\vartheta,\mu)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \tag{4.39}$$

where Σ'' is extended over the 8 quadruplets $(\zeta, \eta, \vartheta, \mu)$ such that $|\{\zeta, \eta, \vartheta, \mu\}| = 3$ and (4.29) holds, and $\Sigma_3^{(\zeta,\eta,\vartheta,\mu)}$ is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (4.27) such that

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu).$$

Let $(\zeta, \eta, \vartheta, \mu)$ be one of these 8 quadruplets. Without loss of generality, we can assume that $\zeta = \vartheta$. Let (see (4.29))

$$\{v\} = \{\alpha, \beta, \gamma, \delta\} \setminus \{\zeta = \vartheta, \eta, \mu\} = \{\alpha, \beta\} \setminus \{\zeta\}.$$

Then by Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| \Sigma_3^{(\zeta,\eta,\vartheta,\mu)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \\
&= \left| \sum_{\psi((\mathbf{r}_\alpha,\mathbf{r}_\beta,\mathbf{r}_\gamma,\mathbf{r}_\delta)) = (\zeta,\eta,\vartheta,\mu)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \\
&= \left| \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \int_{U^3} g_{\mathbf{r}_\zeta}(\mathbf{x}) g_{\mathbf{r}_\eta}(\mathbf{x}) g_{\mathbf{r}_\mu}(\mathbf{x}) \left(\sum_{r_{v1}=a}'' g_{\mathbf{r}_v}(\mathbf{x}) \right) d\mathbf{x} \right| \\
&\leqslant \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \left(\int_{U^3} \left(\sum_{r_{v1}=a}'' g_{\mathbf{r}_v}(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2}
\end{aligned} \tag{4.40}$$

where Σ'' is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ with

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu)$$

such that (see (4.27), (4.28))

$$r_{\alpha 1} = r_{\beta 1} = a, \quad r_{\gamma 1} = r_{\delta 1} = b, \quad r_{\zeta 2} = r_{\eta 2} = c.$$

Note that Σ'' is actually extended over the coordinate $r_{\nu 2}$ only, since by $\zeta = \vartheta$,

$$r_{\zeta 3} = r_{\vartheta 3} = r_{\mu 3} = d = n - r_{\zeta 1} - r_{\zeta 2} = n - a - c,$$

and so the quadruplet $(a, b, c, r_{\nu 2})$ uniquely determines $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$.

By Lemma 4.6 with $m = 1$,

$$\int_{U^3} \Bigl(\sum_{r_{\nu 1} = a}'' g_{\mathbf{r}_\nu}(\mathbf{x}) \Bigr)^2 \, d\mathbf{x} \leqslant n + 1,$$

and so from (4.39) and (4.40), we have

$$\Bigl| \tilde{\Sigma}_3 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, d\mathbf{x} \Bigr|$$

$$\leqslant 8 \sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} (n+1)^{1/2} \leqslant 8(n+1)^{7/2}. \tag{4.41}$$

Finally, we have

$$\tilde{\Sigma}_2 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, d\mathbf{x}$$

$$= \sum_{(\zeta, \eta, \vartheta, \mu)}''' \Sigma_2^{(\zeta, \eta)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, d\mathbf{x} \tag{4.42}$$

where Σ''' is extended over the 4 quadruplets $(\zeta, \eta, \vartheta, \mu)$ such that $\zeta = \vartheta \in \{\alpha, \beta\}$ and $\eta = \mu \in \{\gamma, \delta\}$, and $\Sigma_2^{(\zeta, \eta)}$ is extended over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (4.27) such that

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = (\zeta, \eta, \vartheta, \mu) = (\zeta, \eta, \zeta, \eta).$$

Let $(\zeta, \eta, \zeta, \eta)$ be one of these 4 quadruplets. Let

$$\{\nu, \tau\} = \{\alpha, \beta, \gamma, \delta\} \setminus \{\zeta, \eta\}.$$

By (4.28),

$$r_{\zeta 2} = r_{\eta 2} \quad \text{and} \quad r_{\zeta 3} = r_{\vartheta 3} = r_{\mu 3} = r_{\eta 3},$$

and because $|\mathbf{r}_\zeta| = |\mathbf{r}_\eta| = n$, we get $r_{\zeta 1} = r_{\eta 1}$. Hence by (4.27) and (4.29),

$$r_{\alpha 1} = r_{\beta 1} = r_{\gamma 1} = r_{\delta 1} \quad \text{and} \quad \mathbf{r}_\zeta = \mathbf{r}_\eta. \tag{4.43}$$

Therefore,

$$g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = (g_{\mathbf{r}_\zeta}(\mathbf{x}))^2 \cdot g_{\mathbf{r}_\nu}(\mathbf{x}) \cdot g_{\mathbf{r}_\tau}(\mathbf{x}) = g_{\mathbf{r}_\nu}(\mathbf{x}) \cdot g_{\mathbf{r}_\tau}(\mathbf{x}).$$

By Lemma 2.2(c), the integral

$$\int_{U^3} g_{\mathbf{r}_\nu}(\mathbf{x}) g_{\mathbf{r}_\tau}(\mathbf{x})\, d\mathbf{x}$$

is 0 unless $\mathbf{r}_\nu = \mathbf{r}_\tau$. Hence by (4.43),

$$\Sigma_2^{(\zeta,\eta)} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} = \sum_{\mathbf{r}_\zeta = \mathbf{r}_\eta} \sum_{\substack{\mathbf{r}_\nu = \mathbf{r}_\tau \\ r_{\nu 1} = r_{\zeta 1}}} 1 \leqslant (n+1)^3,$$

and so by (4.42), we have

$$\left| \tilde{\Sigma}_2 \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \leqslant 4(n+1)^3. \tag{4.44}$$

Summarizing, from (4.26), (4.30), (4.38), (4.41) and (4.44), we have

$$\begin{aligned}(Q_0)^2 &\leqslant \varrho^4 (4(n+1)^{7/2} + 8(n+1)^{7/2} + 4(n+1)^3) \\ &\leqslant \varrho^4 \cdot 16(n+1)^{7/2} = \left(\frac{q^{(1/4)-\varepsilon}}{n+1} \right)^4 \cdot 16(n+1)^{7/2} \leqslant \frac{16q}{(n+1)^{1/2}},\end{aligned}$$

and Lemma 4.5(b) follows. □

We are now able to complete the proof of Lemma 4.2. From (4.18), (4.22), (4.23), Lemma 4.5(a) and (b), we have, noting that $p \geqslant 2t$ and $p \geqslant q$,

$$\begin{aligned}&\int_{U^3} |S(\tilde{K} \supseteq, \supseteq G; \mathbf{x})|\, d\mathbf{x} \\ &\quad \leqslant \frac{4q^{1/2}}{(n+1)^{1/4}} \cdot q^{t/2} (16p)^t \cdot p^{p/4} \cdot (4p)^{2p} \leqslant \frac{(8p)^{4p}}{(n+1)^{1/4}}.\end{aligned}$$

This proves Lemma 4.2. □

5. Proof of Lemma 4.3

Consider the decomposition of G into maximal row-connected subgraphs

$$G = G^{(1)} \cup G^{(2)} \cup \cdots \cup G^{(h)}.$$

Since G is special, we obtain that $G^{(i)}, i = 1, 2, \ldots, h$ are special as well.

Reordering the indices, we can assume that the row-graphs $G_0^{(1)}, \ldots, G_0^{(k)}$ have maximum degree $\geqslant 2$ (i.e. the number $|V(G_0^{(i)})|$ of (non-isolated) vertices of $G_0^{(i)}$ is $\geqslant 3$ for $i = 1, 2, \ldots, k$) and $G_0^{(k+1)}, \ldots, G_0^{(h)}$ have maximum degree 1 (i.e. $G^{(k+1)} \cup \cdots \cup G^{(h)}$ is a union of vertex-disjoint edges). By hypothesis, $k \geqslant 1$. Let

$$V(G_0^{(i)}) = \{j_1^{(i)}, \ldots, j_{l(i)}^{(i)}\} \subset W_0 = \{1, \ldots, p\} \quad (1 \leqslant i \leqslant k).$$

Let

$$G^{(k+1)} \cup \cdots \cup G^{(h)} = \{e^{(1)}, \ldots, e^{(t)}\} \cup \{e_1^{(t+1)}, e_2^{(t+1)}, e_3^{(t+1)}, \ldots, e_1^{(t+s)}, e_2^{(t+s)}, e_3^{(t+s)}\}$$

where $e_1^{(j)}, e_2^{(j)}, e_3^{(j)} (t + 1 \leqslant j \leqslant t + s)$ are row-parallel edges. Let

$$e^{(j)} = \{(m_j . i_j), (m'_j . i_j)\} \quad (1 \leqslant j \leqslant t)$$

and

$$e_i^{(j)} = \{(m_j . i), (m'_j . i)\} \quad (t + 1 \leqslant j \leqslant t + s).$$

We recall (4.8) (see also (4.14) and (4.15))

$$S(\supseteq G; \mathbf{x}) = P_0(\mathbf{x}) \cdot P_1(\mathbf{x}) \cdot P_2(\mathbf{x}) \cdot (P_3(\mathbf{x}) \tag{5.1}$$

where

$$P_0(\mathbf{x}) = \prod_{i=1}^{k} \left(\varrho^{l_i} \sum_{G(\mathbf{r}_1^{(i)}, j_1^{(i)}; \ldots; \mathbf{r}_{l(i)}^{(i)}, j_{l(i)}^{(i)}) \supseteq G^{(i)}} g_{\mathbf{r}_1^{(i)}}(\mathbf{x}) \ldots g_{\mathbf{r}_{l(i)}^{(i)}}(\mathbf{x}) \right),$$

$$P_1(\mathbf{x}) = \prod_{j=1}^{t} \left(\varrho^2 \sum_{\mathbf{r}_\alpha \in \mathscr{B}_{m_j}} \sum_{\substack{\mathbf{r}_\beta \in \mathscr{B}_{m'_j} \\ r_{\beta i_j} = r_{\alpha i_j}}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) \right),$$

$$P_2(\mathbf{x}) = \prod_{j=t+1}^{t+s} \left(\varrho^2 \sum_{\mathbf{r} \in \mathscr{B}_{m_j} \cap \mathscr{B}_{m'_j}} 1 \right),$$

$$P_3(\mathbf{x}) = \tilde{\prod_{\mathbf{v}}} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{B}_{\mathbf{v}}} g_{\mathbf{r}}(\mathbf{x}) \right)$$

where the product $\tilde{\prod}$ is taken over all

$$v \in W_0 \backslash V(G) = \{1,\ldots,p\} \backslash \left(\bigcup_{i=1}^{k} \{j_1^{(i)},\ldots,j_{l(i)}^{(i)}\} \cup \{m_1, m_1', \ldots, m_{t+s}, m_{t+s}'\} \right).$$

Applying Lemma 4.4(b) with $k = p$ to (5.1), we have

$$\int_{U^3} |S(\supseteq G; \mathbf{x})| \, d\mathbf{x} \leqslant Q_0' \cdot Q_0'' \cdot Q_1 \cdot Q_2 \cdot Q_3 \tag{5.2}$$

where

$$Q_0' = \left(\int_{U^3} \left(\varrho^{l_1} \sum_{G(\mathbf{r}_1^{(1)}, j_1^{(1)};\ldots;\mathbf{r}_{l(1)}^{(1)}, j_{l(1)}^{(1)}) \supseteq G^{(1)}} g_{\mathbf{r}_1^{(1)}}(\mathbf{x}) \ldots g_{\mathbf{r}_{l(1)}^{(1)}}(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2},$$

$$Q_0'' = \prod_{i=2}^{k} \left(\int_{U^3} \left(\varrho^{l_i} \sum_{G(\mathbf{r}_1^{(i)}, j_1^{(i)};\ldots;\mathbf{r}_{l(i)}^{(i)}, j_{l(i)}^{(i)}) \supseteq G^{(i)}} g_{\mathbf{r}_1^{(i)}}(\mathbf{x}) \ldots g_{\mathbf{r}_{l(i)}^{(i)}}(\mathbf{x}) \right)^{4p} d\mathbf{x} \right)^{1/4p},$$

$$Q_1 = \varrho^{2t} \prod_{j=1}^{t} \left(\int_{U^3} \left(\sum_{\mathbf{r}_\alpha \in \mathscr{B}_{m_j}} \sum_{\substack{\mathbf{r}_\beta \in \mathscr{B}_{m_j'} \\ r_{\beta i_j} = r_{\alpha i_j}}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) \right)^{4p} d\mathbf{x} \right)^{1/4p},$$

$$Q_2 = \prod_{j=t+1}^{t+s} \left(\varrho^2 \sum_{\mathbf{r} \in \mathscr{B}_{m_j} \cap \mathscr{B}_{m_j'}} 1 \right),$$

$$Q_3 = \tilde{\prod_{v}} \left(\int_{U^3} \left(1 + \varrho \sum_{\mathbf{r} \in \mathscr{B}_v} g_{\mathbf{r}}(\mathbf{x}) \right)^{4p} d\mathbf{x} \right)^{1/4p}.$$

Analogously to (4.22), (4.23) and Lemma 4.5(a), we have

$$Q_3 \leqslant (4p)^{2p}, \quad Q_2 \leqslant p^{p/4} \quad \text{and} \quad Q_1 \leqslant q^{t/2}(16p)^t \leqslant p^{p/4}(16p)^{p/2}. \tag{5.3}$$

In order to estimate Q_0' and Q_0'', we distinguish 4 cases according to the following 4 lemmas.

LEMMA 5.1. *Let $H \subseteq K$ be a row-connected special subgraph. Suppose that the maximum degree of H is $\geqslant 2$, and $|V(H_0)| \geqslant 3$. Then, with $V(H_0) = \{j_1, j_2, \ldots, j_l\}$, we have for every $m \geqslant 1$,*

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, j_1;\ldots;\mathbf{r}_l, j_l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x} \leqslant (4lm)^{lm} q^{lm/2} (n+1)^{-m}.$$

LEMMA 5.2. *Let $H \subseteq K$ be a row-connected special subgraph. Suppose that H is a matching, and the maximum degree of H_0 is $\geqslant 3$. Then, with $V(H_0) = \{j_1, j_2, \ldots, j_l\}$, we have for every $m \geqslant 1$,*

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, j_1; \ldots; \mathbf{r}_l, j_l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} \leqslant (4lm)^{lm} q^{lm/2} (n+1)^{-m}.$$

LEMMA 5.3. *Let $H \subseteq K$ be a row-connected special subgraph. Suppose that H is a matching, and H_0 forms a circuit of length $l \geqslant 3$. Then, with $V(H_0) = \{j_1, j_2, \ldots, j_l\}$, we have for every $m \geqslant 1$,*

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, j_1; \ldots; \mathbf{r}_l, j_l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} \leqslant 2 q^{lm/2} (n+1)^{-1/4}.$$

LEMMA 5.4. *Let $H \subseteq K$ be a row-connected special subgraph. Suppose that H is a matching, and the row-graph H_0 of H forms a path of length $l \geqslant 3$. Then, with*

$$V(H_0) = \{j_1, j_2, \ldots, j_l\} \subset W_0 = \{1, \ldots, p\},$$

we have for every $m \geqslant 1$,

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, j_1; \ldots; \mathbf{r}_l, j_l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} \mathrm{d}\mathbf{x} \leqslant 3(16m)^{2m} q^{lm/2} (n+1)^{-1/4}.$$

First we derive Lemma 4.3 from Lemmas 5.1–5.4. By using Lemmas 5.1–5.4 with $m = 1$, $p \geqslant q$ and $p \geqslant l$, we get

$$\begin{aligned} Q_0' &\leqslant (2(4p)^p p^{p/2} (n+1)^{-1} + \\ &\qquad + 3(16)^2 p^{p/2} (n+1)^{-1/4} + 2p^{p/2}(n+1)^{-1/4})^{1/2} \\ &\leqslant 2^5 (4p)^p (n+1)^{-1/8}. \end{aligned} \tag{5.4}$$

By using Lemmas 5.1–5.4 with

$$m = 2p, \quad p > l_2 + l_3 + \cdots + l_k, \quad p \geqslant q,$$

we get

$$\begin{aligned} Q_0'' &\leqslant \prod_{i=2}^{k} (2(8l_i p)^{2l_i p} p^{l_i p} (n+1)^{-2p} + \\ &\qquad + 3(32p)^{4p} p^{l_i p} (n+1)^{-1/4} + 2p^{l_i p}(n+1)^{-1/4})^{1/4p} \\ &\leqslant \prod_{i=2}^{k} (64p)^{2(l_i+1)} \leqslant (64p)^{4p}. \end{aligned} \tag{5.5}$$

Combining (5.2)–(5.5), Lemma 4.3 follows.

It remains to prove Lemmas 5.1–5.4.

Proof of Lemma 5.1. Without loss of generality, we can assume that $V(H_0) = \{1, 2, \ldots, l\}$. Note that H is the union of vertex-disjoint cliques (an edge is considered as a clique of 2 vertices). By hypothesis, one of these cliques, say $K^{(1)}$, has ≥ 3 vertices. Let F be a maximal subgraph of H such that $F \supseteq K^{(1)}$ and

$$\{(k.1), (k.2), (k.3)\} \not\subset V(F) \quad \forall k (1 \leq k \leq l). \tag{5.6}$$

Observe that F is also the union of vertex-disjoint cliques. Let, say

$$F = K^{(1)} \cup K^{(2)} \cup \cdots \cup K^{(f)}.$$

Hence

$$|V(F)| = |V(K^{(1)})| + |V(K^{(2)})| + \cdots + |V(K^{(f)})| \\ \geq 3 + (f-1)2 = 2f + 1. \tag{5.7}$$

For every $j = 1, \ldots, f$, let $v(K^{(j)}) = (k_j . i_j)$ be an arbitrary but fixed vertex of the clique $K^{(j)}$. For every $a_1 \in \{0, 1, \ldots, n\}, a_2 \in \{0, 1, \ldots, n\}, \ldots, a_f \in \{0, 1, \ldots, n\}$, let

$$\mathscr{L}(a_1, a_2, \ldots, a_f)$$

denote the family of l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ such that

$$\mathbf{r}_k = (r_{k1}, r_{k2}, r_{k3}) \in \mathscr{B}_k (1 \leq k \leq l), \quad G(\mathbf{r}_1, 1; \ldots; \mathbf{r}_l, l) \supseteq H, r_{k_j i_j} = a_j \\ (1 \leq j \leq f).$$

Let $v = (k.i) \in V(H)$ be an arbitrary vertex of H. We claim that for all l-tuples

$$(\mathbf{r}_1, \ldots, \mathbf{r}_l) \in \mathscr{L}(a_1, \ldots, a_f), \\ r_{ki} = \text{const} = \text{const}(a_1, \ldots, a_f), \tag{5.8}$$

i.e. the ith coordinate of $\mathbf{r}_k$ depends only on $a_1, a_2, \ldots, a_f$.

In order to prove (5.8), we decompose H into vertex-disjoint cliques. Let $\hat{F}$ be the union of those cliques which have common vertex with F (note that $\hat{F} \supseteq F$), and let $K^{(f+1)}, \ldots, K^{(h)}$ be those cliques which are vertex-disjoint from F. For every $j = f + 1, \ldots, h$, let

$$H^{(j)} = \hat{F} \cup K^{(f+1)} \cup \cdots \cup K^{(j)}.$$

By using the maximality of F and (5.6), and reordering the indices of the cliques $K^{(v)}$, if necessary, we have, for every $j = f+1, \ldots, h-1$, that

$$\exists k(1 \leqslant k \leqslant l)\colon \{(k.1), (k.2), (k.3)\} \subset V(H^{(j)}) \text{ and } \{(k.1), (k.2), (k.3)\} \cap V(K^{(j)}) \neq \emptyset. \tag{5.9}$$

By using (5.9) and the fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$, we conclude, by induction on

$$j = f+1, \ldots, h-1,$$

that for every vertex $v = (k.i) \in V(K^{(j)}), r_{ki} = \text{const}(a_1, \ldots, a_f)$ for all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l) \in \mathscr{L}(a_1, \ldots, a_f)$. This proves (5.8).

Let

$$|\{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(H)| = 1\}| = t$$

and

$$|\{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(H)| \geqslant 2\}| = s = l - t.$$

By (5.6),

$$|V(F)| \leqslant t + 2s = t + 2(l - t) = 2l - t. \tag{5.10}$$

By (5.7) and (5.10), $2f + 1 \leqslant 2l - t$, that is,

$$2f + t \leqslant 2l - 1. \tag{5.11}$$

Without loss of generality, we can assume that

$$\{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(H)| = 1\} = \{1, 2, \ldots, t\}.$$

Let

$$(k.v_k) \in V(H), \quad (k = 1, 2, \ldots, t; v_k \in \{1, 2, 3\}).$$

Note that the last $s = l - t$ index-vectors $\mathbf{r}_{t+1}, \ldots, \mathbf{r}_l$ in any l-tuple $(\mathbf{r}_1, \ldots, \mathbf{r}_l) \in \mathscr{L}(a_1, \ldots, a_f)$ are uniquely determined, i.e. depend only on $a_1, \ldots, a_f$. Then from (5.8) we have

$$\sum_{G(\mathbf{r}_1, 1; \ldots; \mathbf{r}_l, l) \supseteq H} g_{\mathbf{r}_1} \cdots g_{\mathbf{r}_l}$$

$$= \sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_f=0}^{n} \sum_{(\mathbf{r}_1,\ldots,\mathbf{r}_l)\in\mathscr{L}(a_1,\ldots,a_f)} g_{\mathbf{r}_1}\cdots g_{\mathbf{r}_l}$$

$$= \sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_f=0}^{n} \Pi,$$

where

$$\Pi = \left(\sum_{\substack{\mathbf{r}_1\in\mathscr{B}_1\\ r_{1v_1}=c_1}} g_{\mathbf{r}_1}\right)\left(\sum_{\substack{\mathbf{r}_2\in\mathscr{B}_2\\ r_{2v_2}=c_2}} g_{\mathbf{r}_2}\right)\cdots\left(\sum_{\substack{\mathbf{r}_t\in\mathscr{B}_t\\ r_{tv_t}=c_t}} g_{\mathbf{r}_t}\right) g_{\mathbf{r}_{t+1}} g_{\mathbf{r}_{t+2}}\cdots g_{\mathbf{r}_l},$$

and where $\mathbf{r}_k = (c_{k1}, c_{k2}, c_{k3})$ for all $t + 1 \leqslant k \leqslant l$, and

$$c_j = c_j(a_1,\ldots,a_f), \quad 1 \leqslant j \leqslant t, \qquad c_{ki} = c_{ki}(a_1,\ldots,a_f), \quad t+1 \leqslant k \leqslant l, \quad 1 \leqslant i \leqslant 3,$$

are constants depending only on $a_1,\ldots,a_f$. Therefore,

$$\left(\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}\cdots g_{\mathbf{r}_l}\right)^{2m}$$

$$\leqslant \sum_{a_1=0}^{n}\cdots\sum_{a_f=0}^{n}\sum_{a_{f+1}=0}^{n}\cdots\sum_{a_{2f}=0}^{n}\cdots\sum_{a_{(2m-1)f+1}=0}^{n}\cdots\sum_{a_{2mf}=0}^{n}\Gamma(a_1,\ldots,a_{2mf})$$

where

$$\Gamma(a_1,\ldots,a_{2mf}) = \prod_{y=1}^{2m}\left(\prod_{z=1}^{t}\left(\sum_{\substack{\mathbf{r}_{yl+z}\in\mathscr{B}_z\\ r_{(yl+z)v_z}=c_{yt+z}}} g_{\mathbf{r}_{yl+z}}\right)\cdot g_{\mathbf{r}_{yl+t+1}} g_{\mathbf{r}_{yl+t+2}}\cdots g_{\mathbf{r}_{yl+l}}\right)$$

and $c_j = c_j(a_1,\ldots,a_{2mf})$ are constants depending only on $a_1,\ldots,a_{2mf}$. Thus by Lemma 4.4(a) with $k = 2mt$, we have

$$\int_{U^3}\left(\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x})\cdots g_{\mathbf{r}_l}(\mathbf{x})\right)^{2m} d\mathbf{x}$$

$$\leqslant (n+1)^{2mf}\prod_{y=1}^{2m}\prod_{z=1}^{t}\left(\int_{U^3}\left(\sum_{\substack{\mathbf{r}_{yl+z}\in\mathscr{B}_z\\ r_{(yl+z)v_z}=c_{yt+z}}} g_{\mathbf{r}_{yl+z}}(\mathbf{x})\right)^{4mt} d\mathbf{x}\right)^{1/4mt}.$$

Now applying Lemma 4.6, with $m = 2mt$, and (5.11), we obtain

$$\int_{U^3} \left(\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x} \leqslant (n+1)^{2mf} \cdot (4mt(n+1))^{2mt/2}$$

$$= (4mt)^{mt}(n+1)^{(2f+t)m} \leqslant (4mt)^{mt}(n+1)^{(2l-1)m} \leqslant (4ml)^{ml}(n+1)^{(2l-1)m}.$$

Therefore, we have

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x}$$

$$\leqslant \varrho^{2ml}(4ml)^{ml}(n+1)^{(2l-1)m} = \left(\frac{q^{(1/4)-\varepsilon}}{n+1} \right)^{2lm} (4lm)^{lm}(n+1)^{(2l-1)m}$$

$$\leqslant (4lm)^{lm} q^{lm/2} (n+1)^{-m},$$

and Lemma 5.1 follows. □

Proof of Lemma 5.2. We can assume that $V(H_0) = \{1, 2, \ldots, l\}$. Since the maximum degree of H_0 is $\geqslant 3$, we have $l \geqslant 4$. Let F be a maximal subgraph of H such that

$$\forall k (1 \leqslant k \leqslant l) \colon |\{(k.1), (k.2), (k.3)\} \cap V(F)| \leqslant 2 \tag{5.12}$$

and

$$\text{all edges} \quad e = \{(k'.i'), (k''.i'')\} \in H \backslash F \quad \text{satisfy}$$
$$|\{(k'.1), (k'.2), (k'.3), (k''.1), (k''.2), (k''.3)\} \cap V(F)| \leqslant 3. \tag{5.13}$$

Since H is a matching and some vertex of H_0 has degree $\geqslant 3$, from (5.12) we have that $F \neq H$, i.e. F is a proper subgraph of H.

Let $|F| = f$, that is, F has f edges. Since $F \subset H$, we get that F is also a matching, and so

$$|V(F)| = 2|F| = 2f. \tag{5.14}$$

Let

$$F = \{e^{(1)}, e^{(2)}, \ldots, e^{(f)}\},$$

and for every $j = 1, 2, \ldots, f$, let $v^{(j)} = (k_j . i_j)$ be one of the endvertices of the edge $e^{(j)}$.

For every $a_1 \in \{0, 1, \ldots, n\}, \ldots, a_f \in \{0, 1, \ldots, n\}$, let $\mathscr{L}(a_1, \ldots, a_f)$ denote the family of l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l)$ such that

$$\mathbf{r}_k = (r_{k1}, r_{k2}, r_{k3}) \in \mathscr{B}_k (1 \leqslant k \leqslant l), \quad G(\mathbf{r}_1, 1; \ldots; \mathbf{r}_l, l) \supseteq H, r_{k_j i_j} = a_j$$
$$(1 \leqslant j \leqslant f).$$

Let $v = (k.i) \in V(H)$ be an arbitrary vertex of H. We claim that for all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l) \in \mathscr{L}(a_1, \ldots, a_f)$,

$$r_{ki} = \text{const} = \text{const}(a_1, \ldots, a_f), \tag{5.15}$$

i.e. the ith coordinate of $\mathbf{r}_k$ depends only on $a_1, \ldots, a_f$.

Let $e = \{(k'.i'), (k''.i'')\}$ be an arbitrary edge in $H \backslash F$. If

$$|\{(k'.1), (k'.2), (k'.3)\} \cap V(F)| = 2 \quad \text{or} \quad |\{(k''.1), (k''.2), (k''.3)\} \cap V(F)| = 2$$

holds, then by using the fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$, we have

$$r_{k'i'} = r_{k''i''} = \text{const}(a_1, \ldots, a_f).$$

We can therefore assume that

$$|\{(k'.1), (k'.2), (k'.3)\} \cap V(F)| \leqslant 1 \quad \text{and} \quad |\{(k''.1), (k''.2), (k''.3)\} \cap V(F)| \leqslant 1. \tag{5.16}$$

Consider the graph $F^* = F \cup \{e\}$. From the maximality of F, we have that either

$$\exists k (1 \leqslant k \leqslant l): \{(k.1), (k.2), (k.3)\} \subset V(F^*), \tag{5.17}$$

or

$$\exists \text{ edge } e^* = \{(k^*.v^*), (k^{**}.v^{**})\} \in H \backslash F^* \tag{5.18}$$

such that

$$|\{(k^*.1), (k^*.2), (k^*.3)\} \cap V(F^*)| = 2 \quad \text{and}$$
$$|\{(k^{**}.1), (k^{**}.2), (k^{**}.3)\} \cap V(F^*)| = 2.$$

Case (5.17) is impossible, since if $k \in \{k', k'')$, then (5.16) contradicts (5.17), and if $k \in \{1, 2, \ldots, l\} \backslash \{k', k''\}$, then (5.12) contradicts (5.17).

Hence, we can assume that (5.18) holds. We show that the case $\{k', k''\} = \{k^*,$

$k^{**}\}$ is impossible. Indeed, in this case we have that e and e^* are parallel edges of H, and since H is special, H must contain 3 parallel edges. This contradicts to the hypothesis that H is a row-connected matching and $|V(H_0)| > 2$.

Therefore, $|\{k', k''\} \cap \{k^*, k^{**}\}| \leqslant 1$. The case $\{k', k''\} \cap \{k^*, k^{**}\} = \emptyset$ is also impossible, since then (5.13) contradicts (5.18).

Thus, we have $|\{k', k''\} \cap \{k^*, k^{**}\}| = 1$.Let, say, $k' = k^*$. Then by (5.18),

$$|\{(k^*.1), (k^*.2), (k^*.3)\} \cap V(F)| = 1, \tag{5.19}$$

$$\{(k^*.1), (k^*.2), (k^*.3)\} \subset V(F \cup \{e, e^*\}), \tag{5.20}$$

$$|\{(k^{**}.1), (k^{**}.2), (k^{**}.3)\} \cap V(F)| = 2. \tag{5.21}$$

By using the fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$ and (5.21), we conclude that for all l-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_l) \in \mathscr{L}(a_1, \ldots, a_f)$,

$$r_{k^*v^*} = r_{k^{**}v^{**}} = \text{const}(a_1, \ldots, a_f). \tag{5.22}$$

Again by using the fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$ and (5.19), (5.20), (5.22),

$$r_{k'i'} = r_{k''i''} = \text{const}(a_1, \ldots, a_f, r_{k^*v^*}) = \text{const}(a_1, \ldots, a_f),$$

which proves (5.15).

Let

$$T = \{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(H)| = 1\},$$

and $|T| = t$. Let

$$S = \{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(H)| \geqslant 2\},$$

and $|S| = s = l - t$. Next, let

$$Z_i = \{k\colon 1 \leqslant k \leqslant l, |\{(k.1), (k.2), (k.3)\} \cap V(F)| = i\},$$

and $|Z_i| = z_i$ $(i = 0, 1, 2)$. Note that $T \subseteq Z_0 \cup Z_1$, and so $t \leqslant z_0 + z_1$.

By (5.12),

$$|V(F)| = z_1 + 2z_2. \tag{5.23}$$

By using $z_0 + z_1 + z_2 = l$, (5.14) and (5.23),

$$2f = |V(F)| = z_1 + 2z_2 = 2l - 2z_0 - z_1,$$

that is,

$$2f + (2z_0 + z_1) = 2l. \tag{5.24}$$

Since H is a matching and the maximum degree of H_0 is $\geqslant 3$, from (5.12) we have that there is an edge $e = \{(k'.i'), (k''.i'')\} \in H \backslash F$ such that

$$\{(k'.1), (k'.2), (k'.3)\} \subset V(H) \quad \text{or} \quad \{(k''.1), (k''.2), (k''.3)\} \subset V(H).$$

Let, say,

$$\{(k'.1), (k'.2), (k'.3)\} \subset V(H) \quad \text{and} \quad |\{(k''.1), (k''.2), (k''.3)\} \cap V(H)| = y.$$

Clearly $1 \leqslant y \leqslant 3$.

If $y = 1$, then since $F \subset H \backslash \{e\}, k'' \in Z_0$. Hence, $z_0 \geqslant 1$. Since $t \leqslant z_0 + z_1$, by (5.24) we have

$$2l = 2f + (2z_0 + z_1) \geqslant 2f + (z_0 + z_1) + z_0 \geqslant 2f + t + 1,$$

that is,

$$2f + t \leqslant 2l - 1.$$

If $y = 2$, then $k'' \in S$ and, since $F \subset H \backslash \{e\}, k'' \in Z_0 \cup Z_1$. Hence $k'' \in (Z_0 \cup Z_1) \backslash T$, and so $z_0 + z_1 > t$. Thus by (5.24),

$$2l = 2f + (2z_0 + z_1) \geqslant 2f + (z_0 + z_1) > 2f + t,$$

that is,

$$2f + t \leqslant 2l - 1.$$

If $y = 3$, then $\{k', k''\} \subset S$, and by (5.13),

$$k' \in Z_0 \cup Z_1 \quad \text{or} \quad k'' \in Z_0 \cup Z_1.$$

Hence $(Z_0 \cup Z_1) \cap S \neq \emptyset$, and again we have $z_0 + z_1 > t$. Thus by (5.24), again we have

$$2f + t \leqslant 2l - 1.$$

Summarizing, we always have

$$2f + t \leqslant 2l - 1. \tag{5.25}$$

We can now complete the proof of Lemma 5.2 by using (5.25) along exactly the same lines as we completed the proof of Lemma 5.1 by using (5.11). □

Proof of Lemma 5.3. For notational convenience, assume that

$$V(H_0) = \{1, \dots, l\}, \quad H_0 = \{\{1,2\},\{2,3\},\dots,\{l-1,l\},\{l,1\}\} \quad \text{and} \quad \mathbf{r}_k \in \mathscr{B}_k \quad (1 \leqslant k \leqslant l).$$

We distinguish two cases.

Case 1: $\exists j \in \{1,2,\dots,l\}$ such that

$$\{(j.i_1),(j+1.i_1)\} \in H, \{(j+1.i_2),(j+2.i_2)\} \in H, \{(j+2.i_3),(j+3.i_3)\} \in H, |\{i_1,i_2,i_3\}| = 2 \tag{5.26}$$

Case 2: $\forall j (1 \leqslant j \leqslant l): |\{i_1,i_2,i_3\}| = 3$ where i_1, i_2, i_3 are defined by (5.26).

(Note that in (5.26) we use the convention $l+1=1, l+2=2, l+3=3$; and note also that the case $|\{i_1,i_2,i_3\}|=1$ is impossible, since H does not contain a path of length 2.)

We begin with Case 1. Note that in this case $l \geqslant 4$. Without loss of generality, we can assume that

$$j = l, \quad i_1 = i_3 = 1, \quad i_2 = 2, \tag{5.27}$$

or equivalently,

$$\{\{(l.1),(1.1)\},\{(1.2),(2.2)\},\{(2.1),(3.1)\}\} \subset H.$$

Expanding the product in Lemma 5.3, we have

$$\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1,1;\dots;\mathbf{r}_l,l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x}$$

$$= \varrho^{2ml} \tilde{\sum} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x})\, d\mathbf{x} \tag{5.28}$$

where $\tilde{\Sigma}$ is extended over all $2ml$-tuples $(\mathbf{r}_1, \dots, \mathbf{r}_{2ml})$ of index-vectors $\mathbf{r}_j = (r_{j1}, r_{j2}, r_{j3})$ such that

$$\text{if } j \equiv \nu \pmod{l} \quad \text{then} \quad \mathbf{r}_j \in \mathscr{B}_\nu \quad (1 \leqslant \nu \leqslant l),$$
$$r_{ji_\nu} = r_{(j+1)i_\nu} \text{ if } \nu \neq l \quad \text{and} \quad r_{ji_\nu} = r_{(j+1-l)i_\nu} \quad \text{if} \quad \nu = l; \tag{5.29}$$

and by (5.27), $i_l = i_2 = 1, i_1 = 2$.

We have

$$\tilde{\sum} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x})\, d\mathbf{x}$$

$$= \sum_{a_2=0}^{n} \sum_{a_3=0}^{n} \cdots \sum_{a_{2ml}=0}^{n} \int_{U^3} \left(\sum{}' g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right) d\mathbf{x} \tag{5.30}$$

where Σ' is extended over all $2ml$-tuples $(\mathbf{r}_1, \dots, \mathbf{r}_{2ml})$ satisfying (5.29) such that for all $j = 2, 3, \dots, 2ml$,

$$\begin{aligned} &\text{if } j \equiv v \pmod{l} \quad \text{and} \quad 1 \leqslant v \leqslant l-1 \quad \text{then} \quad r_{ji_v} = r_{(j+1)i_v} = a_j, \\ &\text{if } j \equiv 0 \pmod{l} \quad \text{then} \quad r_{ji_v} = r_{(j+1-l)i_v} = a_j. \end{aligned} \tag{5.31}$$

Note that the summation Σ' is actually extended over the coordinate r_{12} only, since the vector $(a_1, \dots, a_{2ml})$ with $a_1 = r_{12} = r_{22}$, uniquely determines $(\mathbf{r}_1, \dots, \mathbf{r}_{2ml})$. Therefore, by (5.29) and (5.31),

$$\sum_{(5.31)}{}' g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x})$$

$$= \left(\prod_{k=3}^{2ml} g_{\mathbf{r}_k}(\mathbf{x}) \right) \cdot \left(\sum_{\substack{\mathbf{r}_1 \in \mathscr{B}_1 \\ r_{11} = a_l}} \sum_{\substack{\mathbf{r}_2 \in \mathscr{B}_2, r_{22} = r_{12} \\ r_{21} = a_2}} g_{\mathbf{r}_1}(\mathbf{x}) g_{\mathbf{r}_2}(\mathbf{x}) \right). \tag{5.32}$$

If $a_l \neq a_2$, then by Cauchy-Schwarz inequality, (5.32) and Lemma 4.7,

$$\left| \int_{U^3} \left(\sum_{(5.31)}{}' g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right) d\mathbf{x} \right.$$

$$\leqslant \left(\int_{U^3} \left(\sum_{\substack{\mathbf{r}_1 \in \mathscr{B}_1 \\ r_{11} = a_l}} \sum_{\substack{\mathbf{r}_2 \in \mathscr{B}_2, r_{22} = r_{12} \\ r_{21} = a_2}} g_{\mathbf{r}_1}(\mathbf{x}) g_{\mathbf{r}_2}(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2} \leqslant (n+1)^{1/2}. \tag{5.33}$$

If $a_l = a_2$, then by (5.32),

$$\left| \int_{U^3} \left(\sum_{(5.31)}{}' g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right) d\mathbf{x} \right| \leqslant \sum_{r_{12} = r_{22}} 1 \leqslant n+1. \tag{5.34}$$

Summarizing, by (5.28), (5.30), (5.33) and (5.34),

$$\int_{U^3}\left(\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\right)^{2m}d\mathbf{x}$$

$$\leqslant \sum_{a_2=0}^{n}\cdots\sum_{a_{2ml}=0}^{n}(n+1)^{1/2}+\sum_{a_2=a_l=0}^{n}\cdots\sum_{a_{l-1}=0}^{n}\sum_{a_{l+1}=0}^{n}\cdots\sum_{a_{2ml}=0}^{n}(n+1)$$

$$\leqslant (n+1)^{2ml-(1/2)}+(n+1)^{2ml-1}\leqslant 2(n+1)^{2ml-(1/2)}. \tag{5.35}$$

Hence

$$\int_{U^3}\left(\varrho^l\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\right)^{2m}d\mathbf{x}$$

$$\leqslant \varrho^{2ml}\cdot 2(n+1)^{2ml-(1/2)}=2\left(\frac{q^{(1/4)-\varepsilon}}{n+1}\right)^{2ml}\cdot(n+1)^{2ml-(1/2)}$$

$$\leqslant 2q^{ml/2}(n+1)^{-1/2}, \tag{5.36}$$

which was to be proved.

Next consider Case 2. Without loss of generality, we can assume that

$$\{(1.1),(2.1)\}\in H \quad\text{and}\quad \{(2.2),(3.2)\}\in H.$$

Then H is uniquely determined, and we have

$$H=\{\{(3k+i.i),(3k+i+1.i)\}: k=0,1,\ldots,l/3-1; i=1,2,3\}.$$

Note that in Case 2, l is divisible by 3.

Expanding the product in Lemma 5.3, we have

$$\int_{U^3}\left(\varrho^l\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_l}(\mathbf{x})\right)^{2m}d\mathbf{x}$$

$$=\varrho^{2ml}\tilde{\sum}\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\,d\mathbf{x}, \tag{5.37}$$

where $\tilde{\Sigma}$ is taken over all $2ml$-tuples $(\mathbf{r}_1,\ldots,\mathbf{r}_{2ml})$ of index-vectors $\mathbf{r}_j=(r_{j1},r_{j2},r_{j3})$

such that

$$\begin{aligned}&\text{if } j \equiv v \pmod{l}\ (1 \leqslant v \leqslant l) \quad \text{then} \quad \mathbf{r}_j \in \mathscr{B}_v,\\ &\text{if } j \not\equiv 0 \pmod{l} \quad \text{and} \quad j \equiv i \pmod{3} \quad \text{then} \quad r_{ji} = r_{(j+1)i},\\ &\text{if } j \equiv 0 \pmod{l} \quad \text{then} \quad r_{j3} = r_{(j+1)3}.\end{aligned} \tag{5.38}$$

Before applying Lemma 4.7, as we did in Case 1, we shall use the following "trick" (essentially a double application of Cauchy-Schwarz inequality and Lemma 2.2(a)). We have

$$\begin{aligned}&\tilde{\sum_{(5.38)}} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\, d\mathbf{x}\\ &= \sum_{a_1=0}^{n} \cdots \sum_{a_{2ml/3}=0}^{n} \sum_{b_1=0}^{n} \cdots \sum_{b_{2ml/3}=0}^{n} \int_{U^3} \left(\sum\nolimits^{*} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right) d\mathbf{x}\end{aligned} \tag{5.39}$$

where Σ^* is taken over all $2ml$-tuples $(\mathbf{r}_1, \ldots, \mathbf{r}_{2ml})$ satisfying (5.38) and (5.40) below

$$\begin{aligned}r_{(3k+1)1} &= r_{(3k+2)1} = a_{k+1} \quad (0 \leqslant k \leqslant 2ml/3 - 1),\\ r_{(3k+2)2} &= r_{(3k+3)2} = b_{k+1} \quad (0 \leqslant k \leqslant 2ml/3 - 1).\end{aligned} \tag{5.40}$$

By Cauchy-Schwarz inequality and (5.39),

$$\begin{aligned}&\tilde{\sum_{(5.38)}} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\, d\mathbf{x}\\ &\leqslant \sum_{a_1=0}^{n} \cdots \sum_{a_{2ml/3}=0}^{n} \sum_{b_1=0}^{n} \cdots \sum_{b_{2ml/3}=0}^{n} \left(\oint \right)^{1/2}\end{aligned} \tag{5.41}$$

where

$$\oint = \int_{U^3} \left(\sum_{\substack{(5.38)\\(5.40)}}^{*} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right)^2 d\mathbf{x}.$$

By using the discrete form of the Cauchy-Schwarz inequality

$$\sum_{i=1}^{t} y_i \leqslant t^{1/2} \cdot \left(\sum_{i=1}^{t} (y_i)^2 \right)^{1/2},$$

from (5.41) we have

$$\tilde{\sum_{(5.38)}} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x})\, d\mathbf{x}$$

$$\leqslant (n+1)^{2ml/3} \left(\sum_{a_1=0}^{n} \cdots \sum_{a_{2ml/3}=0}^{n} \sum_{b_1=0}^{n} \cdots \sum_{b_{2ml/3}=0}^{n} \oint \right)^{1/2}, \tag{5.42}$$

where

$$\oint = \int_{U^3} \left(\sum_{\substack{(5.38)\\(5.40)}}^{*} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right)^2 d\mathbf{x}.$$

Note that

$$\sum_{a_1=0}^{n} \cdots \sum_{a_{2ml/3}=0}^{n} \sum_{b_1=0}^{n} \cdots \sum_{b_{2ml/3}=0}^{n} \oint$$

$$= \sum_{a_1=0}^{n} \cdots \sum_{a_{2ml/3}=0}^{n} \sum_{b_1=0}^{n} \cdots \sum_{b_{2ml/3}=0}^{n} \sum_{c_1=0}^{n} \cdots \sum_{c_{2ml/3}=0}^{n} \sum_{d_1=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} \int \tag{5.43}$$

where

$$\int = \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{4ml}}(\mathbf{x})\, d\mathbf{x}$$

and

$$\text{if } k \equiv v (\operatorname{mod} l)\ (1 \leqslant v \leqslant l) \quad \text{then} \quad \mathbf{r}_k \in \mathscr{B}_v (1 \leqslant k \leqslant 4ml), \tag{5.44_1}$$

$$r_{(3k+1)1} = r_{(3k+2)1} = r_{(2ml+3k+1)1} = r_{(2ml+3k+2)1} = a_{k+1} \quad (0 \leqslant k < 2ml/3),$$
$$r_{(3k+2)2} = r_{(3k+3)2} = r_{(2ml+3k+2)2} = r_{(2ml+3k+3)2} = b_{k+1} \quad (0 \leqslant k < 2ml/3);$$

$$\text{if } 0 \leqslant k < 2lm/3 \quad \text{and} \quad 3k+3 \not\equiv 0 \ (\operatorname{mod} l) \quad \text{then} \tag{5.44_2}$$

$$r_{(3k+3)3} = r_{(3k+4)3} = c_{k+1},\ r_{(2ml+3k+3)3} = r_{(2ml+3k+4)3} = d_{k+1};$$

$$\text{if } 0 \leqslant k < 2lm/3 \quad \text{and} \quad 3k+3 \equiv 0 \ (\operatorname{mod} l) \quad \text{then} \tag{5.44_3}$$

$$r_{(3k+3)3} = r_{(3k+4-l)3} = c_{k+1},\ r_{(2ml+3k+3)3} = r_{(2ml+3k+4-l)3} = d_{k+1}.$$

It follows from (5.44) that $\mathbf{r}_{3k+2} = \mathbf{r}_{2ml+3k+2}$ for all $0 \leqslant k < 2ml/3$, and so

$$g_{\mathbf{r}_{k+2}} \equiv g_{\mathbf{r}_{2ml+3k+2}}$$

for all $0 \leqslant k < 2ml/3$. Using the fact $(g_{\mathbf{r}})^2 \equiv 1$ we get

$$\prod_{j=1}^{4ml} g_{\mathbf{r}_j} \equiv \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}. \tag{5.45}$$

From (5.45) we have

$$\sum_{a_1=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} \int_{U^3} \left(\prod_{j=1}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) \mathrm{d}\mathbf{x}$$

$$= \sum_{a_1=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} \int_{U^3} \left(\prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) \mathrm{d}\mathbf{x}$$

$$= \sum_{a_2=0}^{n} \sum_{a_3=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} \int_{U^3} \left(\sum_{a_1}{}' \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) \mathrm{d}\mathbf{x} \tag{5.46}$$

where Σ' is extended over all $8ml/3$-tuples $(\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_6, \ldots, \mathbf{r}_{4ml-2}, \mathbf{r}_{4ml})$ satisfying (5.44) except of the requirement $r_{11} = r_{(2ml+1)1} = a_1$.

We are now able to complete Case 2 as we did in Case 1. Note that for every fixed $a_2 \in \{0, \ldots, n\}$, $a_3 \in \{0, \ldots, n\}, \ldots, d_{2ml/3} \in \{0, \ldots, n\}$, we have

$$\sum_{a_1}{}' \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j} = \left(\prod_{\substack{j=2 \\ j \neq 2ml+1,\, j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j} \right) \cdot \Sigma_0 \tag{5.47}$$

where

$$\Sigma_0(\mathbf{x}) = \sum_{\substack{\mathbf{r}_1 \in \mathscr{B}_1 \\ r_{13} = c_1}} \sum_{\substack{\mathbf{r}_{2ml+1} \in \mathscr{B}_1,\, r_{(2ml+1)1} = r_{11} \\ r_{(2ml+1)3} = d_1}} g_{\mathbf{r}_1}(\mathbf{x}) g_{\mathbf{r}_{2ml+1}}(\mathbf{x}).$$

Suppose that $c_1 \neq d_1$. Applying Cauchy-Schwarz inequality to (5.47),

$$\int_{U^3} \left(\sum_{a_1}{}' \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) \mathrm{d}\mathbf{x} \leqslant \left(\int_{U^3} (\Sigma_0(\mathbf{x}))^2 \, \mathrm{d}\mathbf{x} \right)^{1/2}.$$

By Lemma 4.7,

$$\int_{U^3} (\Sigma_0(\mathbf{x}))^2 \, d\mathbf{x} \leqslant (n+1),$$

and so

$$\int_{U^3} \left(\sum_{a_1}{}' \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) d\mathbf{x} \leqslant (n+1)^{1/2}. \tag{5.48}$$

If $c_1 = d_1$, then by (5.47) we trivially have

$$\int_{U^3} \left(\sum_{a_1}{}' \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod 3}}^{4ml} g_{\mathbf{r}_j}(\mathbf{x}) \right) d\mathbf{x} \leqslant \sum_{a_1}{}' 1 \leqslant n+1. \tag{5.49}$$

Now from (5.46), (5.48) and (5.49),

$$\begin{aligned} &\sum_{a_1=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{4ml}}(\mathbf{x}) \, d\mathbf{x} \\ &\quad \leqslant \left(\sum_{a_2=0}^{n} \cdots \sum_{c_1=0}^{n} \cdots \sum_{\substack{d_1=0 \\ d_1 \neq c_1}}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} (n+1)^{1/2} + \right. \\ &\quad\quad \left. + \sum_{a_2=0}^{n} \cdots \sum_{c_1=d_1=0}^{n} \cdots \sum_{c_{2ml/3}=0}^{n} \sum_{d_2=0}^{n} \cdots \sum_{d_{2ml/3}=0}^{n} (n+1) \right) \\ &\quad \leqslant (n+1)^{8ml/3-(1/2)} + (n+1)^{8ml/3-1} \leqslant 2(n+1)^{8ml/3-(1/2)}. \end{aligned} \tag{5.50}$$

By (5.39), (5.42), (5.43) and (5.50),

$$\begin{aligned} &\tilde{\sum_{(5.38)}} \int_{U^3} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \, d\mathbf{x} \\ &\quad \leqslant (n+1)^{2ml/3} (2(n+1)^{8ml/3-(1/2)})^{1/2} \leqslant 2(n+1)^{2ml-(1/4)}. \end{aligned} \tag{5.51}$$

Therefore, by (5.37) and (5.51),

$$\begin{aligned} &\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, 1; \dots; \mathbf{r}_l, l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \dots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x} \leqslant \varrho^{2ml} 2(n+1)^{2ml-(1/4)} \\ &\quad = 2 \left(\frac{q^{(1/4)-\varepsilon}}{n+1} \right)^{2ml} (n+1)^{2ml-(1/4)} \leqslant 2q^{ml/2}(n+1)^{-1/4}, \end{aligned}$$

and Lemma 5.3 follows. □

Proof of Lemma 5.4. For notational convenience, assume that

$$V(H_0)=\{1,2,\ldots,l\},\quad H_0=\{\{1,2\},\{2,3\},\ldots,\{l-1,l\}\},\quad \mathbf{r}_j\in\mathscr{B}_j\quad(1\leqslant j\leqslant l).$$

Let

$$H=\{\{(j.i_j),(j+1.i_j)\}: j=1,2,\ldots,l-1\}$$

where $i_j\in\{1,2,3\}$. Without loss of generality, we can assume that

$$i_1=1, i_2=2,\quad \text{i.e.}\quad \{\{(1.1),(2.1)\},\{(2.2),(3.2)\}\}\subset H. \tag{5.52}$$

Expanding the product in Lemma 5.4, we have

$$\int_{U^3}\left(\varrho^l\sum_{G(\mathbf{r}_1,1;\ldots;\mathbf{r}_l,l)\supseteq H} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_l}(\mathbf{x})\right)^{2m}\mathrm{d}\mathbf{x}$$

$$=\varrho^{2ml}\tilde{\sum}\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{5.53}$$

where $\tilde{\Sigma}$ is taken over all $2ml$-tuples $(\mathbf{r}_1,\ldots,\mathbf{r}_{2ml})$ of index-vectors $\mathbf{r}_j=(\mathbf{r}_{j1},\mathbf{r}_{j2},\mathbf{r}_{j3})$ such that

$$\text{if } j\equiv v \ (\mathrm{mod}\ l),\ 1\leqslant v\leqslant l,\quad \text{then}\quad \mathbf{r}_j\in\mathscr{B}_v, \tag{5.54}$$

and if $1\leqslant v\leqslant l-1$ then $r_{ji_v}=r_{(j+1)i_v}$; and by (5.52), $i_1=1$, $i_2=2$.

We have

$$\tilde{\sum_{(5.54)}}\int_{U^3} g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

$$=\sum_{a_2=0}^{n}\cdots\sum_{a_{l-1}=0}^{n}\sum_{a_{l+1}=0}^{n}\cdots\sum_{a_{2l-1}=0}^{n}\sum_{a_{2l+1}=0}^{n}\cdots\sum_{a_{2ml-1}=0}^{n}\oint$$

where

$$\oint=\int_{U^3}\sum{}' g_{\mathbf{r}_1}(\mathbf{x})\ldots g_{\mathbf{r}_{2ml}}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

and the summation Σ' is taken over all $2ml$-tuples $(\mathbf{r}_1,\ldots,\mathbf{r}_{2ml})$ satisfying (5.54) such that

$$\text{if } j \equiv v \,(\text{mod } l),\ 1 \leqslant v \leqslant l-1 \quad \text{and} \quad 2 \leqslant j < 2ml \quad \text{then} \quad r_{ji_v} = r_{(j+1)i_v} = a_j. \tag{5.56}$$

It follows from (5.56) and from the fact $|\mathbf{r}| = r_1 + r_2 + r_3 = n$, that the vector

$$(a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{2l-1}, a_{2l+1}, \ldots, a_{2ml-1})$$

uniquely determines the sequence $\mathbf{r}_j$, $3 \leqslant j < 2ml$, $j \not\equiv 0, 1(\text{mod } l)$ of index-vectors. Thus, for every fixed vector

$$(a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{2l-1}, a_{2l+1}, \ldots, a_{2ml-1})$$

we have

$$\sum_{(5.56)}' g_{\mathbf{r}_1} \cdots g_{\mathbf{r}_{2ml}} = \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \cdot \Pi_4, \tag{5.57}$$

where

$$\Pi_1 = \sum_{\mathbf{r}_1 \in \mathscr{B}_1} \sum_{\substack{\mathbf{r}_2 \in \mathscr{B}_2, r_{21} = r_{11} \\ r_{22} = a_2}} g_{\mathbf{r}_1} g_{\mathbf{r}_2},$$

$$\Pi_2 = \prod_{t=1}^{2m-1} \left(\left(\sum_{\substack{\mathbf{r}_{tl} \in \mathscr{B}_l \\ r_{(tl)i_{l-1}} = a_{tl-1}}} g_{\mathbf{r}_{tl}} \right) \left(\sum_{\substack{\mathbf{r}_{tl+1} \in \mathscr{B}_1 \\ r_{(tl+1)i_1} = a_{tl+1}}} g_{\mathbf{r}_{tl+1}} \right) \right),$$

$$\Pi_3 = \sum_{\substack{\mathbf{r}_{2ml} \in \mathscr{B}_l \\ r_{(2ml)i_{l-1}} = a_{2ml-1}}} g_{\mathbf{r}_{2ml}},$$

$$\Pi_4 = \prod_{\substack{3 \leqslant j < 2ml \\ j \not\equiv 0, 1 (\text{mod } l)}} g_{\mathbf{r}_j}.$$

By using (5.57), Lemma 4.4(b) with $k = 4m$, and Lemma 4.6 with $m = 8m$, we get

$$\left| \int_{U^3} \left(\sum_{(5.56)}' g_{\mathbf{r}_1}(\mathbf{x}) \cdots g_{\mathbf{r}_{2ml}}(\mathbf{x}) \right) d\mathbf{x} \right|$$
$$\leqslant \left(\int_{U^3} (\Pi_1(\mathbf{x}))^2 \, d\mathbf{x} \right)^{1/2} (16m(n+1))^{(4m-1)/2}. \tag{5.58}$$

It follows from Lemma 2.2(c) that

$$\int_{U^3} (\Pi_1(\mathbf{x}))^2 \, d\mathbf{x}$$

$$= \tilde{\sum} \int_{U^3} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}) \, d\mathbf{x} \tag{5.59}$$

where the summation $\tilde{\Sigma}$ is taken over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ such that

$$\{\mathbf{r}_\alpha, \mathbf{r}_\gamma\} \subset \mathscr{B}_1, \quad \{\mathbf{r}_\beta, \mathbf{r}_\delta\} \subset \mathscr{B}_2, \quad r_{\beta 2} = r_{\delta 2} = a_2, \quad r_{\alpha 1} = r_{\beta 1}, \quad r_{\gamma 1} = r_{\delta 1}, \tag{5.60}$$

and there are $\zeta, \eta \in \{\alpha, \beta, \gamma, \delta\}$ such that $\zeta \neq \eta, r_{\zeta 3} = r_{\eta 3}$.

In this way, we can associate with every quadruplet $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ satisfying (5.60) a pair $\{\zeta, \eta\}$, i.e.

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = \{\zeta, \eta\}.$$

(If the mapping ψ is not uniquely determined, then we choose among the possible pairs $\{\zeta, \eta\}$ arbitrarily.)

We distinguish 6 cases as follows (we shall essentially repeat the proof of Lemma 4.5(b)).

Case 1: $\{\zeta, \eta\} = \{\alpha, \gamma\}$
Case 2.1: $\zeta = \alpha, \eta = \delta$
Case 2.2: $\zeta = \beta, \eta = \gamma$
Case 3: $\{\zeta, \eta\} = \{\beta, \delta\}$
Case 4.1: $\zeta = \alpha, \eta = \beta$
Case 4.2: $\zeta = \gamma, \eta = \delta$

For notational convenience, let

$$g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\beta}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}).$$

We have

$$\tilde{\sum_{\text{Case 1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, d\mathbf{x}$$

$$= \sum_{\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = \{\alpha, \gamma\}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \, d\mathbf{x}$$

$$= \sum_{b=0}^{n} \sum_{c=0}^{n} \int_{U^3} \left(\sum_{r_{\alpha 3} = r_{\gamma 3}}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \right) d\mathbf{x} \tag{5.61}$$

where Σ' is taken over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ with

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = \{\alpha, \gamma\}$$

such that (see (5.60))

$$r_{\beta 2} = r_{\delta 2} = a_2, \quad r_{\alpha 1} = r_{\beta 1} = b, \quad r_{\gamma 1} = r_{\delta 1} = c, \quad r_{\alpha 3} = r_{\gamma 3}. \tag{5.62}$$

Note that Σ' is actually extended over the coordinate $r_{\alpha 3}$ only, since the quadruplet (a_2, b, c, d) with $d = r_{\alpha 3} = r_{\gamma 3}$, uniquely determines $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$.

For every fixed a_2, b, c with $b \neq c$, by Cauchy-Schwarz inequality and Lemma 4.7, we have

$$\left| \int_{U^3} \left(\sum_{(5.62)}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \right) d\mathbf{x} \right|$$

$$= \left| \int_{U^3} g_{\mathbf{r}_\beta}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}) \left(\sum_{\substack{\mathbf{r}_\alpha \in \mathscr{B}_1 \\ r_{\alpha 1} = b}} \sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1, r_{\gamma 3} = r_{\alpha 3} \\ r_{\gamma 1} = c}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) \right) d\mathbf{x} \right|$$

$$\leqslant \left(\int_{U^3} \left(\sum_{\substack{\mathbf{r}_\alpha \in \mathscr{B}_1 \\ r_{\alpha 1} = b}} \sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1, r_{\gamma 3} = r_{\alpha 3} \\ r_{\gamma 1} = c}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2} \leqslant (n+1)^{1/2}. \tag{5.63}$$

If $b = c$, then clearly

$$\left| \int_{U^3} \left(\sum_{(5.62)}' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \right) d\mathbf{x} \right| \leqslant \sum_{(5.62)}' 1 \leqslant n + 1. \tag{5.64}$$

Hence, by (5.61), (5.63), (5.64),

$$\left| \tilde{\sum_{\text{Case 1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right|$$

$$\leqslant \sum_{b=0}^{n} \sum_{\substack{c=0 \\ c \neq b}}^{n} (n+1)^{1/2} + \sum_{b=c=0}^{n} (n+1) \leqslant 2(n+1)^{5/2}. \tag{5.65}$$

Next consider

$$\tilde{\sum_{\text{Case 2.1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= \sum_{\psi((\mathbf{r}_\alpha,\mathbf{r}_\beta,\mathbf{r}_\gamma,\mathbf{r}_\delta))=\{\alpha,\delta\}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= \sum_{b=0}^{n} \sum_{c=0}^{n} \int_{U^3} \left(\sum_{r_{\gamma 2}}'' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \right) d\mathbf{x} \tag{5.66}$$

where Σ'' is taken over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ with

$$\psi((\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)) = \{\alpha, \delta\}$$

such that (see (5.60))

$$r_{\beta 2} = r_{\delta 2} = a_2, \quad r_{\alpha 1} = r_{\beta 1} = b, \quad r_{\gamma 1} = r_{\delta 1} = c, \quad r_{\alpha 3} = r_{\delta 3}. \tag{5.67}$$

Note that

$$r_{\alpha 3} = r_{\delta 3} = n - r_{\delta 1} - r_{\delta 2} = n - c - a_2,$$

so Σ'' is actually extended over the coordinate $r_{\gamma 2}$ only.

For every fixed a_2, b, c, we have

$$\sum_{(5.67)}'' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = g_{\mathbf{r}_\alpha}(\mathbf{x}) \cdot g_{\mathbf{r}_\beta}(\mathbf{x}) \cdot g_{\mathbf{r}_\delta}(\mathbf{x}) \left(\sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1 \\ r_{\gamma 1} = c}} g_{\mathbf{r}_\gamma}(\mathbf{x}) \right).$$

Thus, from Lemma 4.6 with $m = 1$, we have

$$\left| \int_{U^3} \left(\sum_{(5.67)}'' g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) \right) d\mathbf{x} \right|$$

$$\leqslant \left(\int_{U^3} \left(\sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1 \\ r_{\gamma 1} = c}} g_{\mathbf{r}_\gamma}(\mathbf{x}) \right)^2 d\mathbf{x} \right)^{1/2} \leqslant (n+1)^{1/2}. \tag{5.68}$$

Combining (5.66) and (5.68),

$$\left| \tilde{\sum_{\text{Case 2.1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \leqslant \sum_{b=0}^{n} \sum_{c=0}^{n} (n+1)^{1/2} = (n+1)^{5/2}. \tag{5.69}$$

Similarly,

$$\left| \tilde{\sum_{\text{Case 2.2}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \leqslant (n+1)^{5/2}. \tag{5.70}$$

Next, consider the integral

$$\tilde{\sum_{\text{Case 3}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

where the summation extends over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ such that (see (5.60))

$$r_{\beta 2} = r_{\delta 2} = a_2, \quad r_{\alpha 1} = r_{\beta 1}, \quad r_{\gamma 1} = r_{\delta 1}, \quad r_{\beta 3} = r_{\delta 3}.$$

These give that

$$r_{\beta 1} = n - r_{\beta 2} - r_{\beta 3} = n - a_2 - r_{\delta 3} = n - r_{\delta 2} - r_{\delta 3} = r_{\delta 1}.$$

Hence $r_{\alpha 1} = r_{\beta 1} = r_{\gamma 1} = r_{\delta 1}$. Since $r_{\beta 3} = r_{\delta 3}$ and $|\mathbf{r}_\beta| = |\mathbf{r}_\delta| = n$, we conclude that $\mathbf{r}_\beta = \mathbf{r}_\delta$, and so $g_{\mathbf{r}_\beta} \equiv g_{\mathbf{r}_\delta}$. Therefore,

$$g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = g_{\mathbf{r}_\alpha}(\mathbf{x}) \cdot (g_{\mathbf{r}_\beta}(\mathbf{x}))^2 \cdot g_{\mathbf{r}_\gamma}(\mathbf{x}) = g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}).$$

Thus, by Lemma 2.2(c), we have (note that $r_{\beta 2} = r_{\delta 2} = a_2$ is fixed)

$$\tilde{\sum_{\text{Case 3}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

$$= \sum_{b=0}^{n} \int_{U^3} \left(\sum_{\mathbf{r}_\alpha \in \mathscr{B}_1} \sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1 \\ r_{\alpha 1} = r_{\gamma 1} = b}} g_{\mathbf{r}_\alpha}(\mathbf{x}) g_{\mathbf{r}_\gamma}(\mathbf{x}) \right) d\mathbf{x} = \sum_{b=0}^{n} \left(\sum_{\substack{\mathbf{r}_\alpha = \mathbf{r}_\gamma \in \mathscr{B}_1 \\ r_{\alpha 1} = r_{\gamma 1} = b}} 1 \right).$$

Hence

$$\left| \tilde{\sum_{\text{Case 3}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \leqslant (n+1)^2. \tag{5.71}$$

Finally, consider the integral

$$\tilde{\sum_{\text{Case 4.1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x}$$

where the summation is taken over all quadruplets $(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta)$ such that (see (5.60))

$$r_{\beta 2} = r_{\delta 2} = a_2, \quad r_{\alpha 1} = r_{\beta 1}, \quad r_{\gamma 1} = r_{\delta 1}, \quad r_{\alpha 3} = r_{\beta 3}.$$

These give that $\mathbf{r}_\alpha = \mathbf{r}_\beta$, and so

$$g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x}) = (g_{\mathbf{r}_\alpha}(\mathbf{x}))^2 \cdot g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}) = g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}).$$

Thus, by Lemma 2.2(c), we have

$$\begin{aligned}
&\tilde{\sum_{\text{Case 4.1}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \\
&\quad = \sum_{\substack{\mathbf{r}_\alpha = \mathbf{r}_\beta \in \mathscr{B}_1 \cap \mathscr{B}_2 \\ r_{\beta 2} = a_2 \text{ fixed}}} \int_{U^3} \left(\sum_{\substack{\mathbf{r}_\gamma \in \mathscr{B}_1 \\ r_{\delta 2} = a_2}} \sum_{\substack{\mathbf{r}_\delta \in \mathscr{B}_2 \\ r_{\delta 1} = r_{\gamma 1}}} g_{\mathbf{r}_\gamma}(\mathbf{x}) g_{\mathbf{r}_\delta}(\mathbf{x}) \right) d\mathbf{x} \\
&\quad = \sum_{\substack{\mathbf{r}_\alpha = \mathbf{r}_\beta \in \mathscr{B}_1 \cap \mathscr{B}_2 \\ r_{\beta 2} = a_2}} \left(\sum_{\substack{\mathbf{r}_\gamma = \mathbf{r}_\delta \in \mathscr{B}_1 \cap \mathscr{B}_2 \\ r_{\delta 2} = a_2}} 1 \right) \leqslant (n+1)^2. \qquad (5.72)
\end{aligned}$$

Similarly,

$$\left| \tilde{\sum_{\text{Case 4.2}}} \int_{U^3} g(\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma, \mathbf{r}_\delta; \mathbf{x})\, d\mathbf{x} \right| \leqslant (n+1)^2. \qquad (5.73)$$

Summarizing, from (5.59), (5.65), (5.69)–(5.72),

$$\begin{aligned}
&\int_{U^3} (\Pi_1(\mathbf{x}))^2\, d\mathbf{x} \\
&\quad \leqslant (2(n+1)^{5/2} + 2(n+1)^{5/2} + (n+1)^2 + 2(n+1)^2) \leqslant 7(n+1)^{5/2}. \qquad (5.74)
\end{aligned}$$

Returning now to (5.53), by (5.55), (5.58) and (5.74), we obtain

$$\begin{aligned}
&\int_{U^3} \left(\varrho^l \sum_{G(\mathbf{r}_1, 1; \ldots; \mathbf{r}_l, l) \supseteq H} g_{\mathbf{r}_1}(\mathbf{x}) \ldots g_{\mathbf{r}_l}(\mathbf{x}) \right)^{2m} d\mathbf{x} \\
&\quad \leqslant \varrho^{2ml} \cdot (n+1)^{2m(l-1)-1} \cdot (16(n+1))^{(4m-1)/2} \cdot (7(n+1)^{5/2})^{1/2} \\
&\quad < \varrho^{2ml} \cdot 3(16m)^{2m} \cdot (n+1)^{2ml-(1/4)} \\
&\quad = \left(\frac{q^{(1/4)-\varepsilon}}{n+1} \right)^{2ml} \cdot 3(16m)^{2m} \cdot (n+1)^{2ml-(1/4)} \\
&\quad \leqslant q^{ml/2} \cdot 3(16m)^{2m} \cdot (n+1)^{-1/4},
\end{aligned}$$

and Lemma 5.4 follows. □

The proof of Theorem 1.2 is complete.

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