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Reciprocity laws on curves

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Introduction

In this paper we will present a p-adic analogue of the reciprocity law for Green's functions on a Riemann surface, Theorem 1.4 below. In the process we will give a new proof of this classical result. The p-adic result may be used to prove the symmetry of p-adic heights.

The p-adic Green's functions are described in [CG] and are obtained using the results of [CdS] by integrating normalized differentials of the third kind. Our results are proven for any collection of functions on a curve satisfying certain (mainly formal) properties described in Sections 1 and 5. The method of proof is to first push the proof of Weil reciprocity given in Serre's *Groupes Algebriques et Corps de Classes*, as far as it will go (§1). This is already far enough to give proof of the reciprocity law for classical Green's functions which avoids the use of Cauchy's Theorem; but not far enough to obtain the p-adic analogue. To finish the proof in the p-adic case we imitate the classical proof and apply a p-adic analogue of Cauchy's Theorem, Proposition 2.3.

In Section 1, we first recall Weil's reciprocity law and the reciprocity law for harmonic functions on a curve with logarithmic singularities. We then state the p-adic Reciprocity law for Abelian integrals of the third kind. Afterwards we set up a general formalism from which these laws can be deduced. Sections 2–4, are devoted to establishing the necessary results in p-adic geometry to state and prove the p-adic reciprocity law. They are also necessary for the results of [CdS]. In Section 5 we prove the p-adic reciprocity law and construct the integrals.

It is interesting to note that the key fact we need about a Jacobian over $\mathbb{C}$ is that its group of $\mathbb{C}$-valued points is compact and the key fact we need about one over $\mathbb{C}_p$ is that a quotient of its group of $\mathbb{C}_p$-valued points by an open subgroup is a torsion group.

NOTATION. By a curve over a field K, we mean a one-dimensional smooth connected proper scheme over K.

I. A general reciprocity law

Let C be a curve over an algebraically closed field K. For a divisor D on C, $|D|$ will denote is support. Weil's reciprocity law asserts

$$f((g)) = g((f)) \tag{1}$$

for any two non-zero rational functions f and g in $K(C)$ such that $|(f)| \cap |(g)| = \emptyset$. In this example,

$$f(D) = \prod f(P)^{\mathrm{ord}_P(D)}$$

for a divisor D on C.

Suppose now that $K = \mathbb{C}$. Let F be a harmonic function F on C with logarithmic singularities. This means that F is a harmonic function on a Zariski open subset of C such that for each point P of $C(\mathbb{C})$ there exists an integer n_P such that for any local uniformizing parameter z_P at P

$$F - n_P \operatorname{Log}|z_P|$$

extends to a continuous function in a neighborhood of P. If we set

$$[F] = \sum n_P P.$$

where the sum runs over $P \in C(\mathbb{C})$, then the classical reciprocity law asserts

$$F([G]) = G([F]) \tag{2}$$

where F and G are harmonic functions on $C(\mathbb{C})$ such that $|[F]| \cap |[G]| = \emptyset$. In this example and the next,

$$f(D) = \sum \mathrm{ord}_P(D) f(P)$$

for a divisor D on C.

Finally we state the p-adic reciprocity law. By an algebraic differential on C we mean an element of the stalk of the sheaf of Kahler differentials on C at the generic point. Let $K = \mathbb{C}_p$ and suppose C has arboreal reduction (see [CdS] §2). In Section 5 Abelian integrals of the third kind are defined on C. These are, in particular, locally analytic functions F on $C(\mathbb{C}_p)$ such that dF is an algebraic differential on C with simple poles and integral residues. For such a function F we define the divisor

$$[F] = \sum(\mathrm{Res}_P\,\mathrm{d}F)P$$

where the sum runs over $P \in C(\mathbb{C}_p)$. In Section 4 (see Theorem 4.10), we will define a linear map ψ from the space of algebraic differentials on C into the first algebraic de Rham cohomology group of C. (This linear map can be interpreted as the logarithm from the universal vectorial extension of the Jacobian on C to its Lie algebra (see [C-3]).) The formula we will prove in Section 5 is:

$$G([F]) - F([G]) = (\psi(\mathrm{d}G), \psi(\mathrm{d}F))$$

where F and G are Abelian integrals of the third kind on $C(\mathbb{C}_p)$ such that $|[F]| \cap |[G]| = \emptyset$ and the pairing on the right hand side is the cup product on de Rham cohomology.

Now we set up the general formalism. Let H be an abelian group and

$$l\colon K^* \to H$$

a homomorphism. Suppose $\mathscr{F}$ is a sheaf of H valued functions on C with the following properties:

(i) If $f \in \mathcal{O}(U)^*$ then $l \circ f \in \mathscr{F}(U)$.

(ii) If $\mathscr{F}_g$ denotes the stalk of $\mathscr{F}$ at the generic point of C then the map $\mathscr{F}(U) \to \mathscr{F}_g$ is an injection and there exists a homomorphism

$$[\ \ ]\colon \mathscr{F}_g \to \text{Divisors of degree zero on } C$$

such that

(a) If $f \in \mathcal{O}_g$, then $[l \circ f] = (f)$

(b) If $F \in \mathscr{F}_g$, $P \in C(K)$ then $F \in \mathrm{Image}(\mathscr{F}_P \to \mathscr{F}_g)$ iff $\mathrm{ord}_P[F] = 0$.

(iii) Suppose $h\colon C \to \mathbb{P}^1$ is a non-constant morphism. Let U be a non-empty Zariski-open subset of $\mathbb{P}^1$ and suppose $F \in \mathscr{F}(h^{-1}(U))$. Then there exists an $f \in \mathcal{O}_{\mathbb{P}^1}(U)^*$, $a \in H$ such that

$$T_hF(P) = \Sigma_H F(Q) = l \circ f(P) +_H a$$

where the summation runs over $h^{-1}(P)$, counting multiplicities.

We say an element of $\mathscr{F}_g$ is principal if it is of the form $l \circ f +_H a$ for some $f \in K(C)^*$ and $a \in H$. If $F \in \mathscr{F}_g$ such that $\mathrm{ord}_P[F] = 0$, then by (ii)b, there exists a unique $L \in \mathscr{F}_P$ such that $L \mapsto F$. We set

$$F(P) = L(P).$$

For $F \in \mathscr{F}_g$ and D a divisor on C such that $|[F]| \cap |D| = \emptyset$, we set

$$F([D]) = \Sigma_H \operatorname{ord}_P(D)P.$$

REMARK. We note that given l and H as above the quadruple

$$(H, \mathscr{F}, l, [\ \ ]) = (K^*, l \circ (\mathcal{O}_C^*) + H, l, (\ \ ))$$

satisfies all of the above conditions.

Our basic reciprocity law is:

THEOREM 1.1. *Suppose $F, G \in \mathscr{F}_g$ one of which is principal and are such that $|[F]| \cap |[G]| = \emptyset$, then,*

$$F([G]) = G([F]).$$

Before we begin the proof, we need a couple of lemmas. First, for F and G in $\mathscr{F}_g$ and $P \in C(K)$, set

$$(F, G)_P = (F, G)_{\mathscr{F},P} = (mF - nG)|_P + {}_H mnl(-1)$$

where $m = \operatorname{ord}_P[G]$ and $n = \operatorname{ord}_P[F]$. This is well defined by (ii)b. We note that in the standard case $\mathscr{F} = \mathcal{O}^*$, $l = \mathrm{id}$,

$$(f, g)_{\mathcal{O}^*,P} = (-1)^{mn}(f^m/g^n)|_P$$

is the tame symbol and also that

$$(l \circ f, l \circ g)_{\mathscr{F},P} = l((f, g)_{\mathcal{O}^*,P}). \tag{1.1}$$

We set $(\ ,\ )_{C,P} = (\ ,\ )_{\mathcal{O}^*,P}$. Now fix $\mathscr{F}$ on C as above and set $\mathscr{S} = l \circ (\mathcal{O}_{\mathbb{P}^1}^*) + H$ on $\mathbb{P}^1$.

LEMMA 1.2. *For all $F \in \mathscr{F}_g$, $f \in K(C)^*$ and $P \in \mathbb{P}^1(K)$ we have*

$$\sum_{f(Q)=P} (F, l \circ f)_Q = (T_f F, l \circ z)_{\mathscr{S},P} \tag{2.1}$$

if f is non-constant and z is the standard parameter on $\mathbb{P}^1$.

Proof.
Case (*i*). $\operatorname{ord}_Q[F] = 0$ for all $Q \in f^{-1}(P)$.
Then LHS of (2.1) equals

$$\begin{aligned}\sum \operatorname{ord}_Q(f)F(Q) &= \operatorname{ord}_P(z)\sum F(Q)\\ &= \operatorname{ord}_P(z)T_f F(P)\\ &= \text{RHS of (2.1)}\end{aligned}$$

Case (*ii*). $F = l \circ h$ for some $h \in K(C)^*$.
Then LHS =

$$\begin{aligned} l\left(\prod_{f(Q)=P} (h,f)_{C,Q}\right) &= l(N_f h, z)_{\mathbb{P}^1,P}\\ &= (T_f(l\circ h), l\circ z)_{\mathcal{S},P}\end{aligned}$$

using Lemma 1, of ([S], §4, Chapt. III) and the fact that

$$T_f(l\circ h) = l\circ H_f h.$$

The general case now follows from these two because for each $F \in \mathcal{F}_g$, we can find an $h \in K(C)^*$ such that the support of $[F] - (h)$ is disjoint from $f^{-1}(P)$. □

LEMMA 1.3. *Let G be a section of $\mathcal{S}$ at the generic point of $\mathbb{P}^1$. Then,*

$$\sum\nolimits_{H\atop P\in\mathbb{P}^1(K)} (G, l\circ z)_{\mathcal{S},P} = 0.$$

Proof. For G of the form $l\circ f$, $f \in K(\mathbb{P}^1)^*$ this follows from Lemma 2 of (S§4, Chapt. 3) and (1.1). For $G \in H$, we have,

$$\text{LHS} = \left(\sum\nolimits_{H\atop P\in\mathbb{P}^1(K)} \operatorname{ord}_P(z)\right)d = 0.$$ □

Proof of Theorem. Suppose G is principal. As the theorem is trivial when $G \in H$ and as

$$F([G]) - G([F]) = \sum\nolimits_{H\atop Q\in C(K)} (F,G)_Q$$

the theorem is an immediate consequence of Lemmas 1.2 and 1.3. □

Examples.

(i) As mentioned above we can take

$$(H, \mathscr{F}, l, [\ \ ]) = (K^*, \mathcal{O}_C^*, \mathrm{id}, (\ \ ))$$

in which case the previous theorem is just the Weil-reciprocity law, (1) above.

(ii) Suppose $K = \mathbb{C}$, $H = \mathbb{R}$, $l = \mathrm{Log} \circ |\ \ |$, and $\mathscr{F}$ is the sheaf of harmonic functions on $C(\mathbb{C})$ with logarithmic singularities (Green's functions). We defined the notion of a harmonic function with logarithmic singularities above. The sheaf $\mathscr{F}$ is defined by setting $\mathscr{F}(U)$ equal to the set of such functions harmonic on U for each Zariski open U. Finally the symbol $[\ \ ]$ is that used in (2) above. It is not hard to check that this quadruple satisfies the required properties. As an application of the previous theorem we will prove the following classical reciprocity law ((2) above).

THEOREM 1.4. *Let $G, F \in \mathscr{F}_g$ such that $|[F]| \cap |[G]| = \emptyset$. Then*

$$F([G]) = G([F]).$$

Proof. It is well known that for each divisor D of degree zero on C, there exists a $G_D \in \mathscr{F}(C - |D|)$ such that

$$[G_D] = D.$$

Moreover, G_D is defined up to a real constant. Let e, d be two divisor classes on C and E and D divisors of degree zero representing them such that $|D| \cap |E| = \emptyset$. Then set

$$(e, d) = G_D(E) - G_E(D).$$

Theorem 1.1 shows that this is well defined. Hence we obtain a pairing

$$J(\mathbb{C}) \times J(\mathbb{C}) \to \mathbb{R}$$

where J is the Jacobian of C. As one may choose, locally, G_D, to vary continuously (and even real analytically) with D, it follows that this pairing is bi-continuous. Since $J(\mathbb{C})$ is compact, the pairing must be trivial. $\square$

The remainder of this paper will be devoted to proving a p-adic analogue of this theorem.

II. Residues and annuli

Henceforth we will always be working over $\mathbb{C}_p$. Let $\mathscr{R} = |\mathbb{C}_p|$ and $\mathscr{R}^* = \mathscr{R} - \{0\}$. The notation and terminology will be as in [CM, §1]. In particular, for a rigid space W, $A(W)$ will denote the ring of rigid analytic functions on W and $CC(W)$ will denote the set of connected components of W with respect to its Grothendieck topology. An open disk is a rigid space isomorphic to the rigid subspace of $\mathbb{C}_p\{x \in \mathbb{C}_p: |x| < 1\}$ and an open annulus is one isomorphic to $\{x \in \mathbb{C}_p: r < |x| < s\}$ for some $r < s \in \mathscr{R}^*$. Henceforth the unmodified expression "analytic" will mean rigid analytic. We also let Ω_W denote the $A(W)$ module of analytic differentials on W.

Let V be an open annulus over $\mathbb{C}_p$. By a uniformizing parameter z on V we mean an analytic isomorphism of V onto a subspace of $\mathbb{C}_p^*$ of the form $A(r, s)$ where $0 < r < s$ are in $\mathscr{R}^*$. We first recall, [C-1, Lemma 2.2], that if $g \in A(V)^*$ then $g(z) = cz^n f(z)$ where $c \in \mathbb{C}_p^*$, $n \in \mathbb{Z}$ and $f \in A(V)^*$ such that $|f - 1|_V < 1$. Moreover n is unique. We have immediately;

LEMMA 2.1. *Given a uniformizing parameter z on an open annulus V there is a unique $\mathbb{C}_p$-linear map, $R: \Omega_V \to \mathbb{C}_p$, characterized by*

(i) $R(\mathrm{d}g) = 0$ (ii) $R(\mathrm{d}z/z) = 1$.

Moreover if z' is any other uniformizing parameter then $R(\mathrm{d}z'/z') = \pm 1$.

Proof. Let $\omega \in \Omega_V$. We may expand it in z as follows:

$$\omega = \sum_{n=-\infty}^{n=\infty} a_n z^n \, \mathrm{d}z$$

with $a_n \in \mathbb{C}_p$ such that the Laurent series converges on V. Then, we set

$$R(\omega) = a_{-1}.$$

This clearly satisfies (i) and (ii) and is characterized by them.

Now as above any unit g in $A(V)$ may be written in the form $cz^n f(z)$ where $c \in \mathbb{C}_p^*, n \in \mathbb{Z}$ and $|f - 1|_V < 1$. Then,

$$\begin{aligned} \mathrm{d}g/g &= n\,\mathrm{d}z/z + \mathrm{d}f(z)/f(z) \\ &= n\,\mathrm{d}z/z + \mathrm{d}\,\mathrm{Log}(f(z)) \end{aligned}$$

where here $\mathrm{Log}(T)$ is the p-adic logarithm which is analytic on the open unit disk of radius one about 1. It follows that

$$R(\mathrm{d}g/g) = n. \tag{2.1}$$

In particular $z' = cz^n f(z)$ and $z = c'z^{n'} f'(z')$ where c, c', n, n' and f, f' satisfy the appropriate conditions. It follows in particular that $nn' = 1$ so that $n = \pm 1$. Hence the final conclusion of the lemma follows from (2.1). □

We see that there are two natural homomorphisms $R\colon \Omega_V \to \mathbb{C}_p$. We call a choice of one an orientation on V. Moreover if V is an oriented annulus, Res_V will denote the corresponding homomorphism. For a unit f in $A(V)$ we set

$$\mathrm{ord}_V f = \mathrm{Res}_V \, \mathrm{d}f/f$$

From the proof of the previous lemma we see that $\mathrm{ord}_V f = \mathrm{ord}_z f$ for any parameter z on V such that $\mathrm{Res}_V \, \mathrm{d}z/z = 1$. It follows easily from the above discussion that:

LEMMA 2.2. *Suppose V is an oriented annulus and $g \in A(V)^*$, then*

(i) $|g|_{z^{-1}A[t]}$ *is strictly increasing as a function of $t \in (r, s)$ if* $\mathrm{ord}_V g > 0$
(ii) $|g|_{z^{-1}A[t]}$ *is strictly decreasing if* $\mathrm{ord}_V g < 0$ *and*
(iii) *there exists a $c \in \mathbb{C}_p$ such that $|cg - 1| < 1$ if* $\mathrm{ord}_V g = 0$.

By a uniformizing parameter z on an open disk, U, we mean an isomorphism onto $B(r)$, for some $r \in \mathscr{R}^*$. If D is a closed disk in U, then $V = U - D$ is an open annulus and it has a natural orientation Res_V so that

$$\mathrm{Res}_V \, \mathrm{d}z/z = 1$$

for any uniformizing parameter z on U such that $0 \in z(D)$.

Suppose W is a one-dimensional smooth rigid space and S is a discrete subset of W. Then $W - S$ is a rigid space. We say an analytic differential on $W - S$ is analytic on W except for isolated singularities. Suppose ω is a differential on W analytic except for isolated singularities. If $P \in W$ and z is a local uniformizing parameter at P then at P we may write

$$\omega = \sum_{n=-\infty}^{\infty} a_n z^n \, \mathrm{d}z$$

and we set

$$\mathrm{Res}_P \, \omega = a_{-1}.$$

It is well known that this is a well defined $\mathbb{C}_p$ linear map. We have the following well known version of Cauchy's theorem:

PROPOSITION 2.3 (Cauchy's Theorem on a Disk). *With notation as above let ω be a differential on U analytic except for finitely many isolated singularities in D. Then*

$$\mathrm{Res}_V(\omega) = \sum \mathrm{Res}_P\, \omega.$$

Proof. Without loss of generality, we may suppose $U = B(1)$. First, using the partial fraction expansion of ω, it is easy to see that

$$\omega = \sum \mathrm{Res}_P\, \omega\, \mathrm{d}z/(z - P) + \mathrm{d}g$$

where g is analytic on U except for finitely many isolated singularities on D. Hence

$$\mathrm{Res}_V\, \omega = \sum \mathrm{Res}_P\, \omega \cdot \mathrm{Res}_V(\mathrm{d}(z - P)/(z - P)).$$

But $z - P$ is a parameter on U such that $0 \in (z - P)(D)$. Hence $\mathrm{Res}_V(\mathrm{d}(z - P)/(z - P)) = 1$ from which the proposition immediately follows. □

III. Wide open spaces

By a model of a scheme X over $\mathbb{C}_p$, we mean a scheme $\mathscr{X}$ over the ring of integers in $\mathbb{C}_p$ whose extension of scalars to $\mathbb{C}_p$ is X. We will use the notation and conventions of [CM, §1]. In the following, the letter C will always denote a curve over $\mathbb{C}_p$ unless otherwise indicated.

We call a subdomain of a curve over $\mathbb{C}_p$ *diskoid* if it is non-empty and consists of the union of a finite collection of disjoint closed disks. By a *wide open space* W over $\mathbb{C}_p$ we mean a rigid analytic space isomorphic to the complement in a smooth curve over $\mathbb{C}_p$ of a diskoid subdomain. There are more intrinsic definitions of such spaces but this one suits our purposes. Moreover, there are theorems to be proved before one knows what the right notion of a wide open space should be in higher dimensions. As affinoids may be thought of as compact spaces, wide opens may be thought of as relatively compact spaces. (An alternative definition is that a wide open is a residue class in an affinoid.) Examples of wide open spaces include open disks and open annuli.

To study wide opens, we first need the following general facts about affinoid subdomains of curves.

LEMMA 3.0. *A subset of C is an affinoid subdomain iff there exists a finite set*

M of rational functions on C at least one element of which is non-constant, such that

$$X = \{x \in C : |f(x)| \leqslant 1, \quad f \in M\}.$$

Proof. We will use the following well known fact several times:

Suppose f is a non-constant rational function on C. Then the set

$$V = \{x \in C : |f(x)| \leqslant 1\} \tag{3.0}$$

is an affinoid subdomain of C; moreover, $\mathcal{O}(C\text{-Poles}(f))$ is dense in $A(V)$.

(In fact, V may be seen to be a finite branched cover of $B[1]$ via f.) It follows immediately from this and Proposition 7.2.3.4 of [BGR] that an X as in the lemma is an affinoid subdomain of C.

Now suppose that X is an affinoid subdomain of C. Then since X is not proper, $X \neq C$. Let $P \in C - X$. Let f be a rational function on C with a pole at P and no other poles. Then the restriction of f to X is bounded. Moreover, after adjusting f by a constant we may suppose $|f|_X \leqslant 1$ and the set $U = \{x \in C : |f(x)| > 1\}$ is an open disk in C. Now by (3.0), the set $Y = \{x \in C : |f(x)| \leqslant 1\}$ is an affinoid subdomain of C. Hence by Satz 3 Section 2 of [G] (see also Corollary 7.3.5.3 of [BGR]), there exists a finite subset S of $Y - X$ and a finite set N of analytic functions on $Y - S$ meromorphic on Y, such that

$$X = \{y \in Y : |g(y)| \leqslant 1, g \in N\}. \tag{3.1}$$

Now let h be a function on C with poles at all the elements of S and no other poles. Then, as before, h is bounded on X. Also, since U is a disk and h has finitely many zeros on U and no poles, h is bounded on U. We may suppose $|h|_{X \cup U} \leqslant 1$. Let $k = f + h$. It follows from the non-archimedean triangle inequality that

$$Z = \{x \in C : |k(x)| \leqslant 1\}$$

is contained in Y. Moreover since $S \cap Z = \emptyset$, the elements of N are analytic on Z. Now, it follows from (3.0) that Z is an affinoid subdomain of C and for each element $g \in N$, there exists an element $g' \in \mathcal{O}(C\text{-poles}(f))$ such that $|g - g'|_Z \leqslant 1$. Let $M = \{k\} \cup \{g' : g \in N\}$. It follows from (3.1) and Proposition 7.2.3.3 of [BGR] that

$$X = \{x \in C : |f(x)| \leqslant 1, f \in M\}$$

as required. $\square$

LEMMA 3.1. *Suppose X and Y are affinoid subdomains of a curve C. Then,*

(i) *$X \cap Y$ is an affinoid subdomain of C*
(ii) *If $X \cap Y = \emptyset$ then X and Y are disconnected from each other.*

Proof. Part (i) follows immediately from Lemma 3.0.
For part (ii) we may suppose $X = \{x: |f(x)| \leqslant 1, f \in S\}$ as in Lemma 3.0. For $f \in S$, let

$$Y_f = \{y \in Y: |g(y)| \leqslant 1, g \in S, g \neq f\}$$

and

$$Y' = \{y \in Y: |g(y)| \geqslant 1, g \in S\}.$$

Then Y_f, $f \in S$, and Y' are affinoid subdomains of Y, $Y = Y' \cup \bigcup Y_f$ and since $X \cap Y = \emptyset$ and meromorphic functions on affinoids achieve their minima,

$$\min\{|f(y)|: y \in Y_f\} > 1$$

and

$$\min\left\{\left|\prod f(y)\right|: y \in Y'\right\} > 1.$$

It follows from this that X disconnected from Y_f, $f \in S$, and Y'. Since $\{X, Y', Y_f: f \in S\}$ is an admissible cover of $X \cup Y$, this implies our lemma. $\square$

We will use the following result of [BL] (Lemmas 2.2, 2.3 and Proposition 6.2) repeatedly in this paper.

LEMMA 3.2. *Suppose Z is a semi-stable model for C. Then*

(i) *if $P \in \tilde{Z}$, $\bar{P}$ is an open disk if P is a smooth point and an open annulus otherwise,*
(ii) *if every irreducible component of $\tilde{Z}$ of genus zero intersects the other components in at least two points, then any open disk in C is contained in a residue class,*
(iii) *moreover if every component as in* (ii) *intersects the other components in at least three points every open annulus is contained in a residue class.*

PROPOSITION 3.3. *Suppose Z is a proper model for C with reduced reduction. Then*

(i) *if Y is a connected open subscheme of $\tilde{Z}$ not equal to $\tilde{Z}$, then $\bar{Y}$ is a connected affinoid subdomain of C,*

(ii) *if Y is a connected closed subscheme of $\tilde{Z}$ not equal to $\tilde{Z}$, then $\bar{Y}$ is a connected wide open subspace of C,*

(iii) *if X and Y are disjoint closed subschemes of $\tilde{Z}$ then $\bar{X}$ and $\bar{Y}$ are disconnected from each other.*

Proof. (i) We may replace Z with the model whose reduction consists of the scheme obtained from $\tilde{Z}$ by blowing down each irreducible component which does not contain a non-empty Zariski open of Y to a point. It now follows that Y is a connected affine open subscheme of $\tilde{Z}$ and as $\bar{Y}$ is the formal completion of Z along Y, $\bar{Y}$ is an affinoid subdomain of C.

(ii) First we replace Z by the model whose reduction is obtained from $\tilde{Z}$ by blowing down Y to a point. It now follows that $\bar{Y}$ is a residue class and so is connected by Satz 6.1 of [B]. It also follows from the discussion before Proposition 4.5 in the proof of the stable reduction theorem in [BL] that $\bar{Y}$ is a wide open; however we would like to sketch how that proof can be made simpler using the stable reduction theorem. There exists a semi-stable model for C which dominates Z and such that the inverse image of Y in $\tilde{Z}$ is one-dimensional. Hence we may assume Z is semi-stable and Y is one-dimensional. Let S denote the finite set of points on Y which also lie on components not on Y. Then for each $P \in S$, $\bar{P}$ is an open annulus by Lemma 3.2(i). As in the proof of Lemma 2.3 of [BL] there exists a function z_P on C which is regular on an affine open of Z containing P, whose reduction is non-zero on an affine open of the irreducible component in Y containing P and is such that $z_P: \bar{P} \to A(r_P, 1)$ is a uniformizing parameter. Now for each P we attach a copy of $B(1)$ using z_P. Then one can show that the resulting space is a smooth proper one-dimensional rigid space and hence is an algebraic curve using the direct image theorem [K-1]. Part (ii) follows immediately.

(iii) By blowing down as above we can assume X and Y are points on an affine open in $\tilde{Z}$. But then there exists a function f on C such that $|f(x)| < 1$ for $x \in \bar{X}$ and $|f(x) - 1| < 1$ for $x \in \bar{Y}$. From this it follows that $\bar{X}$ and $\bar{Y}$ are disconnected from each other. □

REMARK. The question was raised in [BL], whether the algebraic curve constructed in the proof of (ii) depended on the choice of z_P. One can show that it does. Infact, if Y is smooth and one is permitted to shrink the annuli used in the proof of (ii), then one can show (using the lifting theorem of [C-2], for instance) that every curve over $\mathbb{C}_p$ whose reduction is isomorphic to Y may be obtained in this way.

Let W be a wide open space. Let

$$\mathscr{E}(W) = \varprojlim CC(W - X)$$

where X runs over all affinoid subdomains of W. By analogy with non-compact topological spaces we call the elements of $\mathscr{E}(W)$ the *ends* of W. As we will see, the set of ends of W is finite. If $e \in \mathscr{E}(W)$, and X is an affinoid subdomain of W we call the image of e in $CC(W - X)$ an open neighborhood of e.

PROPOSITION 3.4. *Let C be a smooth curve over $\mathbb{C}_p$. Let X be a diskoid subdomain of C. There exists a semi-stable model Z for C such that the inclusion of X into C extends to a morphism of the canonical model of X into Z and the morphism from $\tilde{X}$ to $\tilde{Z}$ is a closed immersion.*

Proof. This is an immediate consequence of the proof of Lemma 1.5 of [CM]. □

COROLLARY 3.4a. *If C is connected then so is $C - X$.*

Proof. This follows from the proposition together with Proposition 3.3(ii). □

Now let W be the wide open $C - X$ in the above proposition. Let Y denote the closed subscheme of $\tilde{Z}$ equal to the complement of the image of $\tilde{X}$. Then $W = \bar{Y}$. Let Y^0 denote the interior of Y. Then $W^0 = \bar{Y}^0$ is an affinoid subdomain of W such that $W - W^0$ is a finite disjoint union of annuli. We will show that the natural map from $\mathscr{E}(W)$ onto $CC(W - W^0)$ is one to one.

Let D be any connected component of X. Then the image of $\tilde{D}$ in $\tilde{Z}$ is a copy of $\mathbb{A}^1$ and its closure is a copy of $\mathbb{P}^1$ which intersects exactly one other component normally at one point. Let Z' denote the model obtained from Z' by blowing down the copies of $\mathbb{P}^1$ corresponding to elements of $CC(X)$ to points. Let T_D denote the component of Z' which contains the image of $\tilde{D}$ which is now a smooth point P_D. Choose an affine open subscheme S of Z' whose reduction contains an affine open of T_D which in turn contains P_D. Now let z_D be a function on S which vanishes at a point in D and whose reduction to $\tilde{S}$ is a uniformizing parameter at P_D. Let $U_D = \bar{P}_D$ and $V_D = U_D - D$. Then V_D is an open annulus in $CC(W - W^0)$. It follows that $z_D(U_D) \cong B(1)$ and so $z_D(V_D) = A(r_D, 1)$ for some $0 < r_D < 1$ in $\mathscr{R}$ since z_D has a zero in D. Now suppose $s\colon CC(X) \to \mathscr{R}$ such that $r_D < s(D) \leqslant 1$ for each D in $CC(X)$. Let $W_s = W - z_D^{-1}(A(r_D, s(D))$. Then by a suitable blowing up argument one can show that W_s is an affinoid subdomain of W. Note that it follows from Lemma 3.1 that if $r_D < s'(D) < s(D) < 1$, then $z_D^{-1}(A(r_D, s'(D))$ is disconnected from W_s since $z_D^{-1}(B[s'(D)])$ and W_s are disjoint affinoids in C.

LEMMA 3.5. *Let R be any affinoid subdomain of W. Then there exists an s as above such that $R \subseteq W_s$.*

Proof. Let $D \in CC(X)$. Let T' denote a Zariski open subset of $\tilde{Z}$ contained in T_D including P_D and on which $\tilde{z}_D$ vanishes only at P_D. Then $\bar{T}'$ is an affinoid subdomain of C and $R' = \bar{T}' \cap R$ is an affinoid subdomain of $\bar{T}'$. Now since

$R \subseteq W$, and $|z_D(x)| = 1$ for all x in $\bar{T}' - U_D$ we see that $z_D(R') \subseteq \mathbb{C}_p - B[r_D]$. Since R' is an affinoid, it follows that z_D achieves its minimum on R' and so $z_D(R') \subseteq \mathbb{C}_p - B(s(D))$ for some $r_D < s(D) \leqslant 1$. This proves the lemma. □

If Y is an affinoid subdomain of W such that the elements of $CC(W - Y)$ are open annuli and the natural map $\mathscr{E}(W) \to CC(W - Y)$ is bijective, then we call Y an *underlying affinoid* of W. (Note that as long as W is not an open disk or annulus one can show that there exists a unique minimal underlying affinoid. Moreover this can be used to give another proof of the existence of a semi-stable model.)

COROLLARY 3.5a. *With notation as above, W_s is an underlying affinoid of W. Moreover, for each $D \in CC(X)$ there exists an open disk U_D in C containing D and no other elements of $CC(X)$ such that*

$$CC(W - W^0) = \{U_D - D : D \in CC(X)\}.$$

Proof. The previous lemma shows that the affinoids W_s are cofinal in the collection of all affinoids contained in W. This implies the map $\mathscr{E}(W) \to CC(W - W_s)$ is bijective. The remainder of the corollary follows if we take $U_D = \bar{P}_D$. □

COROLLARY 3.5b. *Suppose W is a wide space, then W is covered by a nested family of underlying affinoids, $\{W_n\}$, such that the elements of $CC(W - W_n)$ are proper subsets of the corresponding elements of $CC(W - W_m)$ for $m < n$.*

LEMMA 3.6. *Suppose U is a non-empty admissible open rigid subspace of C isomorphic to a finite disjoint union of open disks. Then there exists a model Z of C such that the connected components of U are residue classes of Z.*

Proof. Claim, there exists a semi-stable model Z' of C such that each open disk in U is contained in a residue class of Z'. If the genus of C is positive, any semi-stable model of C such that each irreducible component of $\tilde{Z}'$ isomorphic to $\mathbb{P}^1$ intersects the other components in at least two points, will do. If C is $\mathbb{P}^1$ we can assume one of the connected components of U is $B(1)$ and then the standard model will do. Now each residue class of Z' is either an open disk or an open annulus. It follows that each connected component of U is either a residue class of Z' or a residue class of a unique closed disk contained in a residue class of Z'. Moreover the union of these closed disks is a diskoid subdomain. As in the proof of Proposition 3.4 we can replace Z' with a semi-stable model which dominates it, so that each of these closed disks is of the form $\bar{Y}^0$ where Y is an irreducible component of $\tilde{Z}$. The lemma follows. □

COROLLARY 3.6a.

(i) *$C - U$ is a connected affinoid.*
(ii) *If U is just one open disk and V is an open annulus in C and $U \cap V \neq \emptyset$, $U \nsubseteq V$ then, $U \cup V$ is an open disk in C or $U \cup V = C$ and C has genus zero.*

Proof. (i) follows immediately from the lemma and Lemma 3.3 (ii).

Now (ii) is obviously true for $\mathbb{P}^1$; it is also true for Tate elliptic curves using Tate's parametrization to reduce to the case of $\mathbb{P}^1$. Therefore suppose C has a stable model Z. Then by Lemma 3.2(ii), U and V are each contained in residue classes of Z. Since they intersect they must be contained in the same residue class which is either a disk or an annulus. This allows us to again reduce to the case of $\mathbb{P}^1$. □

Suppose W is a wide open space which contains an underlying affinoid with good reduction. Then we call W a *basic wide open*.

LEMMA 3.7. *Suppose W is a wide open space and $W_1 \subseteq W_2$ are underlying affinoids. Let $V_1 \in CC(W - W_1)$, $V_2 \in CC(W - W_2)$ correspond to the same end and suppose $V_1 \neq V_2$. Then*

(i) *$V_2 \subseteq V_1$ and $V_1 - V_2$ is a half open annulus.*
(ii) *V_2 is disconnected from W_1 and $V_1 - V_2$ is connected to W_1.*

Proof. (i) Clearly $V_2 \subseteq V_1$. If $V_1 - V_2$ is not a half open annulus, then V_2 must be contained in an affinoid in V_1 and so cannot be the image of an end, but this contradicts the hypothesis that W_2 is an underlying affinoid.

(ii) We may suppose $W = C - X$ for some diskoid X in a smooth curve C. Then as we have seen above there exists an underlying affinoid W_3 containing W_2 such that if $V_3 \in CC(W - W_3)$ contains V_3 then there exists a $D \in CC(X)$ and an open disk U contained in C such that $V_3 = U - D$. Since $V_3 \subseteq V_2$ and $V_1 - V_2$ and $V_1 - V_3$ are both half open annuli in V_1, it follows that $U \cup V_1$ is an open disk by Corollary 3.7a(ii), and so there exists a closed disk D' containing $D \cup V_2$ in C. Since D' and W_1 are disjoint, it follows from Lemma 3.1 that D' and W_1 are disconnected from each other. Hence W_1 and V_2 are disconnected from each other, as required. Finally, by arguments similar to the above, we can show that $W_1 \cup (V_1 - V_2)$ is the complement of finitely many disjoint open disks in C. It follows from Corollary 3.6a(i) that $W_1 \cup (V_1 - V_2)$ is a connected affinoid, which yields (ii). □

COROLLARY 3.7a. *Suppose W is a wide open space, W^0 is an underlying affinoid and $V \in CC(W - W^0)$. Then there exists a unique orientation on V, such that if W' is an underlying affinoid containing W^0, $V' \in CC(W - W')$ such that $V' \subseteq V$*

and z is a uniformizing parameter on V such that $\mathrm{ord}_V z = 1$ *we have*

$$|z|_V \geqslant |z|_{V'}.$$

Moreover, if $z(V) = A(r,s)$, then $z^{-1}(A(r,t))$ is disconnected from W^0 and $z^{-1}(A(t,s))$ is connected to W^0.

Proof. With notation as above, suppose $V' \neq V$. Then part (i) of the lemma implies that there exists a parameter z on V such that $z(V) = A(r,s)$ and $z(V') = A(r,t)$ for some $r < t < s$. Now if $W'' \supseteq W'$ is another underlying affinoid, and V'' is the element of $CC(W - W'')$ contained in V', then $V - V''$ and $V' - V''$ are both half open annuli. It follows that $z(V'') = A(r,u)$ for some $r < u < t$. From this it follows that the orientation such that $\mathrm{ord}_V z = 1$ satisfies the conditions of the corollary. The uniqueness follows from the fact that there exists a W' such that $V' \neq V$. This in turn follows from Corollary 3.5b. The final conclusion of the corollary follows from part (ii) of the lemma. □

Now let Ω_e denote set space of differentials analytic on some open neighborhood of e. Let $\omega \in \Omega_e$. Then we set

$$\mathrm{Res}_e\, \omega = \mathrm{Res}_V\, \omega$$

where V is the element of $CC(W - Y)$ corresponding to e for a sufficiently large underlying affinoid Y and the orientation on V is that described in the lemma.

COROLLARY 3.7b. *Suppose W is a wide open and W^0 is an underlying affinoid with irreducible reduction. Let $f \in A(W)^*$ and $e \in \mathscr{E}(W)$ and V is the corresponding element of $CC(W - W^0)$. Suppose z is a uniformizing parameter on V such that $z(V) = A(r,s)$ and* $\mathrm{ord}_V z = 1$, *then*

$$\lim_{t \to s} |f|_{z^{-1}(A[t])} = |f|_{W^0}.$$

Proof. We may suppose $|f|_{W^0} = 1$. By the previous corollary, $V_t = z^{-1}(A[t,s))$ is connected to W^0 for all $r < t < s$. Hence, $f(V_t)$ is connected to $f(W^0)$. By Lemma 2.2, the above limit does indeed converge and in order for $f(V_t)$ to be connected to $f(W^0)$ it had better converge to a number less than or equal to one. Now, since W^0 and $A[t]$ have irreducible reduction $|f^{-1}|_{W^0} = 1$ and $|f^{-1}|_{z^{-1}(A[t])} = (|f|_{z^{-1}(A[t])})^{-1}$. Hence, applying the above argument to f^{-1} in place of f we see that the above limit must converge to a number greater than or equal to one. This concludes the proof. □

LEMMA 3.8. *Suppose Z is a semi-stable model for C and Y is closed subscheme of*

$\tilde{Z}$ not equal to $\tilde{Z}$. Then $\bar{Y}^0$ is an underlying affinoid of $\bar{Y}$ and

$$CC(\bar{Y} - \bar{Y}^0) = \{\bar{P}: P \in Y - Y^0\}.$$

Proof. This follows from Proposition 3.3(ii) and (iii) and its proof, Corollary 3.5a and Lemma 3.2(i). □

Suppose Z is a semi-stable model for C and X and Y are two distint irreducible components of $\tilde{Z}$. Then, by Lemma 3.6,

$$CC(\bar{X} \cap \bar{Y}) = \{\bar{P}: P \in X \cap Y\} = CC(\bar{X} - \bar{X}^0) \cap CC(\bar{Y} - \bar{Y}^0).$$

Hence an annulus $V \in CC(\bar{X} \cap \bar{Y})$ acquires two orientations, one from X, which we call, $\mathrm{Res}_{X,V}$ and one from $\bar{Y}$, which we call, $\mathrm{Res}_{\bar{Y},V}$.

LEMMA 3.9. *With notation as above,*

$$\mathrm{Res}_{\bar{X},V} = -\mathrm{Res}_{\bar{Y},V}.$$

Proof. We may suppose that $V = \bar{P}$ for some $P \in X \cap Y$. Let Z' denote any semi-stable model of C which dominates Z and is such that $\tilde{Z}'$ has exactly one more irreducible component, T, than $\tilde{Z}$ which is a copy of $\mathbb{P}^1$ and maps onto P. Let X' and Y' denote the irreducible components of $\tilde{Z}'$ which contain the inverse images of X^0 and Y^0. Then, $\bar{X} = \bar{X}' \cup \bar{T}$, $\bar{Y} = \bar{Y}' \cup \bar{T}$ and $V = \bar{T}$. Let $Q = T \cap X'$ and $R = T \cap Y'$. It follows from Lemma 3.2 and Proposition 3.3(ii), that $\bar{Q}$ is an open annulus in V, disconnected from $\bar{Y}'$ whose complement is a half open annulus and similarly $\bar{R}$ is an open annulus in V, disconnected from $\bar{X}'$, whose complement is a half open annulus. The lemma now follows from Corollary 3.7a. □

The following result will not be used in this paper but is necessary for [CdS].

LEMMA 3.10. *Suppose W is a basic wide open in C. Then*

(i) *Suppose $C \cong \mathbb{P}^1$. By applying a linear fractional transformation W may be put in the form*

$$B(0, r) - \cup B[\varepsilon, s_\varepsilon]$$

where the union runs over ε in a finite subset S of $B[0, 1] = X$ such that the natural map $S \to \tilde{X}$ is an injection, $r, s_\varepsilon \in \mathscr{R}^*, r > 1$ and $s_\varepsilon < 1$.

(ii) *Suppose $f: \mathbb{C}_p^*/\langle q \rangle \to C$ is an isomorphism where $q \in \mathbb{C}_p^*$, $|q| < 1$. Then W is the image of a basic wide open contained in $A(a, qa)$ for some $a \in \mathbb{C}_p^*$.*

(iii) *Suppose C_0 is a stable model for C. Let $\mathscr{C} = \{\mathrm{red}^{-1}(X)$: X is an irreducible component of $\tilde{C}_0\}$. Then W is contained in X for some X in $\mathscr{C}$ and either W is contained in a residue class of X or $W = X - Y$ where Y is a rigid subspace of X with finitely many connected components each contained in a residue class of X. Moreover if V is a residue class of X and $Z = V \cap U$ then*

(a) *if V is an open disk in X^0, Z is a closed disk, or empty.*
(b) *if V is an open annulus in X^0 and z: $V \to A(1, r)$ is a uniformizing parameter $r > 1, r \in |\mathbb{C}_p|$ then $Z = z^{-1}(A[s, t])$ for some $s, t \in |\mathbb{C}_p|, 1 < s \leqslant t < r$.*
(c) *if $V \in CC(X - X^0)$ and z: $V \to A(1, r)$ is a uniformizing parameter such that $\mathrm{ord}_V z = 1$ for some $r \in |\mathbb{C}_p|, 1 < r$ then $Z = z^{-1}(A[s, r))$ for some $s \in |\mathbb{C}_p|, 1 < s < r$ or Z is empty.*

Note that in case (ii)b Z is never empty.

Proof. Let W^0 be an underlying affinoid of W with good reduction.

Case (i). It follows from the classification theorem [BGR 9.7.2 Theorem 2] that by applying a linear fractional transformation we may asssume

$$W^0 = B[0, 1] - \cup B(\varepsilon, 1)$$

where the union runs over all ε in a finite set S no two elements of which are contained in the same residue class of $B[0, 1]$. Now since each element of $CC(W - W^0)$ is an open annulus it is the image of a set of the form $A(r, s)$ for some $r, s \in |\mathbb{C}_p| \cup \{\infty\}, r < s$. Using the fact that W is connected it follows easily that W is of the form claimed.

Case (ii). Let $X[r] = f(A[r])$ for $r \in \mathscr{R}^*$.

Claim. W^0 is contained in $X[r]$ for some r. Let $Y_1 = f(A[q, q^{1/2}]) \cap W^0$ and $Y_2 = f(A[q^{1/2}, 1]) \cap W^0$. By Lemma 3.1 $\{Y_1, Y_2\}$ is an admissible cover of W^0 by affinoid subdomains. It follows that either Y_1 or Y_2 contains a Zariski open U of W^0. Say Y_1 does. If we let $r = \max\{|a|, a \in A[q, q^{1/2}], f(a) \in Y_1\}$ it follows that $X[r]$ contains a Zariski open of W^0. Now, using Lemma 3.1 and (i) each open disk in C is contained in a residue disk of $X[s]$ for some $s \in \mathscr{R}^*$. Since W^0 is connected, it follows that W^0 is contained in $X[r]$ which establishes our claim. Since W^0 has good reduction it follows from (i) that either W^0 is a Zariski open of $X[r]$ or is contained in a residue class of $X[r]$.

Suppose W^0 is a Zariski open of $X[r]$. Then it follows that each element of $CC(W - W^0)$ is either contained in a residue class of $X[r]$ or in $C - X[r]$. Now $C - X[r]$ is an open annulus. As the open annuli in $C - X[r]$ connected to Zariski opens in $X[r]$ are easily described using (i) the result follows easily.

Suppose next that W is an open annulus. Let z: $W \to A(s, t)$ be a uniformizing parameter. Let $W[u, v] = z^{-1}A[u, v]$ for $s < u \leqslant v < t$. Then $W[u, v] \cap X[r]$ is a subaffinoid of $X[r]$ by Lemma 3.1. If it does not contain a Zariski open of $X[r]$ it is contained in a residue disk since $W[u, v]$ is connected. If $W[u, v]$ is contained in a residue disk of $X[r]$ for all u and v then W is contained in this residue disk. If $W[u, v]$ contains a Zariski open for some u, v then arguing as in the proof of the above claim $W[w, w]$ contains a Zariski open of $X[r]$ for some $s < w < t$. By the claim, $X[r] = W[w, w]$. It then follows from the previous paragraph that W is of the form $f(A(x, y))$ for some $x, y \in \mathscr{R}^*$ such that $|x/y| < |q|$.

Finally suppose W^0 is contained in a residue class of $X[r]$. Then by the previous paragraph, each element of $CC(W - W^0)$ is contained in a residue class of $X[s]$ for some s or in $C - X[r]$. Since W is connected the second possibility does not occur and all the elements of $CC(W - W^0)$ must be contained in the same residue class of $X[r]$ as W^0. This concludes the proof of (ii).

Case (iii). This follows easily from [CM Lemma 1.4] and Lemma 3.2. □

One may deduce easily from this that,

COROLLARY 3.10a. *The intersection of two basic wide opens in C has a finite admissible cover by basic wide opens.*

IV. De Rham cohomology on wide open spaces

THEOREM 4.1. *Suppose A is an Abelian variety over $\mathbb{C}_p$ and H is an open subgroup in the canonical topology of $A(\mathbb{C}_p)$. Then $A(\mathbb{C}_p)/H$ is a torsion group.*

Proof. By [R], there exists an exact sequence

$$0 \to T \to G \to B \to 0$$

where T is an algebraic torus G is an algebraic group and B is an Abelian variety with good reduction, together with a lattice $\Gamma \subseteq G(\mathbb{C}_p)$ and an exact sequence of rigid analytic groups

$$0 \to \Gamma \to G \to A \to 0.$$

Moreover G has a canonical model which is a group scheme and whose reduction is an extension of $\tilde{B}$ by a torus. Let $\hat{G}$ denote the formal completion of G along its special fiber ($\hat{G}$ may be regarded as a formal scheme or as a rigid space). Finally $\Gamma \cap \hat{G}(\mathbb{C}_p) = 0$ and we have an exact sequence

$$0 \to \hat{G}(\mathbb{C}_p) \to G(\mathbb{C}_p) \to \mathbb{Q}^r \to 0$$

where r is the dimension of T. Since Γ is a lattice, the image of Γ in $\mathbb{Q}^r$ is a lattice.

It follows from this, that $\hat{G}(\mathbb{C}_p)$ injects into $A(\mathbb{C}_p)$ and the quotient of this image is a torsion group isomorphic to $(\mathbb{Q}/\mathbb{Z})^r$. Also $\hat{G}(\mathbb{F})$ is a torsion group. Finally, the endomorphism p is a contraction on the kernel of reduction in $\hat{G}(\mathbb{C}_p)$. The theorem follows. □

COROLLARY 4.1a. *Suppose C is a curve over $\mathbb{C}_p$, and U is an open subset of $C(\mathbb{C}_p)$ in the canonical topology. Suppose that D is a divisor of degree zero on C. Then there exists a divisor, E, of degree zero supported on U and a positive integer n such that $nD - E$ is principal.*

Proof. Apply the previous theorem to the Jacobian of C and the subgroup of the Jacobian generated by the classes of degree zero divisors supported on U. □

REMARK. This will be the only use of Jacobians to be made in the remainder of this paper.

For a scheme S over $\mathbb{C}_p$, we let $H^1_{DR}(S)^{\text{alg}}$ denote the first de Rham cohomology group of S. If W is a wide open space, we set

$$H^1_{DR}(W) = \Omega_W/dA(W).$$

We have the following basic comparison theorem. Let C be a smooth curve over $\mathbb{C}_p$ and let Y be a diskoid subdomain of C. Suppose S is a finite subset of points in C, exactly one of which lies in each connected component of Y.

THEOREM 4.2. *The natural restriction map*

$$H^1_{DR}(C - S)^{\text{alg}} \to H^1_{DR}(C - Y)$$

is an isomorphism.

Proof. By Theorem 2.3 of [K-2], one knows that the natural map

$$H^1_{DR}(C - S)^{\text{alg}} \to H^1_{DR}(C - S)^{\text{an}}$$

is an isomorphism. To complete the proof we need to use excision in an appropriate local cohomology theory as in [Gr]. We will only sketch the arguments, for which we are indebted to *A*. Ogus, since everything in [Gr] Section 1 carries over once one knows what an open neighborhood of a closed subspace is.

First for a closed subspace (i.e. the complement of an admissible open) Z of a rigid space X we define an admissible neighborhood of Z to be any admissible open subset U of X containing Z such that $\{U, X - Z\}$ is an admissible open cover of X. For a sheaf of abelian groups F on X, we define $\Gamma_Z(X, F)$ to be the sections in $F(U)$ supported in Z for any admissible neighborhood U of Z in X. For a complex of sheaves $F^\bullet$ on X we define the hypercohomology of $F^\bullet$ with supports in Z, $\mathbb{H}^i_Z(X, F^\bullet)$ as follows (see [H] §1.3): Let $I^\bullet$ be an injective resolution of $F^\bullet$, then we set

$$\mathbb{H}^i_Z(X, F^\bullet) = h^i(\Gamma_Z(X, I^\bullet))$$

in the notation of [H] Section 1.3. We automatically have excision as in [Gr];

$$\mathbb{H}^i_Z(X, F^\bullet) \cong \mathbb{H}^i_Z(U, F^\bullet|_U)$$

for any admissible neighborhood U of Z in X. As in [Gr] any injective is flabby and so, as in Corollary 1.9 of [Gr], we have a long exact sequence:

$$0 \to \mathbb{H}^0_Z(X, F^\bullet) \to \mathbb{H}^0(X, F^\bullet) \to \mathbb{H}^0(X - Z, F^\bullet) \to \mathbb{H}^1_Z(X, F^\bullet) \to \mathbb{H}^1(X, F^\bullet) \to \cdots$$

Now we apply this to the special case where $X = C - S, Z = Y - S$ and $F^\bullet = \Omega^\bullet$. Then from the long exact sequence we have an exact sequence;

$$\mathbb{H}^1_Z(X, \Omega^\bullet) \to \mathbb{H}^1(X, \Omega^\bullet) \to \mathbb{H}^1(X - Z, \Omega^\bullet) \to \mathbb{H}^2_Z(X, \Omega^\bullet). \qquad (*)$$

Let V be a collection of disjoint open disks in C such that exactly one connected component of Y lies in each connected component of V. The existence of V follows from Corollary 3.5a. It is also easy to deduce from Proposition 3.4 that $U = V - S$ is an admissible neighborhood of Z in X. Hence, by excision we have;

$$\mathbb{H}^i_Z(X, \Omega^\bullet) \cong \mathbb{H}^i_Z(U, \Omega^\bullet|_U) \qquad (**)$$

Now we observe that if T is any of the rigid spaces $X, Z, X - Z$ or U, then

$$\mathbb{H}^i(T, \Omega^\bullet) = h^i(\Omega^\bullet(T))$$

by the degeneration of the E_1 spectral sequence for de Rham cohomology (see [H] §1.3) since T has an admissible cover by an increasing collection of affinoids by Corollary 3.5b and $\Omega^\bullet$ is a complex of coherent sheaves. This means, for one thing, that

$$\mathbb{H}^1(C - Y, \Omega^\bullet) = H^1_{DR}(C - Y)$$

as we defined it above. Also, it is easy to see that

$$h^i(\Omega^\cdot(U)) \cong h^i(\Omega^\cdot(U - Z))$$

under the restriction mapping since we know the de Rham cohomology of an annulus and of a punctured disk. From the long exact sequence above with X replaced by U together with (**) we deduce that $\mathbb{H}^i_Z(X, \Omega^\cdot) = 0$ which together with (*) implies the theorem. □

From this theorem we see that wide open spaces have finite dimensional de Rham cohomology. This is one of the key advantages wide opens have over affinoids.

PROPOSITION 4.3. *Let W be a wide open space. Suppose ω is an analytic differential on W. Then*

$$\sum \mathrm{Res}_e\, \omega = 0.$$

Proof. We may take $W = C - X$ as above. Then by Theorem 4.2 there exists an algebraic differential υ on C regular on W such that

$$\omega = \upsilon + \mathrm{d}g$$

where g is analytic on W. Since the residues of an exact differential are zero, it suffices to prove the proposition for υ.

Now as above for each $D \in CC(X)$ there exists an open disk U in C containing D such that υ is regular on $V = U - D$ and

$$\mathrm{Res}_e\, \upsilon = \mathrm{Res}_V\, \upsilon.$$

Now it follows from Cauchy's Theorem on a disk, Proposition 2.3 that

$$\mathrm{Res}_V\, \upsilon = \sum \mathrm{Res}_P\, \upsilon.$$

Hence the proposition follows from the corresponding result for algebraic curves. □

PROPOSITION 4.4. *Let $W = C - X$ for a diskoid subdomain X of C. Then the image of $H^1_{DR}(C)^{\mathrm{alg}}$ in $H^1_{DR}(W)$ is the subspace of classes represented by differentials ω such that* $\mathrm{Res}_e \omega = 0$ *for all ends e of W.*

Proof. Let S be a finite subset of C as in Theorem 4.2. Let ω be an algebraic differential on C regular outside S. Then, it follows from Cauchy's Theorem on a disk that ω has zero residues at the points of S iff $\mathrm{Res}_e\,\omega = 0$ for all ends e of W. The proposition follows immediately. □

COROLLARY 4.4a. *Let $\omega \in \Omega_W$ whose class in $H^1_{DR}(W)$ lies in the image of $H^1_{DR}(C)^{\mathrm{alg}}$. Let $e \in \mathscr{E}(W)$. Let D_e be the corresponding element of $CC(X)$. There exists an open disk U_e containing D_e and no other components of X and an analytic function λ_e on $V_e = U_e - D_e$ such that $\mathrm{d}\lambda_e = \omega$ on V_e.*

Now let

$$(,): H^1_{DR}(C)^{\mathrm{alg}} \times H^1_{DR}(C)^{\mathrm{alg}} \to \mathbb{C}_p$$

denote the cup product. Recall, this may be computed as follows: Suppose ω and υ are two differential of the second kind on C. Then

$$([\omega], [\upsilon]) = \sum \mathrm{Res}_P(\lambda_P \upsilon)$$

where the sum runs over all $P \in C(\mathbb{C}_p)$ and λ_P denotes any formal primitive of ω at P.

PROPOSITION 4.5. *Let $W = C - X$ be a wide open as above. Let c and d denote two elements of $H^1_{DR}(C)^{\mathrm{alg}}$. Let ω and υ be two analytic differentials on W whose classes in $H^1_{DR}(W)$ are the images of c and d. Let λ_e be as above, then*

$$(c, d) = \sum \mathrm{Res}_e\, \lambda_e\, \upsilon$$

where the sum runs over all ends e of W.

Proof. Let ω' and υ' be two differentials of the second kind on C regular on W such that $[\omega'] = c$ and $[\upsilon'] = d$. We have in particular that $\omega' = \mathrm{d}\lambda'_e$ on U_e for some meromorphic function λ'_e on U_e. Hence

$$(c, d) = \sum (\mathrm{Res}_P(\lambda'_e \upsilon') = \mathrm{Res}_e(\lambda'_e \upsilon').$$

Now $\omega = \omega' + \mathrm{d}f$ and $\upsilon = \upsilon' + \mathrm{d}g$ on W where f and g are two analytic functions on W. Hence $\lambda_e = \lambda'_e + f + c_e$ for some constants c_e. Hence

$$\begin{aligned}\mathrm{Res}_e(\lambda_e\, \upsilon) &= \mathrm{Res}_e(\lambda'_e \upsilon) + \mathrm{Res}_e(f\upsilon)\\ &= \mathrm{Res}_e(\lambda'_e \upsilon') + \mathrm{Res}_e(\lambda'_e\, \mathrm{d}g) + \mathrm{Res}_e(f\upsilon') + \mathrm{Res}_e(f\, \mathrm{d}g)\\ &= (c, d) - \mathrm{Res}_e(g\omega) + \mathrm{Res}_e(f\upsilon') + \mathrm{Res}_e(f\, \mathrm{d}g)\end{aligned}$$

since $d(g\lambda'_e) = \lambda'_e\, \mathrm{d}g + g\omega$. The proposition now follows from Proposition 4.3. □

For a wide open W, $e \in \mathscr{E}(W)$, and an analytic function f on some neighborhood of e where it has no poles or zeros, we set

$$\mathrm{ord}_e\, f = \mathrm{Res}_e\, \mathrm{d}f/f.$$

LEMMA 4.6. *If W is a basic wide open, then $W \cong C - X$ where C has a smooth model Z and X is a diskoid subdomain of C each component of which lies in a different residue class of Z.*

Proof. Let W^0 be an underlying affinoid of W with good reduction. By definition $W = C - X$ for some curve C and diskoid subdomain X. Moreover, we know, there exists a subspace U of C isomorphic to a finite disjoint union of open disks each of which contains one element of $CC(X)$ and such that $W^0 = C - X$ and $CC(W - W^0) = CC(U - X)$. We claim that C has a smooth model Z and the elements of $CC(U)$ are residue classes of Z. This follows from the classification theorem, [BGR], Theorem 9.7.2.2, when C has genus zero and from this theorem combined with the Tate parametrization when C is a Tate curve. Therefore, we may suppose C has a stable model, Z. It follows from Lemma 1.4 of [CM] that W^0 is either contained in a residue class of Z or is a formal fiber in $\bar{Y}^0$ for some irreducible component Y of $\tilde{Z}$. Moreover, it follows from Lemma 3.2(ii) that each element of $CC(U)$ is contained in a residue class of Z. Since $U \cup W^0 = C$, the only way this can be true is if $\tilde{Z}$ has one smooth component and U is a finite disjoint union of residue classes. □

LEMMA 4.7. *With notation as in the previous lemma suppose P is a closed point of $\tilde{Z}$ such that $\bar{P}$ contains an element of $CC(X)$. Let e be the corresponding end of W. Suppose $f \in \mathbb{C}_p(C) \cap A(W)^*$ regular on a Zariski open of Z with non-empty reduction. Then $\mathrm{ord}_e\, f = \mathrm{ord}_P\, \tilde{f}$.*

Proof. As is well known,

$$\mathrm{ord}_P\, \tilde{f} = \sum_{x \in \bar{P}} \mathrm{ord}_x f$$

The result now follows from Cauchy's Theorem. □

LEMMA 4.8. *Let W be a basic wide open, suppose $f \in A(W)^*$ and suppose*

$$\mathrm{ord}_e\, f = 0 \tag{4.1}$$

for all ends e of W. Then there exists a $c \in \mathbb{C}_p^*$ such that $|cf - 1|_W < 1$.

Proof. We may suppose that W and the notation are as in the previous lemma. Let $c' \in \mathbb{C}_p^*$ such that $|c'|^{-1} = |f|_{W^0}$. Let $g = c'f$ so that $|g|_{W^0} = 1$. It follows from (4.1) that the reduction of g to $\tilde{W}^0$ is constant. Hence there exists a constant $c'' \in \mathbb{C}_p^*, |c''| = 1$, such that $|c''g - 1|_{W^0} < 1$. Let U denote the largest subset of

W such that $|c''g-1|_U<1$. Now let $V\in CC(W-W^0)$. Since V is connected to W^0, it follows that $U\cap V\neq\emptyset$. It follows from (4.1) and Lemma 2.2(iii) that there exists a $c_V\in\mathbb{C}_p^*$ such that $|c_Vg-1|<1$ on V. Hence on $U\cap V, |g|=1$ and $|c''g-c_Vg|<1$. It follows that $|c''-c_V|<1$. Hence we may take $c=c'c''$ to complete the proof. □

PROPOSITION 4.9. *Let $W=C-X$ where X is a diskoid subdomain of C. Suppose f is an element of $\mathbb{C}_p(C)^*$ such that*

$$\sum_{P\in D}\operatorname{ord}_P f=0 \tag{4.2}$$

for all $D\in CC(X)$. Then there exists a $c\in\mathbb{C}_p^*$ such that $|cf-1|_W<1$.

Proof. Let Z be as in Proposition 3.4. After blowing up we may assume all the irreducible components of $\tilde{Z}$ are smooth and any two components intersect in at most one point. Let $\mathscr{C}$ denote the collection of $\bar{Y}$ where Y is an irreducible component of $\tilde{Z}$. Then $\mathscr{C}$ is a covering of C by basic wide open sets. Let $\mathscr{C}'$ denote the subset of $\mathscr{C}$ consisting of elements U disjoint from X. For $U=\bar{Y}$ in $\mathscr{C}$ let $U^0=\bar{Y}^0$. If U and V are in $\mathscr{C}$, $U\neq V$, then $U\cap V$ is either empty or an annulus in $CC(U-U^0)$. Let $\operatorname{ord}_{U,V}=\operatorname{ord}_{U,U\cap V}$, if $U\neq V$ and $U\cap V\neq\emptyset$ and zero otherwise. It follows that for each $U\in\mathscr{C}'$

$$\sum_{V\in\mathscr{C}}\operatorname{ord}_{U,V} f=0 \tag{4.3}$$

and if U, $V\in\mathscr{C}$, then by Lemma 3.9

$$\operatorname{ord}_{U,V} f=-\operatorname{ord}_{V,U} f$$

and if in addition $V\notin\mathscr{C}'$, then by (4.2) and Cauchy's Theorem, Proposition 2.3,

$$\operatorname{ord}_{U,V} f=0.$$

For $U\in\mathscr{C}'$, let $r_U=|f|_{U^0}$. Now it follows from Corollary 3.7b and Lemma 2.2 that if U and V are in $\mathscr{C}'$ and $\operatorname{ord}_{U,V}f>0$ then $r_V>r_U$. Now we claim that $\operatorname{ord}_{U,V}f\leqslant 0$ for all $U\in\mathscr{C}$. Suppose r_V is the maximum of all the r_U's. Hence $\operatorname{ord}_{V,U}f\leqslant 0$ for all $U\in\mathscr{C}$. But this together with (4.3) implies that $\operatorname{ord}_{V,U}f=0$ for all $U\in\mathscr{C}$. We can apply the same argument to the second largest r_V and so on to get our claim. We also see that all the r_U's are equal. Let r denote their common value. It follows from this and Lemma 2.2 that $|f|_{U\cap V}=r$ if $U\cap V\neq\emptyset$. By dividing f by a constant, we may suppose $r=1$.

It follows from Lemma 4.8 that for each $U\in\mathscr{C}'$ there exists a $C_U\in\mathscr{C}^*$ such that

$|C_U f - 1|_U < 1$. Now if $U \cap V \neq \emptyset$ then $|f|_{U \cap V} = 1$ and $|C_U f - C_V f| < 1$. Since $\tilde{Z}'$ is connected, it follows that we may take all the c_U's to be the same. This completes the proof. □

From the above we see that if X is a diskoid subdomain of a curve C and ω is an algebra8c differential on C, regular on $W = C - X$ such that

$$\sum_{P \in D} \mathrm{Res}_P\, \omega = 0 \tag{4.5}$$

for each $D \in CC(X)$. Then the image of ω in $H^1_{DR}(W)$, lies in the image of $H^1_{DR}(C)^{\mathrm{alg}}$. Since $H^1_{DR}(C)^{\mathrm{alg}}$ injects into $H^1_{\mathrm{DR}}(W)$ we can associate to ω a unique class $\psi_X(\omega)$ in $H^1_{DR}(C)^{\mathrm{alg}}$.

THEOREM 4.10. *There exists a unique $\mathbb{C}_p$-linear map ψ from the space of algebraic differentials on C into $H^1_{DR}(C)^{\mathrm{alg}}$ such that*

(i) $\psi(\omega) = \psi_X(\omega)$ *if X and ω are as in* (4.5).
(ii) $\psi(\mathrm{d}g) = 0$ *for all* $g \in \mathbb{C}_p(C)$.
(iii) $\psi(\mathrm{d}f/f) = 0$ *for all* $f \in \mathscr{C}_p(C)^*$.

Proof. It suffices to show that existence of a homomorphism ψ defined on the group of algebraic differentials with integral residues satisfying (i)–(iii) for then we may extend this homomorphism by $\mathbb{C}_p$-linearity. We also may assume C has positive genus for otherwise ψ will be trivial. Let D be any closed disk in C. Let ω be any algebraic differential on C with integral residues. Then by Corollary 4.1a and Riemann-Roch, there exists a positive integer n, a $g \in \mathbb{C}_p(C)$ and an $f \in \mathbb{C}_p(C)^*$ such that

$$\omega' = n\omega + \mathrm{d}g + \mathrm{d}f/f$$

is regular on $C - D$. Moreover,

$$\sum \mathrm{Res}_P \omega' = \sum \mathrm{Res}_P \omega' = 0.$$

Hence we must set $\psi(\omega) = 1/n\psi_D(\omega')$. It remains to check that this satisfies (i), (ii) and (iii) and is well defined, i.e. is independent of the choices of D, g and f. First observe that (ii) and (iii) will follow immediately from the well definedness of the map we just introduced. Second, well definedness and (i) will follow from the following claim: Suppose X and X' are diskoid subspaces of C, ω and ω' are differentials on C regular on C-X and C-X' respectively such that

$$\sum_{P \in D} \mathrm{Res}_P\, \omega = 0$$

for each $D \in X$ and

$$\sum_{P \in D} \mathrm{Res}_P\, \omega' = 0$$

for each $D \in X'$ and finally

$$\omega' = \omega + \mathrm{d}f/f + \mathrm{d}g.$$

Then $\psi_X(\omega) = \psi_{X'}(\omega')$. Indeed, first since C has positive genus it follows from Lemma 3.2(ii) together with the classification theorem that any two closed disks in C are either disjoint or one contains the other. It follows from this that $X \cup X'$ is diskoid and we may replace both X and X' with $X \cup X'$ so we may assume $X = X'$. Now let $W = C\text{-}X$. Then g is regular on W and

$$\sum \mathrm{ord}_P f = \sum \mathrm{res}_P(\omega - \omega') = 0$$

for $D \in CC(X)$. It follows from Proposition 4.9 that $\mathrm{d}f/f$ is exact on W. Thus $\omega - \omega'$ is exact on W and so $\psi_X(\omega) = \psi_{X'}(\omega')$ which proves the theorem. □

COROLLARY 4.10a. *Suppose σ is a continuous automorphism of $\mathbb{C}_p$ and ω is an algebraic differential on C. Then*

$$\psi(\omega^\sigma) = \psi(\omega)^\sigma$$

on $C\sigma$.

Proof. This follows immediately from the characterization of ψ given in the theorem. □

COROLLARY 4.10b. *Suppose $f\colon C' \to C$ is a morphism of curves. Then*

$$\psi_{C'} f^*(\omega) = f^* \psi_C(\omega).$$

Proof. This follows immediately from the characterization of ψ given in the theorem, the proof of the theorem and the fact that there exists a closed disk D on C such that $f^{-1}(D)$ is a closed disk on C'. □

From the previous two corollaries, we deduce:

COROLLARY 4.10c. *Suppose C is defined over a closed subfield K of $\mathbb{C}_p$ and ω is* *an algebraic differential on C defined over K, then*

$$\psi(\omega) \in H^1_{DR}(C/K)^{\mathrm{alg}},$$

regarding the latter as a subgroup of $H^1_{DR}(C/\mathbb{C}_p)^{\mathrm{alg}}$

V. A Reciprocity Law for Integrals of the Third Kind over $\mathbb{C}_p$

With notation as in Section 1 suppose now $K = \mathbb{C}_p$, $H = \mathbb{C}_p^+$, $l = \log$ is a branch of the p-adic logarithm (in the sense of [C – 1]) and that $\mathscr{F}$ satisfies, in addition to (i)–(iii), the following property:

(iv) Suppose U is a Zariski open in C. Then if $F \in \mathscr{F}(U)$,
  (a) F is analytic on each open disk in U.
  (b) $\mathrm{d}F$ is an algebraic differential on C regular on U.

LEMMA 5.1. *With F as above,* $\mathrm{d}F$ *is a differential of the third kind on C, such that,*

$$[F] = \mathrm{Res}(\mathrm{d}F) =: \sum (\mathrm{Res}_P\, \mathrm{d}F)P.$$

Proof. Let $P \in C(\mathbb{C}_p)$. Let f be an algebraic function of C such that

$$\mathrm{ord}_P f = \mathrm{ord}_P [F].$$

Then $F - \log f$ is defined in a Zariski open neighborhood of P, hence on a disk containing P and hence is analytic at P. If follows that

$$d(F - \log f) = \mathrm{d}F - \mathrm{d}f/f$$

is analytic at P. The lemma follows. □

THEOREM 5.2. *Suppose* $F, G \in \mathscr{F}_g$ *such that* $|[F]| \cap |[G]| = \emptyset$, *then*

$$G([F]) - F([G]) = (\psi(\mathrm{d}G), \psi(\mathrm{d}F)) \tag{5.1}$$

where the pairing on the right hand side is the cup product as in Section 4.

Proof. The theorem is true when either G or F is principal by Theorem 1.1 and the definition of ψ. In particular, it holds when $C = \mathbb{P}^1$ and $\mathscr{F} = \mathscr{S}$ since every generic section of $\mathscr{S}$ is principal.

Now since both sides of (5.1) are bilinear we may use Corollary 4.1a to adjust

F and G by appropriate principal elements of $\mathscr{F}_g$ so that

$$|[F]| \cup |[G]| \subseteq U$$

for some open disk U. Let D be a closed disk in U containing $|[F]| \cup |[G]|$. Let V denote the oriented annulus $U - D$. Claim, F and G are analytic on V. Indeed, after replacing F by a suitable integral multiple (which causes no harm), by Corollary 4.1a there exists an f in $\mathbb{C}_p(C)^*$ such that

$$(f)|_U = [F].$$

Then, by (ii) of Section 1 and (iv) above, $F - \log f$ is analytic on U. Now, by Proposition 4.3, $\mathrm{Res}_V\, \mathrm{d}F = 0$. It follows that $\mathrm{d}f/f$ is exact on V and so $\log f$ is analytic on V. The claim is now immediate.

Applying Proposition 4.5, we have

$$\mathrm{Res}_V\, G\, \mathrm{d}F = (\psi(\mathrm{d}G), \psi(\mathrm{d}F)).$$

We now must identify the left hand side of this formula with the left hand side of (5.1). We are in luck because this will be analysis purely on the disk U. Let z be a uniformizing parameter on U. Let

$$g = \prod (z - z(P))^{\mathrm{ord}_P[G]}, \quad f = \prod (z - z(P))^{\mathrm{ord}_P[F]}$$
$$G' = G - \log g,\ F' = F - \log f.$$

As in the proof of the above claim we see that G' and F' are analytic on U. Hence,

$$\mathrm{Res}_V\, G\, \mathrm{d}F = \mathrm{Res}_V\, G'\, \mathrm{d}F' + \mathrm{Res}_V\, G'\, \mathrm{d}f/f + \mathrm{Res}_V \log(g)\mathrm{d}F' + \mathrm{Res}_V \log(g)\mathrm{d}f/f.$$

Now since G' and F' are analytic on U, $\mathrm{Res}_V\, G'\, \mathrm{d}F' = 0$ by Cauchy's Theorem. Next,

$$\mathrm{Res}_V\, G'\, \mathrm{d}f/f = \sum \mathrm{Res}_P\, G'\, \mathrm{d}f/f = \sum \mathrm{ord}_P(f) G'(P) = G'([F])$$

again by Cauchy's Theorem, and

$$\mathrm{Res}_V \log(g)\, \mathrm{d}F' = -\mathrm{Res}_V(F'\, \mathrm{d}g/g) = -F'([G])$$

by Cauchy's Theorem and the fact that the residue of an exact differential is zero. Finally, we may identify U with a disk in the z-projective line, so that f and g are rational functions on $\mathbb{P}^1$. Then as above

$$\mathrm{Res}_V(\log(g)\,\mathrm{d}f/f) = (\psi_{\mathbb{P}^1}(\mathrm{d}g/g), \psi_{\mathbb{P}^1}(\mathrm{d}f/f))$$

which is of course zero, but more relevantly, it is equal to

$$\log(g((f))) - \log(f((g))) = \log(g([F]) - \log(f([G])$$

using the fact that we know that (5.1) holds for $\mathbb{P}^1$ (alternatively, we could use the Weil reciprocity law here). Adding up all the above formulas yields the theorem. □

Suppose B is a subspace of $H^1_{DR}(C)$ complimentary to $H^0(C, \Omega_C)$. Let $\mathscr{F}_B$ denote the subsheaf of $\mathscr{F}$ such that $\mathscr{F}_B(U) = \{F \in \mathscr{F}(U): \psi(\mathrm{d}F) \in U\}$. The following is the precise analogue of Theorem 4.3.

COROLLARY 5.2a. *Suppose $B = B^\perp$ with respect to the cup product. Then if $F, G \in \mathscr{F}_{B,g}$ such that $|[F]| \cap |[G]| = \emptyset$,*

$$G([F]) = F([G]).$$

REMARK. The hypotheses of this corollary are satisfied, for instance, if C has ordinary reduction and B is the unit root subspace of $H^1_{DR}(C)$.

Finally, we will give an example of a quadruple satisfying (i)–(iv).

Let C be a curve with arboreal reduction (see [CdS] §2). Let $\mathscr{F}$ be the subsheaf of the sheaf $\mathscr{A}_1$ constructed in [CdS] defined by setting $\mathscr{F}(U)$ equal the F in $\mathscr{A}_1(U)$ such that dF is a differential of the third kind with integral residues. If we set $[F] = \mathrm{Res}(\mathrm{d}F)$, then the quadruple $(\mathbb{C}_p^+, \mathscr{F}, \log, [\ ])$ satisfies conditions (ii) and (iv)b by definition, (i) by Corollary 2.5.2 and Lemma 2.6.8 of [CdS], (iii) by Corollary 2.6.6 of [CdS] and (iv)a using 2.5.4 and the fact that $A_1(U) = A(U)$ for an open disk U.

We remark that the construction of the sheaf $\mathscr{F}$ just completed only used the geometry of the curve. It is possible, however, to construct a sheaf $\mathscr{F}$ as above on an arbitrary curve if one uses the Lie algebra structure of its Jacobian.

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