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The Clifford dimension of a projective curve

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0. Introduction

Let C be a smooth connected projective curve over the complex numbers (that is, a compact Riemann surface) of genus g.

What is the "most unusual" line bundle on C? One good way to give this question a sense is to ask for the line bundle that has "the most sections for its degree". A numerical measure introduced by H. Martens, which has lately prominence in the work of Green and Lazarsfeld ([G–L 1, 2]) is the *Clifford index* of the bundle: it is defined for a line bundle L as $\operatorname{Cliff} L = \operatorname{degree} L - 2h^0 L + 2$. The smaller the number is, the more sections L has for its degree. However, this is only a measure of how unusual a line bundle is under certain conditions: for example, line bundles of degree 1 with 1 section are not unusual at all, although their Clifford index is so low. To avoid this and similar trivial cases it is necessary to restrict attention to line bundles with $h^0 L \geqslant 2$. But L and $K \otimes L^{-1}$, where K denotes the canonical bundle of C, have the same Clifford index; in fact, by Riemann-Roch,

$$\operatorname{Cliff} L = g + 1 - h^0 L - h^1 L = g + 1 - h^0 L - h^0(K \otimes L^{-1}).$$

Thus we must restrict attention to bundles with both $h^0 L \geqslant 2$ and $h^1 L \geqslant 2$, and we say that these bundles *contribute to the Clifford index of* C, which is defined as the minimum of their Clifford indices. We say that L *computes* the Clifford index of C if L contributes and the value $\operatorname{Cliff} L$ is this minimum. The notion of the Clifford index of C refines the notion of the *gonality* of C, which is, the smallest degree of a map from C to $\mathbb{P}^1$.

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How much do the notions of Clifford index and the gonality differ? Cliff $C \geqslant 0$ by Clifford's Theorem, and equality holds if and only if C is hyperelliptic, i.e. if C is 2-gonal. On the other hand it is known from Brill-Noether Theory that a general curve of genus g possesses a line bundle L with $h^0 L \geqslant h^0$ and $h^1 L \geqslant h^1$ if and only if $h^0 h^1 \leqslant g$ [ACGH]. It follows at once that a general curve has Clifford index $[(g-1)/2]$ and is $[(g+3)/2]$-gonal (this was previously proved by Meis [M]). Is a curve of Clifford index c always $(c+2)$-gonal?

We define the *Clifford dimension* of C as

$$r = \min\{h^0 L - 1 \mid L \text{ computes the Clifford index of } C\}.$$

A line bundle L, which achieves the minimum (and computes the Clifford index), is said to *compute* the Clifford dimension. A curve of Clifford index c is $(c+2)$-gonal if and only if it has Clifford dimension 1.

In this paper we study curves of Clifford dimension $r > 1$. First, it is easy to show that if $r > 1$ any line bundle computing the Clifford dimension is very ample (Lemma 1.1. below), so that we may think of a curve of Clifford dimension r as a curve embedded in $\mathbb{P}^r$ (conceivably in different ways, corresponding to the different bundles computing the Clifford dimension); we call such an embedding a Clifford embedding.

The case $r = 2$ is classical: The curves of Clifford dimension 2 are exactly the smooth plane curves of degree $\geqslant 5$. Two of the striking features of this situation are:

(a) The Clifford embedding is unique; and
(b) C is $(c+3)$-gonal, and there is a one-dimensional family of pencils computing the gonality, all obtained by projecting from points of the curve.

The case of Clifford dimension 3 was studied by G. Martens [Ma2]. The main result of that paper, and the one that motivates our work, is that curves of Clifford dimension 3 are extremely rare. In fact the curves of Clifford dimension 3 are precisely the complete intersections of pairs of cubics in $\mathbb{P}^3$. It is not too hard to show that such curves are "like" plane curves in that they satisfy the properties (a) and (b) above – in fact (a) can be sharpened to say that there is only one bundle on C computing the Clifford index. It is easy to compute the genus and degree of such a curve, which are uniquely determined: the genus is 10 and the Clifford index is 3. Further the square of the line bundle that embeds such a curve in $\mathbb{P}^3$ is the canonical bundle, and, since it is a complete intersection, it is of course arithmetically Cohen–Macaulay.

Generalizing this situation, Martens conjectured [GL, Problem 3.10] the existence of curves of Clifford dimension r having genus $4r-2$ and Clifford index $2r-3$ for every $r \geqslant 2$ (for $r = 2$, this corresponds to plane quintics). We will prove this conjecture below.

We conjecture that the curves of Clifford dimension $r \geqslant 3$ are all similar to the curves of Clifford dimension 3 and plane quintics:

Conjecture. If C has Clifford dimension $r \geqslant 3$ then:
(1) C has genus $g = 4r - 2$ and Clifford index $2r - 3$ (and thus degree $g - 1$).
(2) C has a unique line bundle L computing the Clifford index.
(3) L^2 is the canonical bundle of C, and L embeds C as an arithmetically Cohen-Macaulay curve in $\mathbb{P}^r$.
(4) C is $2r$-gonal, and there is a one-dimensional family of pencils of degree $2r$, all of the form $|L(-D)|$ where D is the divisor of $2r - 3$ points of C.

We prove this conjecture completely for $r \leqslant 9$, and we will prove a large part of it in general. Specifically, we prove parts 2–4 for all curves satisfying part (1). We prove that (1) is always satisfied for $r \leqslant 9$. For arbitrary r, we show on the one hand that the genus and the Clifford index of a curve of Clifford dimension r are bounded by (explicit) constants depending on r alone, and on the other hand that if the curve does not satisfy condition (1), then its degree is $\geqslant 6r - 6$ and its genus is $\geqslant 8r - 7$. We also show that the conjecture follows from a conjecture of Eisenbud and Harris [E-H] on bounds for the genus of curves of a given degree, which are not contained on a surface of small degree, together with a plausible conjecture about the behavior of Castelnuovo's formula for the (virtual) number $C(d, g, r)$ of $(2r - 2)$-secant $r - 2$ planes of a curve of degree d and genus g in $\mathbb{P}^r$.

One pleasant consequence of the conjecture would be to show that at most 3 different Clifford dimensions are possible for curves of a given genus, and that for many genera – those neither of the form $(d - 2)(d - 1)/2$ nor $4r - 2$ – the only possible Clifford dimension is 1, so that the notions of gonality and Clifford index are equivalent. Further, as G. Martens has noted [Ma4], it would "nearly" prove a conjecture of Harris and Mumford. The (unpublished) conjecture was that the gonality of all curves in a given linear series on a $K3$-surface is the same. Donagi pointed out that this conjecture fails on a $K3$ which is a double cover of the plane, branched over a smooth sextic for the linear series which contains the preimage of the smooth plane cubic transverse to the sextic: this curve has a g^1_4, while a general member has a g^2_6 but no g^1_4 – that is, it has the same Clifford index, but a different Clifford dimension. Green and Lazarsfeld then proved that the Clifford index is indeed constant in such linear series; the conjecture above would imply that Donagi's counterexample to the Harris-Mumford conjecture is the only one.

We also prove a result that allows us to recognize a curve satisfying (1) in its Clifford embedding:

RECOGNITION THEOREM. *Let C be a smooth nondegenerate linearly normal curve of genus $g = 4r - 2$ and degree $g - 1$ in $\mathbb{P}^r$. The following are equivalent:*
(i) *C has Clifford dimension r*

(ii) *C is not contained in any quadric of rank* $\leqslant 4$
(iii) *C is 2r-gonal.*
If these conditons are satisfied, then C is semi-canonical and arithmetically Cohen–Macaulay, and $L = \mathcal{O}_C(1)$ computes the Clifford dimension of C.

This picture of Clifford curves is still not as explicit as the picture obtained in $\mathbb{P}^2$ and $\mathbb{P}^3$. For example, we do not know whether the curves of Clifford dimension $r \geqslant 6$, satisfying Conjecture 1, say, form an irreducible family in the Hilbert scheme, and we do not have any explicit description of the ideal of such a curve in its Clifford embedding. The best we can do is to describe the examples we have. First, the curves we use to prove Martens' Conjecture:

THEOREM (Clifford Curves on $K3$'s). *Let $X \subseteq \mathbb{P}^r$ be a $K3$-surface whose Picard group is generated by a hyperplane section D and a line E contained in X. The linear series $|2D + E|$ contains smooth irreducible curves, and every such is a curve of Clifford dimension r, genus $4r - 2$, and Clifford index $2r - 3$ in $\mathbb{P}^r$. Conversely, if a curve of Clifford dimension $r \geqslant 5$, genus $4r - 2$, and Clifford index $2r - 3$ can be abstractly embedded in a $K3$-surface X', then the intersection X of the quadrics containing the curve in $\mathbb{P}^r$ is a surface birational to X', which contains a line.*

The existence of $K3$-surfaces in $\mathbb{P}^r$ satisfying the conditions of the first part of the theorem follows easily from the surjectivity of the period map for $K3$-surfaces. We reproduce a proof due to David Morrison, for which we are grateful to him.

Every Curve of Clifford dimension $\leqslant 5$ does indeed lie on a $K3$ surface in this way, but there exist curves of Clifford dimension 6 which do not, and it seems a plausible conjecture that the general curve of Clifford dimension 7 will even be cut out by quadrics.

The material of this paper is divided as follows: In section 1 we give a brief exposition of our main tool, the formula of Castelnuovo already mentioned. This is useful to us because $C(d, g, r) = 0$ if there exists a curve of genus g whose Clifford embedding $C \subseteq \mathbb{P}^r$ has degree d. Unfortunately the converse does not hold: for example $C(41, 190, 4) = 0$ and there do exist curves of degree 41 and genus 190 in $\mathbb{P}^4$, but none of them has Clifford dimension 4. Nevertheless, the vanishing of $C(d, g, r)$ does give a rather strong restriction, which we explore using binomial coefficient identities in Section 2. We are grateful to Ira Gessel and V. Strehl for help with these combinatorial matters.

In Section 3 we apply these results, along with some more geometric ideas, to prove the Recognition Theorem, Theorem 3.6. We also show how to derive 2, 3, and 4 of the Conjecture above from part 1, and we prove the lower bound on the degree and genus of curves not satisfying part 1.

In Section 4 we turn to the examples. The first results of the section prove the assertions in the "Clifford curves on $K3$'s" theorem above. Then we give Morrison's argument for the existence of the necessary $K3$-surfaces and prove

Martens' conjecture. Finally we discuss an example of a curve of Clifford dimension 6 which is not contained in a $K3$-surface.

In section 5 we show how the unproved part 1 of our conjecture is related to the conjecture of Eisenbud and Harris. On the basis of partial results on this conjecture obtained by Eisenbud and Harris we establish an upper bound for the genus and the Clifford index of curves of Clifford dimension r. For this we identify the asymptotic term of Castelnuovo's formula with a Laguerre polynomial, a fact pointed out to us by G. Schmeisser. Finally, we explain a method for checking part 1 for any given r. The method only operates with some luck, but we have used it successfully to prove the conjecture for all $r \leqslant 9$.

In addition to the contributions of Ira Gessel, David Morrison, G. Schmeisser and V. Strehl which we have mentioned above, we profited greatly from discussions of this material with Mark Green, Joe Harris, Rob Lazarsfeld, and C. Peters. To all of them our thanks.

1. $(2r-2)$-secant $(r-2)$-planes.

Let C be a curve of Clifford dimension $r \geqslant 2$, and let L be a line bundle which computes the Clifford dimension. Consider the map $\varphi_L: C \to \mathbb{P}^r$ defined by L. The basic observation is

LEMMA 1.1. *L is very ample. Further the image $C \subseteq \mathbb{P}^r$ has no $(2s+2)$-secant s-plane for $1 \leqslant s \leqslant r-2$.*

Proof. If L is not very ample then there are 2 points $p, q \in C$ such that $h^0(L(-p-q)) \geqslant h^0(L) - 1$ $(\,= r \geqslant 2)$. So $L(-p-q)$ would contribute to the Clifford index and moreover $\mathrm{Cliff}(L(-p-q)) \leqslant \mathrm{Cliff}(L)$. By the definition of the Clifford index and the Clifford dimension this cannot happen. So L is very ample.

Similarly if the images of $2s+2$ points $P_1, \ldots, P_{2s+2} \in C$ under φ_L in $\mathbb{P}^r$ are contained in a $\mathbb{P}^s$ for some $s \leqslant r-2$ then $h^0(L(-p_1 \ldots . - p_{2s+2})) \geqslant r - s \geqslant 2$, and $L(-p_1 \ldots - p_{2s+2})$ would contribute to the Clifford index with $\mathrm{Cliff}(L(-p_1 \ldots - p_{2s+2})) \leqslant \mathrm{Cliff}(L)$. Again this cannot happen. □

To contain a point of a curve $C \subseteq \mathbb{P}^r$ imposes one condition on $(r-2)$ planes $\mathbb{P}^{r-2} \subseteq \mathbb{P}^r$. To contain $2r-2$ points imposes in general $2r-2$ conditions. Since the Grassmannian $\mathbb{G}(r-1, r+1)$ of $(r-2)$-planes in $\mathbb{P}^r$ has dimension $2r-2$, we might expect that a "general" curve in $\mathbb{P}^r$ has only a finite number of $(2r-2)$-secant $(r-2)$-planes.

THEOREM 1.2. (Castelnuovo [Cal]). *Let C be an irreducible curve of degree d and geometric genus g in $\mathbb{P}^r$. If C has only finitely many $(2r-2)$-secant*

$(r-2)$-planes then their number (counted with multiplicities) is

$$C(d,g,r)=\sum_{i=0}^{r-1}\frac{(-1)^i}{r-i}\binom{d-r-i+1}{r-1-i}\binom{d-r-i}{r-1-i}\binom{g}{i}.$$

EXAMPLES.

$$C(d,g,2)=\tfrac{1}{2}(d-1)(d-2)-g$$

is the number of double points of a plane curve of degree d and geometric genus g.

$$C(d,g,3)=\frac{(d-2)(d-3)^2(d-4)}{12}-\frac{g(d^2-7d+13-g)}{2}$$

is Cayley's number of 4-secant lines of a space curve.

Combining Lemma 1.1 and Theorem 1.2 we get:

COROLLARY 1.3. *If a line bundle of degree d on a curve of genus g computes the Clifford dimension $r \geqslant 2$ then $C(d,g,r)=0$.* □

Proof of the theorem. A modern proof of a more general but less explicit formula is due to Macdonald (cf. [ACGH, VIII Prop. 4.2]). We derive Castelnuovo's formula from Macdonald's result. It says that the virtual number of $(2r-2)$-secant $(r-2)$-planes is the coefficient of the monomial $(t_1t_2)^r$ in the power series expansion of the term in the brackets

$$C(d,g,r)=\left[-\tfrac{1}{2}(1+t_1+t_2+t_1t_2)^{d-g-r}(1+t_1+t_2)^g(t_1-t_2)^2\right]_{(t_1t_2)^r}$$

(Notice that there are some misprints in [ACGH]).

Writing $u=t_1+t_2$ and $v=t_1t_2$ we obtain

$$C(d,g,r)=-\frac{1}{2}\left[\sum_{j=0}^{r-1}\binom{d-g-r}{j}(1+u)^{d-r-j}v^j(u^2-4v)\right]_{(t_1t_2)^r}$$

$$=-\frac{1}{2}\left[\sum_{j=0}^{r-1}\binom{d-g-r}{j}\binom{d-r-j}{2r-2-2j}u^{2r-2-2j}v^j(u^2-4v)\right]_{(t_1t_2)^r}$$

since the omitted terms do not give a contribution. Moreover we have

$$
\begin{aligned}
[u^{2r-2-2j}v^j(u^2-4v)]_{(t_1t_2)^r} &= \binom{2r-2j}{r-j} - 4\binom{2r-2j-2}{r-j-1} \\
&= \frac{(2r-2-2j)!}{[(r-j)!]^2}[(2r-2j)(2r-2j-1)-4(r-j)^2] \\
&= \frac{-2}{r-j}\binom{2r-2j-2}{r-j-1}.
\end{aligned}
$$

Finally we obtain

$$
C(d,g,r) = \sum_{i=0}^{r-1} \frac{1}{i+1}\binom{2i}{i}\binom{d-2r+1+i}{2i}\binom{d-g-r}{r-1-i}.
$$

To complete the proof it remains to show the following polynomial identity:

$$
\begin{aligned}
&\sum_{i=0}^{r-1} \frac{1}{i+1}\binom{2i}{i}\binom{d-2r+1+i}{2i}\binom{d-g-r}{r-1-i} \\
&\quad = \sum_{i=0}^{r-1} \frac{(-1)^i}{r-i}\binom{d-r-i+1}{r-i-1}\binom{d-r-i}{r-i-1}\binom{g}{i}.
\end{aligned} \tag{1}
$$

The following was shown to us by Ira Gessel: Consider the double sum

$$
D = \sum_{0\leqslant j\leqslant i\leqslant r-1} \frac{(-1)^i(d-2r+i+1)!}{(d-2r-j+1)!(i-j)!j!(j+1)!}\binom{g}{r-i-1}.
$$

Summing first over j and using Vandermonde's identity (cf. [Ri. I, (3)] we get

$$
\begin{aligned}
D &= \sum_{i=0}^{r-1}(-1)^i\binom{g}{r-i-1}\frac{(d-2r+i+1)!}{i!(d-2r+2)!}\sum_{j=0}^{i}\binom{i}{j}\binom{d-2r+2}{d-2r-j+1}, \\
&= \sum_{i=0}^{r-1}(-1)^i\binom{g}{r-i-1}\frac{(d-2r+i+1)!}{i!(d-2r+2)!}\binom{d-2r+i+2}{d-2r+1}.
\end{aligned}
$$

If we set $i = r-1-k$ this yields

$$
D = (-1)^{r-1}\sum_{k=0}^{r-1}\frac{(-1)^k}{r-k}\binom{d-r-k+1}{r-k-1}\binom{d-r-k}{r-k-1}\binom{g}{k},
$$

the right hand side of (1). Summing on i first and again applying Vandermonde's identity (in a slightly different form [Ri, I, (5)]) we obtain

$$D = \sum_{j=0}^{r-1} \frac{(d-2r+j+1)!}{(d-2r-j+1)!\,j!\,(j+1)!} \sum_{i=j}^{r-1} (-1)^i \binom{g}{r-i-1}\binom{d-2r+1+i}{i-j},$$

$$= \sum_{j=0}^{r-1} \frac{(d-2r+j+1)!}{(d-2r-j+1)!\,j!\,(j+1)!} (-1)^{r-1} \binom{d-g-r}{r-j-1},$$

$$= (-1)^{r-1} \sum_{j=0}^{r-1} \frac{1}{j+1} \binom{2j}{j}\binom{d-2r+1+j}{2j}\binom{d-g-r}{r-1-j},$$

the left hand side of (1). This completes the proof of Theorem 1.1. □

REMARK 1.4. Corollary 1.3 helps to explain why curves of Clifford dimension $r \geqslant 3$ are rare. The pair (d, g) has to satisfy the diophantine equation $C(d, g, r) = 0$. By a famous theorem of Siegel [Si] this equation has only finitely many integer-valued solutions for a given r unless the algebraic curve

$$C_r = \{(d, g) \in \mathbb{C}^2 \mid C(d, g, r) = 0\}$$

has a rational component. Since the degree of C_r is $2r - 2$, C_r presumably has no rational component for $r \geqslant 3$.

2. Some vanishing properties of $C(d, g, r)$

In this section we use purely combinatorial techniques to search for solutions of $C(d, g, r) = 0$ close to the bound $\mathrm{Cliff}(C) \leqslant (g-1)/2$. Consider for all integers m and n the numbers

$$r_{nm} := \frac{1}{m+1}\binom{n-1}{m}\binom{n}{m}.$$

They have the following properties (cf. [Ri, I, Example 8]):

$$0 \leqslant r_{nm} \in \mathbb{Z}$$

$$r_{0m} = \delta_{0m}, r_{n0} = 1 \quad (\delta = \text{Kronecker's symbol}),$$

$$r_{nm} = 0 \text{ if } m < 0, \quad r_{nm} = 0 \quad \text{if } m > n \geqslant 0,$$

$$r_{nn} = \delta_{n0},$$

$$r_{nm} = r_{n,n-1-m} \quad \text{if } n \geqslant 1,$$

$$r_{nm} = \sum_{k=0}^{m} r_{mk} \binom{n+k}{2m} \quad \text{(Kreweras' identity).}$$

The first property is a consequence of the last one by recurrence. The numbers r_{nm} are sometimes called Runyon's or Narayana's numbers.

Since we want to study the Clifford index of a curve, we are interested in values of $C(d, g, r)$ only for

$$d \geqslant 2r,$$

which we assume subsequently. We have

$$\begin{aligned} C(d, g, r) &= \sum_{i=0}^{r-1} (-1)^i r_{d-r-i+1, r-1-i} \binom{g}{i}, \\ &= \sum_{i=0}^{r-1} (-1)^i r_{d-r-i+1, d-2r+1} \binom{g}{i}, \\ &= \sum_{j=0}^{d-2r+1} r_{d-2r+1, j} \sum_{i=0}^{r-1} (-1)^i \binom{d-r-i+1+j}{2d-4r+2} \binom{g}{i}, \\ &= C_1(g, d, r) + C_2(g, d, r). \end{aligned}$$

with

$$C_1(d, g, r) = \sum_{j=d-3r+1}^{g-d+r-2} r_{d-2r+1, j} \sum_{i=0}^{r-1} (-1)^i \binom{d-r-i+1+j}{2d-4r+2} \binom{g}{i} \tag{1}$$

and

$$C_2(d, g, r) = \sum_{j=g-d+r-1}^{d-2r} r_{d-2r+1, j} \sum_{i=0}^{r-1} (-1)^i \binom{d-r-i+1+j}{2d-4r+2} \binom{g}{i},$$

since for $j \leqslant d - 3r$ we have $0 \leqslant d - r - i + 1 + j < 2d - 4r + 2$ for all $i = 0, \ldots, r - 1$ and thus the coefficient of $r_{d-2r+1,j}$ vanishes for all $j \leqslant d - 3r$, and because $r_{d-2r+1,d-2r+1} = 0$. For the second sum we note that if $g - d + r - 1 \leqslant j \leqslant d - 2r$ and $r \leqslant i \leqslant g$ we get $0 \leqslant d - r - i + 1 + j \leqslant 2d - 4r + 1$ and hence $\binom{d-r-i+1+j}{2d-4r+2} = 0$. This and the remark that $\binom{g}{i} = 0$ for $0 \leqslant g \leqslant i - 1$ gives

$$C_2(d, g, r) = \sum_{j=g-d+r-1}^{d-2r} r_{d-2r+1, j} \sum_{i=0}^{g} (-1)^i \binom{d-r-i+1+j}{2d-4r+2} \binom{g}{i}.$$

Next we apply Vandermonde's identity (cf. [Ri,I,(5a)]):

$$\sum_{i=0}^{g} (-1)^i \binom{d-r-i+1+j}{2d-4r+2} \binom{g}{i} = \binom{d-r+1+j-g}{2d-4r+2-g}$$

and obtain

$$C_2(d,g,r) = \sum_{j=g-d+r-1}^{d-2r} r_{d-2r+1,j} \binom{d-r+1+j-g}{2d-4r+2-g}. \tag{2}$$

PROPOSITION 2.1. *If $C(d,g,r) = 0$ for some integers d, g, r with $4 \leqslant 2r \leqslant d \leqslant g-1$ then*

$$g \geqslant 2d - 4r + 4$$

with equality if and only if $g = 4r - 2$ and $d = g - 1$.

Proof. If $g \leqslant 2d - 4r + 2$ then $g - d + r - 2 \leqslant d - 3r$ hence $C_1(d,g,r) = 0$ by (1) and $C(d,g,r) = C_2(d,g,r) > 0$. If $g \geqslant 2d - 4r + 3$ then $C_2(d,g,r) = 0$ by (2) hence $C(d,g,r) = C_1(d,g,r)$. So for $g = 2d - 4r + 3$ we obtain

$$C(d, 2d-4r+3, r) = r_{d-2r+1,d-3r+1}$$

and this is zero if and only if $d \leqslant 3r - 2$. Similarly for $g = 2d - 4r + 4$ we have

$$C(d, 2d-4r+4, r) = r_{d-2r+1,d-3r+1} - r_{d-2r+1,d-3r+2}$$

and $r_{d-2r+1,d-3r+1} = r_{d-2r+1,d-3r+2} = 0$ if and only if $d \leqslant 3r - 3$. For $d \geqslant 3r - 2$ the equation $r_{d-2r+1,d-3r+1} = r_{d-2r+1,d-3r+2}$ is equivalent to

$$d^2 - (6r-5)d + (8r^2 - 14r + 6) = 0$$

i.e. $d = 4r - 3$ (or $d = 2r - 2$). In the range $4 \leqslant 2r \leqslant d \leqslant g - 1 \leqslant 2d - 4r + 3$ we have $d \geqslant 4r - 3 \geqslant 3r - 1$. Consequently the equation $C(d,g,r) = 0$ has the single solution $g = 4r - 2$ and $d = 4r - 3$ in this range. □

3. Properties of curves of Clifford dimension r and index $2r - 3$.

In this section we characterize curves of Clifford dimension r and Clifford index $2r - 3$. We also show that if there exists a curve of Clifford dimension r and Clifford index $c \neq 2r - 3$, then $d \geqslant 6r - 6$, $g \geqslant 8r - 7$, and $c \geqslant 4r - 6$. We start with a generalization of a Lemma due to Accola [Ac].

LEMMA 3.1. *Let D and E be divisors of degrees d and e on a curve C of genus g and suppose that $|E|$ is base-point free. Then*

$$h^0(C, \mathcal{O}(D)) - h^0(C, \mathcal{O}(D-E)) \leqslant e/2,$$

equivalently, Cliff $D \geqslant$ Cliff$(D - E)$, *as long as one of the following holds:*

(i) $2D - E$ is special, i.e. $h^1(2D - E) \geqslant 1$; *or*
(ii) *D computes the Clifford index of C, $d \leqslant g - 1$, and $e \neq 2h^0(C, \mathcal{O}(D)) - 1$ if $d = g - 1$.*

REMARKS (1) Accola's Lemma is the case where D is semi-canonical. Clifford's inequality Cliff $E \geqslant 0$ for a divisor E – which may without loss of generality be assumed base-point free – is obtained by taking $D = E$.
(2) The base-point free pencil trick [ACGH.III,ex. B-4] gives

$$h^0(C, \mathcal{O}(D)) - h^0(C, \mathcal{O}(D - E)) \leqslant h^0(C, \mathcal{O}(D + E)) - h^0(C, \mathcal{O}(D)) \tag{1}$$

whenever E is base-point free, so the equivalent conditions

$$h^0(C, \mathcal{O}(D + E)) - h^0(C, \mathcal{O}(D)) \leqslant e/2$$

or

$$\text{Cliff}(D + E) \geqslant \text{Cliff}\, D$$

imply the assertion of the Lemma.

Proof. The equivalence of the two assertions is obvious. We will prove the first under the assumption (i): (This follows the idea of Accola's original proof quite closely.) Since E moves and $2D - E$ is special, i.e. $h^0(K - 2D + E) \neq 0$, we have $h^0(K - 2D) \neq h^0(K - 2D + E)$, cf. [CGH,III,§1]. Hence, by Riemann-Roch, E fails to impose independent conditions on $|2D|$.

Since $|E|$ is base-point free, the monodromy action on the points of a reduced divisor in $|E|$ is transitive, so we may assume that E is reduced and that any divisor of $|2D|$ containing $e - 1$ points of E contains E (Here we use that our ground field has characteristic 0).

We may assume that $h^0(D) > e/2$, since otherwise there is nothing to prove. Let $E' \subseteq E$ be a maximal subset of the points of E imposing linearly independent conditions on $|D|$, so that $\deg E' = h^0 D - h^0(D - E)$. We will show that E imposes at most $\deg(E - E')$ conditions on D, so that

$$\deg E' \leqslant \deg(E - E').$$

whence $\deg E' \leqslant e/2$, and the desired conclusion.

For this it suffices to show that any divisor D_0 of $|D|$ containing $E - E'$ actually contains E. Since the points of E' impose independent conditions on D, we may for each point p of E' choose a divisor D_p of D containing $E' - p$ but not containing E'. It follows that $D_0 + D_p$ contains all but one point of E, and thus by the second paragraph of the proof $D_0 + D_p$ contains E, so that D_0 contains p. Thus D_0 contains E as required. This completes the proof under the assumption (i).

If (ii) holds and $h^1(D+E) \geqslant 2$ then by definition of the Clifford index, $\mathrm{Cliff}(D+E) \geqslant \mathrm{Cliff}(D)$, and we are done by the second remark above.

If $h^1(D+E) \leqslant 1$ then, by Riemann-Roch, $h^0(C, \mathcal{O}(D+E)) \leqslant d+e+2-g$ and it suffices to show

$$d + e + 2 - g \leqslant h^0(C, \mathcal{O}(D)) + e/2$$

i.e. we need $e \leqslant 2h^0(C, \mathcal{O}(D))$ if $d \leqslant g-2$ and $e \leqslant 2h^0(C, \mathcal{O}(D)) - 2$ if $d = g - 1$. We may assume $e = 2h^0(C, \mathcal{O}(D)) - 1$, since otherwise there is nothing to prove. Since $e = 2h^0(C, \leqslant(D)) - 1$ is excluded by our assumption (ii) for $d = g-1$, we are done. □

PROPOSITION 3.2. *If C has Clifford dimension r then any pencil of divisors on C has degree $e \geqslant 2r$.*

Proof. Let D denote a divisor which computes the Clifford dimension, and suppose that E is a base-point free pencil of divisors of degree $e \leqslant 2r-1$. Then $r + 1 - h^0(C, \mathcal{O}(D-E)) \leqslant r-1$ by Lemma 3.1, and hence $D - E$ contributes to the Clifford index. But Lemma 3.1 also gives

$$\mathrm{cliff}(D - E) \leqslant \mathrm{Cliff}\, D;$$

this is impossible by the definition of the Clifford dimension. □

THEOREM 3.3 *If $C \subset \mathbb{P}^r$ is a Clifford embedding, then C is not contained in any quadric of rank $\leqslant 4$.*

Proof. Suppose C is contained in a quadric Q of rank $s \leqslant 4$. Since C is non-degenerate in $\mathbb{P}^r$, we have $s \geqslant 3$. The two pencils of $(r-2)$-planes on Q (which coincide if $s = 3$) induce two pencils of divisors g_a^1 and g_b^1 of degree a and b on C with

$$a + b \leqslant d,$$

where d denotes the degree of C. By Proposition 3.2 we have $a \geqslant 2r$ and $b \geqslant 2r$, so

$$4r \leqslant a + b \leqslant d.$$

On the other hand both pencils contribute to the Clifford index, which implies

$a - 2 \geqslant d - 2r$ and $b - 2 \geqslant d - 2r$. Thus

$$2(d - 2r) \leqslant a + b - 4 \leqslant d - 4,$$

which gives the contradiction $4r \leqslant d \leqslant 4r - 4$. □

COROLLARY 3.4. *Let C be a curve of Clifford dimension $r \geqslant 3$ in $\mathbb{P}^r$. Then $h^0(C, \mathcal{O}_C(2)) \geqslant 4r - 2$.*

Proof. The quadrics of rank $\leqslant 4$ in $\mathbb{P}^r$-form a closed subvariety of codimension $\binom{r-2}{2}$ in the projective space of all quadrics in $\mathbb{P}^r$. By Theorem 3.3 the space of quadrics containing C does not meet this subvariety. Hence

$$h^0(\mathbb{P}^r, I_C(2)) \leqslant \binom{r-2}{2}$$

and

$$\begin{aligned} h^0(C, \mathcal{O}_C(2)) &\geqslant h^0(\mathbb{P}^r, \mathcal{O}(2)) - h^0(\mathbb{P}^r, I_C(2)) \\ &\geqslant \binom{r+2}{2} - \binom{r-2}{2} \geqslant 4r - 2. \end{aligned}$$ □

COROLLARY 3.5. *Let C be a curve of genus g, degree d, and Clifford dimension r in $\mathbb{P}^r$. If $(d, g) \neq (4r - 3, 4r - 2)$ then*

$$d \geqslant 6r - 6 \quad \textit{and} \quad g \geqslant 8r - 7.$$

Proof. Applying Riemann-Roch we get

$$\begin{aligned} h^1(C, \mathcal{O}_C(2)) &= h^0(C, \mathcal{O}_C(2)) - 2d + 1 - g \\ &\geqslant (4r - 2) - 2d + g - 1 && \text{(by Corollary 3.4)} \\ &\geqslant (4r - 2) - 2d + (2d - 4r + 5) - 1 = 2 && \text{(by Proposition 2.1)} \end{aligned}$$

Hence $\mathcal{O}_C(2)$ contributes to the Clifford index. Using again Corollary 3.4 this implies

$$d - 2r \leqslant \mathrm{Cliff}(\mathcal{O}_C(2)) = 2D - 2h^0(C, \mathcal{O}_C(2)) + 2 \leqslant 2d - 8r + 6$$

that is

$$d \geqslant 6r - 6.$$

Once more Proposition 2.1 gives

$$g \geqslant 2d - 4r + 5 \geqslant 8r - 7. \qquad \square$$

THEOREM 3.6. (Recognition Theorem) *Let C be a nondegenerate linearly normal curve of genus $g = 4r - 2$ and degree $d = g - 1$ in $\mathbb{P}^r$ for $r \geqslant 3$. The following are equivalent*:

(i) *C has Clifford dimension r*
(ii) *C is not contained in any quadric of rank $\leqslant 4$*
(iii) *C is $2r$-gonal.*

If these conditions are satisfied then C is semi-canonical and projectively normal, and $L = \mathcal{O}_C(1)$ computes the Clifford dimension.

Proof. Let D denote the hyperlane class of C.

(i) $\Rightarrow$ (ii): Suppose C has Clifford dimension r. Since $g = 4r - 2$ any line bundle computing the Clifford dimension has degree $d = 4r - 3$ by Corollary 3.4. In particular C has Clifford index $2r - 3$. Since C is nondegenerate $h^0(C, \mathcal{O}_C(1)) = r + 1$ for the hyperplane section D of $C \subseteq \mathbb{P}^r$ and D computes the Clifford dimension. Now (ii) follows from Theorem 3.3.

(ii) $\Rightarrow$ (iii): We first show that (ii) implies that C is semi-canonically embedded and projectively normal. Since C is not contained in any rank 4 quadric we obtain

$$h^0(C, \mathcal{O}_C(2)) \geqslant h^0(\mathbb{P}^r, \mathcal{O}(2)) - h^0(\mathbb{P}^r, I_C(2)) \geqslant 4r - 2$$

as in the proof of Corollary 3.4. By Riemann-Roch this gives

$$h^1(C, \mathcal{O}_C(2)) \geqslant 4r - 2 + g - 1 - 2d = 1.$$

Since C has degree $g - 1$ this is only possible if $2D$ is a canonical divisor and equality holds. In particular C is semi-canonical and

$$H^0(\mathbb{P}^r, \mathcal{O}(n)) \to H^0(C, \mathcal{O}_C(n))$$

is surjective for $n = 2$. For $n = 1$ the surjectivity is part of the assumption. For $n = 3$ surjectivity follows from Green's $\mathcal{K}_{p,1}$ theorem and duality [G, 3.c.1 and 2.c.10]: Using Green's notation we have to show $\mathcal{K}_{0,3}(C, \mathcal{O}_C(1)) = 0$. Since C is not a rational normal curve, we obtain:

$$\mathcal{K}_{0,3}(C, \mathcal{O}_C(1)) \cong \mathcal{K}_{0,1}(C, \mathcal{O}_C(2), \mathcal{O}_C(1)) \cong \mathcal{K}_{r-1,1}(C, \mathcal{O}_C(1))^* = 0.$$

For $n \geqslant 4$ we can use the H^0-Lemma [G.4.e.1], since

$$h^1(C, \mathcal{O}_C(n-2)) \leqslant 1 \leqslant r-1 \quad \text{for } n \geqslant 4.$$

This proves that C is projectively normal.

To prove (iii) let C be p-gonal. Since every curve of genus $4r-2$ is p-gonal for some $p \leqslant 2r$ by the Theorem of Meis [Me] it suffices to derive a contradiction if $p \leqslant 2r-1$. If E computes the gonality then since D is semi-canonical we have

$$h^0(C, \mathcal{O}(D-E) \geqslant r+1-(2r-1)/2 = 3/2$$

by Lemma 3.1(i). So actually this space is at least two-dimensional and the multiplication map

$$H^0(C, \mathcal{O}_C(D-E)) \times H^0(C, \mathcal{O}_C(E)) \to H^0(C, \mathcal{O}_C(D)) \cong H^0(\mathbb{P}^r, \mathcal{O}(1))$$

gives us at least one 2×2 matrix of linear forms whose determinant vanishes on C, that is, a rank $\leqslant 4$ quadric in the ideal of C, a contradiction.

(iii) $\Rightarrow$ (i): Suppose C is $2r$-gonal. Since $\operatorname{Cliff}(D) = 2r-3 < 2r-2$, C has some Clifford dimension $r' \geqslant 2$. Suppose D' of degree $d' \leqslant g-1$ computes the Clifford dimension. Define $t = d-2r-(d'-2r')\,(\geqslant 0)$. *If* $t \geqslant r'-1$ then the projection of $C \subseteq \mathbb{P}^{r'}$ to $\mathbb{P}^1$ from $r'-1$ general points of C gives a pencil of degree $d'-r'+1 = d-2r+r'+1-t \leqslant d-2r+2 = 2r-1$, a contradiction. So $s = r'-t \geqslant 2$. Thus the projection of C from t points into $\mathbb{P}^s$ gives a linear series of degree $d'-t = d-2r+2s$ with the same Clifford index as D. Since C is $2r$-gonal the image curve of this projection of C has no $(2s-2)$-secant $(s-2)$-planes. In particular we have

$$C(d-2r+2s, 4r-2, s) = 0$$

by Theorem 1.2. Since $g = 4r-2 = 2(d-2r+2s)-4s+4$ we have $d'-t = g-1$ by Proposition 2.1. So $t = 0$, $d = d'$ and $r = r'$. □

We conclude this section noting two similarities between curves of Clifford dimension r and genus $4r-2$ and smooth plane curves:

THEOREM 3.7. *If C satisfies the equivalent conditions of Theorem* 3.6, *then*

(i) *C has infinitely many pencils of minimal degree, all of which arise as the projections from* $(2r-3)$*-secant* $(r-2)$*-planes.*
(ii) *Only the hyperplane bundle computes the Clifford index of C.*

Proof. (i) Let D denote a hyperplane divisor and let $|E|$ be a pencil of minimal degree $2r$. By lemma 3.1 $D - E$ is linearly equivalent to an effective divisor with $\operatorname{Cliff}(D - E) \leqslant \operatorname{Cliff}(D)$. So $D - E$ does not move. The pencil $|E|$ is given by the projection from the linear span of those points. This establishes a 1–1 correspondence between pencils of minimal degree and $(2r - 3)$-secant $(r - 2)$-planes of $C \subseteq \mathbb{P}^r$. To show that this set is infinite we show that there is a $(2r - 3)$-secant $(r - 2)$-plane containing any point P of C. By Proposition 2.1 $C(d - 1, g, r - 1) = C(4r - 4, 4r - 2, r - 1) \neq 0$. Hence the projection of C from p in $\mathbb{P}^{r-1}$ has some $(2r - 4)$-secant $(r - 3)$-planes by Theorem 2.1. The cone over such a plane with vertex in p is a $(2r - 3)$-secant $(r - 2)$-plane to the original curve.

For (ii), suppose that D' is any divisor computing the Clifford index. Then $\deg D' = g - 1 = \deg D$ since otherwise D' or $K - D'$ would compute a smaller Clifford dimension. Thus both D and D' compute the Clifford dimension. Fix a pencil $|E|$ of degree $2r$. By (i) both $D - E$ and $D' - E$ are linearly equivalent to effective divisors. Thus we may assume that D and D' have been chosen so that $E \leqslant \inf(D, D')$.

With this hypothesis we will show that $D = D'$ as divisors by showing that otherwise one of the divisors

$$F = \inf(D, D')$$

and

$$G = \sup(D, D')$$

contributes to the Clifford index of C and has a smaller Clifford index than D. For convenience we set

$$\delta = \deg F - \deg E,$$
$$\varepsilon = h^1(E) - h^1(F).$$

F contributes to the Clifford index of C since $E \subseteq F \subseteq D$. From the definition of δ and ε,

$$\operatorname{Cliff}(F) = \operatorname{Cliff}(E) - (\delta - 2\varepsilon) = \operatorname{Cliff}(D) + 1 - (\delta - 2\varepsilon).$$

If $D \neq D'$ then $\deg F < \deg D$, and thus $\operatorname{Cliff}(F) > \operatorname{Cliff}(D)$, whence $\delta - 2\varepsilon \leqslant 0$.

Turning to G we note that $g - h^1(G)$, the number of conditions that G imposes on elements of the canonical series, is at most the number of conditions imposed by D' plus the number of conditions imposed by D minus those imposed by F.

Thus

$$\begin{aligned} h^1(G) &\geqslant g - 2(3r - 3) + (2r - 1 + \varepsilon) \\ &= 3 + \varepsilon \geqslant 3. \end{aligned}$$

So since $h^0(G) \geqslant h^0(D) \geqslant 2$, we see that G contributes to the Clifford index of C. From the estimate for $h^1(G)$ and the formula

$$\deg G = 2 \deg D - \deg F = 6r - 6 - \delta$$

we compute

$$\begin{aligned} \mathrm{Cliff}(G) &= \mathrm{Cliff}(K - G) = (2g - 2 - (6r - 6 - \delta)) - 2(h^1(G) - 1) \\ &\leqslant 2r - 4 + (\delta - 2\varepsilon) = \mathrm{Cliff}(D) - 1 + (\delta - 2\varepsilon). \end{aligned}$$

Since we have already shown $\delta - 2\varepsilon \leqslant 0$, this says that

$$\mathrm{Cliff}(G) < \mathrm{Cliff}(D),$$

the desired contradiction. □

REMARK 3.8. By 3.7(i) curves C satisfying the eqivalent conditions of 3.6 are exceptional in the sense of dimension theorems of Martens-Mumford type for the varieties of special linear series (cf. [Ma3]):

$$\dim W^1_p(C) = 1 \quad \text{and } W^1_{p-1}(C) = \emptyset \quad \text{for } p = 2r.$$

We only know the following further examples of curves which are p-gonal with an infinite number of g^1_p's: smooth plane curves of degree $p + 1$, curves of odd genus with maximal gonality, and finite covers of these examples. It would be interesting to know whether there are any other examples.

4. Examples of curves of higher Clifford dimension

In this section we construct for each $r \geqslant 3$ curves of Clifford dimension r. In our examples the curves have genus $g = 4r - 2$ and Clifford index $2r - 3$. By Theorem 3.6 we have to construct semi-canonical curves $C \subseteq \mathbb{P}^r$ of degree $4r - 3$ which are not contained in any quadric of rank $\leqslant 4$. The main difficulty is of course to get a hold on the rank 4 quadrics. We will find our examples on certain $K3$-surfaces which contain a line. Our first Proposition motivates our construction.

PROPOSITION 4.1 *Let $C \subseteq \mathbb{P}^r$ be a curve of degree $d = g - 1 = 4r - 3$, Clifford index $2r - 3$, and Clifford dimension $r \geqslant 3$, which is abstractly contained in a K3-surface X. Then*

$$C \sim 2D + E \text{ in } \operatorname{Pic} X$$

where D is a birationally very ample divisor which maps X into $\mathbb{P}^r$ and cuts out on C the hyperplane series, and E is a line. If $r \geqslant 5$ then the image of X in $\mathbb{P}^r$ is the intersection of all quadrics containing C.

Proof. By the Theorem of Green and Lazarsfeld [GL2] there is a divisor D on X which computes the Clifford index on C. So by Theorem 3.7 D cuts out on C the hyperplane class. Clearly we may assume that $|D|$ has no fixed component. As C does not lie on any rank 4 quadric, the image X' of X in $\mathbb{P}^r$ doesn't either. Hence X' is not a surface of minimal degree and we conclude from the theory of [S–D] that D is birationally very ample. Again by [S–D, §7] we see that there are $\binom{r-2}{2}$ quadrics containing X', which generate the homogeneous ideal of X' if $r \geqslant 5$, since otherwise X' would be contained in a 3-fold scroll, whose ideal is generated by rank $\leqslant 4$ quadrics. By Theorem 3.6 C lies on the same number of quadrics. So we can recover X' from C if $r \geqslant 5$, and since the quadrics in $\mathbb{P}^r$ cut out the complete linear series $|2D|$ on X we find that $2D - C$ is not effective. So $E = C - 2D$ is effective and has degree $E.D = C.D - 2D^2 = 4r - 3 - (4r - 4) = 1$, i.e. E is a line. $\square$

REMARK. This shows that for $r \geqslant 5$ a curve of Clifford dimension r and genus $4r - 2$ can be embedded in at most one $K3$-surface.

LEMMA 4.2. *For each $r \geqslant 3$ there exist K3-surfaces whose Picard group is generated by a very ample divisor D and an irreducible curve E with intersection matrix*

$$\begin{pmatrix} D^2 & E.D \\ D.E & E^2 \end{pmatrix} = \begin{pmatrix} 2r-2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Proof (Morrison). The integral quadratic form defined by the intersection matrix has a primitive embedding into the $K3$ lattice. Choose a polarized Hodge structure on the complement of the image with Hodge numbers $h^{2,0} = 1$ and $h^{1,1} = 18$, and extend this Hodge structure to the entire $K3$ lattice by adjoining the image of the embedding to $H^{1,1}$. For the general such Hodge structure, the integral (1, 1) classes correspond exactly to the image of the embedding.

Let δ and ε be the classes in the $K3$ lattice with the given intersection matrix. Since $\delta^2 > 0$, δ determines a component of the set of elements in $H^{1,1}_{\mathbb{R}}$ with positive square and lies in the closure of some open fundamental domain for the action of the Weyl group of Hodge structures on that set. The surjectivity of the period mapping [BPV] now guarantees the existence of a $K3$-surface X with this Hodge structure, such that the specified open fundamental domain is the (real) cone generated by the ample classes on X (and the closure of the fundamental domain is the cone generated by the nef classes). Now by Riemann–Roch, $\pm\delta$ and $\pm\varepsilon$ are represented by effective divisors. Since δ is a nef class, it must be δ which is represented by a nef effective divisor D. Then since $D.\varepsilon > 0$, it is ε which is represented by an effective divisor E.

By a theorem of Mayer [May], since D is nef the linear system $|D|$ has neither base points nor fixed components unless $D \sim C + kF$ where $C^2 = -2, C.F = 1$, and $F^2 = 0$. In this latter case C and F generate a subgroup of the Picard group whose intersection form has discriminant -1 (= determinant of the intersection matrix). Since $-4r + 3$, the discriminant of Pic X, divides the discriminant of any rank 2 sublattice, this is impossible. So $|D|$ is base-point free. By a theorem of Saint-Donat [S–D], if $|D|$ is not birational, then either $D \sim k\Delta$ with $k = 1$ or 2 and $\Delta^2 = 2$, or there is a curve F with $D.F = 2$ and $F^2 = 0$. The first case is excluded since D is not divisible in Pic X and $D^2 = 2r - 2 > 2$, the second since $-4r + 3$ does not divide -4, the discriminant of $\langle D, F\rangle \subseteq \text{Pic}\, X$. So $|D|$ is birationally ample. By [May] D is even very ample unless there exists a curve F with $D.F = 0$ and $F^2 = -2$. Again this cannot occur since $-4r + 3$ does not divide $-4r + 4$. Finally, E is irreducible since its intersection number with the very ample divisor D is 1. □

THEOREM 4.3. *Let $X \subseteq \mathbb{P}^r$ be a K3-surface as in Lemma* 4.2, *embedded by $|D|$. X is not contained in any rank $\leqslant 4$ quadric, the general element of $|2D + E|$ is smooth, and every smooth curve $C \in |2D + E|$ is a semi-canonically embedded curve of Clifford dimension r, Clifford index* $2r - 3$, *and genus* $g = 4r - 2$.

Proof. Suppose X is contained in a rank $\leqslant 4$ quadric. Then the two rulings induce a decomposition of $D \sim D_1 + D_2$ into moving classes i.e. with $h^0 D_i \geqslant 2$. Writing

$$D_i \sim a_i D + b_i E$$

we have $a_1 + a_2 = 1$. So we may assume $a_1 \leqslant 0$. Then $D_1 - a_1 D \sim b_1 E$ is still a moving class and $b_1 \geqslant 0$. If $b_1 > 0$ then $(D_1 - a_1 D).E < 0$, and E is a fixed component of $|D_1 - a_1 D|$, which we may subtract and still have a moving class. Repeating this we end up with the contradiction $h^0(\mathcal{O}) \geqslant 2$.

This shows that X and consequently any $C \in |2D + E|$ is not contained in any rank $\leqslant 4$ quadric. A smooth C in this class has genus $g = \frac{1}{2}(2D + E)^2 + 1 = 4r - 2$, by adjunction, and degree $d = C.D = 4r - 3$. So a smooth C has Clifford dimension r and Clifford index $2r - 3$ by Theorem 3.6. C is indeed semi-canonically embedded (as also follows from that Theorem), since C does not intersect E. A general element of $|2D + E|$ is smooth since $|2D + E|$ is base-point free. $\square$

Curves of degree $4r - 3$, genus $4r - 2$ and Clifford dimension r do not necessarily lie on a $K3$-surface except for $r \leqslant 5$. We looked for examples on $K3$-surfaces because in this way we can control the rank 4 quadrics in the ideal. In the remaining part of this section we describe the ideal of the curves in our example for $r = 3, 4, 5$, and we construct for $r = 6$ an example which does not lie on a $K3$-surface.

The case $r = 3$ is Martens' example. Any projectively normal semi-canonical space curve of degree g and genus 10 is the complete intersection of two cubics, and any smooth intersection of two cubics in $\mathbb{P}^3$ has Clifford dimension 3 by Theorem 3.6. Every smooth hypersurface of degree 4 which contains such curves is a $K3$ surface which contains a line.

Turning to the case $r = 4$ we have that the ideal of a semi-canonical projectively normal curve C of degree 13 and genus 14 in $\mathbb{P}^4$ is generated by a single quadric and four cubics. More precisely by the structure theorem of Gorenstein rings in codimension 3 [BE], the ideal is generated by the 4×4 pfaffians of a skew symmetric matrix

$$(a_{ij})_{1 \leqslant i,j \leqslant 5} \quad \text{with } \deg a_{ij} = \begin{cases} 2 & \text{if } i = 1 \quad \text{or } j = 1 \\ 1 & \text{if } i \geqslant 2 \quad \text{and } j \geqslant 2 \end{cases}.$$

A smooth intersection of the quadric with one of the cubics is a $K3$-surface which contains a line. C has Clifford dimension 4 if the quadric $a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$ has maximal rank.

In case $r = 5$ the ideal of a semi-canonical projectively normal curve C of degree 17 and genus 18 in $\mathbb{P}^5$ contains 3 quadrics which intersect in a $K3$-surface or a three-dimensional rational normal scroll. The second possibility does not occur in our case since such a scroll is contained in a net of rank 4 quadrics. So our C lies on a (possibly singular) $K3$-surface X. By Proposition 4.1 X contains a line. Hence we may write the quadrics q_i in the ideal as

$$(q_1, q_2, q_3) = (x_1, x_2, x_3, x_4) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix},$$

where the x_i's and a_{ij}'s are some linear forms. From the Hilbert function of C and the self-duality of its resolution it follows that the ideal of C is generated by the quadrics and four further cubic generators. One can check that for a suitable choice of the matrix (a_{ij}) these are precisely the 3×3 minors of this matrix. Indeed for the presentation of (q_1, q_2, q_3) we may alter each of the 3 columns of (a_{ij}) by any of the 6 Koszul relations among the x_i's. Hence this gives us a 18-dimensional family of curves on a fixed $K3$-surface, and this is the dimension of $|C|$.

For $r = 6$ it is no longer true that all semi-canonical projectively normal curves of degree 21 and genus 22 lie on a $K3$-surface. We give an example of a curve of Clifford dimension 6 which does not lie on a $K3$-surface:

Consider the variety $V \subseteq \mathbb{P}^{20}$ defined by 6 quadratic and 1 cubic equations

$$\begin{aligned}
&a_{11}a_{22} - a_{12}a_{21} + b_{11}c_{31} + b_{12}c_{32} + b_{13}c_{33} = 0\\
-&a_{11}a_{23} + a_{13}a_{21} + b_{11}c_{21} + b_{12}c_{22} + b_{13}c_{23} = 0\\
&a_{12}a_{23} - a_{13}a_{22} + b_{11}c_{11} + b_{12}c_{12} + b_{13}c_{13} = 0\\
&b_{11}b_{22} - b_{12}b_{21} + a_{11}c_{13} + a_{12}c_{23} + a_{13}c_{33} = 0\\
-&b_{11}b_{23} + b_{13}b_{21} + a_{11}c_{12} + a_{12}c_{22} + a_{13}c_{32} = 0\\
&b_{12}b_{23} - b_{13}b_{22} + a_{11}c_{11} + a_{12}c_{21} + a_{13}c_{31} = 0\\
&a_{21}(b_{21}c_{11} + b_{22}c_{12} + b_{23}c_{13}) + a_{22}(b_{21}c_{21} + b_{22}c_{22} + b_{23}c_{23})\\
&+a_{23}(b_{21}c_{31} + b_{22}c_{32} + b_{23}c_{33}) + c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32}\\
&- c_{11}c_{23}c_{32} - c_{13}c_{22}c_{31} - c_{12}c_{21}c_{33} = 0
\end{aligned}$$

where $a_{11}, \ldots, a_{23}, b_{11}, \ldots, b_{23}, c_{11}, \ldots, c_{33}$ are homogenous coordinates on $\mathbb{P}^{20}$.

THEOREM 4.4. *If $H \cong \mathbb{P}^6$ is a generic 6-plane in $\mathbb{P}^{20}$ then $C = V \cap H$ is a curve of genus 22, degree 21, and Clifford dimension 6 which is not contained in any $K3$-surface. The quadrics containing C meet in C and a further point $H \cap \{a_{11} = a_{12} = a_{13} = b_{11} = b_{12} = b_{13} = 0\}$.*

Sketch of the proof. All assertions which follow can be easily checked using the computer program Macaulay [BS]. V has codimension 5 and is arithmetically Cohen-Macaulay. The intersection $C = V \cap \mathbb{P}^5$ with a general $\mathbb{P}^6 \subseteq \mathbb{P}^{20}$ is a semi-canonical curve of degree 21 and genus 22. For an open set of 6-planes H in the Grassmannian $\mathbb{G}(7. 21)$ the curve C is smooth and not contained in any rank $\leqslant 4$ quadric, hence C has Clifford dimension 6 by Theorem 3.6. So it suffices to prove that this set is nonempty. The 6-plane section defined by

$$\begin{aligned}
&a_{11} = x_0, && a_{12} = x_1, && a_{13} = x_2,\\
&a_{21} = 7x_1, && a_{22} = 7x_2, && a_{23} = 7x_3,\\
&b_{11} = x_3, && b_{12} = x_4, && b_{13} = x_5,\\
&b_{21} = x_4, && b_{22} = x_5, && b_{23} = 7x_6\\
&c_{11} = x_6, && c_{12} = x_4, && c_{13} = x_0,\\
&c_{21} = x_1, && c_{22} = x_6, && c_{23} = x_5,\\
&c_{31} = x_3, && c_{32} = x_2, && c_{33} = x_6
\end{aligned}$$

i.e. the curve defined by the equations

$$
\begin{aligned}
&-7x_1^2 + 7x_0x_2 + x_3^2 + x_2x_4 + x_5x_6 = 0\\
&7x_1x_2 - 7x_0x_3 + x_1x_3 + x_5^2 + x_4x_6 = 0\\
&-7x_2^2 + 7x_1x_3 + x_4^2 + x_0x_5 + x_3x_6 = 0\\
&-x_4^2 + x_3x_5 + x_0^2 + x_1x_5 + x_2x_6 = 0\\
&x_4x_5 - 7x_3x_6 + x_2^2 + x_0x_4 + x_1x_6 = 0\\
&-x_5^2 + 7x_4x_6 + x_1^2 + x_2x_3 + x_0x_6 = 0\\
&x_0x_1x_2 + 7x_1x_2x_4 + 7x_3^2x_4 + 7x_2x_3x_5 + 7x_1x_4x_5 + x_3x_4x_5\\
&+49x_0x_1x_6 - x_0x_3x_6 + 6x_1x_4x_6 + 55x_2x_5x_6 + 49x_3x_6^2 + x_6^3 = 0
\end{aligned}
$$

gives an explicit example. The smoothness of this curve and the absence of rank 4 quadrics can be verified by a straightforward but messy computation for which we used the computer program Macaulay.

C does not lie on a $K3$-surface since the quadrics defining V intersect in V and the linear space $\{a_{11} = a_{12} = a_{13} = b_{11} = b_{12} = b_{13} = 0\}$. □

We do not present the computer computation here. Instead we sketch how this example was constructed.

Let X be the smooth transversal intersection of two cubic rational normal scrolls A an B in $\mathbb{P}^6$. X is a rational normal surface of degree 9. It is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the rational map defined by the linear series of curves of type (3.3) passing through 9 assigned base points, cf. eg. [Sch]. Let K denote the canonical divisor on X and H the hyperplane class. A general $C \in |2H - K|$ is a smooth semi-canonically embedded curve of degree 21 and genus 22. C is not of Clifford dimension 6 but a general enough small deformation C' of C out of the surface will be. We construct C' via explicit equations. In order to do this we review the equations and syzygies of X and C. Let

$$
a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}
$$

denote the 2×3 matrices with linear entries whose 2×2 minors define A and B respectively. The sysygies of $\mathcal{O}_A$ respectively $\mathcal{O}_B$ as an $\mathcal{O} = \mathcal{O}_{\mathbb{P}^6}$ module are given by the Eagon-Northcott complex $\mathscr{C}_A^0$ respectively $\mathscr{C}_B^0$:

$$
0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \leftarrow 3\mathcal{O}(-2) \xleftarrow{a} 2\mathcal{O}(-3) \leftarrow 0.
$$

Moreover we have explicit descriptions of the syzygies for many of the line bundles on A resp. B, the complexes $\mathscr{C}_A^k$ due to Buchsbaum and Eisenbud, see

[Sch] for many details. The syzygies of $\mathcal{O}_X = \mathcal{O}_A \otimes \mathcal{O}_B$ are given by $\mathscr{C}_A^0 \otimes \mathscr{C}_B^0$:

$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O} \leftarrow 6\mathcal{O}(-2) \leftarrow 4\mathcal{O}(-3) \oplus 9\mathcal{O}(-4) \leftarrow 12\mathcal{O}(-5) \leftarrow 4\mathcal{O}(-6) \leftarrow 0$$

The dual of this complex suitably twisted gives a resolution of $\mathcal{O}_X(K)$:

$$\begin{aligned} 0 &\leftarrow \mathcal{O}_X(K) \leftarrow 4\mathcal{O}(-1) \leftarrow 12\mathcal{O}(-2) \leftarrow 9\mathcal{O}(-3) \oplus 4\mathcal{O}(-4) \\ &\leftarrow 6\mathcal{O}(-5) \leftarrow \mathcal{O}(-7) \leftarrow 0 \end{aligned}$$

Of course this is just the complex $\mathscr{C}_A^1 \otimes \mathscr{C}_B^1(-1)$. From the sequence

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_X(K - 2H) \leftarrow 0$$

we obtain syzygies of $\mathcal{O}_C$ via a mapping cone $\mathscr{C}_A^0 \otimes \mathscr{C}_B^0 \leftarrow \mathscr{C}_A^1 \otimes \mathscr{C}_B^1(-3)$:

$$\begin{aligned} 0 &\leftarrow \mathcal{O}_C \leftarrow \mathcal{O} \leftarrow 6\mathcal{O}(-2) \oplus 4\mathcal{O}(-3) \leftarrow 4\mathcal{O}(-3) \oplus 21\mathcal{O}(-4) \\ &\leftarrow 21\mathcal{O}(-5) \oplus 4\mathcal{O}(-6) \leftarrow 4\mathcal{O}(-6) \oplus 6\mathcal{O}(-7) \leftarrow 0(-9) \leftarrow 0. \end{aligned}$$

To get better hold on the 4 cubic equations of C we note that

$$\mu\colon H^0(X, \mathcal{O}_X(H - K)) \otimes H^0(X, \mathcal{O}_X(H)) \to H^0(X, \mathcal{O}_X(2H - K))$$

is surjective. Indeed $\mathcal{O}_X(H - K)$ is resolved by the complex $\mathscr{C}_A^{-1} \otimes \mathscr{C}_B^{-1}$:

$$0 \leftarrow \mathcal{O}_X(H - K) \leftarrow 9\mathcal{O} \leftarrow 36\mathcal{O}(-1) \leftarrow 54\mathcal{O}(-2) \leftarrow 36\mathcal{O}(-3) \leftarrow 9\mathcal{O}(-4) \leftarrow 0$$

and this complex is exact on global sections for any twist $\otimes\, \mathcal{O}(m)$. So choosing a preimage for the equations of C under μ we find we can represent C by 9 linear forms which are naturally arranged in a 3×3 matrix

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

It turns out that the cubic generators of $\mathscr{I}_C$ are given by the 4 entries of the 2×2 matrix

$$acb^{\mathrm{tr}}$$

We ommit the calculation which checks that these 4 cubics really lift to a mapping cone between $\mathscr{C}_A^0 \otimes \mathscr{C}_B^0$ and $\mathscr{C}_A^1 \otimes \mathscr{C}_B^1(-3)$.

We are now going to deform C. Actually this is possible and easier in the generic setting. Let

$$S = \mathbb{C}[a_{ij}, b_{ij}, c_{ij}]$$

denote a polynomial ring in $6 + 6 + 9 = 21$ variables. Let

$$I = (\wedge^2 a, \wedge^2 b, I_1(acb^{\mathrm{tr}}))$$

be the ideal generated by the 2×2 minors of a and b and by the 4 cubic entries of the 2×2 matrix acb^{tr}. The variety $V(I) \subseteq \mathbb{P}^{20}$ is a generic version of C. A general six-dimensional linear subspace $\mathbb{P}^6 \subseteq \mathbb{P}^{20}$ intersects $V(I)$ in a curve C of degree 21 and genus 22 which lies on two cubic scrolls A and B. Moreover by what we have said before any of these curves arises this way. Next we introduce two 1×2 matrices

$$\varepsilon = (\varepsilon_1, \varepsilon_2) \text{ and } \delta = (\delta_1, \delta_2)$$

whose entries will serve as deformation parameters. The deformation we are looking for is then given by the ideal $I(\varepsilon, \delta)$ generated by the entries of

$$\wedge^2 a + \varepsilon bc^{\mathrm{tr}}.\ \wedge^2 b + \delta ac \quad \text{and}$$

$$acb^{\mathrm{tr}} + \det(c)\begin{pmatrix} \delta_2\varepsilon_2 & -\delta_1\varepsilon_2 \\ -\delta_2\varepsilon_1 & \delta_1\varepsilon_1 \end{pmatrix}$$

This makes sense since $\wedge^2 a$ and $\varepsilon bc^{\mathrm{tr}}$ both are 1×3 matrices and acb^{tr} is a 2×2 matrix while $\det(c)$ is a scalar. So we can add these matrices. To check the flatness of

$$\mathbb{C}[\varepsilon_1, \delta_i] \to S[\varepsilon_i, \delta_i]/I(\varepsilon, \delta)$$

involves a computation which we do not present here. We now specialize to the case $\varepsilon = (1, 0)$ and $\delta = (1, 0)$. Then

$$I_V = I((1,0),(1,0)) \subseteq S$$

defines the variety $V \subseteq \mathbb{P}^{20}$ above. In particular the intersection with a generic $\mathbb{P}^6 \subseteq \mathbb{P}^{20}$ is a smooth semi-canonical curve C' of degree 21 and genus 22. The curve C' never lies on a $K3$-surface: To see this note that its syzygies have the type

$$\begin{aligned} 0 &\leftarrow \mathcal{O}_{C'} \leftarrow \mathcal{O} \leftarrow 6\mathcal{O}(-2) \oplus \mathcal{O}(-3) \leftarrow \mathcal{O}(-3) \oplus 21\mathcal{O}(-4) \\ &\leftarrow 21\mathcal{O}(-5) \oplus \mathcal{O}(-6) \leftarrow \mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow \mathcal{O}(-9) \leftarrow 0 \end{aligned}$$

So the ranks of the syzygy modules are two small for C' to be contained on a $K3$-surface. C' actually lies on no surface of small degree at all. The quadrics containing C' intersect in C' and a single additional point $p = V(a_{11}, a_{12}, a_{13}, b_{11}, b_{12}, b_{13})$ which lies on the rational surface X. So C' remembers a little bit of its construction. The point p may also be interpreted as the image of $((\varepsilon_1, \varepsilon_2), (\delta_1, \delta_2)) = ((1,0),(1,0)) \in \mathbb{P}^1 \times \mathbb{P}^1$ under the rational map

$$\mathbb{P}^1 \times \mathbb{P}^1 \to X \subseteq \mathbb{P}^6$$

defined by the curves of type (3.3) passing through the 9 assigned base points.

5. A bound on the degree

In this section we bound the degree of a curve of Clifford dimension r following the method of Martens [Ma2]. For small r our results allow to verify our Conjecture.

PROPOSITION 5.1. *A curve of Clifford dimension $r \geqslant 4$ in $\mathbb{P}^r$ of degree $d \neq 4r - 3$ is not contained in a surface of degree $\leqslant 2r - 3$.*

Proof. It is well known that a surface X of degree

$$\deg X = r - 1 + \alpha \quad \text{for} \quad \alpha \leqslant r - 2$$

in $\mathbb{P}^r$ is birationally ruled. Let

$$\varphi\colon Y \to X \subseteq \mathbb{P}^r$$

be a desingularisation. Suppose X contains a curve C of Clifford dimension r and degree $d \neq 4r - 3$. By Corollary 3.5 we have $d \geqslant 6r - 6$. Because $\deg C > \deg X$ and C is linearly normal, $\varphi\colon Y \to \mathbb{P}^r$ is defined by a complete linear series.

We distinguish two cases, according to whether X is rational or irregular.

If $q = h^1 \mathcal{O}_Y \geqslant 1$ then according to the Enriques-Kodaira classification of surfaces there is a morphism $Y \to E$ onto a curve E of genus q, whose general fibre is a smooth rational curve. Hence X is ruled by rational curves of degree s, where s is some integer bounded by

$$1 \leqslant s \leqslant \begin{cases} 1 + 2\frac{\alpha - 1}{r + 1 - \alpha} & \text{if} \quad q = 1 \\ \sqrt{\frac{\alpha - 1}{q - 1}} & \phantom{\text{if}} \quad q \geqslant 2 \end{cases}$$

by [Ho, Proposition 1.13]. In particular q is also bounded and $s \leqslant r - 2$. The rulings span at most s-dimensional linear subspaces of $\mathbb{P}^r$. Since C has no $(2s + 2)$-secant s-planes by in Lemma 1.1, we obtain that C is an at most

$(2s+1)$-sheeted cover of E. E is m-gonal for some $m \leqslant [(q+3)/2]$ by the theorem of Meis [Me]. Hence C is at most $(2s+1)[(q+3)/2]$-gonal, and

$$d - 2r = \mathrm{Cliff}(C) \leqslant (2s+1)[(q+3)/2] - 3.$$

since C has Clifford dimension r. Putting all estimates together we find that d cannot be greater or equal then $6r-6$. For example, if $s \geqslant 2$ and $q \geqslant 2$ we have $2s \leqslant 2s(q-1) \leqslant s^2(q-1) \leqslant \alpha - 1 \leqslant r-3$ whence $(2s+1)[(q+3)/2] \leqslant (2s+1)q \leqslant 2s(q-1) + 2s + q \leqslant \frac{9}{4}(r-3)$ and $d \leqslant 2r + \frac{9}{4}(r-3) - 3 \leqslant 5r - 9$. The remaining cases are even simpler.

For rational surfaces a good bound on the degree of a ruling is not known. We apply Theorem 3.3 and the following Proposition.

PROPOSITION 5.2. *A nondegenerate linearly normal rational surface X in $\mathbb{P}^r$ for $r \geqslant 4$ of degree $\leqslant 2r-3$ is contained in a quadric of rank $\leqslant 4$.*
Here nondegenerate and linearly normal means that the map

$$\varphi\colon Y \to X \subseteq \mathbb{P}^r$$

from a desingularisation is given by a complete linear series. In particular X has at most isolated singularities - see the beginning of the proof below.

Proof. Let X have degree $r-1+\alpha$. A general hyperplane section H of X is a smooth non-special curve of degree $r-1+\alpha$ and genus α. Indeed, writing $\tilde{H}$ for the strict transform of H to Y, $\tilde{H}$ is non-special by Clifford's theorem, linearly normal since $h^1(\mathcal{O}_Y) = 0$, and has genus α by the Riemann-Roch formula. Since $r - 1 + \alpha \geqslant 2\alpha + 1$, $\tilde{H}|_{\tilde{H}}$ is very ample, so H is smooth and X has only isolated singularities.

Since X is nondegenerate and linearly normal, we obtain the first equality in the following:

$$\begin{aligned} h^0(\mathbb{P}^r, \mathcal{I}_X(2)) &= h^0(\mathbb{P}^{r-1}, \mathcal{I}_H(2)) \\ &\geqslant h^0(\mathbb{P}^{r-1}, \mathcal{O}(2)) - h^0(H, \mathcal{O}_H(2)) \\ &= \binom{r+1}{2} - (2(r-1+\alpha) + 1 - \alpha) = \binom{r-2}{2} + r - 2 - \alpha. \end{aligned}$$

Since the quadrics of rank $\leqslant 4$ form a closed subvariety of codimension $\binom{r-2}{2}$ in $\mathbb{P}(H^0(\mathbb{P}^r, \mathcal{O}(2)))$ we see that $H^0\mathcal{I}_X(2)$ contains some of these if $\alpha \leqslant r-3$.

In case $\alpha = r-2$ we make an induction on r. If X is singular, then we can project from a singular point. The image is a rational surface by Lüroth's theorem, to which we may apply induction since it has degree $\leqslant 2r-5$. So we may assume that X is non-singular. We apply the adjunction mapping (cf. [So], [*VdV*]). Let K denote a canonical divisor on X. Since $h^1(\mathcal{O}_X(H)) = 0$ the Riemann-Roch formula gives

$$r + 1 = h^0(\mathcal{O}_X(H)) = \chi(\mathcal{O}_X(H)) = \tfrac{1}{2}H.(H - K) + 1$$

So

$$H^2 = 2r - 3 \text{ and } H.K = -3.$$

Adjunction gives $h^0(\mathcal{O}_X(H + K) = \alpha = r - 2 (\geqslant 2)$.

$$h^0(\mathcal{O}_X(-K)) \geqslant \chi(\mathcal{O}_X(-K)) = K^2 + 1.$$

If $K^2 \geqslant 1$ we obtain some rank $\leqslant 4$ quadrics from the multiplication

$$H^0(\mathcal{O}_X(-K)) \times H^0(\mathcal{O}_X(H + K)) \to H^0(\mathcal{O}_X(H)) \cong H^0(\mathbb{P}^r, \mathcal{O}(1)).$$

If $K^2 \leqslant 0$ we consider the adjunction map

$$\varphi_{H+K}: X \to X' \subseteq \mathbb{P}^{r-3}.$$

φ_{H+K} is birational contracting maybe some (-1) curves, unless X is ruled by lines or conics, in which cases X is obviously contained in a rank $\leqslant 4$ quadric. X' has degree $(H + K)^2 = 2(r - 3) - 3 + K^2$. So if $K^2 \leqslant -1$ then by what we have already proved X' is contained in a rank $\leqslant 4$ quadric Q. The 2 rulings of Q give a decomposition $H + K = D_1 + D_2$ where D_i is a divisor with $h^0(\mathcal{O}_X(D_i)) \geqslant 2$ for $i = 1.2$. Since $(D_1 + D_2).H = (H + K).H = 2r - 6$ we may assume that $D_1.H \leqslant r - 3$ and consequently the image of D_1 in $\mathbb{P}^r$ spans an at most $(r - 3)$-dimensional subspace. The multiplication map

$$H^0(\mathcal{O}_X(D_1)) \times H^0(\mathcal{O}_X(H - D_1)) \to H^0(\mathcal{O}_X(H))$$

gives us some rank $\leqslant 4$ quadrics. Finally, if $K^2 = 0$ then $-K$ is effective and we can apply induction on r since the inclusion $H^0(\mathcal{O}_X(H + K)) \subseteq H^0(\mathcal{O}_X(H))$ carries rank 4 quadrics along. To get the induction started we have to check the cases $r = 4, 5$ and 6.

If $r = 4$ then X is a (possibly singular) Castelnuovo surface, which is well known to lie on a unique rank 4 quadric, cf. [0]. $\varphi_{H+K}: X \to \mathbb{P}^1$ exhibits X as a conic bundle.

If $r = 5$ then, because $K^2 = 0$, $\varphi_{H+K}: X \to \mathbb{P}^2$ is a blow-up of nine (possibly infinitesimally near) points in $\mathbb{P}^2$, and $|H| = |L - K| = |4L - p_1 \ldots - p_g|$ is the linear series of quartics passing through the nine points. A curve in the pencil $|L - p_i|$ is embedded as a cubic in $\mathbb{P}^5$, hence spans a $\mathbb{P}^3$. Thus the pencils $|L - p_i|$ sweep out rank $\leqslant 4$ quadrics.

If $r = 6$ then $\varphi_{H+K}: X \to X' \subseteq \mathbb{P}^3$ exhibits X as the blow-up of a cubic surface, which in turn is the blow-up of $\mathbb{P}^2$ in 6 points. Because $K^2 = 0$, X is the blow-up of 3 additional points and $|H| = |6L - 2p_1 \ldots - 2p_6 - p_7 - p_8 - p_9|$. The pencils $|L - p_i|$ for $i = 1, \ldots, 6$ sweep out rank $\leqslant 4$ quadrics. This completes the proof of Proposition 5.2 and hence Proposition 5.1. □

Proposition 5.1 allows us to bound the genus of a curve of Clifford dimension r for a given degree. Consider the functions

$$\pi_\alpha(d, r) \approx \frac{d^2}{2(r - 1 + \alpha)} + 0(r)$$

for $0 \leqslant \alpha \leqslant r - 1$ introduced in [EH, pp. 116].

THEOREM 5.3. (Eisenbud-Harris, [EH, 3.22]). *There exists a constant $d_0 = d_0(r)$ such that every reduced, irreducible and non-degenerate curve in $\mathbb{P}^r$ of degree $d \geqslant d_0$ and geometric genus*

$$g > \pi_\alpha(d, r)$$

for some $\alpha \leqslant r - 1$ lies on a surface of degree $r - 2 + \alpha$ or less. Moreover, we may take

$$d_0 = \begin{cases} 36r, & r \leqslant 6 \\ 288, & r = 7 \\ 2^{r+1}, & r \geqslant 8. \end{cases}$$

However the bound on d_0 given in the theorem does not seem to be the best possible.

CONJECTURE 5.4. (Eisenbud-Harris, loc. cit p. 132) $d_0(r) = 4r - 3$ *suffices.*

For $\alpha = 0$ Theorem 5.3 holds for arbitrary degrees. It is Castelnuovo's bound on the genus of a curve of degree d in $\mathbb{P}^r$. For $\alpha = 1$ Eisenbud and Harris prove Theorem 5.3 for any $d \geqslant 2r + 1$ (loc. cit. 3.15).

COROLLARY 5.5. *The genus of a curve of Clifford dimension r and degree $d \geqslant \max(d_0(r), 4r - 2)$ in $\mathbb{P}^r$ is bounded by*

$$g \leqslant \pi_{r-1}(d, r).$$

Proof. Apply Proposition 5.1. □

On the other hand,

PROPOSITION 5.6. *There is a constant* $d_1 = d_1(r)$ *such that*

$$C(d, g, r) > 0 \text{ for all } d \geqslant d_1 \text{ and } g \leqslant \pi_{r-2}(d, r).$$

Proof. Write $g = xd^2$. $C(d, xd^2, r)/d^{2r-2}$ converges for $d \to \infty$ to

$$F(x) = \sum_{i=0}^{r-1} \frac{(-1)^i x^i}{(r-i)!(r-1-i)!i!}$$

uniformly in bounded domains for x. Since $\pi_{r-2}(d, r)/d^2 \to 1/(4r-6)$ for $d \to \infty$, it suffices to prove that $F(x) > 0$ for $0 \leqslant x \leqslant 1/(4r-6)$. We thank G. Schmeisser for pointing out to us that $F(x)$ is a classically studied polynomial:

$$G(y) = (-1)^{r-1} r! y^{r-1} F\left(\frac{1}{y}\right)$$
$$= \sum_{i=0}^{r-1} (-1)^{r-1+i} \frac{r!}{(r-i)!i!} \frac{y^{r-1-i}}{(r-i-1)!} = \sum_{j=0}^{r-1} (-1)^j \binom{r}{j+1} \frac{y^j}{j!}$$

is a Laguerre polynomial (of index 1), i.e. $G(y) = L_{r-1}^{(1)}(y)$ in a common termonology, cf. [Sz, (5.1.6)]. The zeros of this classical polynomial are well studied, [Sz, Chap. VI], they are distinct and real, and for the largest zero y_1 of $G(y)$ we obtain from [Sz, (6.32.2) and (6.32.7)]

$$y_1 \leqslant 4r\left(1 - \frac{1.8}{(4r)^{2/3}}\right)^2,$$

which is $\leqslant 4r - 6$ for $r \geqslant 3$. Hence $F(x) > 0$ for $0 \leqslant x \leqslant 1/(4r-6)$. □

COROLLARY 5.7. *The degree and the genus of a curve of Clifford dimension r in* $\mathbb{P}^r$ *are bounded by constants depending only on r.*

Proof. Combining Corollary 1.3 with Corollary 5.5 and Proposition 5.6 we see that $d < \max\{d_0(r), 6r - 6, d_1(r)\}$. □

We do not try to compute $d_1(r)$ explicitly. However

CONJECTURE 5.8. $C(d, g, r) > 0$ *for all* $d \geqslant 6r - 6$ *and* $g \leqslant \pi_{r-1}(d, r)$.

Conjecture 5.4 and 5.8 imply the Conjecture of the introduction. For large

d Conjecture 5.8 holds by Proposition 5.6. The following Lemma gives some evidence that 5.8 is true for all $d \geqslant 6r - 6$.

LEMMA 5.9. *Let $d(n) = (2r - 2)n$ and $g(n) = (r - 1)n^2 + 1$. Then $C(d(n), g(n), r) \geqslant 0$ for all integers $n \geqslant 3$.*

Proof. Let X be a generic $K3$-surface in $\mathbb{P}^r$. So Pic $X = \mathbb{Z}H$. Then $d(n) = nH.H$ and $g(n) = p_a(nH)$. Since X contains no curves which are contained in a $\mathbb{P}^{r-2}$ every $(r - 2)$-plane intersects X in a subscheme of length $2r - 2$ by Bezout's theorem. Consider the incidence correspondence

$$I = \{(\mathbb{P}^{r-2}, C) \in \mathbb{G}(r - 1, r + 1) \times |nH| \mid X \cap \mathbb{P}^{r-2} \subseteq C\}.$$

The fibers of $pr_2: I \to |nH|$ are precisely the $(2r - 2)$-secant $(r - 2)$-planes to the corresponding curve. On the other hand, the first projection

$$I \subseteq \mathbb{G}(r - 1, r + 1) \times |nH| \to \mathbb{G}(r - 1, r + 1)$$

makes I into a $\mathbb{P}^N$-subbundle of codimension $2r - 2$: Because X is arithmetically Cohen-Macaulay, and X does not contain any curve which spans a $\mathbb{P}^{r-2}$, the intersection $\mathbb{P}^{r-2} \cap X$ has Hilbert function $(1, r - 1, 2r - 3, 2r - 2, 2r - 2, \ldots)$ for any $\mathbb{P}^{r-2} \in \mathbb{G}(r - 1, r + 1)$. Hence $\mathbb{P}^{r-2} \cap X$ imposes independent conditions on hypersurfaces of degree $n \geqslant 3$ for any $\mathbb{P}^{r-2}$.

Since $\dim \mathbb{G}(r - 1, r + 1) = 2(r - 1)$ we get $\dim I = \dim |nH)$ and the projection onto the second factor $pr_2: I \to |nH|$ has generically finite fibers. So $C(d(n), g(n), r) \geqslant 0$ for $n \geqslant 3$ by Theorem 1.2. $\square$

REMARK 5.10. (1) $d(n), g(n)$ lies on the boundary of $g \leqslant \pi_{r-1}(d, r)$. It is the maximal genus of a curve lying on a $K3$-surface: The intersection matrix of H and any curve C of degree d and genus g on a $K3$-surface is

$$\begin{pmatrix} 2r - 2 & d \\ d & 2g - 2 \end{pmatrix}.$$

So $g \leqslant d^2/2(2r - 2) + 1$ by the Hodge-index theorem, and equality holds if and only if C and H are rationally dependent.

(2) Notice that the proof leaves us with the following alternative: Either $C(6r - 6, 9r - 8, r) > 0$ or a general curve of class $3H$ on a generic $K3$-surface in $\mathbb{P}^r$ has no $(2r - 2)$-secant $(r - 2)$-plane. We think that the second alternative never holds. In any case we have $C(d(n), g(n), r) > 0$ for large enough n.

(3) By [GL2] the Clifford index of a smooth curve C of class nH for $n \geqslant 2$ on a generic $K3$-surface is computed by $|H|$, hence Cliff$(C) = d(n) - 2r$.

For small r the results of this section give a method to check our Conjecture of the introduction.

First we check that Conjecture 5.8 holds up to r. To do this we substitute $g = \pi_{r-1}(d, r)$ in $C(d, g, r)$ and consider its Taylor expansion as a function of d at $d = 6r - 6$. We should find that all its coefficients are strictly positive. Then since

$$C(d, g, r) + C(d - 2, g - 1, r - 1) = C(d, g - 1, r)$$

(as one can check), and $g \leqslant \pi_{r-1}(d, r) \leqslant \pi_{r-2}(d - 2, r - 1)$, we can conclude $C(d, g, r) > 0$ for all $g \leqslant \pi_{r-1}(d, r)$ and $d \geqslant 6r - 6$ by induction on r and descending induction on g.

In a second step we determine all integer solutions of $C(d, g, r) = 0$ in the range $6r - 6 \leqslant d \leqslant d_0(r)$. Hopefuly we simply find no solution.

We checked our Conjecture of the introduction with this method for $r \leqslant 9$. There are only three integer solutions of $C(d, g, r) = 0$ in the range $3 \leqslant r \leqslant 9$ and $6r - 6 \leqslant d \leqslant d_0(r)$:

$$\begin{aligned} r &= 3, \quad d = 13, \quad g = 25 \\ r &= 3, \quad d = 13, \quad g = 66 \\ r &= 4, \quad d = 41, \quad g = 190. \end{aligned}$$

For $d = 13$ the genus $g = 25$ or 66 does not occur for a smooth curve in $\mathbb{P}^3$. However $d = 41$ and $g = 190$ is possible in $\mathbb{P}^4$. For example a smooth plane curve of degree 21 projected from a point in the Veronese embedding of $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ has degree 41 and genus 190. Fortunately we can apply [EH, Theorem 3.15]: $190 = \pi_1(41, 4)$, so any of these curves lies on a surface of degree $\leqslant 4$. Thus $\mathcal{O}(1)$ does not compute the Clifford dimension by Proposition 5.1. In summary:

For $3 \leqslant r \leqslant 9$ *a curve of Clifford dimension r has genus* $4r - 2$, *Clifford index* $2r - 3$, *and satisfies the equivalent conditions of Theorem* 3.6.

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