COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 71, nº 2 (1989), p. 181-227
<http://www.numdam.org/item?id=CM_1989__71_2_181_0>

Compositio Mathematica **71**: 181–227, 1989.

Invariant ideals of polynomial algebras with multiplicity free group action

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Received 16 August 1988; accepted 9 November 1988

Introduction

Let H be an algebraic group over the complex numbers with an irreducible linear representation V as given in one of the following cases.

I. SO_{n+1} with the standard representation on $\mathbb{C}^{n+1}$.
II. SL_{n+1} with the action on $S^2\mathbb{C}^{n+1}$ = symmetric $n+1 \times n+1$-matrices induced from the standard representation on $\mathbb{C}^{n+1}$.
III. $SL_{n+1} \times SL_{n+1}$ with the product representation on $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} =$ $n+1 \times n+1$-matrices.
IV. SL_{2n+2} with the action on $\wedge^2\mathbb{C}^{2n+2}$ = antisymmetric $2n+2 \times 2n+2$ matrices induced from the standard representation on $\mathbb{C}^{2n+2}$.
V. The group of type E_6 with the standard 27-dimensional representation.

The simply connected form G of H acts via the quotient map $G \to H$ also on V. We will study the G-action on the projectivized space $\mathbb{P}(V)$ and the induced action on the polynomial algebra $\mathbb{C}[V]$ given by

$$(gf)(v) = f(g^{-1}v), \qquad g \in G, \qquad f \in \mathbb{C}[V], \quad v \in V.$$

This results in a description of the G-orbits and their closures in $\mathbb{P}(V)$ and a classification of all graded G-invariant ideals in $\mathbb{C}[V]$.

The case I is trivial. The cases II, III, IV have been studied in [*Ab*], [*CEP*] and [*ADF*] respectively. In their method a basis, explicitly chosen case by case, is used in order to describe the G-module structure and invariant ideals of $\mathbb{C}[V]$. A disadvantage of the method is that a great deal of the work has to be done in each case again, while the obtained results are of a similar nature. It will appear that in our approach all five cases can be studied simultaneously. In order to study the invariant ideals we will follow the line of [*CEP*]. Several proofs in this paper do not use the explicit basis and can be used in our method.

In all five cases G has an open orbit in $\mathbb{P}(V)$ and there is an involutive automorphism θ on G and subgroup $K = G^\theta$ such that $G/N_G(K)$ maps isomorphically onto the open orbit via a G-equivariant map. In Section 2 we establish that our list is complete with respect to this property. From this we obtain an injective graded G-equivariant $\mathbb{C}$-algebra homomorphism

$$\phi^*\colon \bigoplus_{d\geqslant 0} \mathbb{C}[V]_d \hookrightarrow \bigoplus_{d\geqslant 0} \mathbb{C}[G/K]\cdot T^d.$$

It is a well known fact that as G-modules

$$\mathbb{C}[G/K] \cong \bigoplus_{\mu\in(G,K)^\wedge} V_\mu,$$

where $(G,K)^\wedge$ is the set of all finite dimensional spherical irreducible representations of G with respect to K (an irreducible representation W of G is said to be spherical for K if $\dim(W^K) = 1$). From this we deduce that $\mathbb{C}[V]$ has as G-module a unique decomposition as sum of homogeneous spherical irreducible representations, which is multiplicity free in each degree. At the end of Section 2 we use results of [CP] in order to describe this decomposition explicitly.

Since the decomposition is multiplicity free in each degree, each graded G-invariant ideal has to be a subsum of the decomposition. Therefore it is useful to have information about the G-span of the product of homogeneous irreducible components in order to describe the graded (prime, primary, radical) ideals and their arithmetic. In Section 3 we prove that a G-submodule spanned by the product of homogeneous G-submodules is already spanned by the product of their K-fixed elements. After that we focus our attention to the algebra of K-fixed elements $\mathbb{C}[V]^K$. Using the morphism ϕ^* above it turns out that we are interested in product formulas for the K-fixed elements Φ_μ in $\mathbb{C}[G/K]$, where

$$\mathbb{C}[G/K]^K = \bigoplus_{\mu\in(G,K)^\wedge} \mathbb{C}\Phi_\mu \qquad \Phi_\mu \in V_\mu^K \text{ non-zero.}$$

More precisely, for $\mu, \nu \in (G,K)^\wedge$ we can write

$$\Phi_\mu\cdot\Phi_\nu = \sum_\lambda d(\mu,\nu,\lambda)\Phi_\lambda$$

and we are interested in the set of λ for which $d(\mu,\nu,\lambda) \neq 0$ since these elements determine the G-span of $\Phi_\mu\cdot\Phi_\nu$.

By general theory there is a torus $A \subseteq G$ – the maximal split torus, see Section 2 – such that the Φ_μ are already completely determined by their restriction to $A/A\cap K \hookrightarrow G/K$. These functions restricted to $A/A\cap K$ are apart from a different normalization precisely the multivariable Jacobi polynomials as used in [H] and some results of that paper are used in the appendix in order to obtain information about the set of λ with $d(\mu, \nu, \lambda) \neq 0$. In the appendix parameters $m_\alpha > 0$ come in, while we need the results only for special values of m_α, see table in Section 2. However we will use explicit expressions for several $d(\mu, \nu, \lambda)$ which are only defined for $m_\alpha > 0$ generic and yield zero divided by zero in our cases. A continuity argument is needed to get the desired results for our parameter values.

The above mentioned decomposition of $\mathbb{C}[V]$ will be indexed by Young diagrams, i.e. by sequences of integers $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n+1} \geqslant 0$, where $n = \dim(A)$ and A the torus mentioned above. If we use the results on products of K-fixed elements in order to describe products of homogeneous G-submodules then we get in terms of Young diagrams for all five cases precisely the same results. This enables us to study the cases simultaneously.

In Section 4 we classify all G-invariant graded (prime, primary, radical) ideals in $\mathbb{C}[V]$, and describe the symbolic powers of prime ideals, and primary decompositions and integral closures of arbitrary ideals. We will work in terms of Young diagrams and it turns out that all problems are combinatorial questions on these diagrams. Since we need the several combinatorial results on Young diagrams on many places in Sections 2, 3, and 4 we have gathered most results in Section 1.

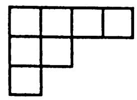
In the last Section we use our results in order to describe $\mathbb{P}(V)$ as G-variety. As full set of closed G-stable subsets we obtain a sequence

$$X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} = \mathbb{P}(V).$$

Each X_i is irreducible and we describe a set of generators of the prime ideal that defines X_i. We also show that X_i can be obtained from X_1 as union of all $i-1$ dimensional projective planes through i points on X_1. Consequently the rank 2 cases of II, III, IV and V (I is always of rank 1) yields precisely the standard Severi-varieties, see [LV].

Section 1. Combinatorics of Young diagrams

A Young diagram σ is a sequence $(\sigma_1, \sigma_2, \sigma_3, \ldots)$ of non-negative integers with $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \ldots$ and $\sigma_i = 0$ for all i sufficiently large. If $\sigma_{n+1} = 0$ we also write $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and D_n denotes the set of all these Young diagrams. A Young diagram in D_n can be represented in the plane by a set of (at most) n rows of boxes; the i-th row consisting of σ_i boxes. For instance if $\sigma = (4, 2, 1) \in D_3$ the picture becomes:



In order to understand combinatorics of Young diagrams it is helpful to keep this representation in mind. By transposing the names 'row' and 'column' we obtain a duality on the pictures of Young diagrams. So there is a corresponding duality on Young diagrams; given a Young diagram $\sigma = (\sigma_1, \ldots, \sigma_n)$, the dual $\sigma^\vee = (\sigma_1^\vee, \sigma_2^\vee, \sigma_3^\vee, \ldots, \sigma_\ell^\vee)$, where $\ell = \sigma_1$, is given by $\sigma_i^\vee = \max\{j \geqslant 1 \mid \sigma_j \geqslant i\}$.

We consider D_n as a subset of $\mathbb{Z}^n$ and provide $\mathbb{Z}^n$, thus via restriction D_n, with some structure. With a sequence $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we associate the support

$$\operatorname{supp}(a) = \{i \mid a_i \neq 0 \qquad i = 1, \ldots, n\}$$

and integers

$$\gamma_k(a) = \sum_{i=k}^{n} a_i \qquad k = 1, \ldots, n.$$

In particular the degree of a is

$$|a| = \gamma_1(a) = \sum_{i=1}^{n} a_i.$$

Furthermore we define three partial orders $\subseteq, \leqslant$ and $\leqslant_\ell$ on $\mathbb{Z}^n$. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ then

$$\begin{aligned}
&a \subseteq b \quad \text{if and only if} \quad a_i \leqslant b_i \quad \text{for all } i = 1, \ldots, n,\\
&a \leqslant b \quad \text{if and only if} \quad \gamma_i(a) \leqslant \gamma_i(b) \quad \text{for all } i = 1, \ldots, n,\\
&a <_\ell b \quad \text{if and only if} \quad a_j < b_j \quad \text{and} \quad a_{j+1} = b_{j+1}, \ldots, a_n = b_n\\
&\quad \text{for some } j \in \{1, \ldots, n\}.
\end{aligned}$$

Clearly $\leqslant$ extends $\subseteq$ and $\leqslant_\ell$ extends $\leqslant$ as partial order. The partial order $\leqslant_\ell$ is even a total order and is also called the lexicographic order. Note that on D_n the partial order $\subseteq$ means inclusion of the corresponding pictures.

Finally we define the set of strips

$$E_n = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid e_i = 0 \text{ or } e_i = 1 \quad \text{for all } i = 1, \ldots, n\}$$

and the subsets of m-strips

$$E_{n,m} = \{e \in E_n \,\|\, e| = m\} \quad \text{for } m = 1,\dots,n.$$

Using the usual addition on $\mathbb{Z}^n$, the strips will be the building blocks of the Young diagrams.

Let $\rho \in D_n$ be a Young diagram. We say that ρ is stratified by the sequence of strips $e^1, e^2, \dots, e^\ell$ if $\Sigma_{i=1}^{j} e^i$ is a Young diagram for all $j = 1,\dots,\ell$ and $\rho = \Sigma_{i=1}^{\ell} e^i$. Given a Young diagram $\sigma \in D_n$, we say that a sequence of strips $e^1,\dots,e^\ell$ is related to σ if $\ell = \sigma_1$ and $|e^{\pi(i)}| = \sigma_i^\vee$ for all $i = 1,\dots,\ell$ for some permutation π on $\{1,\dots,\ell\}$. If ρ is stratified by a sequence of strips related to σ we say that ρ is stratified by σ.

A stratification of a Young diagram ρ by a sequence of strips $e^1,\dots,e^\ell$ can be represented in the plane as follows: We represent ρ as before and for each $1 \leqslant i \leqslant \ell$ and each $j \in \operatorname{supp}(e^i)$ we put the value i in one of the boxes of the jth row of the picture of ρ, such that the numbers in each row are strictly increasing and the numbers in each column are non-decreasing. It turns out that the set of boxes with numbers $\leqslant i$ is precisely the picture corresponding to the Young diagram $\Sigma_{j=1}^{i} e^j$. For example, $(4,2,1)$ is stratified by the sequence $e^1 = (1,1,0)$, $e^2 = (1,0,1)$, $e^3 = (1,1,0)$, $e^4 = (1,0,0)$. In a picture

$$\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
1 & 3 \\
\cline{1-2}
2 \\
\cline{1-1}
\end{array}$$

Also $(4,2,1)$ is stratified by $(2,2,2,1)^\vee = (4,3)$.

Note that each Young diagram σ can be stratified by σ itself in a standard way; since

$$\sigma = \sigma^{\vee\vee} = (\sigma_1^\vee,\dots,\sigma_\ell^\vee)^\vee = (\sigma_1^\vee)^\vee + \cdots + (\sigma_\ell^\vee)^\vee$$

and

$$(\sigma_1^\vee)^\vee + \cdots + (\sigma_j^\vee)^\vee$$

is a Young diagram for all $1 \leqslant j \leqslant \ell = \sigma_1$ the sequence $e^i = (\sigma_i^\vee)^\vee$ satisfies. In the picture of σ it means that we put the number i in all boxes of the i-th column.

We are interested in the set of all $\tau \in D_n$ stratified by some given $\sigma \in D_n$. By definition the degrees of a sequence of strips related to σ are in one to one correspondence with the coordinates of $\sigma^\vee$ via some permutation π. We want to show that it is no restriction to fix the choice of π.

LEMMA 1.1. *Let $\sigma \in D_n$ and $e, f \in E_n$ such that $\sigma + e, \sigma + e + f \in D_n$, and let d be an integer with* $|\mathrm{supp}(e) \cap \mathrm{supp}(f| \leqslant d \leqslant |\mathrm{supp}(e) \cup \mathrm{supp}(f)|$. *Define $\tilde{e} \in E_n$ by $\tilde{e}$ is the minimal element in E_n with respect to the partial order $\leqslant$ such that $|\tilde{e}| = d$ and* $\mathrm{supp}(\tilde{e}) \supseteq \mathrm{supp}(e) \cap \mathrm{supp}(f)$, *and put $\tilde{f} = e + f - \tilde{e} \in E_n$.*
Then $\sigma + \tilde{e}, \sigma + \tilde{e} + \tilde{f} = \sigma + e + f \in D_n$.

Proof. In order to show that $\sigma + \tilde{e}$ is a Young diagram, we have to verify

$$\sigma_i + \tilde{e}_i \geqslant \sigma_{i+1} + \tilde{e}_{i+1} \text{ for } i = 1,\ldots,n-1.$$

Only if $\tilde{e}_i = 0$ and $\tilde{e}_{i+1} = 1$ this needs some verification. In that case it follows from the minimality of $\tilde{e}$ that

$$i \notin \mathrm{supp}(e) \cap \mathrm{supp}(f)$$

but

$$i + 1 \in \mathrm{supp}(e) \cap \mathrm{supp}(f).$$

Hence

$$\sigma_i + 1 \geqslant \sigma_i + e_i + f_i \geqslant \sigma_{i+1} + e_{i+1} + f_{i+1} = \sigma_{i+1} + 2,$$

thus indeed $\sigma_i \geqslant \sigma_{i+1} + 1$.

PROPOSITION 1.2. *Let $\sigma \in D_n$ and π a permutation on $\{1,\ldots,\ell = \sigma_1\}$. The set of Young diagrams τ stratified by sequences of strips $e^1,\ldots,e^\ell$ with $|e^{\pi(i)}| = \sigma_i^\vee$ for all $i = 1,\ldots,\ell$ does not depend on the choice of π.*

Proof. Let τ be stratified by a sequence $e^1,\ldots,e^\ell$. Fix an $1 \leqslant i < \ell$ and put

$$\rho = \sum_{j=1}^{i-1} e^j, \quad e = e^i \quad \text{and} \quad f = e^{i+1}.$$

Let $d = |f|$. By lemma 1.1. τ is also stratified by

$$e^1,\ldots,e^{i-1},\tilde{e},\tilde{f},e^{i+2},\ldots,e^\ell,$$

where

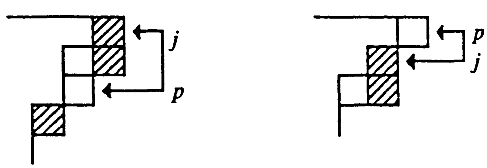
$$|\tilde{e}| = |f| = |e^{i+1}|, |\tilde{f}| = |e^i|.$$

Thus the choice π can be replaced by $\pi \circ (i \quad i+1)$. Since the transpositions $(i \quad i+1)$ generates the permutation group on $\{1,\ldots,\ell\}$ the proposition follows. □

Consequently it is no real restriction if we work with sequences of non decreasing degree in order to describe all τ stratified by some σ. In that case the following lemma says that we can even add an assumption on the last strip in the sequence.

LEMMA 1.3. *Let $\tau \in D_n$ be stratified by a sequence of strips $e^1, \ldots, e^\ell$ of non decreasing degree. Suppose $\tau_p > \tau_{p+1}$ for some $1 \leqslant p \leqslant n$. Then is τ also stratified by a sequence of strips $\tilde{e}^1, \ldots, \tilde{e}^\ell$ with $|\tilde{e}^i| = |e^i|$ for all $i = 1, \ldots, \ell$ and moreover $\tilde{e}^\ell_p = 1$.*

Proof. By induction on ℓ. For $\ell = 1$ is the assertion trivial. Now assume the assertion to be proved up to $\ell - 1 \geqslant 1$. If $e^\ell_p = 1$ the assertion holds for the sequence $e^1, \ldots, e^\ell$ itself, so assume $e^\ell_p = 0$. Write $\rho = \Sigma_{i=1}^{\ell-1} e^i$; this Young diagram is stratified by the sequence $e^1, \ldots, e^{\ell-1}$. Now $\rho_p = \tau_p > \tau_{p+1} \geqslant \rho_{p+1}$, thus by the induction hypothesis we can replace $e^1, \ldots, e^{\ell-1}$ by a sequence as in the lemma. We therefore may assume $e^{\ell-1}_p = 1$. Since $|e^\ell| \geqslant |e^{\ell-1}|$ and $e^\ell_p = 0$, there is a minimal j with $e^\ell_j = 1$ and $e^{\ell-1}_j = 0$, and thus $\rho_j < \rho_{j-1}$. Define $\delta = (\delta_1, \delta_2, \ldots)$ by $\delta_j = -1, \delta_p = 1$ and $\delta_i = 0$ for $i \neq j, p$. We claim that τ is stratified by the sequence $e^1, \ldots, e^{\ell-2}, e^{\ell-1} - \delta, e^\ell + \delta$ and satisfies the desired properties. Picturing the stratification as mentioned before we in fact interchange two boxes of the last two strips $e^{\ell-1}$ and e^ℓ:



Namely the last box in the pth row that belongs to the $(\ell - 1)$-th strip (the non-shaded boxes) is interchanged with the last box in the jth row which belongs to the ℓth strip (the shaded boxes).

By construction it is only necessary to verify whether $\tilde{\rho} = e^1 + \cdots + e^{\ell-2} + e^{\ell-1} - \delta = \rho - \delta$ is a Young diagram. Since $\tilde{\rho}_p = \rho_p - 1, \tilde{\rho}_j = \rho_j + 1$ and $\tilde{\rho}_i = \rho_i$ for $i \neq p, j$ it is sufficient to check $\tilde{\rho}_p \geqslant \tilde{\rho}_{p+1}$ and $\tilde{\rho}_j \leqslant \tilde{\rho}_{j-1}$. Using the (in)equalities above, we get $\tilde{\rho}_p = \rho_p - 1 = \tau_p - 1 \geqslant \tau_{p+1} \geqslant \tilde{\rho}_{p+1}$. If $j = p + 1$ we are ready, while for $j \neq p + 1$ $\tilde{\rho}_j = \rho_j + 1 \leqslant \rho_{j-1} = \tilde{\rho}_{j-1}$. $\square$

We need Proposition 1.2. and Lemma 1.3 in order to prove our main combinatorial result:

PROPOSITION 1.4. *Let σ be a Young diagram.*

(a) *If ρ is a Young diagram with $\rho \geqslant \sigma$ then there exists a Young diagram τ such that $\rho \supseteq \tau, \tau \geqslant \sigma$ and $|\tau| = |\sigma|$.*
(b) *If τ is a Young diagram with $|\tau| = |\sigma|$ then $\tau \geqslant \sigma$ if and only if τ can be stratified by σ.*

Proof. (a) Assume $|\rho| > |\sigma|$. Let j be the minimal with $\rho_j > \rho_{j+1}$ and define $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2, \ldots)$ by $\tilde{\rho}_j = \rho_j - 1$ and $\tilde{\rho}_i = \rho_i$ for $i \neq j$. Clearly $\tilde{\rho} \subseteq \rho$, and we claim that also $\tilde{\rho} \geqslant \sigma$. We prove this by contradiction, so suppose that not $\tilde{\rho} \geqslant \sigma$. Then there is a maximal p such that $\gamma_p(\tilde{\rho}) < \gamma_p(\sigma)$. Of course $p \leqslant j$ and $\tilde{\rho}_p < \sigma_p$, thus

$$\tilde{\rho}_1 = \tilde{\rho}_2 = \cdots = \tilde{\rho}_{p-1} \leqslant \tilde{\rho}_p + 1 \leqslant \sigma_p \leqslant \sigma_{p-1} \leqslant \cdots \leqslant \sigma_1.$$

But then is

$$|\rho| - 1 = |\tilde{\rho}| = \gamma_1(\tilde{\rho}) = \sum_{i=1}^{p-1} \tilde{\rho}_i + \gamma_p(\tilde{\rho}) < \sum_{i=1}^{p-1} \sigma_i + \gamma_p(\sigma) = \gamma_1(\sigma) = |\sigma|,$$

contradicting $|\rho| > |\sigma|$. Now we have found a $\tilde{\rho}$ satisfying $\rho \supseteq \tilde{\rho}$, $|\tilde{\rho}| = |\rho| - 1$ and $\tilde{\rho} \geqslant \sigma$, so repeating the constructing sufficiently many time yields the desired τ. (b) First let τ be stratified by a sequence of strips $e^1, \ldots, e^\ell$ related to σ. Write $\delta_i = (i)^\vee = (1, \ldots, 1)$, i times 1, for $i = 1, 2, \ldots$, then $\delta_i \leqslant \delta$ for all $\delta \in E_{n,i}$. Now by the definition of stratification we have

$$\tau = \sum_{i=1}^{n} e^i \geqslant \sum_{i=1}^{n} \delta_{|e^i|} = \sigma$$

We prove the converse by induction on $\ell = \sigma_1$. For $\ell = 1$ $\tau = \sigma$ is a strip. Now suppose the assertion to be proved up to $\ell - 1 \geqslant 1$. We use transfinite induction on the τ with $|\tau| = |\sigma|$ and $\tau \geqslant \sigma$ (with respect to the order $\leqslant$). If $\tau = \sigma$ the earlier on mentioned standard stratification satisfy. Now let $\tau > \rho$ adjacent, $|\tau| = |\rho| = |\sigma|$, $\rho \geqslant \sigma$ and suppose that ρ is stratified by σ. Fix j maximal with $\gamma_j(\rho) < \gamma_j(\tau)$ and after that a $1 \leqslant i < j$ maximal with $\gamma_i(\rho) = \gamma_i(\tau)$. Then $\rho_i > \tau_i \geqslant \tau_j > \rho_j$, thus $\rho_i \geqslant \rho_j + 2$. Therefore we can find $i \leqslant p < q \leqslant j$ such that

$$\rho_p \geqslant \rho_q + 2, \rho_p > \rho_{p+1} \quad \text{and} \quad \rho_q < \rho_{q-1}.$$

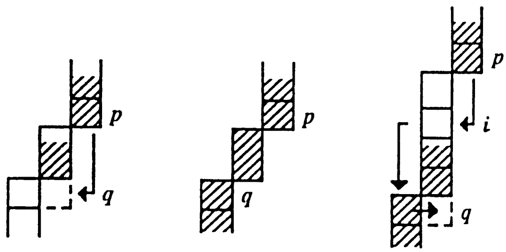
Hence, if we define

$$\delta = (\delta_1, \delta_2, \ldots)$$

with

$$\delta_p = -1, \delta_q = 1 \quad \text{and} \quad \delta_i = 0 \quad \text{for } i \neq p, q,$$

then $\rho + \delta$ is again a Young diagram and $|\rho + \delta| = |\sigma|$. By construction also $\rho < \rho + \delta \leqslant \tau$, thus $\tau = \rho + \delta$ by the adjacency. By assumption ρ can be stratified by σ. Then by Proposition 1.2. and Lemma 1.3. we may assume that ρ can be stratified by a sequence of strips $e^1, \ldots, e^\ell$ related to σ of non-decreasing degree and $e^\ell_p = 1$. With help of this stratification we will find a desired stratification for τ. We distinguish three cases corresponding to the following three pictures:



In the first case we assume $e^\ell_q = 0$. Then $e^\ell + \delta$ is a strip with $|e^\ell + \delta| = |e^\ell|$ so $e^1, \ldots, e^{\ell-1}, e^\ell + \delta$ is a desired stratification of τ.

In the other cases we suppose $e^\ell_q = 1$.

If for all $p \leqslant i \leqslant q$ holds $e^\ell_i = 1$, then $\eta = \Sigma_{i=1}^{\ell-1} e^i$ and $\eta + \delta$ are Young diagrams, $\eta < \eta + \delta$ and η is stratified by $e^1, \ldots, e^{\ell-1}$. Thus by the induction hypothesis $\eta + \delta$ can be stratified by a sequence of strips $\tilde{e}^1, \ldots, \tilde{e}^{\ell-1}$ with $|\tilde{e}^i| = |e^i|$, $i = 1, \ldots, \ell - 1$. Since $\tau = \rho + \delta = \eta + \delta + e^\ell$ it follows that $\tilde{e}^1, \ldots, \tilde{e}^{\ell-1}, e^\ell$ is a desired stratification of τ.

The third case that remains is $e^\ell_p = e^\ell_q = 1$ and $e^\ell_i = 0$ for some $p < i < q$, suppose i to be maximal with this property. Write $\eta = \Sigma_{j=1}^{\ell-1} e^j$, η is stratified by $e^1, \ldots, e^{\ell-1}$. We have $\eta_i = \tau_i$ and

$$\rho_q + 2 = \rho_p > \rho_{p+1} \geqslant \rho_i \geqslant \rho_{q-1} > \rho_q = \eta + 1,$$

thus $\eta_i = \eta_q + 2$, furthermore $\eta_i = \tau_i > \tau_{i+1} - 1 = \eta_{i+1}$ and $\eta_q < \tau_{q-1} - 1 \leqslant \eta_{q-1}$. From these inequalities follows that if we define $\delta^1 = (\delta^1_1, \delta^1_2, \ldots)$ with $\delta^1_i = -1$, $\delta^1_q = 1$ and $\delta^1_j = 0$ for $j \neq i, q$ then $\eta + \delta^1$ is a Young diagram with $|\eta + \delta^1| = |\eta|$ and $\eta + \delta^1 > \eta$. By the induction hypothesis follows that $\eta + \delta^1$ can be stratified by a sequence of strips $\tilde{e}^1, \ldots, \tilde{e}^{\ell-1}$ with $|\tilde{e}^i| = |e^i|$ for $i = 1, \ldots, \ell - 1$. Now define $\delta^2 = (\delta^2_1, \delta^2_2, \ldots)$ by $\delta^2_p = -1, \delta^2_i = 1$ and $\delta^2_j = 0$ for $j \neq p, i$, so $\delta^1 + \delta^2 = \delta$. By construction is $e^\ell + \delta^2$ a strip and $|e^\ell + \delta^2| = |e^\ell|$. Since

$$\tau = \sigma + \delta = \eta + e^\ell + \delta^1 + \delta^2 = (\eta + \delta^1) + (e^\ell + \delta^2)$$

it follows that $\tilde{e}^1, \ldots, \tilde{e}^{\ell-1}, e^\ell + \delta^2$ is a desired stratification for τ. □

From part (b) of the proposition follows that given a sequence of strips $e^1, \ldots, e^\ell$ related to σ such that $\rho + \Sigma_{i=1}^{j} e^i$ is a Young diagram for all $j = 1, \ldots, \ell$ and some Young diagram ρ, then $\tau = \rho + \Sigma_{i=1}^{\ell} e^i \geqslant \rho + \sigma$. Part (b) says that in the special case where ρ is the zero diagram the converse holds too. Unfortunately in general if $\tau \supseteq \rho$, $|\tau| = |\rho + \sigma|$ and $\tau \geqslant \rho + \sigma$ there needs not exist such a stratification. A counter example is already given by $\tau = (2,2,2)$, $\rho = (2,1)$ and $\sigma = (1,1,1)$. However in the following special case, where it is essential that we work inside a set of Young diagrams D_n with n fixed, there is:

PROPOSITION 1.5. *Fix $n > 0$ and let $\sigma \in D_n$, $\ell = \sigma_1$. There exists a $m \geqslant 0$ such that for all $\rho \in D_n$ with $|\rho| = |(m+1)\sigma|$ holds: $\rho \geqslant (m+1)\sigma$ if and only if there is a sequence of strips $e^1, \ldots, e^{m\ell}$ related to $m\sigma$ such that $\sigma + \Sigma_{i=1}^{j} e^i$ is a Young diagram for all $j = 1, \ldots, m \cdot \ell$ and $\rho = \sigma + \Sigma_{i=1}^{m\ell} e^i$.*

Proof. First suppose we have already a $m \geqslant 0$ that satisfies, then $m+1$, and hence all $m' > m$, satisfies. Namely, assume $\tilde{\rho} \geqslant (m+2)\sigma$ for some $\tilde{\rho} \in D_n$. From Proposition 1.4 follows that $\tilde{\rho}$ can be stratified by a sequence of strips

$$\tilde{e}^1, \ldots, \tilde{e}^{(m+2)\ell}$$

related to $(m+2)\sigma$ and by Proposition 1.2 we may assume that

$$\tilde{e}^1, \ldots, \tilde{e}^{(m+1)\ell}$$

is related to $(m+1)\sigma$. Put

$$\rho = \sum_{i=1}^{(m+1)\ell} \tilde{e}^i,$$

by Proposition 1.4 again we have $\rho \geqslant (m+1)\sigma$. By assumption we may apply the proposition, so there is a sequence $e^1, \ldots, e^{m \cdot \ell}$ as stated in the proposition. Now it is obvious that

$$e^1, \ldots, e^{m \cdot \ell}, \tilde{e}^{(m+1)\ell + 1}, \ldots, \tilde{e}^{(m+2)\ell}$$

is a desired sequence for $\tilde{\rho}$.

We now prove by induction on $\ell = \sigma_1$ that the proposition holds if we take $m = n^2$.

For $\ell = 0$ there is nothing to prove. If $\ell = 1$, then $\sigma = (j)^\vee = (1, \ldots, 1)$, j times 1, for some $1 \leqslant j \leqslant n$. If we take in this special case $m = 0$ then the only ρ satisfying the conditions is $\rho = \sigma$, and the assertions become trivial. Thus for $\ell = 1$ the proposition holds for all $m \geqslant 0$.

Next suppose $\ell > 1$ and the proposition to be proved up to $\ell - 1$. Let $\rho \in D_n$ satisfy the conditions. By Propositions 1.4 and 1.2 ρ can be stratified by a sequence of strips $e^1, \ldots, e^{(m+1)\ell}$ related to $(m+1)\sigma$ and of non-decreasing degree.

We claim that we can choose this sequence such that in addition we may assume $e^i = (1,\dots,1)$, j times 1, where $j = \sigma_1^\vee$ is the maximal degree in the sequence, for some $(m+1)(\ell-1) < i \leqslant (m+1)\ell$.

Before we prove this claim, we finish the proof of the proposition. Clearly the Young diagram

$$\tilde{\rho} = \sum_{j=1}^{(m+1)(\ell-1)} e^j$$

is stratified by the sequence of strips

$$e^1,\dots,e^{(m+1)(\ell-1)}$$

related to $\tilde{\sigma} = (m+1)(\sigma - e^i)$. Since $\tilde{\sigma}_1 = \ell - 1$ there is by the induction hypothesis a sequence of strips

$$\tilde{e}^1,\dots,\tilde{e}^{m(\ell-1)}$$

related to $m\tilde{\sigma}$ such that

$$\tilde{\rho} = \tilde{\sigma} + \sum_{j=1}^{m(\ell-1)} \tilde{e}^j$$

and each initial sum

$$\tilde{\sigma} + \sum_{j=1}^{k} \tilde{e}^j \qquad k = 1,\dots,m(\ell-1)$$

of the sequence

$$\tilde{\sigma}, \tilde{e}^1, \tilde{e}^2,\dots,\tilde{e}^{m(\ell-1)}$$

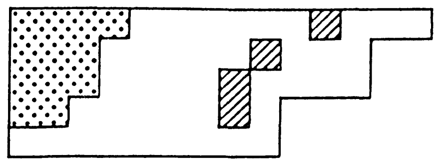
is a Young diagram. It follows that

$$\rho = \tilde{\rho} + \sum_{j=(m+1)\ell-m}^{(m+1)\ell} e^j$$

and each initial sum of the sequence

$$\tilde{\sigma}, \tilde{e}^1,\dots,\tilde{e}^{m(\ell-1)}, e^{(m+1)\ell-m},\dots,e^{(m+1)\ell}$$

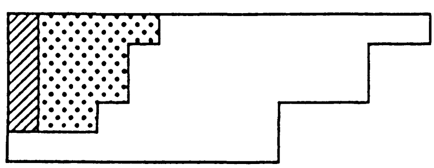
is a Young diagram. Now e^i can be found in the last part of the sequence and is itself a Young diagram. Thus in a picture the stratification looks like



where $\tilde{\sigma}$ and e^i are the dotted and shaded area respectively and the blank part corresponds to the remaining strips.
Because the sum of Young diagrams is again a Young diagram it follows that each initial sum of the sequence

$$\sigma = \tilde{\sigma} + e^i, \tilde{e}^1, \ldots, \tilde{e}^{m(\ell-1)}, e^{(m+1)\ell - m}, \ldots, \hat{e}^i, \ldots, e^{(m+1)\ell}$$

is a Young diagram. In the picture this can be interpreted as shifting the shaded strip into the first position:



Now σ corresponds to the union of the dotted and shaded area and the remaining set of strips yield the blanks area.

Since the total sum equals ρ, we have found a desired sequence.

It remains to prove the claim.

In the first instance we only know that the sequence has non-decreasing degree. We define

$$a_i = (m+1)\ell - n^2 + i.n \quad \text{for } i = 0, 1, \ldots, n.$$

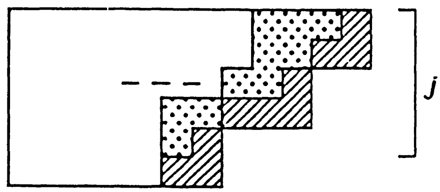
Thus

$$(m+1)(\ell - 1) < (m+1)\ell - m = a_0 < a_1 < \cdots < a_n = (m+1).\ell.$$

We first show that the sequence $e^1, \ldots, e^{(m+1).\ell}$ can be chosen such that in addition

$$\{1, \ldots, j\} \subseteq \bigcup_{k \in [a_{i-1}+1, a_i]} \mathrm{supp}(e^k) \quad \text{for all } i = 1, \ldots, n. \qquad (*)$$

Here $[a, b]$, $a, b \in \mathbb{Z}$, denotes $\{c \in \mathbb{Z} | a \leqslant c \leqslant b\}$ and as before $j = \sigma_1^\vee$. Thus the picture of such a stratification looks like



where the shaded area corresponds to $e^{a_{n-1}+1} + \cdots + e^{a_n}$, the dotted area to $e^{a_{n-2}+1} + \cdots + e^{a_{n-1}}$, etc. Write $\rho^i = \Sigma_{k=1}^i e^k$; each time we change our choice of the sequence $e^1, \ldots, e^{(m+1)\ell}$ below we suppose that the definition of the ρ^i change with it. We need the following fact:

Let $1 \leqslant s < t \leqslant (m+1)\ell$ and suppose

$$p \notin, p+1 \in \bigcup_{k \in [s+1, t]} \operatorname{supp}(e^k)$$

for some $1 \leqslant p < n$, then $\rho_p^s > \rho_{p+1}^s$ thus by Lemma 1.3 we may assume $e_p^s = 1$ for an appropriate chosen sequence $e^1, \ldots, e^s$.

We now prove $(*)$ for $i = n$. Since e^{a_i} has degree j, there is a $j \leqslant q \leqslant n$ with $e_q^{a_i} = 1$. Now fix $t = a_i$ and let s run through the row $a_i - 1, a_i - 2, \ldots$ as long as there is a maximal $p < q$ (now p depends on s) such that

$$p \notin \bigcup_{k \in [s+1, a_i]} \operatorname{supp}(e^k)$$

and replace the sequence $e^1, \ldots, e^s$ by an other choice such that $e_p^s = 1$. Clearly s stays $\geqslant a_i - (n-1)$, thus after the algorithm we get the desired assertion for $i = n$. It is obvious that we can repeat this algorithm for $i = n-1, n-2, \ldots, 1$ (in this order!) such that we get ultimately $(*)$.

We now assume that we have chosen a sequence $e^1, \ldots, e^{(m+1)\ell}$ that in addition satisfies $(*)$. We use Lemma 1.1 in order to alter this sequence into a sequence that satisfies the claim.

Let i run through the sequence $(m+1)\ell - 2, (m+1)\ell - 3, \ldots, (m+1)\ell - n^2$ (in this order), and replace e^{i+1}, e^{i+2} by $\tilde{e}^{i+1}, \tilde{e}^{i+2}$ in accordance with Lemma 1.1., where $\sigma = \rho^i$, $e = e^{i+1}$, $f = e^{i+2}$ and $d = j$. After carrying out the step for $i = a_p$, $p = n-1, n-2, \ldots, 0$ it follows from $(*)$ that

$$\{1, 2, \ldots, \text{minimum } (j, n-p)\} \subseteq \operatorname{supp}(e^{a_p+1}).$$

Hence $\{1,\ldots,j\} \subseteq \mathrm{supp}(e^i)$ for some $i > a_0 = (m+1)\ell - m$, and thus $e^i = (1,\ldots,1)$, j times 1, because $|e^i| = j$. $\square$

Section 2. Structure and representation theory

Let G be a semisimple simply connected algebraic group over the complex numbers with an involutive automorphism θ and fixed point group $K = G^\theta$. By definition the quotient space G/K is a semisimple symmetric space. Among the tori A with $\theta(a) = a^{-1}$ for all $a \in A$ we fix a torus A of maximal dimension. This torus is called a maximal split torus and its dimension the rank of the symmetric space G/K. There always exists a maximal torus T such that $A \subseteq T$ and T is θ-stable. We fix one such T. Let $\mathfrak{g}$ be the Lie algebra of G, $A^\wedge$ the character set of A and $\lambda \in A^\wedge$. Put

$$\begin{aligned} \mathfrak{g}_\lambda &= \{X \in \mathfrak{g} \mid Ad(a)X = a^\lambda X \quad \forall a \in A\}, \\ m_\lambda &= \dim_{\mathbb{C}}(\mathfrak{g}_\lambda), \\ R(\mathfrak{g}, \mathfrak{a}) &= \{\alpha \in A^\wedge \mid \alpha \neq 0 \quad \text{and} \quad m_\alpha > 0\}. \end{aligned}$$

Furthermore let $N_K(A)$ and $C_K(A)$ be the normalizer and centralizer of A in K respectively. Put

$$W = N_K(A)/C_K(A).$$

The set $R(\mathfrak{g}, \mathfrak{a})$ is named the restricted rootsystem. It is a possibly non-reduced rootsystem with Weyl group W. The restricted rootsystem is called the type of the symmetric space G/K. If it is irreducible then we say G/K is irreducible. Let E be the real vectorspace spanned by $R(\mathfrak{g}, \mathfrak{a})$. We define

$$\begin{aligned} P &= \{\lambda \in E \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in R(\mathfrak{g}, \mathfrak{a})\}, \\ P_+ &= \{\lambda \in P \mid (\lambda, \alpha^\vee) \geqslant 0 \quad \forall \alpha \in R_+(\mathfrak{g}, \mathfrak{a})\}, \end{aligned}$$

where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ and $R_+(\mathfrak{g}, \mathfrak{a})$ a set of positive roots in $R(\mathfrak{g}, \mathfrak{a})$ such that the induced order is compatible with the order on all weights. Here $(\cdot,\cdot)$ is a W-invariant inner product on E. Since by assumption G is simply connected the character lattice $A^\wedge$ of A equals P. For any finite dimensional irreducible representation V of G holds $\dim_{\mathbb{C}} V^K \leqslant 1$. If this dimension equals 1 then V is called a spherical representation, and each non-zero K-fixed vector a spherical vector. $(G, K)^\wedge$ denotes the set of all finite dimensional irreducible representations. Helgason's theorem [Hel 2, chap. V] says that there is an one to one correspondence

$$(G, K)^\wedge \to 2 \cdot P_+.$$

Given $\lambda \in P_+$ then the character 2λ on A extends in a unique way to a character $2\tilde{\lambda}$ on T by demanding $2\tilde{\lambda}(t) = 1$ for all $t \in T$ with $\theta(t) = t$. We thus obtain a spherical irrreducible representation V_λ of G with the highest weight $2\tilde{\lambda}$, and this gives the one to one correspondence mentioned above. Since the $\mathbb{C}$-algebra $\mathbb{C}[G/K]$ is as G-module isomorphic to the direct sum of all spherical representations in $(G, K)^\wedge$ we get

$$\mathbb{C}[G/K] \cong \bigoplus_{\lambda \in P_+} V_\lambda.$$

Now take a $\lambda \in P_+$, $\lambda \neq 0$, and a spherical vector $v \in V_\lambda$. In the projectivized space $\mathbb{P}(V_\lambda)$ holds $\mathrm{stab}_G(\bar{v}) = N_G(K)$, see [CP, (1.7)], thus the G-orbit of $\bar{v}$ in $\mathbb{P}(V_\lambda)$ is isomorphic to $G/N_G(K)$. We are interested in the cases where the closure of the orbit $G\bar{v}$ is the whole projective space $\mathbb{P}(V_\lambda)$. In that case the map

$$\phi: G/K \times \mathbb{C}^* \to V_\lambda$$

given by

$$\phi(gK, t) = tgv$$

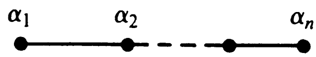
has an open dense image in V_λ. This induces an injective graded G-module homomorphism:

$$\phi^*: \mathbb{C}[V_\lambda] \cong \bigoplus_{d \geqslant 0} \mathbb{C}[V_\lambda]_d \hookrightarrow \bigoplus_{d \in \mathbb{Z}} \mathbb{C}[G/K] T^d \cong \mathbb{C}[G/K \times \mathbb{C}^*].$$

Consequently for any $d \geqslant 0$ $\mathbb{C}[V_\lambda]_d$ is a multiplicity free G-module or equivalently $\mathbb{C}[V_\lambda]$ is a multiplicity free $G \times \mathbb{C}^*$-module. A complete list for irreducible G/K with $\overline{G\bar{v}} = \mathbb{P}(V_\lambda)$ is given in the following table:

	G	K	V_λ	$\dim V_\lambda$	rank	m_α
I	SO_{n+1}	O_n	$\mathbb{C}^{n+1}$	$n+1$	1	$n-1$
II	SL_{n+1}	SO_{n+1}	$S^2\mathbb{C}^{n+1}$	$\binom{n+2}{2}$	n	1
III	SL_{n+1}^2	SL_{n+1} diag	$\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$	$(n+1)^2$	n	2
IV	SL_{2n+2}	Sp_{2n+2}	$\wedge^2\mathbb{C}^{2n+2}$	$(n+1)(2n+1)$	n	4
V	E_6	F_4	$(\mathbb{C}^3 \otimes \mathbb{C}^3)^3$	27	2	8

Strictly speaking we have, in accordance with the assumptions, to replace the pairs G, K by their simply connected forms. Note that all cases are of type A_n, where n is the rank. Let the Dynkin diagram be



where $\alpha_1, \ldots, \alpha_n$ are the simple roots of $R(\mathfrak{g}, \mathfrak{a}) = A_n$. The fundamental weights $\lambda_1, \ldots, \lambda_n$ are the duals of the coroots $\alpha_1^\vee, \ldots, \alpha_n^\vee$ and after eventually transposing the Dynkin diagram we may assume that $\lambda = \lambda_n$.

The table can be obtained as follows: In [Ka, Theorem 3] Kac gives a complete list of multiplicity-free irreducible linear actions of connected reductive algebraic groups, i.e. irreducible linear representations such that $\mathbb{C}[V]$ decomposes multiplicity free. By the above mentioned facts our cases must be contained in this list. A case by case verification using the classification of irreducible symmetric spaces in [Hel 3] yields the table. This table is also obtained by Heckman [personal comm.], who determined all cases where the closure $\overline{Gv}$ in $\mathbb{P}(V)$ has the Betti-numbers of a projective space.

For the rest of this paper we will restrict ourselves to the cases of the table.

We want to describe $\mathbb{C}[V_\lambda]$ as G-module. We already know that ϕ^* embeds for any $d \geqslant 0$ the homogeneous component $\mathbb{C}[V_\lambda]_d$ in $\mathbb{C}[G/K] \cong \oplus_{\lambda \in P_+} V_\lambda$. Work of de Conicini and Procesi gives an explicit decomposition. In fact we also be able to give our own proof, see remark to Corollary 3.9. As usual we provide P and $P_+ = \{n_1\lambda_1 + \cdots + n_n\lambda_n | n_i \geqslant 0\}$ with the partial order

$$\mu \leqslant v \text{ if and only if } v - \mu = \sum_{i=1}^{n} n_i\alpha_i \quad n_i \geqslant 0, n_i \in \mathbb{Z}.$$

THEOREM 2.1. $\mathbb{C}[V_\lambda]_d \cong \oplus_{\mu \leqslant d\lambda_1} V_\mu$.

Proof. Let $X(d\lambda_1)$ be the closure of the G-orbit of the spherical vector in $V_{d\lambda_1}$. Denote by $L_{d\lambda_1}$ the restriction of the trivial line bundel $\mathcal{O}(1)$ on $\mathbb{P}(V_{d\lambda})$ to $X(d\lambda_1)$. The composition of the G-equivariant map

$$\begin{aligned} &V_\lambda \to V_\lambda^{\otimes d}, \\ &v \mapsto v \otimes \cdots \otimes v \end{aligned}$$

and the projection

$$V_\lambda^{\otimes d} \to V_{d\lambda}$$

on the Cartan component induces a natural isomorphism $X(\lambda_1) \to X(d\lambda_1)$ and through this isomorphism the line bundle $X(d\lambda_1)$ corresponds to the linebundle $L_{\lambda_1}^{\otimes d}$ on $X(\lambda_1)$. Since $\mathbb{C}[V_\lambda]_d$ can be interpreted as the sections in the linebundle

$$\mathcal{O}(1)^{\otimes d} \quad \text{on} \quad \mathbb{P}(V_\lambda) \quad \text{we have}$$
$$\mathbb{C}[V_\lambda]_d \cong H^0(\mathbb{P}(V_\lambda), \mathcal{O}(1)^{\otimes d}).$$

Because in our special case $X(\lambda_1) = \mathbb{P}(V_\lambda)$, we get

$$H^0(\mathbb{P}(V_\lambda), \mathcal{O}(1)^{\otimes d}) \cong H^0(X(\lambda_1), L_{\lambda_1}^{\otimes d})$$

and using the isomorphism above

$$H^0(X(\lambda_1), L_{\lambda_1}^{\otimes d}) \cong H^0(X(d\lambda_1), L_{d\lambda_1}).$$

The theorem in [CP, Section 8] says

$$H^0(X(d\lambda_1), L_{d\lambda_1}) \cong \bigoplus_{\mu \leqslant d\lambda_1} V_\mu. \qquad \square$$

The disjoint union $\amalg_{d \geqslant 0}\{\mu \in P_+ \mid \mu \leqslant d\lambda_1\}$ figures as index set for the decomposition of $\mathbb{C}[V_\lambda]$ as G-module. Later on we will study the multiplicative structure and then it is for combinatorial reasons easier to work with Young diagrams. In order to attach to each pair (μ, d) with $\mu \leqslant d\lambda_1$ a Young diagram, we need the following.

LEMMA 2.2. *Let* $\Sigma_{i=1}^n a_i\lambda_i \in P_+$ *and* $d \geqslant 0$, *then*

$$\sum_{i=1}^{n} a_i\lambda_j \leqslant d\lambda_i \quad \textit{if and only if}$$

$$d = \sum_{i=1}^{n} a_i \cdot i + a_{n+1}(n+1) \quad \textit{for some integer} \quad a_{n+1} \geqslant 0.$$

Proof. In order to write the fundamental weights in terms of the fundamental roots one has to invert the Cartan matrix. For the rootsystem A_n we get, see [Hul 1, Section 13]:

$$\lambda_i = \frac{1}{n+1}((n-i+1)\alpha_1 + 2(n-i+1)\alpha_2 + \cdots +$$
$$+ (i-1)(n-i+1)\alpha_{i-1} + i(n-i+1)\alpha_i + i(n-i)\alpha_{i+1} + \cdots i\alpha_n).$$

From this follows that $\lambda_i \leqslant i\lambda_1$ and $0 \leqslant (n+1)\cdot\lambda_1$. But then

$$\sum_{i=1}^{n} a_i\lambda_i \leqslant \sum_{i=1}^{n} a_i i\lambda_1 \leqslant \sum_{i=1}^{n} a_i i\lambda_1 + a_{n+1}(n+1)\lambda_1 = \left(\sum_{i=1}^{n} a_i i + a_{n+1}(n+1)\right)\lambda_1$$

so that "if" part follows. Conversely, if $\Sigma_{i=1}^{n} a_i\lambda_i \leqslant d\lambda_1$, then the coefficient of α_n of $d\lambda_1 \dot{-} \Sigma_{i=1}^{n} a_i\lambda_i$ expressed in terms of the fundamental roots is $a_{n+1} = (d - \Sigma_{i=1}^{n} a_i\cdot i)/(n+1)$. By assumption a_{n+1} must be a nonnegative integer, thus $d = \Sigma_{i=1}^{n} a_i\cdot i + a_{n+1}\,(n+1)$ is of the desired form. □

Now we attach to the pair $\mu = \Sigma_{i=1}^{n} a_i\lambda_i \in P_+$ and $d \geqslant 0$ with $\mu \leqslant d\lambda_1$ the Young diagram $\sigma = \sigma_{d,\mu} = (\sigma_1, \ldots, \sigma_{n+1})$ defined by $\sigma_j = \Sigma_{j=1}^{n+1} a_j$, where a_{n+1} is defined as in the Lemma. Then

$$|\sigma| = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_j = \sum_{j=1}^{n+1} a_j\cdot j = d \quad \text{and} \quad a_i = \sigma_1 - \sigma_{i+1}.$$

Conversely let $\sigma = (\sigma_1, \ldots, \sigma_{n+1})$ be a Young diagram. Define $d = |\sigma|$ and $\mu = \mu_\sigma = \Sigma_{i=1}^{n}(\sigma_i - \sigma_{i+1})\lambda_i$. Now we have

$$\begin{aligned}
&\sum_{j=1}^{n} (\sigma_j - \sigma_{j+1})j + \sigma_{n+1}(n+1) \\
&\quad = \sum_{i=1}^{n} \sum_{j=1}^{n} (\sigma_j - \sigma_{j+1}) + \sigma_{n+1}(n+1) \\
&\quad = \sum_{i=1}^{n} (\sigma_i - \sigma_{n+1}) + \sigma_{n+1}(n+1) \\
&\quad = \sum_{i=1}^{n} \sigma_i = |\sigma| = d.
\end{aligned}$$

Now by the Lemma $\mu \leqslant d\cdot\lambda_1$.

We have that D_{n+1} is the disjoint union of the subsets

$$D_{n+1,d} = \{\sigma \in D_{n+1} \mid |\sigma| = d\} \quad d = 0, 1, 2, \ldots.$$

We just proved a one to one correspondence between the $\mu \in P_+$ with $\mu \leqslant d\lambda_1$ and the elements of $D_{n+1,d}$. Let V_σ denote the irreducible summand V_{μ_σ} in $\mathbb{C}[V_{\lambda_1}]_{|\sigma|}$ then theorem 2.1 translates into

THEOREM 2.3. $\mathbb{C}[V_\lambda] \cong \bigoplus_{\sigma \in D_{n+1}} V_\sigma.$

Section 3. Multiplicative structure

The first purpose of this section is to relate the multiplicative structure of irreducible G-submodules of $\mathbb{C}[V_\lambda]$ and $C[G/K]$ to the multiplicative structure of the K-fixed elements. For each $\lambda \in P_+$ we choose the elementary spherical function $\Phi_\lambda \in V_\lambda^K$ normalized by $\Phi_\lambda(e) = 1$, where e denotes the coset of the unit element of G. These elementary spherical functions form a $\mathbb{C}$-basis for the set of all bi-K-invariant functions $\mathbb{C}[G/K]^K$ on G. So for any $\mu, \nu \in P_+$ we can write

$$\Phi_\mu \cdot \Phi_\nu = \sum_{\lambda \in P_+} d(\mu, \nu, \lambda) \Phi_\lambda.$$

Also for each $\sigma \in D_{n+1}$ is the irreducible G-submodule V_σ of $\mathbb{C}[V_\lambda]$ spherical, thus we can choose a spherical vector Φ_σ in V_σ. The morphism $\phi^*: \mathbb{C}[V_\lambda] \hookrightarrow \bigoplus_{d \in \mathbb{Z}} \mathbb{C}[G/K]T^d$ maps V_σ isomorphically onto $V_{\mu_\sigma} \cdot T^{|\sigma|}$ and we normalize Φ_σ in such a way that it is mapped by ϕ^* to $\Phi_{\mu_{r\sigma}} \cdot T^{|\sigma|}$. These functions Φ_σ, also called spherical functions, form a $\mathbb{C}$-basis for $\mathbb{C}[V_\lambda]^K$. As above we can write for any $\sigma, \tau \in D_{n+1}$

$$\Phi_\sigma \cdot \Phi_\tau = \sum_{\rho \in D_{n+1}} d(\sigma, \tau, \rho) \Phi_\rho.$$

We also define a multiplication of the irreducible G-modules V_σ and V_τ in $\mathbb{C}[V_\lambda]$ by

$$V_\sigma \cdot V_\tau = G\text{-module in } \mathbb{C}[V_\lambda] \text{ spanned by } \{f \cdot g \mid f \in V_\sigma, g \in V_\tau\}.$$

Of course there is for $\mathbb{C}[G/K]$ a similar definition.

THEOREM 3.1. *$V_\sigma \cdot V_\tau = \bigoplus_\rho V_\rho$ where the sum is taken over all $\rho \in D_{n+1}$ with $d(\sigma, \tau, \rho) \neq 0$.*

Proof. Using the morphism ϕ^* we get

$$d(\sigma, \tau, \rho) \neq 0 \text{ if and only if } |\sigma| + |\tau| = |\rho| \text{ and } d(\mu_\sigma, \mu_\tau, \mu_\rho) \neq 0.$$

It is therefore equivalent to prove

THEOREM 3.2. *$V_\mu \cdot V_\nu = \bigoplus_\lambda V_\lambda$, where the sum is taken over all λ with $d(\mu, \nu, \lambda) \neq 0$.*

Proof. We begin with some general theory.

After extending the Zariski-topology on G/K to the $\mathbb{C}$-topology one can take a compact real form G_0/K_0 of it. Define

$$C^\infty(G_0/K_0)^{K_0\text{-}fin} \subseteq L^2(G_0/K_0).$$

the space of all K_0-finite functions $f \in C^\infty(G_0/K_0)$. i.e. $K_0 \cdot f$ is contained in a finite dimensional subspace. The unitary trick says that the restriction map gives an isomorphism

$$r: \mathbb{C}[G/K] \to C^\infty(G_0/K_0)^{K_0\text{-}fin}.$$

The advantage of working in $C^\infty(G_0/K_0)^{K_0\text{-}fin}$ is that the restriction of the G_0-invariant Hermitean innerproduct of the unitary representation $L^2(G_0/K_0)$ provides a Hermitean innerproduct $\langle \cdot,\cdot \rangle$ on it. The decomposition of $\mathbb{C}[G/K]$ carries over to a decomposition in pairwise orthogonal irreducible components of $C^\infty(G_0/K_0)^{K_0\text{-}fin}$ as G_0-module. Write $V^r_\mu = rV_\mu$ and $\Phi^r_\mu = r(\Phi_\mu)$.

For any irreducible unitary spherical representation W of (G_0, K_0) with innerproduct $\langle \cdot,\cdot \rangle$ and $e_w \in W$ a spherical unit vector we now define

$$f_w \in C^\infty(G_0/K_0)^{K_0-fin}, \quad w \in W, \quad \text{as}$$

$$f_w(g) = \langle w, ge_W \rangle, \quad g \in G_0/K_0$$

and a $\mathbb{C}$-linear G_0-equivariant embedding

$$\phi: W \to C^\infty(G_0/K_0)^{K_0-fin}$$

$$w \mapsto f_w.$$

For $W = V^r_\mu$ an irreducible summand ϕ becomes in fact a map of V^r_μ into itself given by multipliction with some scalar $\alpha_\mu \in \mathbb{C}^*$. Thus for $e_\mu = \alpha_\mu^{-1} \cdot \Phi^r_\mu$ and $f \in V^r_\mu$ we get

$$f(g) = \langle f, ge_\mu \rangle g \in G_0.$$

We are now ready to prove the theorem.

Given $\mu, \nu \in P_+$, we provide the vectorspace $V^r_\mu \otimes V^r_\nu$ with a Hermitean innerproduct $\langle \cdot,\cdot \rangle$ by demanding

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle \quad v_1, v_2 \in V^r_\mu, w_1, w_2 \in V^r_\nu.$$

Then there is an orthogonal direct sum decomposition

$$V^r_\mu \otimes V^r_\nu = \bigoplus_{j=1}^{m} W_j \oplus W,$$

where the W_j are irreducible spherical representations of (G_0, K_0) and W do not contain any spherical vector. The orthogonal projection on W_j will be denoted by $\pi_j, j = 1, \ldots, m$. In each W_j we choose a spherical unit vector e_j, then we can write

$$e_\mu \otimes e_\nu = \sum_{j=1}^{m} a_j e_j.$$

Given $f_1 \in V_\mu$ and $f_2 \in V_\nu$ we get for any $g \in G_0/K_0$

$$\begin{aligned}(f_1 \cdot f_2)(g) &= f_1(g) \cdot f_2(g)\\ &= \langle f_1, ge_\mu \rangle \langle f_2, ge_\nu \rangle\\ &= \langle f_1 \otimes f_2, g(e_\mu \otimes e_\nu) \rangle\\ &= \sum_{j=1}^{m} \bar{a}_j \langle f_1 \otimes f_2\, ge_j \rangle\\ &= \sum_{j=1}^{m} \bar{a}_j \langle \pi_j(f_1 \otimes f_2), ge_j \rangle. \end{aligned} \tag{$*$}$$

Since the products $f_1 \cdot f_2$ span $V_\mu^r \cdot V_\nu^r$, this gives that each V_λ^r occurring in $V_\mu^r \cdot V_\nu^r$ must be isomorphic with some W_j with $a_j \neq 0$.

On the other hand if we take $f_1 = e_\mu$ and $f_2 = e_\nu$ then $(*)$ becomes

$$(e_\mu \cdot e_\nu)(g) = \sum_{j=1}^{m} \bar{a}_j \langle a_j e_j, ge_j \rangle = \sum_{j=1}^{m} |a_j|^2 \langle e_j\, ge_j \rangle.$$

For each $j = 1, \ldots, m$ we have an embedding $\phi_j \colon W_j \to C^\infty(G_0/K_0)^{K_0\text{-}fin}$ as defined above, thus $W_j \cong \phi_j(W_j) \cong V_{\lambda_j}^r$ for some λ_j. Moreover

$$\langle e_j, ge_j \rangle = e_{\lambda_j}(g) = \alpha_{\lambda_j}^{-1} \cdot \Phi_{\lambda_j}^r(g),$$

so if $a_j \neq 0$ then occurs $V_{\lambda_j}^r$ in $V_\mu^r \cdot V_\nu^r$.

Reformulating this in terms of spherical functions gives

$$\Phi_\mu \cdot \Phi_\nu = \alpha_\mu \alpha_\nu \sum_{j=1}^{m} \alpha_{\lambda_j}^{-1} |a_j|^2 \Phi_{\lambda_j}$$

and V_λ occurs in $V_\mu \otimes V_\nu$ if and only if $\lambda = \lambda_j$ for some j with $a_j \neq 0$, thus if and only if $d(\mu, \nu, \lambda) \neq 0$. □

We now focus our attention to the spherical functions $\mathbb{C}[G/K]^K$. From general theory, see [R], [V1], we know that $K\dot{A}K \subseteq G$ is a dense subset, thus spherical functions are completely determined by their restrictions to

$$A/A \cap K \hookrightarrow G/K.$$

Each K-orbit in G/K that intersects $A/A \cap K$, intersects in a W-orbit, where as before $W \cong N_K(A)/C_K(A)$ is the Weyl group of $R(\mathfrak{g}, \mathfrak{a})$. Thus there is a restriction isomorphism

$$r: \mathbb{C}[G/K]^K \to \mathbb{C}[A/A \cap K]^W.$$

Since $A \cap K = \{a \in A \mid a = a^{-1}\}$ we have an isomorphism

$$\psi: A/A \cap K \to A$$

defined by $\psi(a) = a^2$, and an induced isomorphism

$$\psi^*: \mathbb{C}[A]^W \to \mathbb{C}[A/A \cap K]^W.$$

Put

$$P(\lambda, a) = \Phi_{\lambda|A/A\cap K}(a^{\frac{1}{2}}) \in \mathbb{C}[A]^W \quad \Phi_\lambda \in \mathbb{C}[G/K]^K.$$

By composing r and $\psi^{*^{-1}}$ we get an isomorphism

$$\phi: \mathbb{C}[G/K]^K \to \mathbb{C}[A]^W \quad \text{and}$$

$$\phi(\Phi_\lambda) = P(\lambda, a).$$

Apart from a different normalization the polynomials $P(\lambda, a)$ are the multi-variable Jacobi polynomials as introduced in [H]. Let T be the real compact form of the complex torus A provided with the $\mathbb{C}$-topology, and provide $\mathbb{C}[A]^W$ with a Hermitean innerproduct $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_T f(t)\overline{g(t)}\delta(t)\,\mathrm{d}t \quad f, g \in \mathbb{C}[A]^W$$

and weight function

$$\delta(t) = \prod_{\alpha \in R_+} |1 - t^\alpha|^{m_\alpha}$$

where the m_α are the multiplicities defined in Section 2, R_+ the set of positive roots in $R(\mathfrak{g}, \mathfrak{a})$ and dt the normalized Haar measure on T. Then the polynomials form an orthogonal basis.

Let $\mu, \nu \in P_+$, we have

$$P(\mu, a) \cdot P(\nu, a) = \sum_{\lambda \in P_+} d(\mu, \nu, \lambda) P(\lambda, a).$$

Define

$$S(\mu, \nu) = \{\lambda \in P_+ \mid d(\mu, \nu, \lambda) \neq 0\} \quad \text{and}$$

$$C(\mu) = \{\eta \in P_+ \mid w(\eta) \leqslant \mu \quad \text{for all } w \in W\}.$$

It follows from [H, Section 7] that

PROPOSITION 3.3. $S(\mu, \nu) \subseteq (\mu + C(\nu)) \cap P_+$ □

The results of the same paper are used in the appendix to make a calculation in order to prove:

PROPOSITION 3.4. *Let $\mu, \nu \in P_+$ and $\mu + w(\nu) \in P_+$ for some $w \in W$, then*

$$\mu + w(\nu) \in S(\mu, \nu).$$ □

For general μ and ν these propositions do not give sufficient information in order to describe $S(\mu, \nu)$, however if we take $\nu = \lambda_i, i = 1, \ldots, n$, a fundamental weight they do. Since $R(\mathfrak{g}, \mathfrak{a}) = A_n$ we know that all fundamental weights are minuscule (see [Hu, ex. 13.4.13]), i.e. $C(\lambda_i) = W\lambda_i$ for $i = 1, \ldots, n$. Thus combining the propositions we get

PROPOSITION 3.5. $S(\mu, \lambda_i) = \{\mu + w(\lambda_i) \mid w \in W\} \cap P_+$. □

In order to employ this proposition we study the W-orbits of the fundamental weights $\lambda_1, \ldots, \lambda_n$.

LEMMA 3.6.

$$W\{\lambda_1, \ldots, \lambda_n\} = \left\{ \pm \sum_{j=1}^{m} (-1)^j \lambda_{m_j} \mid 1 \leqslant m \leqslant n \quad \text{and} \right.$$

$$\left. 1 \leqslant m_1 < m_2 < \cdots < m_m \leqslant n \right\}.$$

Proof. Let $s_{\alpha_i}, i = 1, \ldots, n$, denote the fundamental reflections, thus

$$s_{\alpha_i}\lambda_j = \lambda_j \quad \text{for} \quad i \neq j,$$

$$s_{\alpha_i}\lambda_i = \lambda_{i-1} - \lambda_i + \lambda_{i+1}.$$

Clearly the set on the right hand side is closed under the W-action and contains $\{\lambda_1, \ldots, \lambda_n\}$, thus contains $W\{\lambda_1, \ldots, \lambda_n\}$. It is also clear that it contains $2 \cdot 2^n - 2 = 2^{n+1} - 2$ elements. When we prove that $W\{\lambda_1, \ldots, \lambda_n\}$ contains the same number of elements we are done.

Fix j, then the stabilizer in W of λ_j is generated by the fundamental reflections $s_{\alpha_1}, \ldots, s_{\alpha_{j-1}}, s_{\alpha_{j+1}}, \ldots, s_{\alpha_n}$ and thus contains $j!(n-j+1)!$ elements. Hence $W\lambda_j$ contains $(n+1)!/j!(n-j+1)! = \binom{n+1}{j}$ elements. Since the W-orbits of the fundamental weights are disjoint we get

$$|W\{\lambda_1, \ldots, \lambda_n\}| = \sum_{j=1}^{n} \binom{n+1}{j} = 2^{n+1} - 2. \qquad \square$$

Given $\mu \in W\{\lambda_1, \ldots, \lambda_n\}$, we can write

$$\mu = \sum_{i=1}^{n} a_i^{\mu} \lambda_i$$

with $a_i^{\mu} \in \{-1, 0, 1\}$. We define

$$a_{n+1}^{\mu} = 1 \quad \text{if} \quad a_j^{\mu} = -1 \quad \text{for} \quad j = \max\{i \mid a_i^{\mu} \neq 0\},$$

$$a_{n+1}^{\mu} = 0 \quad \text{otherwise.}$$

A straightforward calculation gives that the number

$$d(\mu) = \sum_{i=1}^{n+1} a_i^{\mu} \cdot i$$

is constant on Weyl group orbits and $d(W\lambda_j) = d(\lambda_j) = j$.

We define

$$\varepsilon \colon W\{\lambda_1, \ldots, \lambda_n\} \to E_{n+1},$$

$$\varepsilon \colon \mu \mapsto \left(\sum_{i=1}^{n+1} a_i^{\mu}, \sum_{i=2}^{n+1} a_i^{\mu}, \ldots, \sum_{i=n+1}^{n+1} a_i^{\mu} \right).$$

Because $a_i^\mu = \varepsilon(\mu)_i - \varepsilon(\mu)_{i+1}$ $i = 1, \ldots, n$ we can, given $\varepsilon(\mu)$, find back μ, thus ε is injective. From the identity

$$|\varepsilon(\mu)| = \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} a_i^\mu = \sum_{j=1}^{n+1} a_j^\mu \cdot j = d(\mu)$$

follows that

$$\varepsilon(\mu) \in E_{n+1,d(\mu)}.$$

For each $j = 1, \ldots, n$ the sets $W\lambda_j$ and $E_{n+1,j}$ contains both $\binom{n+j}{j}$ elements, thus for the restriction of ε holds $\varepsilon\colon W\lambda_j \to E_{n+1,j}$ is a bijection.

We are now ready to state the main result about products of irreducible *G-summands in* $\mathbb{C}[V_\lambda]$. Let $\delta_j = (1, \ldots, 1) \in D_{n+1}$, j times 1, for $j = 1, \ldots, n+1$ and thus V_{δ_j} a unique irreducible summand of $\mathbb{C}[V_\lambda]_j$.

THEOREM 3.7. *Let $\sigma \in D_{n+1}$, then*

$$V_\sigma \cdot V_{\delta_j} = \bigoplus_\rho V_\rho$$

where the sum is taken over all $\rho \in D_{n+1}$ with $|\rho| = |\sigma| + j$ and

$$\rho_i - \sigma_i = 0 \text{ or } 1 \quad \text{for all } i = 1, \ldots, n+1.$$

REMARK. The assertion is in accordance with a special case of the "Littlewood-Richardson-rule", see [M]. In case III of our classification this rule can be used in order to describe the product of $V_\sigma \cdot V_\tau$ for $\sigma, \tau \in D_{n+1}$ arbitrary. In virtue of the many analogies between the cases of our classification we conjecture that the rule be satisfied for all of them.

Proof of 3.7. In the beginning of the proof of Theorem 3.1 we noted that $d(\sigma, \delta_j, \rho) \neq 0$ if and only if $|\sigma| + j = |\rho|$ and $d(\mu_\sigma, \lambda_j, \mu_\rho) \neq 0$. From proposition 3.5 follows that $d(\mu_\sigma, \lambda_j, \lambda) \neq 0$ if and only if $\lambda = \mu_\sigma + \mu \in P_+$ for some $\mu \in W\lambda_j$. Let $\mu \in W\lambda_j$ arbitrary, then

$$\mu_\sigma + \mu = \sum_{i=1}^{n} (\sigma_i - \sigma_{i+1} + a_i^\mu)\lambda_i,$$

so $\mu_\sigma + \mu \in P_+$ if and only if

$$\sigma_i - \sigma_{i+1} + a_i^\mu = \left(\sigma_i + \sum_{j=1}^{n+1} a_j^\mu\right) - \left(\sigma_{i+1} + \sum_{j=i+1}^{n+1} a_j^\mu\right) \geqslant 0 \quad \text{for all } i = 1, \ldots, n.$$

This sequence of inequalities is also equivalent with $\sigma + \varepsilon(\mu)$ is a Young diagram. Because $\mu_{\sigma+\varepsilon(\mu)} = \mu_\sigma + \mu$ and

$$|\sigma + \varepsilon(\mu)| = |\sigma| + |\varepsilon(\mu)| = |\sigma| + d(\mu) = |\sigma| + j$$

we find $d(\sigma, \delta_j, \rho) \neq 0$ if and only if $\rho \in D_{n+1}$ is of the form $\sigma + \varepsilon(\mu)$ for some $\mu \in W\lambda_j$. Since $\varepsilon\colon W\lambda_j \to E_{n+1,j}$ is bijective the theorem follows. □

The special case $j = 1$, thus $V_{\delta_1} = \mathbb{C}[V_\lambda]_1$, plays an important role in the classification of G-invariant ideals. In this case is $|\rho| = |\sigma| + 1$ and $\rho_i - \sigma_i = 0$ or 1 for $i = 1, \ldots, n+1$ equivalent with $\rho \supsetneqq \sigma$ adjacent (i.e. if $\rho \supseteq \tau \supseteq \sigma$ then $\rho = \tau$ or $\tau = \sigma$).

COROLLARY 3.8.

$$\mathbb{C}[V_\lambda]_1 \cdot V_\sigma = \bigoplus_{\substack{\rho \supsetneqq \sigma \\ \text{adjacent}}} V_\rho, \qquad \square$$

Using our combinatorial results on Young diagrams we can prove two other corollaries. Let $\sigma \in D_{n+1}$ and write $\sigma = \Sigma_{i=1}^{n+1} a_i \delta_i$, where the a_i are non-negative integers and as before $\delta_i = (1, \ldots, 1)$, i times 1.

COROLLARY 3.9.

$$V_{\delta_1}^{a_1} \cdot V_{\delta_2}^{a_2} \cdot \ldots \cdot V_{\delta_{n+1}}^{a_{n+1}} = \bigoplus_{\substack{\tau \geqslant \sigma, \tau \in D_{n+1} \\ |\tau| = |\sigma|}} V_\tau.$$

Proof. By Theorem 3.7. V_τ is summand of the left hand side if and only if

$$|\tau| = a_1 \cdot |\delta_1| + \cdots + a_{n+1} |\delta_{n+1}| = |\sigma|$$

and there is a sequence of Young diagrams $\tau_1, \tau_2, \ldots, \tau_a = \tau$ where $a = \Sigma_{i=1}^{n+1} a_i = \sigma_1$ such that

$$\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots, \tau_a - \tau_{a-1}$$

is a sequence of strips with degrees

$$|\delta_1|, \ldots, |\delta_1|, |\delta_2|, \ldots, |\delta_{n+1}| \quad a_1 \text{ times } |\delta_1|, \ldots, a_{n+1} \text{ times } |\delta_{n+1}|.$$

By definition this is equivalent with saying that τ can be stratified by σ. Hence by Proposition 1.4 the corollary follows. □

REMARK. One can prove that for $\sigma, \tau \in D_{n+1}$ with $|\sigma| = |\tau|, \tau \geqslant \sigma$ is equivalent with $\mu_\tau \leqslant \mu_\sigma$ in P_+. Using this one can translate the corollary to $\mathbb{C}[G/K]$. For $a_1, \ldots, a_n \geqslant 0$ the statement becomes:

$$V_{\lambda_1}^{a_1} \cdot \ldots \cdot V_{\lambda_n}^{a_n} = \bigoplus_{\lambda} V_\lambda,$$

where the sum is taken over all $\lambda \in P_+$ with $\lambda \leqslant a_1\lambda_1 + \cdots + a_n\lambda_n$. This fact and the embedding ϕ^* makes it possible to prove Theorems 2.1 and 2.3 in an other way.

Let $\sigma = \Sigma_{i=1}^{n+1} a_i \delta_i$ be as above.

COROLLARY 3.10. *There exists an integer $m > 0$ such that*

$$\bigoplus_{\substack{\tau \geqslant \sigma, \tau \in D_{n+1} \\ |\tau| = |\sigma|}} V_\tau \cdot \bigoplus_{\substack{\tau \geqslant m\cdot\sigma, \tau \in D_{n+1} \\ |\tau| = m\cdot|\sigma|}} V_\tau = V_\sigma \cdot \bigoplus_{\substack{\tau \geqslant m\cdot\sigma, \tau \in D_{n+1} \\ |\tau| = m\cdot|\sigma|}} V_\tau.$$

Proof. By Corollary 3.9 the left hand side is in fact the sum of all V_τ with $\tau \in \tau \in D_{n+1} \tau \geqslant (m+1)\sigma$ and $|\tau| = (m+1)|\sigma|$, and the right hand side is

$$V_\sigma \cdot V_{\lambda_1}^{ma_1} \cdot \ldots \cdot V_{\lambda_1}^{ma_{n+1}}.$$

By a likewise reasoning as in Corollary 3.9 using again Theorem 3.7 one deduces that the right hand side is the sum of all V_τ with $\tau \in D_{n+1}$, and $\tau = \sigma + \Sigma_{i=1}^{m\cdot\ell} e^i$ for some sequence of strips $e^1, \ldots, e^{m\ell}$ related to $m\sigma$ such that $\sigma + \Sigma_{i=1}^{j} e^i$ is a Young diagram for all $j = 1, \ldots, m\ell$. By Proposition 1.5 now follows that both sides are a sum over the same set of τ's. □

Section 4. The invariant ideals

In the preceding sections the main work has been in order to classify the graded G-invariant ideals in $\mathbb{C}[V_{\lambda_1}]$. Let I be such an ideal, then $I = \bigoplus_{d \geqslant 0} I_d$ where $I_d = I \cap \mathbb{C}[V_{\lambda_1}]_d$. Since $\mathbb{C}[V_{\lambda_1}]_d$ has a multiplicity free decomposition as G-module, it follows that I_d is a sum of some V_σ with $\sigma \in D_{n+1}, |\sigma| = d$. Hence $I = \bigoplus_{\sigma \in D_I} V_\sigma$ for some subset $D_I \subseteq D_{n+1}$. Let I_σ denote the graded invariant ideal generated by V_σ. First we describe these minimal ideals:

THEOREM 4.1.

$$I_\sigma = \bigoplus_{\tau \supseteq \sigma} V_\tau$$

Proof. By the definition

$$I_\sigma = \mathbb{C}[V_{\lambda_1}]\cdot V_\sigma = \bigoplus_{d\geqslant 0} \mathbb{C}[V_{\lambda_1}]_d \cdot V_\sigma = \bigoplus_{d\geqslant 0} \mathbb{C}[V_{\lambda_1}]_1^d \cdot V_\sigma.$$

Now the theorem follows by Corollary 3.8. □

A subset $D \subseteq D_{n+1}$ is called a diagrammatic ideal, shortly d-ideal, if $\sigma \in D, \tau \in D_{n+1}$ and $\sigma \subseteq \tau$ implies $\tau \in D$. For each d-ideal D there is a unique minimal finite subset $\{\tau_1, \ldots, \tau_m\}$ such that

$$D = \{\tau \in D_{n+1} \mid \tau \supseteq \tau_i \quad \text{for some } i = 1, \ldots, m\}$$

and we will write

$$D = (\tau_1, \ldots, \tau_m).$$

It is easy to give a direct proof for this, however it follows already from the classification theorem below and the fact that $\mathbb{C}[V_{\lambda_1}]$ is a Noetherian ring.

THEOREM 4.2. *$I \to D_I$ is a bijective map from the set of G-invariant ideals to the set of d-ideals, it preserves containment and commutes with taking intersections.*

Proof. For any subset $D \subseteq D_{n+1}$ the ideal generated by all V_τ with $\tau \in D$ is $\Sigma_{\tau\in D} I_\tau$. From Theorem 4.1 follows $\Sigma_{\tau\in D} I_\tau = \oplus_\rho V_\rho$ where the sum is taken over all $\rho \in D_{n+1}$ that contains some $\tau \in D$, thus the map is bijective. The other assertions are trivial. □

Because the partial order $\leqslant$ extends the partial order $\subseteq$, we have that

$$A_\sigma = \bigoplus_{\tau\geqslant\sigma} V_\tau \quad \text{and} \quad A'_\sigma = \bigoplus_{\tau>\sigma} V_\tau$$

are graded G-invariant ideals for any $\sigma \in D_{n+1}$. We can write $\sigma = \Sigma_{i=1}^{n+1} a_i \delta_i$, where the a_i are non-negative integers and $\delta_i = (i)^\vee = (1, \ldots, 1), i$ times 1. Put

$$I_i = I_{\delta_i} \quad \text{for} \quad i = 1, \ldots, n+1.$$

PROPOSITIONS 4.3. $A_\sigma = I_1^{a_1} \cdot \cdots \cdot I_{n+1}^{a_{n+1}}$.

Proof. By Proposition 1.4a and Corollary 3.9 both sides are generated by the V_τ with $\tau \in \{\rho \in D_{n+1} \mid \rho \geqslant \sigma \text{ and } |\rho| = |\sigma|\}$. □

Now the invariant ideals are classified by d-ideals we want to describe the d-ideals corresponding to the invariant prime, primary and radical ideals. We first introduce the corresponding notions for the d-ideals and after their classification and some preparation we prove in Theorem 4.7 below that they indeed correspond to the usual ones.

Let D be a d-ideal $\neq D_{n+1}$. We say that D is

prime if $\sigma + \tau \in D$ implies $\sigma \in D$ or $\tau \in D$,
primary if $\sigma + \tau \in D$ implies $\sigma \in D$ or $m \cdot \tau \in D$ for some m,
radical if $m \cdot \sigma \in D$ for some m implies $\sigma \in D$.

The radical of any d-ideal D is defined as

$$\sqrt{D} = \{\sigma \in D_{n+1} \mid m \cdot \sigma \in D \quad \text{for some } m\}$$

and is clearly a radical ideal.

THEOREM 4.4.

(a) *The prime d-ideals are* $(\delta_1), (\delta_2), \ldots, (\delta_{n+1})$ *and the empty set.*
(b) *The radical d-ideals are just the prime d-ideals.*
(c) *The primary d-ideals with radical* (δ_j) *are the d-ideals generated by* $m \cdot \delta_j$ *for some* $m > 0$ *together with some elements of the form* $\Sigma_{\ell=j}^{n+1} a_\ell \delta_\ell$, $a_\ell \geqslant 0$ *and not all zero.*

Proof. (a) and (b). Let D be a prime or radical d-ideal, and $\sigma = \Sigma_{i=1}^{j} a_i \delta_i \in D$ with $a_j \neq 0$. Since $(\Sigma_{i=1}^{j} a_i)\delta_j \supseteq \sigma$, a multiple of δ_j lies in D, so $\delta_j \in D$. Thus D must be of the given form. The converse is trivial.

(c). Let D be a primary d-ideal with radical (δ_j). Of course there is some minimal m such that $m \cdot \delta_j \in D$. Now let $\rho \in D$ and write $\rho = \sigma + \tau$ where $\sigma = \Sigma_{i=j}^{n+1} a_i \delta_i$ and $\tau = \Sigma_{i=1}^{j-1} a_i \delta_i$. Since $\tau \notin (\delta_j)$, thus $m\tau \notin D$ for any m, it follows that $\sigma \in D$. Because $\rho \supseteq \sigma$ we see that a set of generators can be chosen of the desired form.

Conversely let D be a d-ideal generated by elements of the given form. It is clear that for $\tau \in D_{n+1}$ holds $k\tau \notin D$ for all k if and only if we can write $\tau = \Sigma_{i=1}^{j-1} a_i \delta_i$. So if $\sigma + \tau \supseteq \rho, \sigma, \tau \in D_{n+1}$, for one of the generators ρ, but $k\tau \notin D$ for all k, it follows that $\sigma \supseteq \rho$, thus D is primary. $\square$

Let $\sigma, \tau \in D_{n+1}$, we have the following inclusions:

$$V_{\sigma+\tau} \subseteq V_\sigma \cdot V_\tau \subseteq \bigoplus_{\substack{\rho \geqslant \sigma+\tau \\ |\rho| = |\sigma| + |\tau|}} V_\rho. \tag{4.5}$$

The first inclusion holds since $\mu_{\sigma+\tau} = \mu_\sigma + \mu_\tau$, thus $V_{\sigma+\tau}$ is the image of the Cartan component of $V_\sigma \otimes V_\tau \to V_\sigma \cdot V_\tau$. The second inclusion is a consequence of Corollary 3.9 if one write σ, τ and $\sigma + \tau$ as sum of $\delta_1, \ldots, \delta_{n+1}$.

Now let $f, g \in \mathbb{C}[V_{\lambda_1}]$ two non-zero elements. Write

$$f = \sum_{\tau \in F} f_\tau,$$

with $f_\tau \in V_\tau$ non-zero and some unique finite set $F \subseteq D_{n+1}$.
Similarly write

$$g = \sum_{\tau \in G} g_\tau.$$

Let $\sigma \in F$ and $\tau \in G$ be the unique minimal elements in these sets with respect to the lexicographic order $\leqslant_\ell$ defined in Section 1.

LEMMA 4.6.

$$f \cdot g = \sum_{\rho \geqslant \sigma + \tau} h_\rho \quad \textit{with} \quad h_\rho \in V_\rho \quad \textit{and} \quad h_{\sigma+\tau} \neq 0.$$

Proof. Because the lexicographic order $\geqslant_\ell$ extends the partial order $\geqslant$, it follows from (4.5) that for any $\rho_1 \in F$ and $\rho_2 \in G$

$$V_{\rho_1} \cdot V_{\rho_2} \subseteq \bigoplus_{\rho \geqslant_\ell \rho_1 + \rho_2} V_\rho.$$

Since $\rho_1 \geqslant_\ell \sigma$ and $\rho_2 \geqslant_\ell \tau$ we get, using the definition of $\geqslant_\ell$, $\rho_1 + \rho_2 \geqslant_\ell \sigma + \tau$ and equality holds only if $\rho_1 = \sigma$ and $\rho_2 = \tau$. From this follows the first assertion of the lemma and also that the only contribution of $f \cdot g$ to $h_{\sigma+\tau}$ comes from $f_\sigma \cdot g_\tau$. This reduces the proof of the second part to the case $f = f_\sigma$ and $g = g_\tau$ in order to prove the second part of the lemma. In other words we have to prove that the G-equivariant projection p: $V_\sigma \cdot V_\tau \to V_{\sigma+\tau}$ on the Cartan component maps $f \cdot g$ to a non-zero element. Suppose we have fixed a Borelsubgroup $B = T \cdot U$ of G, where T is the maximal θ-stable torus of paragraph 2 and U a maximal unipotent subgroup, so that we can talk about (highest) weight vectors. We fix highest weight vectors $h_\sigma \in V_\sigma, h_\tau \in V_\tau$ and $h_{\sigma+\tau} = p(h_\sigma \cdot h_\tau) \in V_{\sigma+\tau}$. Since V_σ is an irreducible representation, there is a non-empty open subset $\mathcal{O}_f \subseteq U$ such that for all $u \in \mathcal{O}_f$ $uf = \alpha_u \cdot h_\sigma$ + terms of lower weight with $\alpha_u \neq 0$. Similarly their is a $\mathcal{O}_g \subseteq U$. Thus for $u \in \mathcal{O}_f \cap \mathcal{O}_g \neq \emptyset$ we get $p(uf \cdot ug) = \beta_u \cdot h_{\sigma+\tau}$ + (terms of lower weight) and $\beta_u \neq 0$. Then $u \cdot p(f \cdot g) = p(uf \cdot ug) \neq 0$, thus $p(f \cdot g) \neq 0$. □

THEOREM 4.7. *The* 1-1 *correspondence* $I \leftrightarrow D_I$ *of Theorem* 4.2 *preserves the notions prime, primary and radical.*

Proof. We first prove that $I \to D_I$ preserves these notions. It needs easy commutative algebra to see that the properties prime, primary and radical of an ideal $I \subseteq \mathbb{C}[V_{\lambda_1}]$ can be characterized by: for all finite dimensional $\mathbb{C}$-vector-spaces V, W of $\mathbb{C}[V_{\lambda_1}]$ holds (prime) if $V \cdot W \subseteq I$ then $V \subseteq I$ or $W \subseteq I$, (primary) if $V \cdot W \subseteq I$ then $V \subseteq I$ or $W^m \subseteq I$ for some m and (radical) if $V^m \subseteq I$ for some m then $V \subseteq I$. Since $V_{\sigma+\tau} \subseteq V_\sigma \cdot V_\tau$ for all $\sigma, \tau \in D_{n+1}$ (4.5), it is obvious that $I \to D_I$ preserves the notions.

Now let D be a d-ideal. We write

$$I_D = \bigoplus_{\sigma \in D} V_\sigma \quad \text{and} \quad I_D^C = \bigoplus_{\sigma \in D_{n+1} \setminus D} V_\sigma.$$

Suppose D is a prime or radical d-ideal. We have to prove for all $f, g \in \mathbb{C}[V_{\lambda_1}]$ with $f, g \notin I_D$ that $f \cdot g \notin I_D$. Write $f = f_1 + f_2$ and $g = g_1 + g_2$, where $f_1, g_1 \in I_D$ and $f_2, g_2 \in I_D^C$. Then

$$f \cdot g = f_1 g_1 + f_1 g_2 + f_2 g_1 + f_2 g_2 \notin I_D \quad \text{if and only if } f_2 \cdot g_2 \notin I_D.$$

So we may assume $f, g \in I^C$. By Lemma 4.6.

$$f \cdot g = \sum_{\rho \geqslant \sigma+\tau} h_\rho \qquad h_\rho \in V_\rho \quad \text{and} \quad h_{\sigma+\tau} \neq 0$$

for some $\sigma, \tau \in D_{n+1} \setminus D$. Since $\sigma, \tau \notin D$ implies $\sigma + \tau \notin D$ it follows that $f \cdot g \notin I_D$.

Now suppose D is a primary d-ideal. From the classification of prime and primary d-ideals follows $\sqrt{D} = (\delta_j)$ and $m\delta_j \in D$ for some m and some $1 \leqslant j \leqslant n+1$. By (4.5) holds

$$V_{\delta_j}^{m(n+1)} \subseteq \bigoplus_{\tau \geqslant m(n+1) \cdot \delta_j} V_\tau,$$

and $\tau \geqslant m(n+1)\delta_j$ means in particular $\gamma_j(\tau) \geqslant m \cdot (n+1)$ from which follows $\tau \supseteq m\delta_j$. Thus $V_{\delta_j} \subseteq \sqrt{I_D}$, so $I_D \subseteq I_{\sqrt{D}} \subseteq \sqrt{I_D}$. Since $I_{\sqrt{D}}$ is a prime ideal we get

$$I_{\sqrt{D}} = \sqrt{I_D}.$$

In order to prove that I_D is primary, it is sufficient now to show for $f \notin I_{\sqrt{D}}$ and $g \in I^C, g \neq 0$, that $f \cdot g \notin I$. Given such f and g we get by Lemma 4.6, using the same definition of σ and τ, $f \cdot g = \Sigma_{\rho \geqslant_\ell \sigma+\tau} h_\rho$ with $h_\rho \in V_\rho$ and $h_{\sigma+\tau} \neq 0$. Because $f \notin I_{\sqrt{D}}$ and $\sqrt{D} = (\delta_j)$ it follows that $\sigma \notin \sqrt{D}$, namely diagrams not in (δ_j) are in the lexicographic order smaller then elements in (δ_j). Then $\sigma + \tau \notin D$, thus $f \cdot g \notin I$. □

In a Noetherian ring each ideal has a primary decomposition, i.e. can be written as intersection of primary ideals. We give an algorithm in order to write each graded G-invariant ideal in $\mathbb{C}[V_{\lambda_1}]$ as intersection of graded G-invariant primary ideals. By Theorem 4.2 and 4.7 we can work with d-ideals.

For $\sigma, \tau \in D_{n+1}$ we define

$$\sigma \cup \tau = (\max(\sigma_1, \tau_1), \ldots, \max(\sigma_{n+1,} \tau_{n+1})) \in D_{n+1}.$$

Clearly $\cup$ is commutative and associative and for any $\tau \in D_{n+1}$ holds

$$\tau = \sum_{i=1}^{n+1} a_i \delta_i$$

$$= \left(\sum_{i=1}^{n+1} a_i\right)\delta_1 \cup \left(\sum_{i=2}^{n+1} a_i\right)\delta_2 \cup \cdots \cup \left(\sum_{i=n+1}^{n+1} a_i\right)\delta_{n+1}, \tag{4.8}$$

for some $a_i \geqslant 0, a_i \in \mathbb{Z}$. Thus each Young diagram can be written as union of so called rectangular Young diagrams.

It is also straight forward to verify that the following identity for d-ideals holds:

$$(\tau_1, \ldots, \tau_m) \cap (\rho_1, \ldots, \rho_\ell) = (\tau_i \cup \rho_j)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant \ell}. \tag{4.9}$$

As a special case we get

$$(\rho_1, \tau_1, \ldots, \tau_m) \cap (\rho_{2,} \tau_1, \ldots, \tau_m) = (\rho_1 \cup \rho_2, \tau_1, \ldots, \tau_m). \tag{4.10}$$

Now let D be any d-ideal. The algorithm in order to obtain the primary decomposition of D runs as follows.

First choose a finite set of generators for D, and write each generator as a union of rectangular diagrams as mentioned in (4.8). Next use (4.10) repeatedly in order to write D as an intersection of d-ideals, all generated by rectangular diagrams only. By Theorem 4.4.c d-ideals generated by rectangular diagrams are primary, so we have obtained a primary decomposition.

The intersection of primary ideals that belong to the same prime ideal is again primary, (4.9) can be used for taking these intersections. Finally we have to remove the superfluous primary ideals in order to obtain an irredundant primary decomposition.

For the minimal ideals $I_\tau, \tau \in D_{n+1}$, we can give an explicit primary decomposition. Namely in accordance with (4.8) we can write

$$\tau = b_1\delta_1 \cup b_2\delta_2 \cup \cdots \cup b_{n+1}\delta_{n+1}$$

where $b_1 \geqslant b_2 \geqslant \cdots b_{n+1}$. Since $b_i\delta_i \subseteq b_j\delta_j$ if $b_i = b_j$ for some $1 \leqslant i \leqslant j \leqslant n+1$, also

$$\tau = b_{m_1}\delta_{m_1} \cup \cdots \cup b_{m_m}\delta_{m_m}.$$

for some subsequence $b_{m_1} > b_{m_2} > \cdots > b_{m_m} > 0$. Now the algorithm above gives

PROPOSITION 4.11.

$$I_\tau = I_{b_{m_1}\cdot\sigma_{m_1}} \cap \cdots \cap I_{b_{m_m}\cdot\sigma_{m_m}}$$

is an irredundant primary decomposition. The associated prime ideals are $I_{m_1}, \ldots, I_{m_m}$.

In order to give an explicit primary decomposition of the $A_\sigma, \sigma \in D_{n+1}$, we need two Lemma's.

For $\sigma \in D_{n+1}$ put

$$J(\sigma) = \{j \mid 1 \leqslant j \leqslant n+1, \sigma_j \neq 0 \quad \text{or} \quad (n+1-j)(\sigma_j - 1) \geqslant \gamma_{j+1}(\sigma)\}$$

LEMMA 4.12. *For any* $\tau \in D_{n+1}$ *holds*

$$\tau \geqslant \sigma \quad \textit{if and only if } \gamma_j(\tau) \geqslant \gamma_j(\sigma) \quad \textit{for all } j \in J(\sigma).$$

$J(\sigma)$ is a minimal subset of $\{1, \ldots, n+1\}$ with this property.

Proof. By definition $\tau \geqslant \sigma$ if and only if $\gamma_j(\tau) \geqslant \gamma_j(\sigma)$ for all $j \geqslant 1$. Clearly the j with $\sigma_j = 0$ are redundant since for such j $\gamma_j(\sigma) = 0$. Now let $1 \leqslant j \leqslant n+1$ and suppose $\sigma_j \neq 0$ and $(n+1-j)(\sigma_j - 1) < \gamma_{j+1}(\sigma)$. *If* $\tau_j \geqslant \sigma_j$ then $\gamma_j(\tau) \geqslant \gamma_j(\sigma)$ will be a consequence of $\gamma_{j+1}(\tau) \geqslant \gamma_{j+1}(\sigma)$, whereas $\tau_j < \sigma_j$ implies $\gamma_{j+1}(\tau) \leqslant (n+1-j)(\sigma_j - 1) < \gamma_{j+1}(\sigma)$. So the test of the inequality is superfluous for j if we test $j+1$. Since for j sufficiently large always holds $\gamma_j(\tau) \geqslant \gamma_j(\sigma)$ it follows that we can restrict ourselves to $J(\sigma)$.

Now let $j \in J$. But

$$\tau = (|\sigma|, \ldots, |\sigma|, \sigma_j - 1, \sigma_{j+1}, \sigma_{j+2}, \ldots, \sigma_{n+1}) \in D_{n+1}.$$

Then $\gamma_j(\tau) < \gamma_j(\sigma)$ and $\gamma_i(\tau) \geqslant \gamma_i(\sigma)$ for all $i \neq j$, thus J is minimal. □

LEMMA 4.13. $D_{j,m} = \{\sigma \in D_{n+1} \mid \gamma_j(\sigma) \geqslant m\}$ *is a primary d-ideal with radical* (δ_j) *for all* $1 \leqslant j \leqslant n+1$ *and* $m > 0$.

Proof. We show that $D_{j,m}$ can be generated by elements of the form as desired in Theorem 4.4c).

$m\cdot\sigma_j \in D_{j,m}$ because $\gamma_j(m\sigma_j) = m$.

If $\tau \in D_{j,m}$ then

$$\tau \supseteq \tau' = (\tau_j, \dots, \tau_j, \tau_{j+1}, \dots, \tau_{n+1}) \in D_{n+1} \quad \text{and} \quad \gamma_j(\tau') = \gamma_j(\tau),$$

thus $\tau' \in D_{j,m}$ and is clearly of the desired form. □

We now give the primary decomposition of A_σ in terms of d-ideals.

PROPOSITON 4.14.

$$\{\tau \in D_{n+1} \mid \tau \geqslant \sigma\} = \bigcap_{j \in J(\sigma)} D_{j,\gamma_j(\sigma)}$$

is an irredundant primary decomposition.

Proof. Combining Lemmas 4.12 and 4.13 yields the decomposition. Since $J(\sigma)$ is minimal the intersection has to be irredundant □

Let P be a prime-ideal in a Noetherian ring. For fixed $m > 0$ occurs in each irredundant primary decomposition of P^m a primary ideal $P^{(m)}$ associated to P (i.e. $\sqrt{P^{(m)}} = P$). $P^{(m)}$ does not depend on the chosen decomposition and is called the m-th symbolic power of P.

In $\mathbb{C}[V_{\lambda_1}]$ the G-invariant prime ideals are $I_i = I_{\delta_i} i = 1, \dots, n+1$. We determine a primary decomposition of their powers and describe the symbolic powers.

PROPOSITION 4.15.

$$I_j^{(m)} = \bigoplus_{\tau \in D_{j,m}} V_\tau$$

and

$$I_j^m = I_j^{(m)} \cap I_{j-1}^{(2m)} \cap \cdots \cap I_\ell^{((1+j-\ell)m)}$$

is an irredundant primary decomposition, where $\ell = \max(1, n+1-(n+1-j)m)$.

Proof. By definition $A_{m\cdot\sigma_j} = I_j^m$ and in Proposition 4.14 an irredundant primary decomposition of $A_{m\cdot\sigma_j}$ is given in terms of d-ideals:

$$\{\tau \in D_{n+1} \mid \tau \geqslant m\cdot\sigma_j\} = \bigcap_{i \in J(m\sigma_j)} D_{i,\gamma_i(m\sigma_j)}.$$

We determine $J(m\cdot\sigma_j)$. Of course $J(m\cdot\sigma_j)\subseteq\{1,\dots,j\}$, so let $1\leqslant \ell\leqslant j$. Then $\ell\in J(m\cdot\sigma_j)$ if and only if

$$(n+1-\ell)(m-1)\geqslant \gamma_{\ell+1}(m\sigma_j)=(j-\ell)m$$

thus if and only if

$$\ell\geqslant (n+1)-(n+1-j)\cdot m.$$

So we get

$$\{\tau\in D_{n+1}\mid \tau\geqslant m\sigma_j\}=D_{j,m}\cap D_{j-1,2m}\cap\cdots\cap D_{\ell,(1+j-\ell)m}.$$

In particular the δ_j-primary component $m\delta_j$ is $D_{j,m}$. Now the proposition follows by translating these facts back to G-invariant ideals. □

Finally we want to describe the integral closures of G-invariant ideals in $\mathbb{C}[V_{\lambda_1}]$. Given a graded G-invariant ideal I, an element $f\in\mathbb{C}[V_{\lambda_1}]$ is said to be integral dependend on I if it satisfies an equation of the form $z^\ell+a_1z^{\ell-1}+\cdots+a_\ell=0$ with $a_i\in I^i$. This is equivalent with $M\cdot f\subseteq M\cdot I$ for some finite dimensional $\mathbb{C}$-vectorspace $M\subseteq\mathbb{C}[V_{\lambda_1}]$, see [ZS, appendix 4]. The integral closure of I is the ideal of all integral dependend elements, and is again a graded G-invariant ideal.

We first determine the integral closures of minimal ideals.

PROPOSITION 4.16. *The integral closure of I_σ is A_σ.*

Proof. By Corollary 3.10. there is a $m>0$ such that for

$$N=\bigoplus_{\substack{\tau\geqslant\sigma,\tau\in D_{n+1}\\ |\tau|=|\sigma|}} V_\tau$$

and

$$M=\bigoplus_{\substack{\tau\geqslant m\sigma,\tau\in D_{n+1}\\ |\tau|=|\sigma|}} V_\tau$$

holds

$$M\cdot N=M\cdot V_\sigma.$$

Because N generates A_σ and V_σ generates I_σ it follows that A_σ is integral over I_σ.

In order to show that A_σ equals the integral closure of I_σ we prove that for any

$\tau \in D_{n+1}$ with not $\tau \geqslant \sigma$, their do not exists a finite dimensional vectorspace M such that $M \cdot V_\tau \subseteq M \cdot I_\sigma$. Clearly we can restrict ourselves to G-invariant vectorspaces M. If not $\tau \geqslant \sigma$, then $\gamma_i(\tau) < \gamma_i(\sigma)$ for some $1 \leqslant i \leqslant n+1$. Now take $\rho \in D_{n+1}$ such that V_ρ is summand of M with $\gamma_i(\rho)$ minimal. By (4.5) $V_{\rho+\tau}$ is a summand of $M \cdot V_\tau$, but for each summand V_η of $M \cdot I_\sigma$ holds

$$\gamma_i(\eta) \geqslant \gamma_i(\rho) + \gamma_i(\sigma) > \gamma_i(\rho) + \gamma_i(\tau),$$

thus $V_{\rho+\tau}$ is not a summand of $M \cdot I_\sigma$. □

Via the 1–1 correspondence $I \leftrightarrow D_I$ we have for d-ideals the notion integral closure. We describe the integral closures of arbitrary invariant ideals in terms of d-ideals.

We extend the partial order $\leqslant$ on D_{n+1} to $\mathbb{Q}^{n+1} \supseteq D_{n+1}$ as follows: let $a = (a_1, \ldots, a_{n+1}), b = (b_1, \ldots, b_{n+1}) \in \mathbb{Q}^{n+1}$ then

$$a \leqslant b \quad \text{if and only if } \sum_{i=j}^{n+1} a_i \leqslant \sum_{i=j}^{n+1} b_i \quad \text{for all } j = 1, \ldots, n+1.$$

PROPOSITION 4.17. *The integral closure of the d-ideal* $(\sigma_1, \ldots, \sigma_p)$ *is*

$$\left\{ \tau \in D_{n+1} \mid \tau \geqslant a_1\sigma_1 + \cdots + a_p\sigma_p \quad \text{for some } a_i \in \mathbb{Q}, a_i \geqslant 0 \text{ and } \sum_{i=1}^{p} a_i = 1 \right\}$$

Proof. First note that from Proposition 4.16 follows that for any σ in the integral closure and any $\tau \in D_{n+1}$ with $\tau \geqslant \sigma$ also τ is in the integral closure.

Now let $\tau \in D_{n+1}$ with $\tau \geqslant a_1\sigma_1 + \cdots + a_p\sigma_p$ for some $a_i \in \mathbb{Q}, a_i \geqslant 0$ and $\Sigma_{i=1}^p a_i = 1$. Choose a positive integer m such that ma_i is integral for all $i = 1, \ldots, p$. Then $m\tau \geqslant ma_1\sigma_1 + \cdots + ma_p\sigma_p$ and $\Sigma_{i=1}^p ma_i = m$. By (4.5) we get

$$V_\tau^m \subseteq \bigoplus_{\rho \geqslant m \cdot \tau} V_\rho$$

and

$$V_{\Sigma_{i=1}^p ma_i\sigma_i} \subseteq V_{\sigma_1}^{ma_1} \cdot \ldots \cdot V_{\sigma_p}^{ma_p} \subseteq I^m_{(\sigma_1, \ldots, \sigma_p)}.$$

Thus by the remark at the beginning of the proof it follows that V_τ^m is contained in the integral closure of $I^m_{(\sigma_1, \ldots, \sigma_p)}$. Using the definition of integral dependence it follows that V_τ is integral over $I_{(\sigma_1, \ldots, \sigma_p)}$.

Conversely suppose V_τ is integral over $I_{(\sigma_1, \ldots, \sigma_p)}$. Then for $m > 0$ sufficiently large is

$$V_\tau^m \subseteq V_\tau^{m-1} \cdot I_{(\sigma_1,\ldots,\sigma_p)} + V_\tau^{m-2} \cdot I_{(\sigma_1,\ldots,\sigma_p)}^2 + \cdots + I_{(\sigma_1,\ldots,\sigma_p)}^m.$$

By (4.5) V_ρ is contained in the right hand side implies $\rho \geqslant \Sigma_{i=1}^p b_i\sigma_i + b_{p+1}\tau$ for some non-negative integers $b_1,\ldots,b_{p+1}$ with $\Sigma_{i=1}^{p+1} b_i = m$ and $b_{p+1} < m$. In particular since $V_{m\tau} \subseteq V_\tau^m$ this holds for $\rho = m\cdot\tau$. In this case put $k = m - b_{p+1} = \Sigma_{i=1}^p b_p > 0$, then $k\tau \geqslant \Sigma_{i=1}^p b_i\sigma_i$ or equivalently $\tau \geqslant \Sigma_{i=1}^p (b_i/k)\sigma_i$, where $b_i/k \in \mathbb{Q}$, $b_i/k \geqslant 0$ and $\Sigma_{i=1}^p b_i/k = 1$. □

Section 5. The G-orbits in $\mathbb{P}(V_{\lambda_1})$.

In Section 4 the graded G-invariant prime ideals of $\mathbb{C}[V_{\lambda_1}]$ have been classified. We found a chain of prime ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{n+1} \supseteq I_{n+2} = (0)$, where I_i is generated by the homogeneous polynomials of degree i in M_{δ_i} for $i = 1,2,\ldots,n+1$. We consider $\mathbb{C}[V_{\lambda_1}]$ as the homogeneous coordinate ring of the projective variety $\mathbb{P}(V_{\lambda_1})$. The ideal I_1 equals the maximal homogeneous ideal and does not play a role. For $i = 1,\ldots,n+1$ we define

$$X_i = \text{zero set of } I_{i+1} \text{ in } \mathbb{P}(V_{\lambda_1}).$$

PROPOSITION 5.1.

(a) $\{X_1,\ldots,X_{n+1}\}$ *is a complete set of G-invariant closed subsets.*
(b) $X_1, X_2 - X_1,\ldots,X_{n+1} - X_n$ *are the G-orbits.*
(c) X_1 *is the orbit of the highest weight vector.*
(d) X_i *is the union of all $(i-1)$-dimensional projective planes through i points of X_1 together with their limit positions.*
(e) $X_1,\ldots,X_n$ *are normal varieties with rational singularities.*

Proof. Since $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{n+1}$ form a complete set of G-invariant irreducible closed subsets and any G-invariant closed subset is a union of them (a) follows immediately. Because any G-orbit is open in its closure [Kr, II2.2] (b) follows from (a). The orbit of the highest weight vector is always closed [Kr, III3.5] and of course G-stable thus, combining (a) and (b), equal to X_1.

Now fix a Borel subgroup $B = TU$ with T the maximal θ-fixed torus defined in Section 2 and U a maximal unipotent subgroup. Choose highest weight vectors $h_i \in V_{\delta_i}^U$, for $i = 1,\ldots,n+1$. Since for any $\sigma,\tau \in D_{n+1}$ $V_{\sigma+\tau}$ corresponds to the Cartan component of $V_\sigma \otimes V_\tau$, see (4.5), it follows that for any $\sigma = \Sigma_{i=1}^{n+1} a_i\delta_i$ the U-invariant element $h_\sigma = h_1^{a_1}\cdot\cdots\cdot h_{n+1}^{a_{n+1}}$ is a highest weight vector in V_σ^U. We also get that $\mathbb{C}[V_{\lambda_1}]^U = \mathbb{C}[h_1,\ldots,h_{n+1}]$ is a polynomial algebra and thus

$(\mathbb{C}[V_{\lambda_1}]/I_{i+1})^U \cong C[h_1, \ldots, h_i], i = 1, \ldots, n+1$. Several geometric properties hold for the affine varieties Y_i corresponding to $\mathbb{C}[V_{\lambda_1}]/I_{i+1}$ if and only if they hold for the affine varieties Y_i/U corresponding to $(\mathbb{C}[V_{\lambda_1}]/I_{i+1})^U$. This is proved for normality [V2] or [Kr] and for having rational singularities only [Br]. So $Y_1, \ldots, Y_{n+1}$ are normal varieties with rational singularities. From this follows (e). It remains to show (d). This will be proved after the case by case study below of our classification given in the table of Section 2.

In order to describe generators for the invariant prime ideals we need

PROPOSITION 5.2. *I_{n+1} is generated by one G-fixed homogeneous element of degree $n+1$. For $i = 1, \ldots, n$ is I_i generated by the set of all partial derivatives of a set of generators of I_{i+1}.*

Proof. Since I_{n+1} is generated by $V_{\delta_{n+1}}$ and $\mu_{\delta_{n+1}} = 0 \in P_+$ the first assertion follows. Now fix $1 \leqslant i \leqslant n$. The symbolic power $I_i^{(m)}, m \geqslant 1$, can be interpreted as the set of functions in $\mathbb{C}[V_{\lambda_1}]$ vanishing to order $\geqslant m$ on Y_{i-1}, see [EH]. By Proposition 4.15 $I_i^{(2)}$ is generated by $V_{2\delta_i}$ and $V_{\delta_{i+1}}$. Given bases $f_1, \ldots, f_p$ of $V_{\delta_{i+1}}$ and $Z_1, \ldots, Z_q$ of $\mathbb{C}[V_{\lambda_1}]_1$ the partial derivatives $(\partial/\partial Z_l)f_k$ vanish to order $\geqslant 1$ on Y_{i-1}, thus are all in V_{δ_i}. Clearly for any $g \in G$ $g(\partial/\partial Z_l)f_k$ can be written as a linear combination of the partial derivatives, so they form a set of generators for V_{δ_i}. Since V_{δ_i} generates I_i and $V_{\delta_{i+1}}$ generates I_{i+1} the proposition follows. □

Now we describe the situation case by case for the classification given in Section 2.

(I) $G = \mathrm{SO}_{m+1}, K = \mathrm{O}_m, V_{\lambda_1} = \mathbb{C}^{m+1}$ the standard representation and rank $n = 1$.

Let $Z_1, \ldots, Z_{n+1}$ denote the coordinate functions, then

$$I_1 = (Z_1, \ldots, Z_{m+1}), I_2 = (Z_1^2 + \cdots + Z_{m+1}^2).$$

(III) $G = \mathrm{SL}_m \times \mathrm{SL}_m, K = \mathrm{SL}_m \hookrightarrow \mathrm{diag}, V_{\lambda_1} = \mathbb{C}^m \otimes \mathbb{C}^m$ and rank $n = m - 1$. V_{λ_1} can be identified with the set of complex $m \times m$-matrices $M_{m,m}$ such that the G-action becomes $(A, B)M = AMB^{-1}, (A, B) \in G, M \in M_{m,m}$. Let $Z_{ij}, 1 \leqslant i,j \leqslant m$, denote the coordinate functions on $M_{m,m}$ and Z the $m \times m$-matrix with i-j entry Z_{ij}. Clearly $\det(Z)$ is a G-invariant homogeneous polynomial of degree m, hence $I_m = (\det(Z))$. Because the partial derivatives of the k-minors of $Z, k \geqslant 1$, are zero or $(k-1)$-minors it follows by Proposition 5.6 that for $i = 1, \ldots, m$ I_i is generated by the i-minors of Z. Consequently the variety Y_i (and X_i) consists of (the classes) of rank $\leqslant i$ matrices.

(II) $G = \mathrm{SL}_m, K = \mathrm{SO}_m, V_{\lambda_1} = S^2\mathbb{C}^m$ and rank $n = m - 1$.

V_{λ_1} can be identified with the set of symmetric complex $m \times m$-matrices $SM_{m,m} \subseteq M_{m,m}$ such that the G-action becomes $A \cdot M = AMA^t$, $A \in G$, $M \in SM_{m,m}$. Let $Z_{ij} = Z_{ji}$, $1 \leqslant i,j \leqslant m$, denote the coordinate functions on $SM_{m,m}$ and Z the $m \times m$-matrix with i-j entry Z_{ij}.

As in case III we get for $i = 1, \ldots, m$:

I_i is generated by the i-minors of Z.

Y_i (and X_i) consists of the (classes) of rank $\leqslant i$ symmetric matrices.

(IV) $G = \mathrm{SL}_m, m$ even, $K = \mathrm{Sp}_m, V_{\lambda_1} = \wedge^2\mathbb{C}^m$ and rank $n = (m/2) - 1$.

V_{λ_1} can be identified with the set of anti-symmetric complex $m \times m$-matrices $AM_{m,m} \subseteq M_{m,m}$ such that the G-action becomes $A \cdot M = AMA^t$, $A \in G$, $M \in AM_{m,m}$. Let $Z_{ij} = -Z_{ji}$, $1 \leqslant i,j \leqslant m$, denote the coordinate functions on $AM_{m,m}$ and Z the $m \times m$-matrix with i-j entry Z_{ij}. Since Z is an antisymmetric matrix we can take its pfaffian $Pf(Z)$, this is a G-invariant polynomial of degree m and thus $I_{(m/2)} = (Pf(Z))$. In order to obtain generators for $I_{(m/2)-1}$ we have to take partial derivatives. Let $1 \leqslant i < j \leqslant m$ then $(\partial/\partial Z_{ij})Pf(Z)$ is precisely the pfaffian of the $(m-2)$-minor obtained from Z by cancelling the i-th and j-th row and column. Repeating this argument yields that for $i = 1, \ldots, n + 1 = m/2$:

I_i is generated by the pfaffians of the $2i$-minors of Z of which the involved row-set and column-set are equal.

Y_i (and X_i) consist of the (classes) of rank $\leqslant 2i$ anti-symmetric matrices.

(V) G of type E_6, K of type F_4, V_{λ_1} the standard 27 dimensional representation and the rank $n = 2$.

V_{λ_1} can be identified with the vectorspace of triples of 3×3 matrices $(M_{3,3})^3$ such that the G-action leaves the cubic form

$$\det(Z^1) + \det(Z^2) + \det(Z^3) - \operatorname{tr}(Z^1 \cdot Z^2 \cdot Z^3) \tag{5.3}$$

invariant, the Dickson representation, see [D] or [F]. Here Z^k denotes the 3×3 matrix with i-j entry Z^k_{ij}, where Z^k_{ij}, $1 \leqslant i,j,k \leqslant 3$ are the obvious coordinate functions. Thus I_3 is generated by the cubic form. In order to obtain the generators of I_2 we have to determine all the partial derivatives of the cubic form. We claim that I_2 is generated by the 27 functions:

$$(\mathrm{adj}(Z^1) - Z^2 \cdot Z^3)_{ij}, \quad (\mathrm{adj}(Z^2) - Z^3 \cdot Z^1)_{ij}, \quad (\mathrm{adj}(Z^3) - Z^1 Z^2)_{ij}, \quad 1 \leqslant i,j \leqslant 3.$$

Here adj(A) denotes the adjoint matrix of A. It is an explicit calculation to obtain this result:

Using the identity

$$\mathrm{tr}(Z^1 \cdot Z^2 \cdot Z^3) = \sum_{i,j,k=1}^{3} Z^1_{ij} Z^2_{jk} Z^3_{ki}$$

one obtains that the partial derivative to $Z^1_{ij}, 1 \leqslant i,j \leqslant 3$ of the cubic form is

$$(Z^1)_{ji} - \sum_{k=1}^{3} Z^2_{jk} Z^3_{ki} = (\mathrm{adj}(Z^1) - Z^2 Z^3)_{ji},$$

where the 'co-factor' $(Z^1)_{ji}$ denotes $(-1)^{i+j}$ times the minor of Z^1 obtained by cancelling the i-th row and j-th column. The other partial derivatives are obtained in a similar way by permuting the Z^1, Z^2, Z^3 in a cyclic way.

We now prove Proposition 5.1(d). Put

T_i = union of all $(i-1)$-dimensional projective planes through i points of X_1 together with their limit positions.

$$S_i = \{\mathbb{C}^* \cdot x \in \mathbb{P}(V_{\lambda_1}) \mid x = x_1 + \cdots + x_i \quad \text{for some } \mathbb{C}^* x_1, \ldots, \mathbb{C}^* x_i \in X_1\}.$$

Clearly S_i and T_i are G-stable, $S_i \subseteq T_i$ and $\bar{S}_i = \bar{T}_i$, thus it is sufficient to prove $X_i = S_i$.

For case I there is nothing to prove. Using the matrix representations above the assertion follows for the cases II, III and IV from the facts: (1) X_1 contains a basis for V_{λ_1} and (2) rank $(A + B) \leqslant$ rank (A) + rank (B) for matrices A, B. It remains to prove case V. Since S_2 is G-stable and $S_2 \supseteq X_1$, $S_2 \neq X_1$ it is sufficient to prove $S_2 \subseteq X_2$. For this purpose we use the description above of the 27 dimensional representation.

Let $(A, B, C), (A', B', C') \in V_{\lambda_1}$ two triples of 3×3-matrices with their equivalence class in X_1. Thus these triples are zero's of the 27 functions of I_2 or equivalently:

$$\mathrm{adj}(A) = BC, \quad \mathrm{adj}(B) = CA, \quad \mathrm{adj}(C) = AB \tag{$*$}$$

and similarly for (A', B', C'). We show that $s\cdot(A, B, C) + t(A', B', C')$ is for any $s, t \in \mathbb{C}$ a zero of the cubic form (5.3), and thus an element of X_2.

We will use that for all 3×3-matrices M and N holds:

$$\det(sM + tN) = s^3 \det(M) + s^2 t \operatorname{tr}(\operatorname{adj}(M)\cdot N) + st^2 \operatorname{tr}(M\cdot\operatorname{adj}(N)) + t^3\cdot\det(N).$$

This identity can be derived from

$$\det(M - tI) = \det(M) - \operatorname{tr}(\operatorname{adj}(M))t + \operatorname{tr}(M)t^2 - t^3$$

by substituting $sM\cdot N^{-1}$ for M and multiplying with $\det(N)$.

Substitute $s(A, B, C) + t(A', B', C')$ in the cubic form (5.7), we obtain

$$\det(sA + tA') + \det(sB + tB') + \det(sC + tC') - \operatorname{tr}((sA + tA')\cdot(sB + tB')(sC + tC')).$$

This is a homogeneous polynomial of degree 3 in the variables s and t. We determine these coefficients:

The coefficient of s^3 is

$$\det(A) + \det(B) + \det(C) - \operatorname{tr}(ABC).$$

Since $I_3 \subseteq I_2$ this coefficient must be zero.

The coefficient of s^2t is

$$\operatorname{tr}(\operatorname{adj}(A)\cdot A') + \operatorname{tr}(\operatorname{adj}(B)\cdot B') + \operatorname{tr}(\operatorname{adj}(C)\cdot C') - \operatorname{tr}(A'BC + AB'C + ABC').$$

Substituting $(*)$ in this expression gives

$$\operatorname{tr}(BCA') + \operatorname{tr}(CAB') + \operatorname{tr}(ABC') - \operatorname{tr}(A'BC + AB'C + ABC')$$

Since the trace function is linear and $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ for all 3×3 matrices M, N, it follows that this coefficient is zero. By symmetry the coefficients of st^2 and t^3 are also zero. This finishes the proof for case V and of assertion 5.1 (d).

Appendix. Multiplication of multivariable Jacobi polynomials

Let R be an irreducible root system, not necessarily reduced, with Weyl group W. Fix a base and denote by R_+, P and P_+ the positive roots in R, the weight lattice of R and the dominant weights of R respectively. Attach to each $\alpha \in R$ a $m_\alpha \in \mathbb{R}_{\geqslant 0}$ such that $m_{w\alpha} = m_\alpha$ for all $w \in W$, and define $m_\alpha = 0$ for $\alpha \in P \backslash R$. Let H be a complex torus with character lattice equal to the weight lattice P, and with compact form T. Define a Hermitean inner product on $\mathbb{C}[H]^W$, the W-invariant polynomial functions on H, by

$$(f, g) \doteq \int_T f(t)\overline{g(t)}\delta(t)\,\mathrm{d}t \qquad f, g \in \mathbb{C}[H]^W$$

where the weight function δ is given by

$$\delta(t) = \prod_{\alpha \in R_+} |1 - t^\alpha|^{m_\alpha}$$

and $\mathrm{d}t$ the normalized Haar measure on T.

One can write $\mathbb{C}[H] \simeq \bigoplus_{\lambda \in P} \mathbb{C} \cdot \chi_\lambda$, with $\chi_\lambda : H \to \mathbb{C}^*$ the character given by $\chi_\lambda : h \mapsto h^\lambda$. We recall some facts from [HO] and [H].

$\mathbb{C}[H]^W$ has a basis of orthogonal polynomials of the form

$$P(\mu, h) = \sum_{\nu \in C(\mu)} \Gamma_\nu(\mu)\chi_\nu \quad \mu \in P_+$$

with

$$\Gamma_\mu(\mu) = 1, \Gamma_{w\nu}(\mu) = \Gamma_\nu(\mu) \quad \text{for all} \quad w \in W \quad \text{and}$$

$$C(\mu) = \{\nu \in P \mid w\nu \leqslant \mu \quad \text{for all} \quad w \in W\}.$$

Here the partial order $\leqslant$ on P is as usual defined by

$$\lambda \leqslant \mu \quad \text{if and only if } \mu - \lambda \in R_+ \cdot \mathbb{Z}_{\geqslant 0}.$$

Our notation is fairly different from that in [HO] and [H]; m_α corresponds to $2k_\alpha$ in [HO] and our $P(\mu, h)$ corresponds to $\phi(w_0\mu, k, h)$ in [HO, (3.11)...(3.14)] and $P(w_0\mu, k; h)$ in [H, (8.2)], where w_0 denotes the longest element in W.

So given μ, $\nu \in P_+$ we can write

$$P(\mu, h) \cdot P(\nu, h) = \sum_{\lambda \leqslant \mu + \nu} d(\mu, \nu, \lambda) \cdot P(\lambda, h).$$

We are interested in the coefficients $d(\mu, \nu, \lambda)$. Note that these coefficients correspond to $d(w_0\mu, w_0(\nu+\rho), w_0(\nu+\rho+w\mu))$ in [H, (7.10)]. From the orthogonality relations follows $d(\mu,\nu,\lambda)=0$ if not $w_0\mu+\nu\leqslant\lambda\leqslant\mu+\nu$, see [H, 8.4)]. Write $m_1=\frac{1}{2}m_\alpha+m_{2\alpha}$ for $\alpha\in R_+$ a short root, $m_2=\frac{1}{2}m_\beta$ if there exists a $\beta\in R_+$ indivisible but not a short root and $m_2=0$ otherwise.

THEOREM. *Let μ, ν and $w\mu+\nu\in P_+$ for some $w\in W$. Then $d(\mu,\nu,w\mu+\nu)>0$ for all $m_\alpha>0$. In fact $d(\mu,\nu,w\mu+\nu)$ can be written as product of non-zero factors $(am_1+bm_2+c)^{\pm 1}$ with a, b, $c\in\mathbb{Z}_{\geqslant 0}$.*

This proposition is an immediate consequence of the two lemmas below. In fact all we have to do is to work out the following identity proved in [H, (7.10)]: For $m_\alpha\geqslant 0$ generic

$$d(\mu,\nu,w\mu+\nu)=\frac{c(w_0w^{-1}(\nu+\rho))\cdot c(w_0(\nu+\rho+w\mu))}{c(w_0(\nu+\rho))\cdot c(w_0(w^{-1}(\nu+\rho)+\mu))}, \tag{1}$$

where the c-function is defined by

$$c(\lambda)=c_0\cdot\sum_{\substack{\alpha\in R+\\ \alpha\text{ indivisible}}}c_\alpha(\lambda) \tag{2}$$

with

$$c_\alpha(\lambda)=2^{(\lambda,\alpha^\vee)+\frac{1}{2}m_\alpha+m_{2\alpha}}\cdot\frac{\Gamma(\frac{1}{2}(1+m_\alpha+m_{2\alpha}))}{\Gamma(\frac{1}{2}(-(\lambda,\alpha^\vee)+\frac{1}{2}m_\alpha+m_{2\alpha}))}\cdot$$

$$\frac{\Gamma(-(\lambda,\alpha^\vee))}{\Gamma(\frac{1}{2}(-(\lambda,\alpha^\vee)+\frac{1}{2}m_\alpha+1))} \tag{3}$$

and $c_0=c_0(m_\alpha)$ a nonzero constant, $\rho=\frac{1}{4}\Sigma_{\alpha\in R_+}m_\alpha\alpha, \alpha^\vee=2\alpha/(\alpha,\alpha)$, and $(\cdot,\cdot)$ a W-invariant inner product on the real vectorspace spanned by R.

First substitute the product formula (2) in (1), we have

$$d(\mu,\nu,w\mu+\nu)=\prod_{\substack{\alpha\in R+\\ \alpha\text{ indivisible}}}\frac{c_\alpha(w_0w^{-1}(\nu+\rho))\cdot c_\alpha(w_0(\nu+\rho+w\mu))}{c_\alpha(w_0(\nu+\rho))\cdot c_\alpha(w_0(w^{-1}(\nu+\rho)+\mu))}.$$

Now use $c_\alpha(w\lambda)=c_{w^{-1}\alpha}(\lambda)$, $w_0R_+=R_-$ and $w_0^2=1$. We get

$$d(\mu,\nu,w\mu+\nu)=\prod_{\substack{\alpha\in R+\\ \alpha\text{ indivisible}}}\frac{c_{-\alpha}(w^{-1}(\nu+\rho))\cdot c_{-\alpha}(\nu+\rho+w\mu)}{c_{-\alpha}(\nu+\rho)\cdot c_{-\alpha}(w^{-1}(\nu+\rho)+\mu)}$$

$$= \prod_{\substack{\alpha \in R_+ \\ \alpha \text{ indivisible}}} \frac{c_{-w\alpha}(\nu + \rho)}{c_{-\alpha}(\nu + \rho)} \cdot \frac{c_{-\alpha}(\nu + \rho + w\mu)}{c_{-w\alpha}(\nu + \rho + w\mu)}$$

$$= \prod_{\substack{\alpha \in R_+ \\ \alpha \text{ indivisible}}} \frac{c_{-\alpha}(\nu + \rho + w\mu)}{c_{-\alpha}(\alpha + \rho)} \cdot \prod_{\substack{\alpha \in wR_+ \\ \alpha \text{ indivisible}}} \frac{c_{-\alpha}(\nu + \rho)}{c_{-\alpha}(\nu + \rho + w\mu)}$$

Now write

$$R_+ = (R_+ \cap wR_+) \cup (R_+ \cap wR_-) \text{ and } wR_+ = (R_+ \cap wR_+) \cup (R_- \cap wR_+),$$

then the factors in both products corresponding to the α in $R_+ \cap wR_+$ cancel out, hence

LEMMA 1.

$$d(\mu, \nu, w\mu + \nu) = \prod_{\substack{\alpha \in R_+ \cap wR_- \\ \alpha \text{ indivisible}}} \frac{c_{-\alpha}(\nu + \rho + w\mu)}{c_{-\alpha}(\nu + \rho)} \cdot \prod_{\substack{\alpha \in R_- \cap wR_+ \\ \alpha \text{ indivisible}}} \frac{c_{-\alpha}(\nu + \rho)}{c_{-\alpha}(\nu + \rho + w\mu)}$$

$$= \prod_{\substack{\alpha \in R_+ \cap wR_- \\ \alpha \text{ indivisible}}} \frac{c_{-\alpha}(\nu + \rho + w\mu) \cdot c_\alpha(\nu + \rho)}{c_{-\alpha}(\nu + \rho) \cdot c_\alpha(\nu + \rho + w\mu)}. \tag{4}$$

□

Now fix $\alpha \in R_+ \cap wR_-$, α indivisible. Then

LEMMA 2.

$$\frac{c_{-\alpha}(\nu + \rho + w\mu)}{c_{-\alpha}(\nu + \rho)} \tag{5}$$

and

$$\frac{c_\alpha(\nu + \rho)}{c_\alpha(\nu + \rho + w\mu)} \tag{6}$$

can be written as product of non-zero factors $(am_1 + bm_2 + c)^{\pm 1}$ *with* $a, b, c \in \mathbb{Z}_{\geqslant 0}$.

In the proof of this lemma we will use the following facts:
$-(w\mu, \alpha^\vee) = (\mu, (w^{-1}\alpha)^\vee) \leqslant 0$ since $w^{-1}\alpha \in R_-$.

- If $2\alpha \in R$ then $\frac{1}{2}(w\mu, \alpha^\vee) = (w\mu, (\alpha/(\alpha, \alpha))) = (w\mu, (2\alpha)^\vee) \in \mathbb{Z}$.
- $(\rho, \alpha^\vee) = am_1 + bm_2$ for some $a, b \in \mathbb{Z}_{\geqslant 0}$ and $a \neq 0$ if α is a short root while $b \neq 0$ if α is not a short root.

To see the last fact, write $\rho = m_1\rho_1 + m_2\rho_2$ with $\rho_1 = \frac{1}{2}\Sigma\alpha$, sum over all short and indivisible roots $\alpha \in R_+$ and $\rho_2 = \frac{1}{2}\Sigma\alpha$, sum over all not short and indivisible roots $\alpha \in R_+$. Now $(\alpha, \rho) = m_1(\alpha, \rho_1) + m_2(\alpha, \rho_2)$ and $(\alpha, \rho_1) \neq 0$ if α short and indivisible, while $(\alpha, \rho_2) \neq 0$ if α not short and indivisible.

Furthermore we denote $(z)_x = \Gamma(z + x)/\Gamma(z)$.

The duplication formula of the Γ-function gives $(2z)_{2x} = (z)_x \cdot (z + \frac{1}{2})_x \cdot 2^{2x}$.

If we take $x = n$ a positive integer then we have the Pochhammer symbol

$$(z)_n = z\cdot(z+1)\cdot\cdot(z+n-1) = (-1)^n\cdot(-z-n+1)_n.$$

Proof of Lemma 2. We begin with the substitution of formula (3). We have

$$\begin{aligned}
\frac{c_\alpha(\nu+\rho)}{c_\alpha(\nu+\rho+w\mu)} &= 2^{-(w\mu,\alpha^\vee)}\cdot\frac{\Gamma(-(\nu+\rho,\alpha^\vee))}{\Gamma(-(\nu+\rho+w\mu,\alpha^\vee))}\cdot\\
&\quad \frac{\Gamma(\frac{1}{2}(-(\nu+\rho+w\mu,\alpha^\vee)+\frac{1}{2}m_\alpha+m_{2\alpha}))}{\Gamma(\frac{1}{2}(-(\nu+\rho,\alpha^\vee)+\frac{1}{2}m_\alpha+m_{2\alpha}))}\cdot\\
&\quad \frac{\Gamma(\frac{1}{2}(-(\nu+\rho+w\mu,\alpha^\vee)+\frac{1}{2}m_\alpha+1))}{\Gamma(\frac{1}{2}(-(\nu+\rho,\alpha^\vee)+\frac{1}{2}m_\alpha+1))}\\
&= 2^{-(w\mu,\alpha^\vee)}(\tfrac{1}{2}(-(\nu+\rho,\alpha^\vee)+\tfrac{1}{2}m_\alpha+m_{2\alpha}))_{-(\frac{1}{2}w\mu,\alpha^\vee)}\cdot\\
&\quad \frac{(\frac{1}{2}(-(\nu+\rho,\alpha^\vee)+\frac{1}{2}m_\alpha+1))_{-\frac{1}{2}(w\mu,\alpha^\vee)}}{(-(\nu+\rho,\alpha^\vee))_{-(w\mu,\alpha^\vee)}}.
\end{aligned}\tag{7}$$

and in a similar way

$$\begin{aligned}
\frac{c_{-\alpha}(\nu+\rho+w\mu)}{c_{1\alpha}(\nu+\rho)} &= 2^{-(w\mu,\alpha^\vee)}(\tfrac{1}{2}((\nu+\rho+w\mu,\alpha^\vee)+\tfrac{1}{2}m_\alpha+m_{2\alpha}))_{-\frac{1}{2}(w\mu,\alpha^\vee)}\cdot\\
&\quad \frac{(\frac{1}{2}((\nu+\rho+w\mu,\alpha^\vee)+\frac{1}{2}m_\alpha+m_{2\alpha}))_{-\frac{1}{2}(w\mu,\alpha^\vee)}}{((\nu+\rho+w\mu,\alpha^\vee))_{-(w\mu,\alpha^\vee)}}.
\end{aligned}\tag{8}$$

Now distinguish the two cases $2\alpha \in R$ and $2\alpha \notin R$.

If $2\alpha \in R$ then we know that $-\frac{1}{2}(w\mu, \alpha^\vee)$ is a positive integer. Then we can rewrite (7) into

$$\frac{c_\alpha(\nu+\rho)}{c_\alpha(\nu+\rho+w\mu)} = 2^{-(w\mu,\alpha^\vee)}\left(\tfrac{1}{2}((\nu+\rho+w\mu,\alpha^\vee)-\tfrac{1}{2}m_\alpha-m_{2\alpha})+1\right)_{-\frac{1}{2}(w\mu,\alpha^\vee)}\cdot$$

$$\frac{\left(\tfrac{1}{2}((\nu+\rho+w\mu,\alpha^\vee)-\tfrac{1}{2}m_\alpha-1)+1\right)_{-\frac{1}{2}(w\mu,\alpha^\vee)}}{((\nu+\rho+w\mu,\alpha^\vee)+1)_{-(w\mu,\alpha^\vee)}}. \tag{9}$$

Next write out the Pochhammer symbols in (8) and (9) in order to obtain the desired product formulas. By the third fact mentioned before the proof it is clear that all coefficients of m_1 and m_2 are non-negative.

Now assume $2\alpha \notin R$. Then $m_{2\alpha} = 0$, so we can use the duplication formula in order to rewrite (7) and (8). We get

$$\frac{(c_\alpha(\nu+\rho)}{c_\alpha(\nu+\rho+w\mu)} = \frac{(-(\nu+\rho,\alpha^\vee)+\tfrac{1}{2}m_\alpha)_{-(w\mu,\alpha^\vee)}}{(-(\nu+\rho,\alpha^\vee))_{-(w\mu,\alpha^\vee)}} \tag{10}$$

$$\frac{c_{-\alpha}(\nu+\rho+w\mu)}{c_{-\alpha}(\nu+\rho)} = \frac{((\nu+\rho+w\mu,\alpha^\vee)+\tfrac{1}{2}m_\alpha)_{-(w\mu,\alpha^\vee)}}{(\nu+\rho+w\mu,\alpha^\vee)_{-(w\mu,\alpha^\vee)}}. \tag{11}$$

We can finish the proof as in the first case. □

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