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FRED DIAMOND

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On congruence modules associated to Λ-adic forms

FRED DIAMOND
Department of Mathematics, Ohio State University, Columbus, OH 43210, U.S.A.

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1. Introduction and notation

In this paper we generalize a result of Ribet [R2] concerning congruences between modular forms. The problem is to raise the level of newform by prime l in the following sense: If f is a newform of level N prime to l, when is there a congruent newform g of the same character and weight, but of level dl with d dividing N?

THEOREM 1 (Ribet). *If* $f = \Sigma\, a_n q^n$ *is a newform in* $S_2(\Gamma_0(N); K)$, *then such a* $g = \Sigma b_n q^n$ *exists if and only if* $a_l^2 \equiv (l+1)^2 \bmod \mathfrak{p}$.
(Here K is a sufficiently large number field, $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ not dividing $\phi(N)Nl$, and f congruent to g means $a_n \equiv b_n \bmod \mathfrak{p}$ for n prime to Nl.)

If two newforms are congruent, then the associated representations $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathcal{O}_K/\mathfrak{p})$ have isomorphic semi-simplifications. The constraint on a_l follows immediately from the properties of the representations. To prove the existence of congruences is more difficult, and Ribet's proof relies on the injectivity of a certain homomorphism of Jacobians of modular curves. This is a result of Ihara [I, Lemma 3.2].

The aim of this paper is to prove an analogue of Theorem 1 for ordinary Λ-adic forms. These are formally q-expansions with coefficients in $\Lambda = \mathbb{Z}_p[[T]]$ which p-adically interpolate classical modular forms. They have been studied by Hida [H4], [H5] and [H6], and by Wiles, whose conjecture [W, §1.6] motivates our main result, Theorem 6. Corollary 6.9 provides a generalization of Theorem 1 to p-ordinary forms of any character and weight $k \geqslant 2$.

Our general approach to the problem is the same as Ribet's, but we appeal extensively to Hida's theory of families of congruent modular forms to make Ihara's lemma effective in the context of Λ-adic forms.

We begin by using the duality between modular forms and Hecke operators in a standard way (e.g. [R1]) to reduce the problem of finding congruences to the study of a certain Hecke module constructed using the cohomology of modular curves. Next, in Chapter 3, we review Ribet's method of computing this module and its annihilator [R2]. A further analysis is aimed at proving the existence of a certain set of newforms congruent to f (Theorem 4c). We sharpen these results for p-ordinary forms (Theorem 5) by proving that Hida's idempotent annihilates the p-part of the Shimura subgroup, which is a source of exceptional primes in Theorem 1. We then use Hida's theory to study an analogue of the cohomology congruence module for ordinary Λ-adic forms and prove Theorem 6. In the last chapter, we prove slightly weaker versions of these results for $p = 2$.

Throughout the paper we fix a rational prime p and a finite extension K of $\mathbb{Q}_p$. We let $\mathcal{O}_K$ denote the integral closure of $\mathbb{Z}_p$ in K and $\mathfrak{p}$ the maximal ideal of $\mathcal{O}_K$. We also fix embeddings of K into the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and of $\bar{\mathbb{Q}}$ into $\mathbb{Q}_p$ and $\mathbb{C}$.

For a positive integer m, let

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod m \right\}$$

and

$$\Gamma_1(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m) \,\middle|\, d \equiv 1 \bmod m \right\}.$$

For a group Γ with $\Gamma_1(m) \subseteq \Gamma \subseteq \Gamma_0(m)$, we let $S_k(\Gamma; \mathbb{C})$ denote the cusp forms of weight k for Γ. We will identify a cusp form with its Fourier expansion at infinity, i.e., its q-expansion. For a subring A of $\mathbb{C}$, we let $S_k(\Gamma; A) = S_k(\Gamma; \mathbb{C}) \cap A[[q]]$. It is a well-known result of Shimura that $S_k(\Gamma; \mathbb{C}) = S_k(\Gamma; \mathbb{Z}) \otimes \mathbb{C}$. Thus for any field F containing $\mathbb{Q}$, we can define $S_k(\Gamma; F)$ as $S_k(\Gamma; \mathbb{Z}) \otimes F$, and for $A \subseteq F$, we let $S_k(\Gamma; A) = S_k(\Gamma; F) \cap A[[q]]$.

We can define an action of the Hecke operator $T_{n,m}$ for $n \geqslant 1$ on $S_k(\Gamma; K)$ which preserves $S_k(\Gamma; \mathcal{O}_K)$ [H3, (4.1)]. This is often denoted $T(n)$. For q prime to m, we write $S_{q,m}$ for the Hecke operator often denoted $T(q, q)$. There is also an action of the Hecke operators on various cohomology groups associated to Γ. We usually write T_n or S_q for any endomorphism defined by $T_{n,m}$ or $S_{q,m}$. We make frequent use of standard properties of Hecke operators which can be found in [S], [H3] or [L].

To any eigenform in $S_k(\Gamma; \mathbb{C})$ of the Hecke operators $T_{n,m}$ for all $n \geqslant 1$, there is associated a unique newform of level d for some d dividing m. This is a normalized eigenform of the Hecke operators $T_{n,d}$ and is often called a primitive form. By a congruence between newforms, we shall always mean a congruence of nth Fourier coefficients for n prime to the levels. Otherwise a congruence between cusp forms refers to all their coefficients.

2. Duality and congruence modules

In this chapter we discuss in some generality a "congruence module" associated to two spaces of cusp forms. We fix a level m and a weight k.

For any K-subspace $S \subseteq S_k(\Gamma_1(m), K)$ which is stable under the Hecke operators T_n for all $n \geqslant 1$, let M_S denote the lattice of forms in S with integral Fourier expansions. Thus,

$$M_S = S(\mathcal{O}_K) = S \cap S_k(\Gamma_1(m), \mathcal{O}_K).$$

Suppose that X, Y and Z are such spaces and that $Z = X \oplus Y$. Then M_Z contains $M_X \oplus M_Y$ and we define the congruence module

$$\frac{C_{X,Y} = M_Z}{(M_X \oplus M_Y).}$$

$C_{X,Y}$ is a Hecke module with only a finite number of elements. It measures congruences between forms in X and Y as follows: for $f \in M_X$, $g \in M_Y$ and $d \in \mathcal{O}_K$, we have

$$f \equiv g \bmod d\mathcal{O}_K \quad \text{if and only if} \quad d^{-1}(f-g) \in M_Z.$$

We now use the duality between modular forms and Hecke operators to relate the congruence module to a quotient of the Hecke algebra. For a space S as above, denote by $\mathbb{T}_S(K)$ the K-algebra of endomorphisms of S generated by the T_n. Similarly let $\mathbb{T}_S$ be the $\mathcal{O}_K$-algebra of endomorphisms of M_S generated by the T_n. We regard $\mathbb{T}_S$ as a subring of $\mathbb{T}_S(K) \cong \mathbb{T}_S \otimes_{\mathcal{O}_K} K$.

The bilinear pairings

$$\mathbb{T}_S(K) \times S \to K \quad \text{and} \quad \mathbb{T}_S \times M_S \to \mathcal{O}_K$$

defined by $(T, f) \mapsto c_1(f \mid T)$, where $c_n\colon S \to K$ sends a form to its nth Fourier coefficient, induce homomorphisms

$$\phi_S\colon S \to \mathrm{Hom}_K(\mathbb{T}_S(K), K) \quad \text{and} \quad \phi_{M_S}\colon M_S \to \mathrm{Hom}_{\mathcal{O}_K}(\mathbb{T}_S, \mathcal{O}_K).$$

PROPOSITION 2.1. *ϕ_S is an isomorphism of $\mathbb{T}_S(K)$-modules, and ϕ_{M_S} is an isomorphism of $\mathbb{T}_S$-modules.*

This duality result is well-known for $S = S_k(\Gamma_1(m), \mathcal{O}_K)$. The proof (e.g. [L, Th. 4.4]) work for any subspace S stable under the Hecke action.

Now suppose again that we have a decompositon $Z = X \oplus Y$ into spaces stable under the Hecke operators. Then restricting Hecke operators to X and Y yields the surjections $e_X: \mathbb{T}_Z(K) \to \mathbb{T}_X(K)$ and $e_Y: \mathbb{T}_Z(K) \to \mathbb{T}_Y(K)$. Their sum is the inclusion $\mathbb{T}_Z(K) \to \mathbb{T}_X(K) \oplus \mathbb{T}_Y(K)$. By the previous proposition, $\mathbb{T}_Z(K)$ and $\mathbb{T}_X(K) \oplus \mathbb{T}_Y(K)$ have equal dimension, so we may identify them and regard $\mathbb{T}_X(K)$ and $\mathbb{T}_Y(K)$ as subrings of $\mathbb{T}_Z(K)$, and e_X and e_Y as idempotents in $\mathbb{T}_Z(K)$. Restricting e_X and e_Y to $\mathbb{T}_Z$ also yields surjections $\mathbb{T}_Z \to \mathbb{T}_X$ and $\mathbb{T}_Z \to \mathbb{T}_Y$, but now the inclusion $\mathbb{T}_Z \to \mathbb{T}_X \oplus \mathbb{T}_Y$ has finite cokernel.

Restricting the pairing $\mathbb{T}_Z(K) \times Z \to K$ to $(\mathbb{T}_X \oplus \mathbb{T}_Y) \times M_Z$ induces the pairing

$$\frac{(\mathbb{T}_X \oplus \mathbb{T}_Y)}{\mathbb{T}_Z} \times \frac{M_Z}{(M_X \oplus M_Y)} \to \frac{K}{\mathcal{O}_K},$$

since $\mathbb{T}_S \times M_S \to \mathcal{O}_K$ for $S = X, Y$ and Z. This pairing induces a $\mathbb{T}_Z$-linear homomorphism

$$\phi_{X,Y}: C_{X,Y} \to \operatorname{Hom}_{\mathcal{O}_K}(\mathbb{T}_{X,Y}, K/\mathcal{O}_K)$$

where we put $\mathbb{T}_{X,Y} = (\mathbb{T}_X \oplus \mathbb{T}_Y)/\mathbb{T}_Z$.

PROPOSITION 2.2. *$\phi_{X,Y}$ is an isomorphism.*

Proof. We have the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & M_X \oplus M_Y & \longrightarrow & M_Z & \longrightarrow & C_{X,Y} & \longrightarrow 0 \\
& \downarrow{\scriptstyle \phi_{M_X} \oplus \phi_{M_Y}} & & \downarrow{\scriptstyle \phi_{M_Z}} & & \downarrow{\scriptstyle \phi_{X,Y}} & \\
0 \longrightarrow & \widehat{\mathbb{T}_X} \oplus \widehat{\mathbb{T}_Y} & \xrightarrow{\alpha} & \widehat{\mathbb{T}_Z} & \xrightarrow{\beta} & \operatorname{Hom}_{\mathcal{O}_K}(\mathbb{T}_{X,Y}, K/\mathcal{O}_K) & \longrightarrow 0.
\end{array}$$

Here α is the transpose of the natural inclusion. Since this has finite cokernel, α is injective. β is defined as follows: Any $s \in \operatorname{Hom}_{\mathcal{O}_K}(\mathbb{T}_Z, \mathcal{O}_K)$ extends uniquely to some $s' \in \operatorname{Hom}_K(\mathbb{T}_Z(K), (K)$; for $T \in \mathbb{T}_X \oplus \mathbb{T}_Y$, define $\beta(s)(T \bmod \mathbb{T}_Z) = s'(T) \bmod \mathcal{O}_K$. Since $\mathbb{T}_X \oplus \mathbb{T}_Y$ is a free $\mathcal{O}_K$-module, β is surjective. Also, $\ker \beta = \operatorname{image} \alpha$. It is easy to see that the diagram commutes. Since the rows are exact, and (by Proposition 2.1) $\phi_{M_X} \oplus \phi_{M_Y}$ and ϕ_{M_Z} are isomorphisms, so is $\phi_{X,Y}$. □

COROLLARY 2.3. *$C_{X,Y}$ is isomorphic to $\mathbb{T}_{X,Y}$ as an $\mathcal{O}_K$-module (non-canonically).*

Proof. This follows immediately from their structure as finitely generated torsion $\mathcal{O}_K$-modules. □

Now denote by I_X and I_Y the annihilators of X and Y in $\mathbb{T}_Z$, i.e., the kernels of

e_X and e_Y. Note that e_X induces isomorphisms

$$\frac{\mathbb{T}_X}{e_X(I_Y)} \xleftarrow{\sim} \frac{\mathbb{T}_Z}{(I_X + I_Y)} \xrightarrow{\sim} \mathbb{T}_{X,Y}.$$

So we can prove the existence of congruences by exhibiting a $\mathbb{T}_Z$-module, say Ω, such that the action of $\mathbb{T}_Z$ factors through $\mathbb{T}_X$ and $\mathbb{T}_Y$:

$$\begin{array}{ccc} & \mathbb{T}_X & \\ \nearrow & & \searrow \\ \mathbb{T}_Z & & \mathrm{End}_{\mathcal{O}_K}(\Omega) \\ \searrow & & \nearrow \\ & \mathbb{T}_Y & \end{array} \tag{2.1}$$

Then $\mathrm{Ann}_{\mathbb{T}_Z}(\Omega) \supseteq I_X + I_Y$, or equivalently, $\mathrm{Ann}_{\mathbb{T}_X}(\Omega) \supseteq e_X(I_Y)$.

For an integer j, and a space S of cusp forms stable under the Hecke operators, we define $\mathbb{T}_S^{(j)}(K)$ to be the K-algebra of endomorphisms of S generated by the T_n for n prime to j. Then let $S_\ell^{(j)}$ be the kernel of the surjection $S \to \mathrm{Hom}_K(\mathbb{T}_S^{(j)}(K), K)$ induced by ϕ_S. Define $S^{(j)}$ as $S/S_\ell^{(j)}$. Then, as $\mathbb{T}_S^{(j)}(K)$-modules, we have the isomorphism $S^{(j)} \xrightarrow{\sim} \mathrm{Hom}_K(\mathbb{T}_S^{(j)}(K), K)$. Similarly, we define $\mathbb{T}_S^{(j)}$ and obtain $M_S^{(j)} \subseteq S^{(j)}$ isomorphic to $\mathrm{Hom}_{\mathcal{O}_K}(\mathbb{T}_S^{(j)}, \mathcal{O}_K)$. For $f \in S$, we write $f^{(j)}$ for the image of f in $S^{(j)}$. Then $f^{(j)} \in M_S^{(j)}$ if and only if $c_1(f|T) \in \mathcal{O}_K$ for all $T \in \mathbb{T}_S^{(j)}$. In particular note that $c_n(f) \in \mathcal{O}_K$ for all n prime to j.

Suppose that $Z = X \oplus Y$ is a decomposition which is stable under the Hecke operators and has the property

$$Z_\ell^{(j)} = X_\ell^{(j)} \oplus Y_\ell^{(j)}. \tag{2.2}$$

Then we obtain a decompositon $Z^{(j)} = X^{(j)} \oplus Y^{(j)}$ with $M_X^{(j)} \oplus M_Y^{(j)} \subseteq M_Z^{(j)}$. We define a more general congruence module

$$\frac{C_{X,Y}^{(j)} = M_Z^{(j)}}{(M_X^{(j)} \oplus M_Y^{(j)})}.$$

This measures congruences of nth coefficients for n prime to j. If we also define $\mathbb{T}_{X,Y}^{(j)}$ to be the cokernel of the injection $\mathbb{T}_Z^{(j)} \to \mathbb{T}_X^{(j)} \oplus \mathbb{T}_Y^{(j)}$, we get an isomorphism as in Proposition 2.2.

$$\phi_{X,Y}^{(j)}: C_{X,Y}^{(j)} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K}(\mathbb{T}_{X,Y}^{(j)}, K/\mathcal{O}_K). \tag{2.3}$$

Note that $\mathbb{T}_{X,Y}^{(j)} \cong \mathbb{T}_Z^{(j)}/(I_X^{(j)} + I_Y^{(j)})$ where $I_X^{(j)} = I_X \cap \mathbb{T}_Z^{(j)}$ and $I_Y^{(j)} = I_Y \cap \mathbb{T}_Z^{(j)}$ are the annihilators of X and Y in $\mathbb{T}_Z^{(j)}$. Thus to prove the existence of congruences "outside j", we construct a module for $\mathbb{T}_Z^{(j)}$ so that the action factors through $\mathbb{T}_X^{(j)}$ and $\mathbb{T}_Y^{(j)}$.

3. The cohomology congruence module

Now we turn to the specific case of a decomposition into the spaces of forms which are new and old at a prime l. In this chapter we review Ribet's method of computing the cohomology congruence module and its annihilator [R2]. We do this for forms of any character.

We fix a level Nl, with $l \nmid N$, and consider cusp forms of weight 2. Then for any field $F \supseteq \mathbb{Q}$, we let

$$Z(F) = S_2(\Gamma_1(N) \cap \Gamma_0(l); F) = S_2(\Gamma_1(N) \cap \Gamma_0(l); \mathbb{Q}) \otimes_{\mathbb{Q}} F.$$

There are the two injections

$$b_1, b_l \colon S_2(\Gamma_1(N); \mathbb{C}) \to Z(\mathbb{C})$$

defined by

$$f \mapsto f \quad \text{and} \quad f \mapsto f \left| \left[\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \right] \right. .$$

Let $X(\mathbb{C})$ be the sum of the images of these two maps, and let $Y(\mathbb{C})$ be the space orthogonal to $X(\mathbb{C})$ under the Petersson inner product.

$X(\mathbb{C})$ and $Y(\mathbb{C})$ have bases consisting of forms with rational Fourier coefficients, so we can decompose

$$Z(\mathbb{Q}) = X(\mathbb{Q}) \oplus Y(\mathbb{Q})$$

where

$$X(\mathbb{Q}) = Z(\mathbb{Q}) \cap X(\mathbb{C}) \quad \text{and} \quad Y(\mathbb{Q}) = Z(\mathbb{Q}) \cap Y(\mathbb{C}),$$

and then

$$Z(F) = X(F) \oplus Y(F)$$

where

$$X(F) = X(\mathbb{Q}) \otimes_{\mathbb{Q}} F \quad \text{and} \quad Y(F) = Y(\mathbb{Q}) \otimes_{\mathbb{Q}} F.$$

We write X, Y and Z for $X(K)$, $Y(K)$ and $Z(K)$, where K is the field we fixed in Chapter 1.

$X(\mathbb{C})$ and $Y(\mathbb{C})$ are stable under the Hecke operators T_n, for $n \geqslant 1$. So in fact $X(F)$ and $Y(F)$ are stable, and we can define $\mathbb{T}_X(F)$, $\mathbb{T}_Y(F)$ and $\mathbb{T}_Z(F)$ as the F-algebras of endomorphisms of $X(F)$, $Y(F)$ and $Z(F)$ generated by the T_n. Then we have natural isomorphisms

$$\mathbb{T}_X(F) \cong \mathbb{T}_X(\mathbb{Q}) \otimes_{\mathbb{Q}} F, \quad \mathbb{T}_Y(F) \cong \mathbb{T}_Y(\mathbb{Q}) \otimes_{\mathbb{Q}} F$$

and

$$\mathbb{T}_Z(F) \cong \mathbb{T}_Z(\mathbb{Q}) \otimes_{\mathbb{Q}} F.$$

Now we use the cohomology of modular curves to construct modules for $\mathbb{T}_X$ and $\mathbb{T}_Y$. Define the curves

$$\mathfrak{X} = (\Gamma_1(N) \cap \Gamma_0(l))\backslash\mathfrak{H}^* \quad \text{and} \quad \mathfrak{X}' = (\Gamma_1(N))\backslash\mathfrak{H}^*.$$

Then there are two coverings $B_1, B_l\colon \mathfrak{X} \to \mathfrak{X}'$ defined by $z \mapsto z$ and $z \mapsto lz$ on $\mathfrak{H}$. For any field $F \supseteq \mathbb{Q}$, define

$$V(F) = \mathrm{H}^1(\mathfrak{X}; F) \cong \mathrm{H}^1(\mathfrak{X}; \mathbb{Q}) \otimes_{\mathbb{Q}} F$$

and

$$V'(F) = \mathrm{H}^1(\mathfrak{X}'; F) \cong \mathrm{H}^1(\mathfrak{X}'; \mathbb{Q}) \otimes_{\mathbb{Q}} F.$$

Then the coverings above induce

$$\alpha_F = B_1^* \oplus B_l^*\colon V'(F)^2 \to V(F). \tag{3.1}$$

Write $A(F)$ for the image of α_F. The cup products define non-degenerate skew-symmetric bilinear pairings

$$C\colon V(F) \times V(F) \to F \quad \text{and} \quad C'\colon V'(F) \times V'(F) \to F. \tag{3.2}$$

These induce isomorphisms

$$\theta_F\colon V(F) \to \widehat{V(F)} \quad \text{and} \quad \theta'_F\colon V'(F) \to \widehat{V'(F)}.$$

Let $B(F)$ be the space orthogonal to $A(F)$ under C. Thus $B(F) = \ker \alpha'_F$ where

$$\alpha'_F = (\theta'_F \oplus \theta'_F)^{-1} \circ {}^t\alpha_F \circ \theta_F\colon V(F) \to V'(F)^2. \tag{3.3}$$

Write simply A, B, V and V' for $A(K)$, $B(K)$, $V(K)$ and $V'(K)$.

There is an action of the Hecke operators T_n on $V(\mathbb{R})$, which commutes with the natural isomorphism $\phi: Z(\mathbb{C}) \xrightarrow{\sim} V(\mathbb{R})$, and which is stable on $V(\mathbb{Q}) \subseteq V(\mathbb{R})$. It follows immediately that $\mathbb{T}_Z(F)$ acts faithfully on $V(F)$.

PROPOSITION 3.1. $A = e_x V$ and $B = e_Y V$.

Proof. First we note the commutativity of

$$\begin{array}{ccc} S_2(\Gamma_1(N); \mathbb{C}) & \xrightarrow{b_i} & Z(C) \\ \phi' \downarrow \wr & & \phi' \downarrow \wr \\ V'(\mathbb{R}) & \xrightarrow{B_i^*} & V(\mathbb{R}) \end{array}$$

for $i = 1$ and l. So $\phi(X(\mathbb{C})) = A(\mathbb{R})$.

Then the equation

$$C(\phi(f), \phi(g)) = \operatorname{Im}(\langle f, g \rangle) \quad \text{for } f, g \in Z(\mathbb{C}) \tag{3.4}$$

(where $\langle , \rangle$ denotes the Peterson inner product) shows that $\phi(Y(\mathbb{C})) = B(\mathbb{R})$.

Now note that since $Z(\mathbb{Q}) = X(\mathbb{Q}) \oplus Y(\mathbb{Q})$, we have $e_X, e_Y \in \mathbb{T}_Z(\mathbb{Q})$ (identifying $\mathbb{T}_Z(\mathbb{Q})$ with its image in $\mathbb{T}_Z(F)$). Since ϕ commutes with e_X and e_Y, we have

$$e_X V(\mathbb{R}) = A(\mathbb{R}) \quad \text{and} \quad e_Y V(\mathbb{R}) = B(\mathbb{R}).$$

It then follows that $A \subseteq e_X V$ and $B \subseteq e_Y V$. Counting dimensions yields the desired equalities. □

We have lattices L in V, and L' in V', defined by the images of the cohomology groups with coefficients in $\mathcal{O}_K$,

$$L \cong \mathrm{H}^1(\mathfrak{X}; \mathcal{O}_K) \quad \text{and} \quad L' \cong \mathrm{H}^1(\mathfrak{X}'; \mathcal{O}_K).$$

The Hecke operators act on these as well, so L is a $\mathbb{T}_Z$-module. Now we can define the cohomology congruence module

$$\Omega = \frac{((L + A) \cap (L + B))}{L}.$$

This is the intersection of the image of A with that of B in V/L. It is a $\mathbb{T}_Z$-module, and by Proposition 3.1, the action factors through $\mathbb{T}_X$ and $\mathbb{T}_Y$ as in (2.1).

The cup product (3.2) induces an isomorphism $\theta_K: V \xrightarrow{\sim} \hat{V}$ which restricts to give $\theta_{\mathcal{O}_K}: L \xrightarrow{\sim} \hat{L}$, thus defining an isomorphism $\theta: (V/L) \xrightarrow{\sim} (\hat{V}/\hat{L})$. Similarly, we define $\theta': (V'/L') \xrightarrow{\sim} (\hat{V}'/\hat{L}')$. Since $\alpha_K(L'^2) \subseteq L$, α_K induces a homomorphism $\alpha: (V'/L)^2 \to (V/L)$. Similarly α'_K induces

$$\alpha' = (\theta' \oplus \theta')^{-1} \circ {}^t\alpha \circ \theta: (V/L) \to (V'/L')^2.$$

These $\mathcal{O}_K$-linear maps are displayed in the diagram

$$\begin{array}{ccc} (V'/L')^2 & \xrightarrow{\alpha} & V/L \\ {\scriptstyle (\theta'\oplus\theta')^{-1}}\uparrow\wr & & \theta\downarrow\wr \\ (\hat{V}'/\hat{L}')^2 & \xleftarrow{{}^t\alpha} & \hat{V}/\hat{L}. \end{array} \tag{3.5}$$

The key to computing the cohomology congruence module is [R2, Th. 4.1] or [I, Lemma 3.2].

LEMMA 3.2. *α is injective.*

Proof. We have by Corollary 4.2 of [R2] the surjectivity of

$$(B_{1*}, B_{l*}): \mathrm{H}_1(\mathfrak{X}; \mathbb{Z}) \to \mathrm{H}_1(\mathfrak{X}'; \mathbb{Z})^2.$$

This implies the injectivity of

$$\begin{array}{cccc} & \mathrm{Hom}(\mathrm{H}_1(\mathfrak{X}'; \mathbb{Z}), K/\mathcal{O}_K)^2 & \longrightarrow & \mathrm{Hom}(\mathrm{H}_1(\mathfrak{X}; \mathbb{Z}), K/\mathcal{O}_K) \\ & \wr\| & & \wr\| \\ \text{and of} & (V'/L')^2 & \xrightarrow{\alpha} & V/L. \end{array}$$

□

We note a consequence of the lemma. Since α is injective we have $\alpha_{\mathcal{O}_K}(L'^2) = A \cap L$. It follows that $\ker \alpha' = (B + L)/L$. This, together with image $\alpha = (A + L)/L$, gives the equation

$$\Omega = \text{image}\, \alpha \cap \ker \alpha'. \tag{3.6}$$

Also note that we can define an action of $\mathbb{T}_Z$ on $(V'/L')^2$ so that α is a homomorphism of $\mathbb{T}_Z$-modules. We can describe the action explicitly in terms of the Hecke operators on $S_2(\Gamma_1(N), K)$. Writing τ_n for $T_{n,N}$ and σ_n for $S_{n,N}$, we have for $f \in V'(\mathbb{Q}) \subseteq V'(\mathbb{R}) \cong S_2(\Gamma_1(N), \mathbb{C})$,

$$\text{if } l \nmid n, \quad \text{then} \quad f|T_n = f|\tau_n \quad \text{and} \quad f\left|\left[\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}\right]\right|T_n = f|\tau_n\left[\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix}\right],$$

while

$$f|T_l = f|\tau_l - f|[\sigma_l]\left[\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\right] \quad \text{and} \quad f\left|\left[\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\right]\right|T_l = lf.$$

In terms of 2×2-matrices acting on the left of $(V'/L')^2$, let T_n act as $\left(\begin{smallmatrix} \tau_n & 0 \\ 0 & \tau_n \end{smallmatrix}\right)$ if $l \nmid n$, and T_l as $\left(\begin{smallmatrix} \tau_l & l \\ -\sigma_l & 0 \end{smallmatrix}\right)$. The equation $T_{nl^r} = T_n T_l^r$ and $\mathcal{O}_K$-linearity complete the definition of the action of $\mathbb{T}_Z$. Now by Lemma 3.2 and (3.6) we have an isomorphism of T_Z-modules

$$\Omega \cong \ker(\alpha' \circ \alpha).$$

It is well-known that $T_l^2 - S_l$ annihilates Y (e.g. [H3, Lemma 3.2]). (We consider S_l in $\mathbb{T}_Z$ by letting S_l S_q for $q \equiv l \bmod N$.) Consequently $\Delta \subseteq \ker \eta$, where $\Delta = \ker(\alpha' \circ \alpha)$, and η is the endomorphism defined by $T_l^2 - S_l$ on $(V'/L')^2$. This is in fact an equality.

PROPOSITION 3.3. $\Delta = \ker \eta$.

Proof. We wish to compute $\alpha' \circ \alpha$, i.e. to chase around the rectangle (3.5). By $\mathcal{O}_K$-linearity, it will suffice to compute $\alpha_{\mathbb{Q}}$ and $\alpha'_{\mathbb{Q}}$.

Writing γ for $\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\alpha_{\mathbb{Q}}\begin{pmatrix} f \\ g \end{pmatrix} = f + g|[\gamma] \quad \text{for } f, g \in V'(\mathbb{Q}).$$

Now take coset decompositions

$$\Gamma_1(N) = \bigcup_{k=1}^{l+1} (\Gamma_1(N) \cap \Gamma_0(l))\alpha_k \quad \text{and} \quad \Gamma_1(N) = \bigcup_{k=1}^{l+1} \gamma(\Gamma_1(N) \cap \Gamma_0(l))\gamma^{-1}\beta_k.$$

Then we have for $f \in V'(\mathbb{Q})$ and $h \in V(\mathbb{Q})$,

$$\langle f, h \rangle_{\Gamma_1(N) \cap \Gamma_0(l)} = \left\langle f, \sum_{k=1}^{l+1} h|[\alpha_k] \right\rangle_{\Gamma_1(N)}$$

$$\text{and } \langle f|[\gamma], h \rangle_{\Gamma_1(N) \cap \Gamma_0(l)} = \left\langle f, \sum_{k=1}^{l+1} h|[\gamma^{-1}\beta_k] \right\rangle_{\Gamma_1(N)}.$$

Relating the cup product to the Peterson inner product (3.4), we have computed ${}^t\alpha$. By the above equations,

$$\alpha'_{\mathbb{Q}}(h) = \sum_{k=1}^{l+1} \begin{pmatrix} h|[\alpha_k] \\ h|[\gamma^{-1}\beta_k] \end{pmatrix} \qquad \text{for } h \in V(\mathbb{Q}).$$

Thus

$$\alpha'_{\mathbb{Q}} \circ \alpha_{\mathbb{Q}} \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{k=1}^{l+1} \begin{pmatrix} f|[\alpha_k] + g|[\gamma\alpha_k] \\ f|[\gamma^{-1}\beta_k] + g|[\beta_k] \end{pmatrix} \quad \text{for } f, g \in V(\mathbb{Q}).$$

Since $\Gamma_1(N) \cap \Gamma_0(l) = \Gamma_1(N) \cap \gamma^{-1}\Gamma_1(N)\gamma$, we have for $f, g \in V'(\mathbb{Q})$,

$$f|\tau_l = \sum_{k=1}^{l+1} f|[\gamma^{-1}\beta_k] \quad \text{and} \quad g|\tau_l^* = \sum_{k=1}^{l+1} g|[\gamma\alpha_k],$$

where τ_l^* is adjoint to τ_l, i.e., $\tau_l^* = \sigma_l^{-1}\tau_l$. This gives

$$\alpha' \circ \alpha = \begin{pmatrix} l+1 & \sigma_l^{-1}\tau_l \\ \tau_l & l+1 \end{pmatrix},$$

so

$$\eta = \begin{pmatrix} \tau_l & l \\ -\sigma_l & 0 \end{pmatrix}^2 - \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_l \end{pmatrix} = \begin{pmatrix} -\sigma_l & \tau_l \\ 0 & -\sigma_l \end{pmatrix} \circ \alpha' \circ \alpha.$$

Since $= \left(\begin{smallmatrix} -\sigma_l & \tau_l \\ 0 & -\sigma_l \end{smallmatrix}\right)$ is an automorphism, we have $\Delta = \ker(\alpha' \circ \alpha) = \ker \eta$. □

Before using this proposition to compute the annihilator of Ω, we begin to restrict our attention to forms with specified characters. For a group Γ with $\Gamma_1(N) \subseteq \Gamma \subseteq \Gamma_0(N)$, let

$$H = \frac{\overline{\Gamma \cap \Gamma_0(l)}}{(\Gamma_1(N) \cap \Gamma_0(l))}.$$

Suppose ψ is a K-valued character on H. For any $\mathcal{O}_K[H]$-module M, define

$$M^{(\psi)} = \{m \in M \mid hm = \psi(h)m \quad \text{for all } h \in H\}.$$

Let e_ψ denote the idempotent $(1/[H:1])\Sigma_{h \in H}\psi(h^{-1})h \in K[H]$. Then $e_\psi M = M^{(\psi)}$ for any $K[H]$-module M.

Z is naturally a $K[H]$-module. In fact, we have the homomorphism

$$\rho: K[H] \to \mathbb{T}_Z(K)$$

defined by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto S_d$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \cap \Gamma_0(l)$. Since e_ψ commutes with the Hecke operators, we have the decomposition

$$S_2(\Gamma \cap \Gamma_0(l), \psi; K) = Z^{(\psi)} = X^{(\psi)} \oplus Y^{(\psi)},$$

into spaces stable under the Hecke operators. V is also a $K[H]$-module via ρ. So $L^{(\psi)}$ and $V^{(\psi)}$ are $\mathbb{T}_{Z^{(\psi)}}$-modules, $A^{(\psi)}$ is a $\mathbb{T}_{X^{(\psi)}}$-module, and $B^{(\psi)}$ is a $\mathbb{T}_{Y^{(\psi)}}$-module. Now define

$$\Omega_\psi = \frac{((A^{(\psi)} + L^{(\psi)}) \cap (B^{(\psi)} + L^{(\psi)}))}{L^{(\psi)}}.$$

This is a $\mathbb{T}_{Z^{(\psi)}}$-module for which the action factors through $\mathbb{T}_{X^{(\psi)}}$ and $\mathbb{T}_{Y^{(\psi)}}$.

The inclusion $\Gamma \cap \Gamma_0(l) \subseteq \Gamma$ induces an isomorphism $H \xrightarrow{\sim} \bar{\Gamma}/\Gamma_1(N)$. This acts naturally on $S_2(\Gamma_1(N); K)$, and the action factors through the Hecke operators of level N, via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \sigma_d$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. So L' and V' are H-modules as well. We see easily that α_K is a homomorphism of H-modules. We need only note that the adjoint of h is h^{-1} under the cup products to conclude that α'_K is also a homomorphism of H-modules. Thus the restrictions of α and α' define homomorphisms

$$\alpha_\psi : \left(\frac{V'^{(\psi)}}{L'^{(\psi)}}\right)^2 \to \frac{V^{(\psi)}}{L^{(\psi)}}$$

and

$$\alpha'_\psi : \frac{V^{(\psi)}}{L^{(\psi)}} \to \left(\frac{V'^{(\psi)}}{L'^{(\psi)}}\right)^2.$$

The image of α_ψ is $(A^{(\psi)} + L^{(\psi)})/L^{(\psi)}$, but the kernel of α'_ψ is not necessarily $(B^{(\psi)} + L^{(\psi)})/L^{(\psi)}$. However if we assume that

$$p \nmid [\bar{\Gamma} : \Gamma_1(N)] = [H : 1], \tag{3.7}$$

then we have $e_\psi \in \mathcal{O}_K[H]$, and consequently

$$\ker \alpha'_\psi = \frac{(B + L)^{(\psi)}}{L^{(\psi)}} = \frac{e_\psi(B + L)}{L^{(\psi)}} = \frac{(B^{(\psi)} + L^{(\psi)})}{L^{(\psi)}}.$$

Therefore, under this assumption

$$\Omega_\psi = \ker \alpha'_\psi \cap \operatorname{image} \alpha_\psi \cong \ker(\alpha'_\psi \circ \alpha_\psi) = \ker \eta_\psi$$

where η_ψ is the restriction of η to $(V'^{(\psi)}/L'^{(\psi)})^2$. Write Δ_ψ for $\ker \eta_\psi$.

Note that $(V'^{(\psi)}/L'^{(\psi)})^2 \cong A^{(\psi)}/(A^{(\psi)} \cap L^{(\psi)})$ is a faithful $\mathbb{T}_{X^{(\psi)}}$-module. We can identify $\mathbb{T}_{X^{(\psi)}}$ with a subring of $\mathrm{End}_{\mathcal{O}_K}(V'^{(\psi)}/L'^{(\psi)})^2$. Let $\mathbb{T}'_{X^{(\psi)}}$ denote the integral closure of $\mathbb{T}_{X^{(\psi)}}$ in $\mathbb{T}_{X^{(\psi)}}(K)$.

PROPOSITION 3.4. *If* $p\nmid[H:1]$, *then* $\eta_\psi \mathbb{T}_{X^{(\psi)}} \subseteq e_{X^{(\psi)}}(I_{Y^{(\psi)}}) \subseteq \eta_\psi \mathbb{T}'_{X^{(\psi)}}$.

Proof. The first inclusion is immediate; we prove the second one. Suppose that $T \in e_{X^{(\psi)}}(I_{Y^{(\psi)}})$. Then $T \in \mathrm{Ann}\, \mathbb{T}_{X^{(\psi)}} \Delta_\psi$. So there is $\varepsilon \in \mathrm{End}_{\mathcal{O}_K}(V'^{(\psi)}/L'^{(\psi)})^2$ with $T = \varepsilon\eta_\psi$. Since Δ_ψ is finite, η_ψ is not a zero-divisor in $\mathbb{T}_{X^{(\psi)}}$. Consequently, there exists $\omega \in \mathbb{T}_{X^{(\psi)}}$ such that $d = \eta_\psi\omega \in \mathcal{O}_K$, and $d \neq 0$. Therefore $d\varepsilon = \varepsilon\eta_\psi\omega = T\omega$, so $\varepsilon \in \mathbb{T}_{X^{(\psi)}}(K)$. Since $\mathrm{End}_{\mathcal{O}_K}(V'^{(\psi)}/L'^{(\psi)})^2$ is a finitely generated $\mathcal{O}_K$-module, we conclude that $\varepsilon \in \mathbb{T}'_{X^{(\psi)}}$. □

4. Congruences to a newform

Suppose $f = \Sigma a_n q^n \in S_2(\Gamma_0(N), \chi; K)$ is a newform of level N. We will associate certain $\mathcal{O}_K$-cyclic congruence modules to f and relate their annihilators to factors of $(a_l^2 - \chi(l)(l+1)^2)$. We assume, in this chapter, that $p \neq 2$, and that Γ is the largest subgroup of $\Gamma_0(N)$ satisfying (3.7). In Chapter 7, we discuss the weaker results obtained by these methods when $p = 2$ or $\Gamma = \Gamma_0(N)$.

If K contains the roots α and β of $x^2 - a_l x + \chi(l)l$, then $f_\alpha = f - \beta f(lz)$ is an eigenform in $S_2(\Gamma \cap \Gamma_0(l), \psi; K)$, where ψ is the restriction of χ. We have the following generalization of Ribet's result in terms of congruence modules. (We write subscript f_α for the one-dimensional Kf_α.)

THEOREM 4A. $C_{f_\alpha, Y^{(\psi)}} \cong (\alpha^2 - \chi(l))^{-1}\mathcal{O}_K/\mathcal{O}_K$.

Proof. Identifying $\mathbb{T}_{f_\alpha}$ with $\mathcal{O}_K$, we have $e_{f_\alpha}(\eta_\psi) = \alpha^2 - \chi(l)$, and $e_{f_\alpha}(\mathbb{T}'_{X^{(\psi)}}) = \mathcal{O}_K$. Therefore, by Proposition 3.4,

$$e_{f_\alpha}(I_{Y^{(\psi)}}) = (\alpha^2 - \chi(l))\mathcal{O}_K.$$

So $\mathbb{T}_{f_\alpha, Y^{(\psi)}} \cong \mathcal{O}_K/(\alpha^2 - \chi(l))\mathcal{O}_K$. We now apply Proposition 2.2 to compute the congruence module. □

Note that $(\alpha^2 - \chi(l))(\beta^2 - \chi(l))\mathcal{O}_K = (a_l^2 - \chi(l)(l+1)^2)\mathcal{O}_K$. At this point we can easily prove the existence of a newform $g \in Y^{(\psi)}$ such that $g \equiv f \bmod \mathfrak{p}$ when $\mathfrak{p} | (a_l^2 - \chi(l)(l+1)^2)$. We will prove a stronger result (Theorem 4c) which is slightly more difficult when $\alpha^2 \equiv \beta^2 \bmod \mathfrak{p}$. Note that this can happen only if $l \equiv \pm 1 \bmod p$.

If K contains a root ζ of $x^2 - \chi(l)$, we can define a Hecke operator R_l such that $R_l^2 = S_l$ on $Z^{(\psi)}$ and $f | R_l = \zeta f$. We let $R_l = \zeta^{-m} S_l^{(m+1)/2}$ where $\phi(N) = 2^r m$ with m odd. Similarly we define ρ_l so $\rho_l^2 = \sigma_l$ on $S_2(\Gamma, \psi; K)$. Since $T_l^2 = S_l$ on $Y^{(\psi)}$, we have the decomposition $Y^{(\psi)} = Y^+ \oplus Y^-$, where $Y^\pm$ is the subspace of $Y^{(\psi)}$ on which $T_l = \pm R_l$. Let $\eta^\pm = T_l \mp R_l \in \mathbb{T}_{X^{(\psi)}}$. Then $\eta^\pm \in e_{X^{(\psi)}}(I_{Y^\pm})$.

PROPOSITION 4.1. $\eta^\pm \mathbb{T}_{X^{(\psi)}} \subseteq e_{X^{(\psi)}}(I_Y{}^\pm) \subseteq \eta^\pm \mathbb{T}'_{X^{(\psi)}}$.

Proof. Suppose $T \in e_{X^{(\psi)}}(I_{Y^\pm})$. Then by Proposition 3.4,

$$\eta^\mp T \in e_{X^{(\psi)}} I_{Y^{(\psi)}} \subseteq \eta \mathbb{T}'_{X^{(\psi)}} = \eta^\mp \eta^\pm \mathbb{T}'_{X^{(\psi)}}.$$

Since η is not a zero-divisor in $\mathbb{T}_{X^{(\psi)}}(K)$, neither is $\eta^\mp$. So we conclude that $T \in \eta^\pm \mathbb{T}'_{X^{(\psi)}}$. □

For a newform f as above, let $X_f = K\{f, f(lz)\}$.

PROPOSITION 4.2. *There is an injection* $C_{X_f, Y^\pm} \to (a_l \pm \zeta(l+1))^{-1} \mathcal{O}_K / \mathcal{O}_K$.

Proof. We have the isomorphism $\mathbb{T}_{X_f} \xrightarrow{\sim} \mathcal{O}_K[T_l]$ where T_l satisfies $T_l^2 - a_l T_l + \chi(l) l = 0$. Since $e_{X_f}(\eta^\pm) \in e_{X_f}(I_{Y^\pm})$, $\mathbb{T}_{X_f, Y^\pm}$ is a quotient of

$$\frac{\mathcal{O}_K[T_l]}{(T_l \pm \zeta)} \cong \frac{\mathcal{O}_K}{(a_l \mp \zeta(l+1))\mathcal{O}_K}.$$

Now duality (Proposition 2.2) completes the proof. □

Since $\mathbb{T}_{X_f}$ is not necessarily integrally closed, we cannot use Proposition 4.1 to prove that this is an isomorphism. We must instead appeal to the weaker congruence module $C^{(l)}_{X_f, Y^\pm}$ which measures congruences of nth coefficients of q-expansions for n prime to l.

We observe that $Z^{(l)}_{\mathscr{C}} = X^{(l)}_{\mathscr{C}}$. First note that $Z^{(l)}_{\mathscr{C}}$ has a basis in $Z(\mathbb{Q})$. Now suppose that $f \in Z^{(l)}_{\mathscr{C}} \cap Z(\mathbb{Q})$. Then $f = g|[\gamma]$ for some g invariant under the action of $\gamma^{-1}(\Gamma_1(N) \cap \Gamma_0(l))\gamma$ and $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. Since these generate $\Gamma_1(N)$, we have $f \in X$. So the decompositions we consider satisfy (2.2).

LEMMA 4.3. *There is an injection* $(a_l \mp \zeta(l+1))^{-1} \mathcal{O}_K / \mathcal{O}_K \to C^{(l)}_{X_f, Y^\pm}$

Proof. We decompose $\Delta_\psi = \Delta^+ \oplus \Delta^-$ where $\Delta^\pm = \eta^\mp \Delta_\psi$ and analyze the components. Note that $\Delta^\pm$ is a module of $\mathbb{T}_{X^{(\psi)}}$ and of $\mathbb{T}_{Y^\pm}$.

Let δ denote the automorphism $\left(\begin{smallmatrix} 1 & \rho_l^{-1} \\ \rho_l & -1 \end{smallmatrix}\right)$ on $(V'^{(\psi)}/L'^{(\psi)})^2$. Then

$$\delta \circ \alpha'_\psi \circ \alpha_\psi \circ \delta^{-1} = \rho_l^{-1} \begin{pmatrix} \tau_l + \rho_l(l+1) & 0 \\ 0 & -\tau_l + \rho_l(l+1) \end{pmatrix}.$$

So

$$\begin{aligned} \Delta_\psi &= \ker(\alpha'_\psi \circ \alpha_\psi) = \delta^{-1} \ker(\delta \circ \alpha'_\psi \circ \alpha_\psi \circ \delta^{-1}) \\ &= \delta^{-1} \ker \varepsilon^- \oplus \delta^{-1} \ker \varepsilon^+, \end{aligned}$$

where

$$\varepsilon^+ = \begin{pmatrix} 1 & 0 \\ 0 & \tau_l - \rho_l(l+1) \end{pmatrix} \quad \text{and} \quad \varepsilon^- = \begin{pmatrix} \tau_l + \rho_l(l+1) & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & \frac{1}{2}\sigma_l^{-1}(\tau_l - \rho_l(l-1)) \\ 0 & -\frac{1}{2}\rho_l^{-1} \end{pmatrix}\eta^- = \varepsilon^-\delta^{-1},$$

we find that

$$\delta^{-1}\ker\varepsilon^- = \ker(\varepsilon^-\delta^{-1}) = \ker\eta^- \subseteq \eta^+ \ker\eta_{(\psi)} = \eta^+\Delta = \Delta^-.$$

Similarly $\delta^{-1}\ker\varepsilon^+ \subseteq \Delta^+$, and we conclude that $\Delta^\pm = \delta^{-1}\ker\varepsilon^\pm$. δ is an isomorphism of $\mathbb{T}^{(l)}_{X^{(\psi)}}$-modules which gives $\Delta^\pm \cong \ker\varepsilon^\pm$. We can regard $\varepsilon^\pm$ as the endomorphism of $V'^{(\psi)}/L'^{(\psi)}$ defined by $\tau_l \mp \rho_l(l+1)$.

Now let $\mathbb{T}^{(\psi)}_N$ be the $\mathcal{O}_K$-algebra of endomorphisms of $S_2(\Gamma, \psi; K)$ generated by the Hecke operators τ_n, for all $n \geqslant 1$. Then $V'^{(\psi)}/L'^{(\psi)}$ is a faithful $\mathbb{T}^{(\psi)}_N$-module. Similarly define $\mathbb{T}^{(\psi)}_N(K)$. By an earlier remark, $X^{(l)} \cong S_2(\Gamma, \psi; K)$ as a $\mathbb{T}^{(l)}_{X^{(\psi)}}$-module, where T_n acts as τ_n for n prime to l. So $\mathbb{T}^{(\psi)}_N(K) \cong \mathbb{T}^{(l)}_{X^{(\psi)}}(K)$, and we regard $\varepsilon^\pm$ as an element of $\mathbb{T}^{(l)\prime}_{X^{(\psi)}}$, the integral closure of $\mathbb{T}^{(l)}_{X^{(\psi)}}$ in $\mathbb{T}^{(l)}_{X^{(\psi)}}(K)$.

Suppose $T \in \mathrm{Ann}_{\mathbb{T}^{(l)}_{X^{(\psi)}}}\Delta^\pm$. Then $T = \omega\varepsilon^\pm$ for some $\omega \in \mathrm{End}_{\mathcal{O}_K}(V'^{(\psi)}/L'^{(\psi)})$. Since $\varepsilon^\pm$ has finite kernel on $V'^{(\psi)}/L'^{(\psi)}$, it has an inverse in $\mathbb{T}^{(l)}_{X^{(\psi)}}(K)$. It follows that $\omega \in \mathbb{T}^{(l)\prime}_{X^{(\psi)}}$. So

$$e_{X^{(\psi)}} I^{(l)}_{Y^\pm} \subseteq \mathrm{Ann}_{\mathbb{T}^{(l)}_{X^{(\psi)}}}\Delta^\pm \subseteq \varepsilon^\pm \mathbb{T}^{(l)\prime}_{X^{(\psi)}}.$$

Restricting to $X^{(l)}_f$, we get $\mathbb{T}^{(l)}_{X_f}(K) \xrightarrow{\sim} K$, with $e_{X_f}(\varepsilon^\pm) \mapsto a_l \mp \zeta(l+1)$ and $e_{X_f}(\mathbb{T}^{(l)\prime}_{X^{(\psi)}}) \xrightarrow{\sim} \mathcal{O}_K$. So the inclusion above gives a surjection

$$\mathbb{T}^{(l)}_{X_f, Y^\pm} \to \frac{\mathcal{O}_K}{(a_l \mp \zeta(l+1))\mathcal{O}_K},$$

and by duality (2.3) an injection

$$C^{(l)}_{X_f, Y^\pm} \to \frac{(a_l \mp \zeta(l+1))^{-1}\mathcal{O}_K}{\mathcal{O}_K}. \qquad \square$$

THEOREM 4B. $C_{X_f, Y^\pm} \cong (a_l \mp \zeta(l+1))^{-1}\mathcal{O}_K/\mathcal{O}_K$.

Proof. By the lemma, there is $g^\pm \in Y^\pm$ such that

$$c_1((f - g^\pm)|T) \equiv 0 \bmod (a_l \mp \zeta(l+1))\mathcal{O}_K \quad \text{for all } T \in \mathbb{T}^{(l)}_{Z^{(\psi)}}.$$

Let $f^{\pm} = f \mp \zeta l f(lz) \in M_{X_f}$. Then

$$f^{\pm}|T_l = (a_l \mp \zeta l)f - \chi(l) l f(lz) \equiv \pm\zeta f^{\pm} \bmod (a_l \mp \zeta(l+1))\mathcal{O}_K.$$

So for n prime to l and $r \geqslant 0$, we have

$$c_1(f^{\pm}|T_{nl^r}) \equiv c_1(f|T_n R_l^r) \bmod (a_l \mp \zeta(l+1))\mathcal{O}_K.$$

Since $g^{\pm}|T_{nl^r} = g^{\pm}|T_n R_l^r$,

$$c_1((f^{\pm} - g^{\pm})|T_{nl^r}) \equiv c_1((f^{\pm} - g^{\pm})|T_n R_l^r) \bmod (a_l \mp \zeta(l+1))\mathcal{O}_K.$$

From its definition, we find that $R_l \in \mathbb{T}^{(l)}_{Z^{(\psi)}}$ and conclude that

$$f^{\pm} \equiv g^{\pm} \bmod (a_l \mp \zeta(l+1))\mathcal{O}_K. \qquad \square$$

COROLLARY 4.4. *If $p \nmid (l+1)$, then $C_{X_f, Y^{(\psi)}} \cong (a_l^2 - \chi(l)(l+1)^2)^{-1}\mathcal{O}_K/\mathcal{O}_K$.*

Proof. If $p \nmid (l+1)$, then $a_l + \zeta(l+1)$ or $a_l - \zeta(l+1)$ is a unit in $\mathcal{O}_K$. $\square$

We can use properties of Fitting ideals to prove the existence of an anticipated degree of congruences between newforms, as in [W, Lemma 1.4.3]. We will write $\mathrm{Fitt}_R M$ for the Fitting ideal of an R-module M. Assume now that K contains the coefficients of all newforms of level dividing Nl. Let $\mathcal{N}$ denote the set of newforms in $Y^{(\psi)}$; let $\mathcal{N}^{\pm} = \mathcal{N} \cap Y^{\pm}$.

We recall that by a congruence between newforms, we mean a congruence of nth coefficients, for n prime to their conductors.

THEOREM 4C. *If $(a_l^2 - \chi(l)(l+1)^2)\mathcal{O}_K = \mathfrak{p}^d$, then there exist $d_i \in \mathbb{Z}$ and distinct $g_i \in \mathcal{N}$ such that $g_i \equiv f \bmod \mathfrak{p}^{d_i}$ and $\Sigma d_i \geqslant d$.*

Proof. We have $X_f^{(Nl)} = X_f^{(l)} \cong K\{f\}$, and $Y^{\pm(Nl)} = Y^{\pm(N)} \cong \oplus_{g \in \mathcal{N}^{\pm}} K\{g\}$. The homomorphism of $\mathcal{O}_K$-algebras $\mathbb{T}^{(Nl)}_{Y^{\pm}, X_f} \to \mathbb{T}_{Y^{\pm}, X_f}$ is surjective since the image is a cyclic $\mathcal{O}_K$-module. So $\mathrm{T}^{(Nl)}_{Y^{\pm}, X_f} \cong \mathcal{O}_K/\mathfrak{p}^{d^{\pm}}$ with $d^+ + d^- \geqslant d$, by Lemma 4.3. Let $M^{\pm} = \oplus_{g \in \mathcal{N}^{\pm}} \mathcal{O}_K\{g\}$. This is a faithful $\mathbb{T}^{(Nl)}_{Y^{\pm}}$-module, so $\mathrm{Fitt}_{\mathbb{T}^{(Nl)}_{Y^{\pm}}}(M^{\pm}) = 0$. Therefore $\mathrm{Fitt}_{\mathbb{T}^{(Nl)}_{Y^{\pm}, X_f}}(M^{\pm}/I^{(Nl)}_{X_f} M^{\pm}) = 0$ and $\mathrm{Fitt}_{\mathcal{O}_K}(M^{\pm}/I^{(Nl)}_{X_f} M^{\pm}) \subseteq \mathfrak{p}^{d^{\pm}}$. This shows that

$$\sum_{g_i \in \mathcal{N}} \mathrm{length}_{\mathcal{O}_K}\left(\frac{\mathcal{O}_K g_i}{I^{(Nl)}_{X_f} g_i}\right) \geqslant d.$$

Since $I^{(Nl)}_{X_f}$ contains $T_n - a_n$ for n prime to Nl, we find that

$$c_n(g_i) \equiv a_n \bmod \mathfrak{p}^{d_i} \quad \text{for } n \text{ prime to } Nl,$$

where $d_i = \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_K g_i / I_{X_f} g_i)$. $\square$

Note that unless p divides $\phi(N)$, we take $\Gamma = \Gamma_0(N)$. For a fixed level N, there are finitely many exceptions to this, and they are independent of l. For the exceptional primes p, we still obtain congruences, but these congruences are to forms whose character need only coincide with χ on Γ.

EXAMPLE 4.5. Let f be the unique newform of level 11 and trivial character. For $l = 2$, we find $a_l = -2$ and $a_l^2 - \chi(l)(l+1)^2 = -5$. So we conclude that there is a newform g of level 22 with conductor divisible by 2, such that $g \equiv f \bmod \mathfrak{p}$, where $\mathfrak{p}$ is a prime dividing 5. But there are no newforms of level 2 or 22 with trivial character. So such a g has character ρ, where ρ is a non-trivial even character mod 11. A genus computation shows that for each such ρ, there is one newform g_ρ of conductor 22, and g_ρ has coefficients in $K = \mathbb{Q}(\zeta_5)$ where ζ_5 is a primitive fifth root of unity. It follows from the theorem that $g_\rho \equiv f \bmod \mathfrak{p}$ where $\mathfrak{p}$ is the prime of K dividing 5. We also have $4\mathrm{Tr}_{K/\mathbb{Q}} g_\rho \equiv f + 2f(lz) \bmod 5$.

5. Congruences to a p-stabilized newform

From now on we assume that the level N is divisible by p. Write $N = N_0 p^r$ with N_0 prime to p and $r \geqslant 1$. Furthermore, suppose that $r \geqslant 2$ if $p = 3$ to ensure that there are no elliptic elements of order p. (For $p = 3$, if N_0 is divisible by a prime congruent to 2 mod 3, we only need $r \geqslant 1$.) We continue to assume that $p \neq 2$, as the case $p = 2$ is treated in Chapter 7.

We say f is ordinary at p if $ef = f$, where e is Hida's operator [H3, (4.3)]. This is an idempotent Hecke operator of level N, which can be defined as the p-adic limit $\lim_{t \to \infty} \tau_p^{t!}$. We also write e for the idempotent in $\mathbb{T}_Z$ attached to T_p. Any $\mathcal{O}_K$-module M of the Hecke operators decomposes into its ordinary and non-ordinary parts, i.e., $M = eM \oplus (1-e)M$.

If $f = \Sigma a_n q^n$ is a normalized eigenform of T_p, then f is ordinary if and only if $a_p \in \mathcal{O}_K^*$. To any newform $h = \Sigma c_n q^n$ of level m prime to p with $c_p \in \mathcal{O}_K^*$, we associate normalized ordinary eigenform of level mp called a p-stabilized newform [H3, Lemma 3.3]. If h is a newform of level mp^s with $s \geqslant 1$, then we say h is p-stabilized if it is ordinary. In this case the power of p dividing the conductor of the character of h is equal to s, unless $s = 1$ and the conductor is prime to p [H3, Lemma 3.2].

We will prove that the results in Chapter 4 hold for

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, d^{p-1} \equiv 1 \bmod p^r \right\}$$

if f is a p-stabilized newform level N. In particular, if $r = 1$, then $\Gamma = \Gamma_0(N)$. Note that Γ does not necessarily satisfy (3.7).

For a group G with $\Gamma_1(N) \subseteq G \subseteq \Gamma$ and a group A on which G acts trivially, we have an action of

$$H = \frac{\bar{\Gamma}}{\Gamma_1(N)} \cong \frac{(\mathbb{Z}/N_0 p\mathbb{Z})^*}{\{\pm 1\}}$$

on $\mathrm{H}^1(\bar{G}, A)$ defined by $u^\sigma(\tau) = u(\sigma\tau\sigma^{-1})$ for $\sigma \in H, \tau \in G$ and $u \in \mathrm{H}^1(\bar{G}, A)$.

Let ψ be any K-valued character of H. We write $K(\psi)$ for the Γ-module K with the action defined by ψ. We similarly define the submodule $\mathcal{O}_K(\psi)$. The restriction homomorphisms of the parabolic cohomology groups

$$\mathrm{H}^1_P(\bar{\Gamma}, K(\psi)) \to \mathrm{H}^1_P(\Gamma_1(N), K)^{(\psi)}$$

is an isomorphism. The restriction

$$\text{res}: \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi)) \to \mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K)^{(\psi)} \tag{5.1}$$

is not necessarily an isomorphism, but does have finite cokernel.

We can relate these parabolic cohomology groups to certain cohomology groups of sheaves. (See [H1, §1] and [S, Ch. 8].) We define the curves

$$\mathfrak{Z} = \Gamma \backslash \mathfrak{H} \quad \text{and} \quad \mathfrak{Y} = \Gamma_1(N) \backslash \mathfrak{H}.$$

Let $\mathfrak{O}_K(\psi)$ denote the sheaf $(\mathfrak{H} \times \mathcal{O}_K(\psi))/\Gamma$ over $\mathfrak{Z}$ and let $\mathfrak{O}_K$ be the constant sheaf defined by $\mathcal{O}_K$ over $\mathfrak{Y}$. Write $\mathrm{H}^1_p(\mathfrak{Z}, \mathfrak{O}_K(\psi))$ for the image of $\mathrm{H}^1_c(\mathfrak{Z}, \mathfrak{O}_K(\psi))$ in $\mathrm{H}^1(\mathfrak{Z}, \mathfrak{O}_K(\psi))$. Similarly define $\mathrm{H}^1_p(\mathfrak{Y}, \mathfrak{O}_K)$. We identify the following groups under the natural isomorphisms

$$\mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K) \cong \mathrm{H}^1_p(\mathfrak{Y}, \mathfrak{O}_K) \cong L'$$

and

$$\mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi)) \cong \mathrm{H}^1_p(\mathfrak{Z}, \mathfrak{O}_K(\psi)).$$

We have a pairing

$$C_\psi: \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi)) \times \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi)) \to \mathcal{O}_K$$

induced by the cup products

$$\mathrm{H}^1_c(\mathfrak{Z}, \mathfrak{O}_K(\psi)) \times \mathrm{H}^1(\mathfrak{Z}, \mathfrak{O}_K(\psi^{-1})) \to \mathcal{O}_K$$

and

$$\mathrm{H}^1(\mathfrak{Z}, \mathfrak{O}_K(\psi)) \times \mathrm{H}^1_c(\mathfrak{Z}, \mathfrak{O}_K(\psi^{-1})) \to \mathcal{O}_K.$$

We can similarly define

$$\mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K) \times \mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K) \to \mathcal{O}_K.$$

By standard properties of cup products, this coincides with (3.2) C': $V' \times V' \to K$ and is related to C_ψ by the equation

$$\tfrac{1}{2}\phi(N_0 p)C_\psi(u, v) = C'(\mathrm{res}(u), \mathrm{res}(v)),$$

for $u \in \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi))$ and $v \in \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi^{-1}))$. Thus if we write L'_ψ for the image of (5.1), we have

$$C'(L'_\psi, L'_{\psi^{-1}}) \subseteq \phi(N_0)\mathcal{O}_K. \tag{5.2}$$

Since e_ψ and $e_{\psi^{-1}}$ are adjoint under the cup products, C' restricts to define a non-degenerate pairing $V'^{(\psi)} \times V'^{(\psi^{-1})} \to K$ under which $L'^{(\psi^{-1})}$ is dual to $e_\psi L'$. We relate the ordinary parts of these lattices in Lemma 5.1, which is the key to improving our results for ordinary forms.

Denote by S_N the Shimura subgroup of level N, i.e., the kernel of the natural homomorphism $J_0(N) \to J_1(N)$ where $J_i(N)$ is the Jacobian of $\Gamma_i(N)\backslash\mathfrak{H}^*$. Recall that for trivial χ the primes dividing the order of S_N furnish exceptions to Theorem 1 (see [R2, §4] and Example 4.5). Note that $S_{N_0 p} \otimes \mathbb{Z}_p$ is naturally a quotient of $L'^{(\psi)}/\phi(N_0)e_\psi L'$ with $r = 1$, ψ trivial and $K = \mathbb{Q}_p$. Thus the lemma says in particular that Hida's operator and its adjoint annihilate $S_{N_0 p} \otimes \mathbb{Z}_p$ and generalizes this notion to non-trivial ψ and $r \geqslant 1$.

LEMMA 5.1. $e\phi(N_0)e_\psi L' = eL'^{(\psi)}$.

Proof. Since $\phi(N_0)e_\psi \in \mathcal{O}_K[H]$, we have $\phi(N_0)e_\psi L' \subseteq L'^{(\psi)}$ and the inclusion $e\phi(N_0)e_\psi L' \subseteq eL'^{(\psi)}$ is immediate. To prove the opposite inclusion we must demonstrate the surjectivity of several restriction homomorphisms.

(i) Let $\bar{\Gamma}_\psi = \ker \psi \subseteq \bar{\Gamma}$. We first show that the restriction map

$$\mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi)) \to \mathrm{H}^1_P(\bar{\Gamma}_\psi, \mathcal{O}_K)^{(\psi)} \tag{5.3}$$

is surjective. We have the inflation-restriction exact sequence

$$\mathrm{H}^1(\bar{\Gamma}, \mathcal{O}_K(\psi)) \to \mathrm{H}^1(\bar{\Gamma}_\psi, \mathcal{O}_K)^{(\psi)} \to \mathrm{H}^2(G, \mathcal{O}_K(\psi))$$

where $G = \bar{\Gamma}/\bar{\Gamma}_\psi$ is cyclic. We find

$$\mathrm{H}^2(G, \mathcal{O}_K(\psi)) \cong \frac{\mathcal{O}_K(\psi)^G}{N_G \mathcal{O}_K(\psi)} = 0.$$

Thus we have the surjectivity of the restriction $\mathrm{H}^1(\bar{\Gamma}, \mathcal{O}_K(\psi)) \to \mathrm{H}^1(\bar{\Gamma}_\psi, \mathcal{O}_K)^{(\psi)}$.

Now suppose $v \in \mathrm{H}^1_P(\bar{\Gamma}_\psi, \mathcal{O}_K)^{(\psi)}$. Take $u \in \mathrm{H}^1(\bar{\Gamma}, \mathcal{O}_K(\psi))$ with $v = \mathrm{res}(u)$. We wish to prove that $u \in \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi))$. So suppose π is a parabolic element of Γ. If $\pi \in \Gamma_\psi$, then $u(\pi) = v(\pi) = 0 \in (\pi - 1)\mathcal{O}_K$. If $\pi \notin \Gamma_\psi$, then we will show that $(\pi - 1)\mathcal{O}_K = \mathcal{O}_K$. π fixes some cusp $s = a/c$ with $a, c \in \mathbb{Z}$. Then

$$\pi = (-1)^i \begin{pmatrix} 1 - ach & c^2 h \\ -c^2 h & 1 + ach \end{pmatrix} \qquad \text{for some } h, i \in \mathbb{Z}.$$

For any prime q dividing N, we have $q | c^2 h$, so $q | ch$ and $1 - ach \equiv 1 \bmod q$. So we find that $\pi \in \Gamma \cap \{\pm 1\}\Gamma_1(\Pi q)$, where the product runs over primes q dividing N. But

$$[(\Gamma \cap \{\pm 1\}\Gamma_1(\Pi q)) : \{\pm 1\}\Gamma_1(N)] | N_0,$$

so $\pi^{N_0} \in \Gamma_\psi$ and $\psi(\pi) - 1 \in \mathcal{O}_K^*$. We conclude $u(\pi) \in \mathcal{O}_K = (\pi - 1)\mathcal{O}_K$, so $u \in \mathrm{H}^1_P(\bar{\Gamma}, \mathcal{O}_K(\psi))$. This proves the surjectivity of (5.3).

(ii) The double coset operator τ_p commutes with the restriction homomorphism

$$\mathrm{H}^1_P(\bar{\Gamma}_\psi, \mathcal{O}_K) \to \mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K)^{\Gamma_\psi} \tag{5.4}$$

and with the action of H. Therefore so does $e = \varinjlim \tau_p^{t!}$, and we have a homomorphism of H-modules

$$e\mathrm{H}^1_P(\bar{\Gamma}_\psi, \mathcal{O}_K) \to e\mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K)^{\Gamma_\psi}. \tag{5.5}$$

We will prove it is surjective as well.

We have the isomorphism $\mathrm{H}^1_P(\bar{\Gamma}_\psi, K) \to \mathrm{H}^1_P(\Gamma_1(N), K)^{\Gamma_\psi}$. Therefore for $v \in e\mathrm{H}^1_P(\Gamma_1(N), \mathcal{O}_K)^{\Gamma_\psi}$, there exists $u \in \mathrm{H}^1_P(\bar{\Gamma}_\psi, K)$ with $\mathrm{res}(u) = v$. We wish to prove that $eu \in e\mathrm{H}^1_P(\bar{\Gamma}_\psi, \mathcal{O}_K)$.

We have the double coset decomposition

$$\Gamma_\psi \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_\psi = \bigcup_{i=0}^{p-1} \Gamma_\psi \alpha_i \qquad \text{where } \alpha_i = \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}.$$

Take any $\delta = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_\psi$. Then $(u|\tau_p)(\delta) = \Sigma_{i=0}^{p-1} u(\alpha_i \delta \alpha_{j(i)}^{-1})$ where $\alpha_i \delta \alpha_{j(i)}^{-1} \in \Gamma_\psi$ for each i. We calculate

$$\alpha_i \delta \alpha_{j(i)}^{-1} = \begin{pmatrix} a + icN & * \\ cpN & d - j(i)cN \end{pmatrix}$$

and conclude that $\alpha_i \delta \alpha_{j(i)}^{-1} \delta^{-1} \in \Gamma_1(N)$. Therefore

$$(u|\tau_p - pu)(\delta) = \sum_{i=0}^{p-1} u(\alpha_i \delta \alpha_{j(i)}^{-1} \delta^{-1}) = \sum_{i=0}^{p-1} v(\alpha_i \delta \alpha_{j(i)}^{-1} \delta^{-1}) \in \mathcal{O}_K,$$

and we have $(u|\tau_p - pu) \in \mathrm{H}_P^1(\bar{\Gamma}_\psi, \mathcal{O}_K)$. This implies $u|\tau_p^M \in \mathrm{H}_P^1(\bar{\Gamma}_\psi, \mathcal{O}_K)$ for some $M \geqslant 0$, and thus $eu \in e\mathrm{H}_P^1(\bar{\Gamma}_\psi, \mathcal{O}_K)$.

Since $\mathrm{res}(eu) = e(\mathrm{res}(u)) = ev = v$, we have the desired surjectivity of (5.5), which combined with that of (5.3) gives

$$eL'_\psi = eL'^{(\psi)}. \tag{5.6}$$

(iii) The double coset operator τ_p^* also commutes with (5.4) and is H-linear. Therefore so does the adjoint of Hida's operator, $e^* = \varinjlim (\tau_p^*)^{t!}$. We show that the homomorphism

$$e^*\mathrm{H}_P^1(\bar{\Gamma}_\psi, \mathcal{O}_K) \to e^*\mathrm{H}_P^1(\Gamma_1(N), \mathcal{O}_K)^{\Gamma_\psi} \tag{5.7}$$

is surjective as well.

Now we have the double coset decomposition

$$\Gamma_\psi \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_\psi = \bigcup_{i=0}^{p-1} \Gamma_\psi \beta_i \qquad \text{where } \beta_i = \begin{pmatrix} p & 0 \\ iN & 1 \end{pmatrix}.$$

For $\delta = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_\psi$, we have

$$\beta_i \delta \beta_{j(i)}^{-1} = \begin{pmatrix} a - bj(i)N & bp \\ * & d + biN \end{pmatrix}$$

and $\beta_i \delta \beta_{j(i)}^{-1} \delta^{-1} \in \Gamma_1(N)$ for the appropriate $j(i)$. As in step (ii) we can deduce the surjectivity of (5.7). Combined with that of (5.3), it gives

$$e^*L'^{(\psi^{-1})} = e^*L'_{\psi^{-1}}. \tag{5.8}$$

The cup product C' restricts to a non-degenerate pairing $eV'^{(\psi)} \times e^*V'^{(\psi^{-1})} \to K$

for which the dual lattice of $ee_\psi L'$ is $L' \cap e^* e_{\psi^{-1}} V' = e^* L'^{(\psi^{-1})}$. Note that the surjectivity of (5.3) implies that $eL'_\psi \subseteq L'_\psi$. (5.6), (5.8) and (5.2) now give

$$C'(eL'^{(\psi)}, e^* L'^{(\psi^{-1})}) = C'(eL'_\psi, e^* L'_{\psi^{-1}}) \subseteq C'(L'_\psi, L'_{\psi^{-1}}) \subseteq \phi(N_0)\mathcal{O}_K.$$

It follows that $eL'^{(\psi)} \subseteq \phi(N_0) ee_\psi L'$. □

We briefly offer another interpretation of the lemma. It in fact computes the ordinary part of the cohomology congruence module corresponding to the decomposition

$$S_2(\Gamma_1(N); K) = T_\psi \oplus U_\psi.$$

where $T_\psi = S_2(\Gamma, \psi; K) = e_\psi S_2(\Gamma_1(N); K)$ and $U_\psi = (1 - e_\psi) S_2(\Gamma_1(N); K)$.

COROLLARY 5.2. *If $f \in eT_\psi$ is an eigenform, then $C_{f,U_\psi} \cong \phi(N_0)^{-1}\mathcal{O}_K/\mathcal{O}_K$.*

Proof. $e_\psi L'/L'^{(\psi)}$ is a module for $\mathbb{T}_{T_\psi}$ and $\mathbb{T}_{U_\psi}$, and

$$\frac{ee_\psi L'}{eL'^{(\psi)}} \cong \left(\frac{\mathcal{O}_K}{\phi(N_0)\mathcal{O}_K}\right) ee_\psi L'$$

is a module for $\mathbb{T}_{eT_\psi}$ and $\mathbb{T}_{eU_\psi}$. Its annihilator in $\mathbb{T}_{eT_\psi}$ is contained in $\phi(N_0)\mathbb{T}'_{eT_\psi}$ so we have a surjection $\mathbb{T}_{f,eU_\psi} \to \mathcal{O}_K/\phi(N_0)\mathcal{O}_K$. Since $\phi(N_0)e_\psi \in \mathcal{O}_K[H]$ it is an isomorphism. Now apply Proposition 2.2. □

We now return to the problem of raising the level of a p-stabilized newform. Since $eA^{(\psi)}$, $eB^{(\psi)}$ and $eL^{(\psi)}$ are modules for $\mathbb{T}_{eX^{(\psi)}}$, $\mathbb{T}_{eY^{(\psi)}}$ and $\mathbb{T}_{eZ^{(\psi)}}$, respectively, we define the cohomology congruence module

$$\Omega^o_\psi = \frac{[(eA^{(\psi)} + eL^{(\psi)}) \cap (eB^{(\psi)} + eL^{(\psi)})]}{eL^{(\psi)}}.$$

This is a module for $\mathbb{T}_{eX^{(\psi)}}$ and $\mathbb{T}_{eY^{(\psi)}}$. We obtain the following expression for it using the preceding lemma and results in Chapter 3.

LEMMA 5.3. *Ω^o_ψ is the kernel of the endomorphism $T_l^2 - S_l$ of $(eA^{(\psi)} + eL^{(\psi)})/eL^{(\psi)}$.*

Proof. We have defined homomorphisms of H-modules (3.1) $\alpha_K\colon V'^2 \to V$ and (3.3) $\alpha'_K\colon V \to V'^2$ with image $\alpha_K = A$ and $\ker \alpha'_K = B$. Since $T_p \circ \alpha_K = \alpha_K \circ \tau_p$, e commutes with α_K. Similarly e^* commutes with α_K, so e commutes with α'_K. Therefore the restrictions of α_K and α'_K define

$$\alpha^o_{K,\psi}\colon (eV'^{(\psi)})^2 \to eV^{(\psi)} \tag{5.9}$$

and

$$\alpha'^{o}_{K,\psi}: eV^{(\psi)} \to (eV'^{(\psi)})^2,$$

with image $\alpha^{o}_{K,\psi} = eA^{(\psi)}$ and $\ker \alpha'^{o}_{K,\psi} = eB^{(\psi)}$. These induce

$$\alpha^{o}_{\psi}: \left(\frac{eV'^{(\psi)}}{eL'^{(\psi)}}\right)^2 \to \frac{eV^{(\psi)}}{eL^{(\psi)}}$$

and

$$\alpha'^{o}_{\psi}: \frac{eV^{(\psi)}}{eL^{(\psi)}} \to \left(\frac{eV'^{(\psi)}}{eL'^{(\psi)}}\right)^2.$$

These are simply restrictions of α and α' (3.5). Recall (Lemma 3.2) that α is injective, and therefore so is α^{o}_{ψ}. Its image is $(eA^{(\psi)} + eL^{(\psi)})/eL^{(\psi)}$. We wish to prove that α'^{o}_{ψ} has kernel $(eB^{(\psi)} + eL^{(\psi)})/eL^{(\psi)}$. We have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\alpha'_{\mathcal{O}_K}} & L'^2 \\ {\scriptstyle \phi(N_0)ee_\psi}\downarrow & & \downarrow{\scriptstyle \phi(N_0)ee_\psi} \\ eL^{(\psi)} & \xrightarrow{\alpha'^{o}_{\mathcal{O}_K,\psi}} & (eL'^{(\psi)})^2. \end{array}$$

The surjectivity of $\alpha'_{\mathcal{O}_K}$ is a consequence of Lemma 3.2 (this is equivalent to $\ker \alpha' = (B + L)/L$), and the surjectivity of $\phi(N_0)ee_\psi$ on the right is Lemma 5.1. Therefore $\alpha'^{o}_{\mathcal{O}_K,\psi}$ is surjective, and this is equivalent to $\ker \alpha'^{o}_{\psi} = (eB^{(\psi)} + eL^{(\psi)}/eL^{(\psi)}$. So $\Omega^{o}_{\psi} \cong \ker(\alpha'^{o}_{\psi} \circ \alpha^{o}_{\psi})$. We now apply Proposition 3.3 to conclude that this is in fact the kernel of $T_l^2 - S_l$ on $(eV^{(\psi)}/eL^{(\psi)})^2 \cong (eA^{(\psi)} + eL^{(\psi)})/eL^{(\psi)}$. □

Now suppose that $f = \Sigma a_n q^n \in S_2(\Gamma_0(N), \chi; K)$ is a p-stabilized newform of level N. Let ψ be the restriction of χ to Γ. Let $\mathcal{N}^o$ denote the set of p-stabilized newforms in $Y^{(\psi)}$. By a congruence between p-stabilized newforms, we mean of course a congruence of nth coefficients for n prime to their levels. We can now apply the methods of Chapter 4 to obtain the following sharper result for ordinary forms.

THEOREM 5.

(A) *If $\alpha \in K$ is a root of $x^2 - a_l x + l\chi(l)$, then*

$$C_{f_\alpha, Y^{(\psi)}} \cong \frac{(\alpha^2 - \chi(l))^{-1}\mathcal{O}_K}{\mathcal{O}_K}.$$

(B) *If* $\zeta \in K$ *is a root of* $x^2 - \chi(l)$, *then*

$$C_{X_f, Y^{\pm(\psi)}} \cong \frac{(a_l \mp \zeta(l+1))^{-1}\mathcal{O}_K}{\mathcal{O}_K}.$$

(C) *For a sufficiently large K, if* $(a_l^2 - \chi(l)(l+1)^2)\mathcal{O}_K = \mathfrak{p}^d$, *then there exist* $d_i \in \mathbb{Z}$ *and distinct* $g_i \in \mathcal{N}^o$ *such that* $g_i \equiv f \bmod \mathfrak{p}^{d_i}$ *and* $\Sigma d_i \geqslant d$.

EXAMPLE 5.4. We reconsider the form f of level 11 in Example 4.5. We have a_5 prime to 5, so we replace f by the associated p-stabilized newform of level 55. Now we have $\Gamma = \Gamma_0(55)$, so there is a newform $g = \Sigma b_n q^n$ of level 110 and trivial character such that $a_n \equiv b_n \bmod \mathfrak{p}$ for n prime to 110, where $\mathfrak{p}$ is a prime over 5.

6. Congruences to a Λ-adic newform

We will now review some elements of Hida's theory of Λ-adic forms ([H4], [H5] and [H6]). Our exposition follows [W]. We continue to assume $p \neq 2$.

Let

$$\Gamma_r = \{v \in \mathbb{Z}_p^* \mid v \equiv 1 \bmod p^r\} \qquad \text{for } r \geqslant 1,$$

and

$$\mu = \{\delta \in \mathbb{Z}_p^* \mid \delta^{p-1} = 1\}.$$

Then $\mathbb{Z}_p^* = \Gamma_1 \times \mu$. Fix $u = 1 + p$, a topological generator of Γ_1. We define the completed group ring

$$\Lambda = \mathcal{O}_K[[\Gamma_1]] = \varprojlim \mathcal{O}_K[\Gamma_1/\Gamma_r].$$

It is isomorphic to the ring of formal power series $\mathcal{O}_K[[T]]$ where the isomorphism is defined by $u \mapsto 1 + T$. We identify these two rings.

For m prime to p, we let $G = (\mathbb{Z}/m\mathbb{Z})^* \times \mu$. For an even K-valued character ψ of G we define $\underline{\psi}\colon G \times \Gamma_1 \to \Lambda$ by $(a, v) \mapsto \psi(a)v$. Note that natural isomorphisms

$$G \times \Gamma_1 \cong \left(\frac{\mathbb{Z}}{mZ}\right)^* \times \mathbb{Z}_p^* \cong \varprojlim \left(\frac{\mathbb{Z}}{mp^r\mathbb{Z}}\right)^*.$$

If $a \in \mathbb{Z}$ is prime to mp, then we can write $a = u^\alpha \delta$ for some $\alpha \in \mathbb{Z}_p$ and $\delta \in \mu$. Then $\underline{\psi}(a) = \psi(a)(1 + T)^\alpha$. We extend this to all of $\mathbb{Z}$ by letting $\underline{\psi}(a) = 0$ if a is not prime to mp.

For $r \geqslant 1$, let $\mathfrak{S}_r = \{(k,\zeta) \in \mathbb{Z} \times \bar{\mathbb{Q}}_p \mid k \geqslant 2 \text{ and } \zeta^{p^{r-1}} = 1\}$, and let $\mathfrak{S} = \bigcup_r \mathfrak{S}_r$. For $(k,\zeta) \in \mathfrak{S}$, we define a homomorphism of $\mathcal{O}_K$-algebras $\nu_{k,\zeta}: \Lambda \to \bar{\mathbb{Q}}_p$ by $T \mapsto \zeta u^{k-2} - 1$, and call the kernel $P_{k,\zeta}$. We extend $\nu_{k,\zeta}$ to a homomorphism $\Lambda[[q]] \to \bar{\mathbb{Q}}_p[[q]]$.

For ζ as above, we define $\rho_\zeta: \Gamma_1/\Gamma_r \to \bar{\mathbb{Q}}_p$ by $u \mapsto \zeta$. We define ω to be the inclusion $\mu \to \bar{\mathbb{Q}}_p$. Via the isomorphism

$$G \times \left(\frac{\Gamma_1}{\Gamma_r}\right) \cong \left(\frac{\mathbb{Z}}{mp^r\mathbb{Z}}\right)^* \cong \frac{\Gamma_0(mp^r)}{\Gamma_1(mp^r)}$$

we regard ρ_ζ, ω and $\underline{\psi}$ as characters of $\Gamma_0(mp^r)$. We are ready to define the Λ-module of ordinary Λ-adic cusp forms of level $\bar{m} = mp^\infty$ and character $\underline{\psi}$.

$$\mathcal{S}^o_\Lambda(\bar{m}, \underline{\psi}) = \left\{ \mathcal{F} \in \Lambda[[q]] \,\middle|\, \begin{array}{l} \nu_{k,\zeta}(\mathcal{F}) \in eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p) \\ \text{for all but finitely many } (k,\zeta) \in \mathfrak{S} \end{array} \right\}. \tag{6.1}$$

For a finite extension L of the field of fractions F_Λ of Λ, let $\mathcal{O}_L$ denote the integral closure of Λ in L. $\mathcal{O}_L$ is a complete local two-dimensional Krull ring [B, Ch. 7], and its localization at any height one prime is a discrete valuation ring. Then we define

$$\mathcal{S}^o_L(\bar{m}, \underline{\psi}) = \mathcal{S}^o_\Lambda(\bar{m}, \underline{\psi}) \otimes_\Lambda L \subseteq L[[q]]$$

and

$$\mathcal{S}^o_A(\bar{m}, \underline{\psi}) = \mathcal{S}^o_L(\bar{m}, \underline{\psi}) \cap A[[q]] \qquad \text{for any ring } A \subseteq L.$$

We know that $\mathcal{F} \in \mathcal{S}^o_{\mathcal{O}_L}(\bar{m}, \underline{\psi})$ if and only if $\nu(\mathcal{F}) \in eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$ for all but finitely many $\nu: \mathcal{O}_L \to \bar{\mathbb{Q}}_p$ which extend a $\nu_{k,\zeta}$. So the definition above is independent of the choice of $\mathcal{O}_K$ and coincides with (6.1) for $\mathcal{O}_L = \Lambda$. It is known that $\mathcal{S}^o_{\mathcal{O}_L}(\bar{m}, \underline{\psi})$ is a finitely generated Λ-module [W, Th. 1.2.2].

For a positive integer n, we define the nth Hecke operator T_n as follows. If $\mathcal{F} = \Sigma a_m q^m \in L[[q]]$, then $T_n\mathcal{F} = \Sigma b_m q^m$ with $b_m = \Sigma_{d|(m,n)} \underline{\psi}(d) d a_{mnd^{-2}}$. Since $\nu(\underline{\psi}(d)) = d^{k-2}(\psi\rho_\zeta\omega^{2-k})(d)$, we find that $\nu(\mathcal{F})|T_n = \nu(T_n\mathcal{F})$ for any $\mathcal{F} \in \mathcal{S}^o_{\mathcal{O}_L}(\bar{m}, \underline{\psi})$ and any ν such that $\nu(\mathcal{F}) \in eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$. So we see that T_n is an endomorphism of $L, \mathcal{S}^o_A(\bar{m}, \underline{\psi})$ if $A \supseteq \Lambda$.

If $\mathcal{F}$ is an eigenform of the Hecke operators, then its eigenvalues are in $\mathcal{O}_L$. We say $\mathcal{F}$ is normalized if $c_1(\mathcal{F}) = 1$, and in that case $T_n(\mathcal{F}) = c_n(\mathcal{F})\mathcal{F}$. We say $\mathcal{F}$ is a Λ-adic newform of level $\bar{m}$ if $\nu(\mathcal{F})$ is a p-stabilized newform of level divisible by m for all but finitely many ν as above. Then we know [W, Prop. 1.5.2] that for sufficiently large L, $\mathcal{S}_L^o(\bar{m}, \underline{\psi})$ is spanned by the set

$$\{\mathcal{F}(az) \mid \mathcal{F} \in \mathcal{S}_{\mathcal{O}_L}^o(\bar{m}, \underline{\psi}) \text{ is a newform of level } \bar{d} \text{ with } da|m\}.$$

We also make use of the following theorem ([W, Th. 1.4.6]).

LIFTING THEOREM. *If f is a p-stabilized newform in $eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$, then there exists an eigenform $\mathcal{F} \in \mathcal{S}_{\mathcal{O}_L}^o(\bar{m}, \underline{\psi})$ for some finite extension L of F_Λ such that $v(\mathcal{F}) = f$ for some v extending $v_{k,\zeta}$.*

We now present a theory of duality and congruence modules analogous to that for classical modular forms. For an L-subspace $\mathcal{S}$ of $\mathcal{S}_L^o(\bar{m}, \underline{\psi})$ which is stable under the Hecke operators and a subring A of L with $\Lambda \subseteq A$, let $\mathcal{S}(A) = \mathcal{S} \cap \mathcal{S}_A^o(\bar{m}, \underline{\psi})$. Let $\mathbb{T}_{\mathcal{S}}(A)$ be the A-algebra of endomorphisms of $\mathcal{S}(A)$ generated by the T_n. We write simply $\mathbb{T}_{\mathcal{S}}$ for $\mathbb{T}_{\mathcal{S}}(\mathcal{O}_L)$. The bilinear pairing

$$\mathbb{T}_{\mathcal{S}}(A) \times \mathcal{S}(A) \to A$$

defined by $(T, f) \mapsto c_1(f \mid T)$ induces a homomorphism

$$\phi_{\mathcal{S}(A)}: \mathcal{S}(A) \to \mathrm{Hom}_A(\mathbb{T}_{\mathcal{S}}(A), A).$$

For any prime ideal P of $\mathcal{O}_L$, write $\mathcal{O}_P$ for the localization of $\mathcal{O}_L$ at P. We have the following analogue of Proposition 2.1.

PROPOSITION 6.1. *$\phi_{\mathcal{S}(\mathcal{O}_P)}$ is an isomorphism.*

Proof. $\phi_{\mathcal{S}(\mathcal{O}_P)}$ is injective with torsion-free cokernel. The transpose

$$\phi'_{\mathcal{S}(\mathcal{O}_P)}: \mathbb{T}_{\mathcal{S}}(\mathcal{O}_P) \to \mathrm{Hom}_{\mathcal{O}_P}(\mathcal{S}(\mathcal{O}_P), \mathcal{O}_P)$$

is also injective. Note that $\mathbb{T}_{\mathcal{S}}(\mathcal{O}_P) \cong (\mathbb{T}_{\mathcal{S}})_P$ is a finitely generated $\mathcal{O}_P$-module. If P has height 0 or 1, $\mathbb{T}_{\mathcal{S}}(\mathcal{O}_P)$ is free, and the injectivity of ϕ' implies the surjectivity of ϕ. If P is maximal ($\mathcal{O}_P = \mathcal{O}_L$), we still find that the localization of the cokernel at a height one prime is trivial and therefore ϕ is surjective. □

Note that the proposition allows us to identify $\mathcal{S}(\mathcal{O}_P)$ with $\mathcal{S}(\mathcal{O}_L)_P$.

If we have a decomposition $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are such spaces, we define the congruence module

$$C_{\mathcal{X},\mathcal{Y}} = \mathcal{Z}(\mathcal{O}_L)/(\mathcal{X}(\mathcal{O}_L) \oplus \mathcal{Y}(\mathcal{O}_L)).$$

It is a finitely generated $\mathcal{O}_L$-torsion module, as is

$$\mathbb{T}_{\mathcal{X},\mathcal{Y}} = (\mathbb{T}_{\mathcal{X}} \oplus \mathbb{T}_{\mathcal{Y}})/\mathbb{T}_{\mathcal{Z}}.$$

$\mathbb{T}_{\mathscr{X},\mathscr{Y}}$ is isomorphic to $\mathbb{T}_{\mathscr{Z}}/(I_{\mathscr{X}} + I_{\mathscr{Y}})$ where $I_{\mathscr{X}}$ and $I_{\mathscr{Y}}$ are the annihilators of $\mathscr{X}$ and $\mathscr{Y}$ in $\mathbb{T}_{\mathscr{Z}}$. We still have a local version of Proposition 2.2.

PROPOSITION 6.22. *If P is a height one prime of $\mathcal{O}_L$, then*

$$(C_{\mathscr{X},\mathscr{Y}})_P \cong \mathrm{Hom}_{\mathcal{O}_P}((\mathbb{T}_{\mathscr{X},\mathscr{Y}})_P, L/\mathcal{O}_P).$$

In certain cases we shall consider, $C_{\mathscr{X},\mathscr{Y}}$ is of the form $\mathfrak{a}/\mathcal{O}_L$, where $\mathfrak{a} \subseteq L$ is a finitely generated $\mathcal{O}_L$-module. We can then use this proposition to compute the divisor of $\mathfrak{a}$ since $\mathrm{length}_P(C_{\mathscr{X},\mathscr{Y}}) = \mathrm{length}_P(\mathbb{T}_{\mathscr{X},\mathscr{Y}})$.

For l prime to $N_0 p$, and $\underline{\psi}$ of conductor dividing $N_0 p$, we can define a decomposition of $\mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi})$ into spaces of forms which are old and new at l [W, (1.6.1)]. We review this definition. We have the two inclusions

$$b_1, b_l \colon \mathscr{S}^o_L(\bar{N}_0, \underline{\psi}) \to \mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi})$$

defined by $\mathscr{F} \mapsto \mathscr{F}$ and $\mathscr{F} \mapsto \mathscr{F}(lz)$. Let $\mathscr{X}(L)$ be the sum of the images. For a field M containing the eigenvalues of all Λ-adic newforms of level dividing $\overline{N_0 l}$, we let $\mathscr{Y}(M)$ be the space spanned by the set

$$\{\mathscr{F}(az) \mid \mathscr{F} \in \mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi}) \text{ is a newform of level } \bar{d} \text{ with } l|d \text{ and } ad|N_0 l\}.$$

For any finite extension L of F_Λ, we let $\mathscr{Y}(L) = \mathscr{Y}(LM) \cap \mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi})$. Then we have $\mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi}) = \mathscr{X}(L) \oplus \mathscr{Y}(L)$. The decomposition is stable under the action of the Hecke operators. We can also characterize $\mathscr{Y}(L)$ as the kernel of $(T_l^2 - \underline{\psi}(l))$ in $\mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi})$. For fixed L, we write $\mathscr{X}$ for $\mathscr{X}(L)$, $\mathscr{Y}$ for $\mathscr{Y}(L)$ and $\mathscr{Z}$ for $\mathscr{S}^o_L(\overline{N_0 l}, \underline{\psi})$.

It is implicit in the work of Hida that in the definition (6.1) "all but finitely many" can be replaced by "all", and that in the lifting theorem "eigenform" can be replaced by "unique newform". We present this as a consequence of [H5, Th. 1.2] (or [H6, Th.3.4] for $p = 3$).

PROPOSITION 6.3. *If $\mathscr{F} \in \mathscr{S}^o_{\mathcal{O}_L}(\bar{m}, \underline{\psi})$, then $v(\mathscr{F}) \in eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$ for any v extending any $v_{k,\zeta}$.*

Proof. Let L be a finite Galois extension of F_Λ containing the eigenvalues of all Λ-adic eigenforms of level dividing $\bar{m}$. It suffices to prove the proposition for this field, because it then follows for F_Λ and consequently for any finite extension of F_Λ.

Write P for the kernel of v, $\mathcal{O}$ for $\mathcal{O}_L$ and M for the field of fractions of $v(\mathcal{O})$. Let

$$\mathscr{A}(v) = \{\mathscr{F} \in \mathscr{S}^o_{\mathcal{O}}(\bar{m}, \underline{\psi}) \mid v(\mathscr{F}) \in eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)\}.$$

Suppose f is a p-stabilized newform in $eS_k(dp^r, \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$ (with d prime to p and the conductor of ψ dividing dp). As a consequence of the lifting theorem, there is $\mathcal{F} \in \mathcal{S}^{o}_{\mathcal{O}}(\bar{d}, \underline{\psi})$ with $v(\mathcal{F}) = f$. Since $eS_k(\Gamma_0(N_0p^r), \psi\rho_\zeta\omega^{2-k}; \bar{\mathbb{Q}}_p)$ is spanned by the set of $f(az)$ for such f with $da|m$, we find that v induces an isomorphism

$$\frac{\mathcal{A}(v)_P}{P\mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})_P} \to eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; M).$$

For some v, we have $\mathcal{A}(v) = \mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})$, and as a consequence of Hida's theorem, $\dim_M eS_k(\Gamma_0(mp^r), \psi\rho_\zeta\omega^{2-k}; M)$ is independent of v. Therefore for any v,

$$\dim_M\left(\frac{\mathcal{A}(v)_P}{P\mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})_P}\right) = \operatorname{rank}_{\mathcal{O}_P}\mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})_P.$$

We conclude that $\mathcal{A}(v)_P = \mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})_P$. It follows that $\mathcal{A}(v) = \mathcal{S}^{o}_{\mathcal{O}}(\bar{m}, \underline{\psi})$. □

COROLLARY 6.4. *v induces isomorphisms*

$$\mathcal{Z}(\mathcal{O}) \otimes \frac{\mathcal{O}_P}{P\mathcal{O}_P} \xrightarrow{\sim} eS_k(N_0lp^r, \psi\rho_\zeta\omega^{2-k}; K)$$

$$\mathcal{X}(\mathcal{O}) \otimes \frac{\mathcal{O}_P}{P\mathcal{O}_P} \xrightarrow{\sim} eS_k(N_0lp^r, \psi\rho_\zeta\omega^{2-k}; K)^{\text{old}}$$

and

$$\mathcal{Y}(\mathcal{O}) \otimes \frac{\mathcal{O}_P}{P\mathcal{O}_P} \xrightarrow{\sim} eS_k(N_0lp^r, \psi\rho_\zeta\omega^{2-k}; K)^{\text{new}},$$

where old *and* new *signify old and new at l.*

We now proceed to construct an analogue of the classical cohomology congruence module. For now assume that $\mathcal{O}_L = \mathcal{O}_K[[T]]$ for a finite extension K of $\mathbb{Q}_p$. We recall that ψ can be regarded as a character of a certain subgroup of $\Gamma_0(N_0p^r)/\Gamma_1(N_0p^r) \cong \Gamma_0(N_0lp^r)/(\Gamma_1(N_0p^r) \cap \Gamma_0(l))$. We define

$$V'_r = e\mathrm{H}^1_P(\Gamma_1(N_0p^r), K)^{(\psi)}$$
$$L'_r = e\mathrm{H}^1_P(\Gamma_1(N_0p^r), \mathcal{O}_K)^{(\psi)}$$
$$V_r = e\mathrm{H}^1_P(\Gamma_1(N_0p^r) \cap \Gamma_0(l), K)^{(\psi)}$$

and

$$L_r = e\mathrm{H}^1_P(\Gamma_1(N_0p^r) \cap \Gamma_0(l), \mathcal{O}_K)^{(\psi)}.$$

We regard L'_r as a lattice in V'_r and L_r as a lattice in V_r. We already defined (5.9) homomorphisms (which we rename, as they depend on r)

$$\alpha_r\colon (V'_r)^2 \to V_r$$

and

$$\alpha'_r\colon V_r \to (V'_r)^2.$$

Let A_r denote the image of α_r; let B_r be the kernel of α'_r. Recall that α_r and α'_r restrict to define maps on the lattices as well. For $s \geqslant r \geqslant 1$, the restriction homomorphisms $V'_r \to V'_s$, $L'_r \to L'_s$, $V_r \to V_s$ are $L_r \to L_s$ and compatible with the inclusions and homomorphisms considered above. We let V'_∞ be the direct limit $\varinjlim V'_r$, and similarly define L'_∞, V_∞, L_∞, A_∞ and B_∞. We then have inclusions $L'_\infty \subseteq V'_\infty$ and $L_\infty, A_\infty, B_\infty \subseteq V_\infty$.

Now we show that V_∞ is naturally a $\mathbb{T}_{\mathscr{X}}$-module. Let $\mathbb{T}_r$ denote the $\mathcal{O}_K$-algebra generated by the Hecke operators on $eS_2(\Gamma_1(N_0p^r) \cap \Gamma_0(l), \mathcal{O}_K)^{(\psi)}$. Then we have a natural map

$$\mathcal{O}_L = \mathcal{O}_K[[\Gamma_1]] \to \mathcal{O}_K[\Gamma_1/\Gamma_r] \to \mathbb{T}_r,$$

which induces a map from the polynomial ring

$$\mathcal{O}_L[T_n]_{n \geqslant 1} \to \mathbb{T}_r.$$

By Corollary 6.4, this map factors through the $\mathbb{T}_{\mathscr{X}}$. We also have the commutativity of

$$\begin{array}{ccc} \mathbb{T}_{\mathscr{X}} & \longrightarrow & \mathbb{T}_s \\ & \searrow & \downarrow \\ & & \mathbb{T}_r \end{array} \qquad \text{for } s \geqslant r \geqslant 1.$$

This makes V_r a $\mathbb{T}_{\mathscr{X}}$-module for $r \geqslant 1$, and the action is compatible with the inclusions $V_r \to V_s$ for $s \geqslant r \geqslant 1$. So V_∞ is a $T_{\mathscr{X}}$-module as are L_∞, A_∞ and B_∞. By Corollary 6.4 we also find that A_∞ is a module for $\mathbb{T}_{\mathscr{X}}$ and that B_∞ is a module for $\mathbb{T}_{\mathscr{Y}}$. We then define

$$\Omega_\infty = \frac{((A_\infty + L_\infty) \cap (B_\infty + L_\infty))}{L_\infty}$$

and have $(I_{\mathscr{X}} + I_{\mathscr{Y}}) \subseteq \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(\Omega_\infty)$ and $e_{\mathscr{X}} I_{\mathscr{Y}} \subseteq \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(\Omega_\infty)$.

Let η_∞ denote the endomorphism $T_l^2 - \underline{\psi}(l)$ of $(A_\infty + L_\infty)/L_\infty \cong (V'_\infty/L'_\infty)^2$.

The following is an easy consequence of Lemma 5.3.

PROPOSITION 6.5. $\Omega_\infty = \ker \eta_\infty$.

Proof. Since direct limits preserve exact sequences, we have

$$\Omega_\infty = \frac{\varinjlim((A_r + L_r) \cap (B_r + L_r))}{L_r}.$$

By Lemma 5.3, this is exactly $\varinjlim(\ker(T_l^2 - S_l)_r)$, where $(T_l^2 - S_l)_r$ denotes the endomorphism of $(A_r + L_r)/L_r$. Since $\underline{\psi}(l) = S_l$ on V_r, we conclude that

$$\varinjlim(\ker(T_l^2 - S_l)_r) = \ker \eta_\infty. \qquad \square$$

Computing the annihilator of Ω_∞ requires a little more work than in the classical setting. We are missing two key ingredients.

(1) $(V'_\infty/L'_\infty)^2$ is a faithful $\mathbb{T}_{\mathscr{X}}$-module.
(2) $\mathrm{End}_{\mathcal{O}_L}(V'_\infty/L'_\infty)^2$ is a finitely generated $\mathcal{O}_L$-module.

Both of these are consequences of the following lemma due to Hida [H5, Th.3.1].

LEMMA 6.6. *For $s \geqslant r \geqslant 2$, the restriction map*

$$e\mathrm{H}^1_P(\Gamma_1(N_0p^r), K/\mathcal{O}_K) \to e\mathrm{H}^1_P(\Gamma_1(N_0p^s), K/\mathcal{O}_K)^{\Gamma_r}$$

is an isomorphism.

Proof. We find that Hida's proof works equally well for $\mathbb{Q}_p/\mathbb{Z}_p$ replaced by $K/\mathcal{O}_K$. The assumption $r \geqslant 2$ ensures that it works for $p = 3$ as well. $\square$

LEMMA 6.7. *$(V'_\infty/L'_\infty)^2$ is a faithful $\mathbb{T}_{\mathscr{X}}$-module.*

Proof. By Lemma 6.6, $V'_r/L'_r \to V'_s/L'_s$ is injective for $s \geqslant r \geqslant 2$, so for $r \geqslant 2$, $V'_r/L'_r \to V'_\infty/L'_\infty$ is injective. Now suppose $T \in \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(V'_\infty/L'_\infty)^2$. Then for $r \geqslant 2$, $T(V'_r)^2 \subseteq (L'_r)^2$; so in fact

$$T \in \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(V'_r)^2 = \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(A_r) \qquad \text{for all } r \geqslant 1.$$

Therefore T annihilates the new (at l) part of $eS_2(\Gamma_1(N_0p^r) \cap \Gamma_0(l); K)^{(\psi)}$. If $\mathscr{F} \in \mathscr{X}(\mathcal{O}_L)$, then

$$\nu_{2,\zeta}(T\mathscr{F}) = \nu_{2,\zeta}(\mathscr{F})|T = 0 \qquad \text{for all } \zeta \in \mu_{p^\infty}.$$

Thus $T\mathscr{F} \in \cap P_{2,\zeta}\mathscr{X}(\mathcal{O}_L) = 0$ and we conclude that $T \in I_{\mathscr{X}}$. $\square$

LEMMA 6.8. *$\mathrm{End}_{\mathcal{O}_L}(V'_\infty/L'_\infty)^2$ is a finitely generated $\mathcal{O}_L$-module.*

Proof. We give V'_∞/L'_∞ the discrete topology, and let $\mathscr{C}$ be its Pontrjagin dual. So $\mathscr{C} = \mathrm{Hom}(V'_\infty/L'_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$ is naturally an $\mathcal{O}_L$-module. Similarly, let $\mathscr{D} = \mathrm{Hom}(D, \mathbb{Q}_p/\mathbb{Z}_p)$, where

$$D = \varinjlim (e\mathrm{H}^1_P(\Gamma_1(N_0 p^r), K/\mathcal{O}_K).$$

Then we have an injection of $\mathcal{O}_L$-modules

$$\mathrm{End}_{\mathcal{O}_L}(V'_\infty/L'_\infty)^2 \to \mathrm{End}_{\mathcal{O}_L}\mathscr{C}^2,$$

and a surjection of $\mathcal{O}_L$-modules $\mathscr{D} \to \mathscr{C}$. Therefore it suffices to prove that $\mathscr{D}$ is a finitely generated $\mathcal{O}_L$-module. In fact, by Nakayama's Lemma, we need only show $\mathscr{D}/M\mathscr{D}$ is finite, where M is the maximal ideal of $\mathcal{O}_L$. But $\mathscr{D}/M\mathscr{D}$ is the Pontrjagin dual of

$$D[M] \subseteq D^{\Gamma_r}[p] \cong e\mathrm{H}^1_P\left(\Gamma_1(N_0 p^r), \frac{K}{\mathcal{O}_K}\right)[p] \qquad \text{for } r \geqslant 2,$$

by Lemma 6.6. This is a finite group. □

Now suppose that $T \in \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}(\Omega_\infty)$. By Lemma 6.7 we can identify $\mathbb{T}_{\mathscr{X}}$ with a subring of $\mathrm{End}_{\mathcal{O}_L}(V'_\infty/L'_\infty)^2$. Then $T = \varepsilon\eta_\infty$ for some $\varepsilon \in \mathrm{End}_{\mathcal{O}_L}(V'_\infty/L'_\infty)^2$. Recall that $T_l^2 - \underline{\psi}(l)$ is an automorphism of A_r for all $r \geqslant 1$, so it is an automorphism of A_∞. Therefore η_∞ is not a zero-divisor in $\mathbb{T}_{\mathscr{X}}$, and $\eta_\infty\omega$ is a non-zero element of $\mathcal{O}_L$ for some $\omega \in \mathbb{T}_{\mathscr{X}}$. So we have $\varepsilon \in \mathbb{T}_{\mathscr{X}}(L)$. Letting $\mathbb{T}'_{\mathscr{X}}$ denote the integral closure of $\mathbb{T}_{\mathscr{X}}$ in $\mathbb{T}_{\mathscr{X}}(L)$, we conclude from Lemma 6.8 that $T \in \eta_\infty\mathbb{T}'_{\mathscr{X}}$. Since $\eta_\infty \in e_{\mathscr{X}}(I_{\mathscr{Y}}) \subseteq \mathrm{Ann}_{\mathbb{T}_{\mathscr{X}}}\Omega_\infty$,

$$\eta_\infty\mathbb{T}_{\mathscr{X}} \subseteq e_{\mathscr{X}}(I_{\mathscr{Y}}) \subseteq \eta_\infty\mathbb{T}'_{\mathscr{X}}.$$

Now let M be any finite extension of L, and Q any height one prime of $\mathcal{O}_M$. For $P = Q \cap \mathcal{O}_M$,

$$0 \to I_{\mathscr{Y}(L),P} \to \mathbb{T}_{\mathscr{X}(L),P} \to \mathbb{T}_{\mathscr{Y}(L),P} \to 0$$

is an exact sequence of free $\mathcal{O}_P$-modules. Therefore $I_{\mathscr{Y}(M),Q} = I_{\mathscr{Y}(L),P} \otimes_{\mathcal{O}_P} \mathcal{O}_Q$. We conclude that

$$\eta_\infty\mathbb{T}_{\mathscr{X},P} \subseteq e_{\mathscr{X}}(I_{\mathscr{Y},P}) \subseteq \eta_\infty\mathbb{T}'_{\mathscr{X},P} \tag{6.2}$$

holds without the restriction that L be of the form $\mathcal{O}_K[[T]]$.

Let $\mathscr{F} = \Sigma a_n q^n$ be a Λ-adic newform in $\mathscr{S}^0_L(\bar{N}_0, \underline{\psi})$. Suppose also that

L contains the roots α and β of $x^2 - a_l x + l\underline{\psi}(l)$. Then $\mathcal{F}_\alpha = \mathcal{F} - \beta\mathcal{F}(lz) \in \mathcal{X}$ is an eigenform of the Hecke operators. The congruence module $C_{\mathcal{F}_\alpha,\mathcal{Y}}$ is isomorphic to $\mathfrak{a}/\mathcal{O}_L$ where $\mathfrak{a}$ is an $\mathcal{O}_L$-lattice in L. For a height one prime P of $\mathcal{O}_L$, $\mathfrak{a}_P = P^{-d_P}\mathcal{O}_P$ where $\operatorname{div}(\mathfrak{a}) = \Sigma - d_P P$. We identify $\mathbb{T}_{\mathcal{F}_\alpha}$ with $\mathcal{O}_L$. Then $e_{\mathcal{F}_\alpha}(I_Y)_P = (\alpha^2 - \underline{\psi}(l))\mathcal{O}_P$ for any height one prime ideal P. (In fact this follows for P maximal as well.) Proposition 6.2 and (6.2) yield a formula for the divisor.

THEOREM 6A. $\operatorname{div}(\mathfrak{a}) = -\operatorname{div}(\alpha^2 - \underline{\psi}(l))\mathcal{O}_L$.

Suppose that $\mathcal{O}_L$ contains a root ξ of $x^2 - \underline{\psi}(l)$. As with classical cusp forms, we decompose $\mathcal{Y} = \mathcal{Y}^+ \oplus \mathcal{Y}^-$ with $\mathcal{Y}^\pm = \ker(T_l \mp \xi)$ in $\mathcal{Y}$. Let $\mathcal{X}_{\mathcal{F}} = L\{\mathcal{F}, \mathcal{F}(lz)\}$. The congruence module $C_{\mathcal{X}_{\mathcal{F}},\mathcal{Y}^+}$ is also of the form $\mathfrak{b}/\mathcal{O}_L$ for some $\mathfrak{b} \subseteq L$. Note that $\xi \in \mathcal{O}_K[[T]]$ for suitable $\mathcal{O}_K$, allowing us to decompose Ω_∞ and compute the divisor of $\mathfrak{b}$ by the method we used to prove Theorem 4b.

THEOREM 6B. $\operatorname{div}(\mathfrak{b}) = -\operatorname{div}(a_l - \xi(l+1))\mathcal{O}_L$.

Let $\mathcal{N}^o_\infty$ be set of newforms in $\mathcal{Y}$. By a congruence between Λ-adic newforms, we mean a congruence of nth coefficients for n prime to their levels. Our method of Fitting ideals yields the following analogue of Theorems 4c and 5c for sufficiently large L.

THEOREM 6C. *There exist ideals I_i in $\mathcal{O}_L$ and distinct $\mathcal{G}_i \in \mathcal{N}^o_\infty$ such that*

$$\mathcal{F} \equiv \mathcal{G}_i \bmod I_i \quad \textit{and} \quad \sum \operatorname{div}(I_i) \geqslant \operatorname{div}(a_l^2 - \underline{\psi}(l)(l+1)^2).$$

We have the following generalization of Ribet's theorem for ordinary forms of any character and weight at least 2. This can be proven directly from Theorem 5c using properties of Λ-adic forms, but we present it as a corollary to Theorem 6c. Let $\mathfrak{m}$ be the maximal ideal of the ring of integers of $\bar{\mathbb{Q}}_p$.

COROLLARY 6.9. *If $f = \Sigma a_n q^n$ is a p-stabilized newform in $S_k(\Gamma_0(N_0 p^r), \chi; \bar{\mathbb{Q}}_p)$ with $a_l^2 - \chi(l)l^{k-2}(l+1)^2 \in \mathfrak{m}$, then there exists a p-stabilized newform $g \in S_k(\Gamma_0(N_0 p^r l), \chi; \bar{\mathbb{Q}}_p)$ of level divisible by l such that $f \equiv g \bmod \mathfrak{m}$.*

Proof. We have $\chi = \psi\rho_\zeta\omega^{2-k}$ for some even character ψ on G, and some $\zeta \in \mu_{p^\infty}$. By the lifting theorem and Proposition 6.3, there is a newform $\mathcal{F} \in \mathcal{S}^o_L(\bar{N}_0, \underline{\psi})$ such that $v(\mathcal{F}) = f$ for some v extending $v_{k,\zeta}$ (and sufficiently large L). If $\mathcal{F} = \Sigma\mathcal{A}_n q^n$, then $\mathcal{A}_l^2 - \underline{\psi}(l)(l+1)^2 \in M$, where M is the maximal ideal of $\mathcal{O}_L$. By Krull's Principal Ideal Theorem, $\mathcal{A}_l^2 - \underline{\psi}(l)(l+1)^2 \in P$, for some height one prime ideal P of $\mathcal{O}_L$. By Theorem 6c, there is a Λ-adic newform $\mathcal{G} \in \mathcal{Y}$ with $\mathcal{G} \equiv \mathcal{F} \bmod P\mathcal{O}_P$. Since $P\mathcal{O}_P \cap \mathcal{O}_L = P \subseteq M$, we have $v(\mathcal{G}) \equiv f \bmod \mathfrak{m}$. $v(\mathcal{G})$ is a p-stabilized newform of conductor divisible by l. □

7. Error terms for $p=2$

We now determine the extent to which our methods and results apply to $p = 2$. Except as noted, we use the notation and proofs of the three preceding chapters.

The first problem we encounter is that we cannot necessarily define a Hecke operator R_l with the desired properties unless l represents a square in $\Gamma_0(N)/\{\pm 1\}\Gamma$. We therefore work with $\Gamma = \Gamma_0(N)$. This discussion pertains to odd primes as well. Since $\frac{1}{2}\phi(N)e_\psi \in \mathcal{O}_K[H]$, we find that $\frac{1}{2}\phi(N)\alpha_\psi(\Delta_\psi) \subseteq \Omega_\psi$. Our expression for the congruence module as in Theorem 4a now contains an error term which is independent of l (and p). We have

$$v_{\mathfrak{p}}(\alpha^2 - \chi(l)) \geqslant \text{length}_{\mathfrak{p}}\, C_{f_\alpha, Y^{(\psi)}} \geqslant v_{\mathfrak{p}}(\alpha^2 - \chi(l)) - v_{\mathfrak{p}}(\tfrac{1}{2}\phi(N)).$$

We also find that $2\phi(N)\ker\eta^\pm$ in $(V'^{(\psi)}/L'^{\psi})^2$ is a module for $\mathbb{T}_{X^{(\psi)\pm}}$ and $\mathbb{T}_{Y^{(\psi)\pm}}$. This gives (cf. Theorem 4b)

$$v_{\mathfrak{p}}(a_l \mp \zeta(l+1)) \geqslant \text{length}_{\mathfrak{p}}\, C_{X_f, Y^{(\psi)\pm}} \geqslant v_{\mathfrak{p}}(a_l \mp \zeta(l+1)) - v_{\mathfrak{p}}(2\phi(N)),$$

and (cf. Theorem 4c) there are distinct newforms $g_i \in \mathcal{N}$ such that

$$g_i \equiv f \bmod \mathfrak{p}^{d_i} \quad \text{with} \quad \sum d_i \geqslant v_{\mathfrak{p}}(a_l^2 - \chi(l)(l+1)^2) - 2v_{\mathfrak{p}}(2\phi(N)).$$

We now turn specifically to the case of $p = 2$. For ordinary forms we would like an error term independent of N since we will vary the power of 2 dividing the level. We write $N = N_0 2^r$ with odd N_0. We assume that $r \geqslant 2$. (We only need $r \geqslant 1$ if N_0 is divisible by a prime congruent to 3 mod 4.) We now let

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, d \equiv \pm 1 \bmod 2^r \right\}.$$

(For $r = 1$ or 2 this is $\Gamma_0(N)$.) We first note that Theorem 5a still holds. If $l \equiv \pm 1$ mod 8 (or if $r = 1$ or 2), then l represents a square in $\Gamma_0(N)/\Gamma$, and we can define R_l. We find in this case that if f is a p-stabilized newform, then (cf. Theorem 5b)

$$v_{\mathfrak{p}}(a_l \mp \zeta(l+1)) \geqslant \text{length}_{\mathfrak{p}}\, C_{X_f, Y^{(\psi),\pm}} \geqslant v_{\mathfrak{p}}(a_l \mp \zeta(l+1)) - v_{\mathfrak{p}}(4). \tag{7.1}$$

For any l prime to N, we then have (cf. Theorem 5c) distinct p-stabilized newforms $g_i \in \mathcal{N}^o$ such that

$$g_i \equiv f \bmod \mathfrak{p}^{d_i} \quad \text{with} \quad \sum d_i \geqslant v_{\mathfrak{p}}(a_l^2 - \chi(l)(l+1)^2) - v_{\mathfrak{p}}(16).$$

This follows from (7.1) if $l \equiv \pm 1 \bmod 8$. Otherwise we observe that $8(\alpha^2 - \chi(l))$ or $8(\beta^2 - \chi(l))$ is in $(a_l^2 - \chi(l)(l+1)^2)\mathcal{O}_K$ and it is a consequence of Theorem 5a. Also note that we can make

$$\sum d_i \geqslant \tfrac{1}{2} v_p(a_l^2 - \chi(l)(l+1)^2).$$

To define Λ-adic forms when $p = 2$, we let

$$\Gamma_r = \{v \in \mathbb{Z}_p^* \mid v \equiv 1 \bmod 2^{r+1}\}$$

and $\mu = \{\pm 1\}$. We can take $u = 5$ as a topological generator of Γ_1. Then our definition (6.1) becomes

$$\mathcal{S}_\Lambda^o(\bar{m}, \underline{\psi}) = \left\{ \mathcal{F} \in \Lambda[[q]] \,\middle|\, \begin{array}{l} v_{k,\zeta}(\mathcal{F}) \in eS_k(\Gamma_0(m2^{r+1}),\ \psi\rho_\zeta\omega^{2-k};\ \bar{\mathbb{Q}}_p) \\ \text{for all but finitely many } (k,\zeta) \in \mathfrak{S} \end{array} \right\}.$$

Theorem 6a holds. If $l \equiv \pm 1 \bmod 8$, then (cf. Theorem 6b)

$$-\mathrm{div}(a_l - \xi(l+1))\mathcal{O}_L \leqslant \mathrm{div}\, \mathfrak{b} \leqslant -\mathrm{div}(a_l - \xi(l+1))\mathcal{O}_L + \mathrm{div}\, 4\mathcal{O}_L.$$

We now have (cf. Theorem 6c) distinct Λ-adic newforms $\mathcal{G}_i \in \mathcal{N}_\infty^o$ such that

$$\mathcal{G}_i \equiv \mathcal{F} \bmod I_i \quad \text{with} \quad \sum \mathrm{div}(I_i) \geqslant \mathrm{div}(a_l^2 - \underline{\psi}(l)(l+1)^2)\mathcal{O}_L - \mathrm{div}\, 16\mathcal{O}_L.$$

We can also ensure by Theorem 6a that

$$\sum \mathrm{div}(I_i) \geqslant \tfrac{1}{2}\mathrm{div}(a_l^2 - \underline{\psi}(l)(l+1)^2)\mathcal{O}_L,$$

which gives Corollary 6.9.

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