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On the classification of irreducible low rank unitary representations of classical groups

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Section 1. Introduction

The theory of low rank representations for the symplectic group Sp_{2n} over a local field was introduced by Howe [5]. In this paper we develop the same theory for all type I classical groups. As far as the ideas are concerned, we have little new to offer, for most of them are already contained in [5]. However, we have a more specific goal here. That is to complete the classification of all irreducible low rank representations of a classical group, as promised in [16]. Given a type I classical group G, we have constructed in [16] a collection of irreducible low rank representations of G, all of them unitary. The main result of the present paper states that these exhaust all irreducible low rank representations of G.

To be a little more precise, let F be a local field of characteristic not equal to 2. Let $\bar{G}$ be a type I classical group over F. (cf. Definition 2.1). If G is Sp_{2n} we let $\bar{G}$ denote the metaplectic two-fold cover of Sp_{2n}. In all other cases let $\bar{G} = G$. Suppose G' is another classical group so that (G, G') is a reductive dual pair in the sense of Howe [6]. Assume further that (G, G') is in the stable range with G' the smaller member (cf. [4]). According to [16] we have an injective map

$$\hat{\bar{G}}'(\varepsilon) \hookrightarrow \hat{\bar{G}}, \quad \sigma \mapsto \pi(\sigma) \tag{1}$$

which arises from Howe's local duality correspondence, and is explicitly described. Here if G' is not Sp_{2n} then the set $\hat{\bar{G}}'(\varepsilon)$ is essentially just the unitary dual of G'. Otherwise it is (roughly) half of the unitary dual of $\widetilde{Sp}_{2n}$. (The metaplectic cover of Sp_{2n}). (see Section 4 for the precise definition of $\hat{\bar{G}}'(\varepsilon)$). We prove

THEOREM. (Theorem 4.8 below) *Let π be an irreducible unitary representation of $\bar{G}$. Assume π is of low rank. Then there is a group G' as above, a representation $\sigma \in \hat{\bar{G}}'(\varepsilon)$, and a character χ of $\bar{G}$ such that*

$$\pi = \chi \otimes \pi(\sigma),$$

Here $\pi(\sigma)$ is as in (1). *Furthermore, the various sets of representations of the above form associated to different G''s are disjoint.*

For $G = \mathrm{Sp}_{2n}$, this was proved by Howe [7] under slightly more restrictive condition. (namely the representation π should have rank not greater than $n - 2$.) The analogous result for GL_n was proved by Scaramuzzi [19].

In Section 5 we define the notion of a distinguished representation. A notion closely related to this was introduced in a global context by Piatetski-Shapiro [17] (see also [3]). Roughly speaking, distinguished representations are the most "singular" ones among representations which are not of low rank. In this regard it is worth noting that low rank representations can never be local constituents of cusp forms [14], while distinguished representations often arise as local constituents of examples of cusp forms contradicting the Ramanujan conjecture. (cf. [9][11]) The main result of Section 5 is a characterization of distinguished representations in certain cases. We remark that this has some global applications [15].

Section 2. Some algebra

(2.1) Throughout this paper, we let F be a local field of characteristic other than 2. There are two types of classical groups over F. Those of type II are the groups of invertible elements in simple algebras over F, i.e. general linear groups with entries in a division algebra over F. These we shall not consider in this paper. (When the division algebra is F itself the low rank representations have been studied by Scaramuzzi [19]). Much more is known for general linear groups than for other classical groups. Thus it seems the theory of low rank representations has less to offer for type II groups than it does for type I.

Hence forward by a classical group G we shall mean the one as given in the following definition.

DEFINITION 2.1. A type I classical group G is explicitly constructed as follows. There is

(a) a division algebra D of F, with involution $\#$;

(b) a vector space V over D, with non-degenerate sesquilinear form $(,)$, which is either hermitian or skew-hermitian; such that

(c) G is the identity component, in the algebraic sense, of the isometry group of $(,)$.

Let Z_0 be the fixed points of the involution $\#$ in the center of D. Nothing will change if we replace F by Z_0. Hence we always assume

(d) $Z_0 = F$.

Since F is fixed, we shall not distinguish an algebraic group witn its F-rational points in our notation. It is important to realize that, according to our definition the group G always comes together with a space V and a form $(,)$. For simplicity we usually just speak of G, and let the data $V, (,)$ be implicitly understood. When necessary we write $G = G(V)$ to indicate the dependence of G on V. If V_0 is a non-degenerate subspace of V we let $G(V_0)$ denote the subgroup of G consisting of elements which act as the identity on the orthogonal complement of V_0.

Because of the last assumption in the definition, D will either be a quadratic extension or a quaternion algebra over F. Let

$$\eta = \begin{cases} 1, & \text{if } (,) \text{ is hermitian} \\ -1, & \text{if not.} \end{cases} \tag{2}$$

We list the possibilities as in [26], No. 27. There are five altogether.

(I_0) $D = F, \eta = -1$; $G = \mathrm{Sp}_{2n}$ is the symplectic group

(I_1) D is a quaternion algebra over F with $\#$ the usual involution, and $\eta = 1$

(I_2) D is a quadratic extension of F and $\#$ is the Galois involution. We may take $\eta = 1$

(I_3) D and $\#$ as in (I_1), $\eta = -1$

(I_4) $D = F, \eta = 1$. The isometry group is denoted O_m if V has dimension m. Then $G = \mathrm{SO}_m$, the special orthogonal group.

In cases (I_0)–(I_2) above, the isometry group of $(,)$ is connected (as an algebraic group). In case (I_3) the isometry group has two connected components. Nevertheless its F-points coincides with G. Thus (I_4) is the only case when G is not the full isometry group of $(,)$. The restriction from O_m to $G = \mathrm{SO}_m$ in this case is hardly necessary, but will save us a lot of trouble.

(2.2) For later purposes, we give a description of a typical maximal parabolic subgroup of G. Fix once for all a maximal set of independent vectors

$$\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\},$$

such that

$$(e_i, e_j) = 0 = (e_i^*, e_j^*), \; (e_i, e_j^*) = \delta_{ij}. \tag{3}$$

The integer n is both the Witt index of $(,)$ and the split rank of G. For each integer k with $1 \leqslant k \leqslant n$ we let X_k be the span of $e_1, \ldots, e_k$ and X_k^* the span of $e_1^*, \ldots, e_k^*$. Set $V_k = X_k \oplus X_k^*$, and let $V_k^\perp$ be its orthogonal complement in V with respect to $(,)$. Let P_k be the parabolic subgroup of G preserving X_k^*. We may take for a Levi factor of P_k the group of elements preserving both X_k and X_k^*. This group is isomorphic to $GL(X_k^*) \cdot G(V_k^\perp)$. Write the Levi decomposition of P_k as

$$P_k = GL(X_k^*) \cdot G(V_k^\perp) \cdot N_k, \tag{4}$$

where N_k is the unipotent radical of P_k. The group N_k is at most two-step unipotent, and will fit into the following exact sequence

$$1 \to ZN_k \to N_k \to \mathrm{Hom}_D(V_k^\perp, X_k^*) \to 1, \tag{5}$$

where ZN_k denotes the center of N_k, except when $k = 1$ and G is of type (I_4). In this exceptional case the group N_1 is itself abelian, and (5) remains valid if we consider ZN_1 to be the trivial group. Let $\eta' = -\eta$. Let $B(X_k, \eta')$ be the space of all sesquilinear forms on X_k having the opposite symmetry as $(\,,)$ under interchange of the two variables. Thus if $(\,,)$ is hermitian then $B(X_k, \eta')$ consists of skew-hermitian forms, and vice-versa. Define $B(X_k^*, \eta')$ analogously. There is a natural isomorphism

$$ZN_k \simeq B(X_k, \eta').$$

The spaces $B(X_k, \eta')$ and $B(X_k^*, \eta')$ are naturally dual to each other. To make this explicit we note that there is an involution

$$*\colon \mathrm{End}_D(V) \to \mathrm{End}_D(V),\ T \mapsto T^*$$

defined by the identity

$$(Tu, v) = (u, T^*v) \qquad (u, v \in V).$$

This involution will preserve $\mathrm{Hom}_D(X_k, X_k^*)$. (Which is to be identified with the endomorphisms of V that vanish on $X_k^* \oplus V_k^\perp$ and have their images contained in X_k^*). Similarly it preserves $\mathrm{Hom}_D(X_k^*, X_k)$. We have natural isomorphisms

$$B(X_k, \eta') \simeq \{z \in \mathrm{Hom}_D(X_k, X_k^*) \mid z^* = \eta' z\}, \tag{6}$$

$$B(X_k^*, \eta') \simeq \{\hat{z} \in \mathrm{Hom}_D(X_k^*, X_k) \mid \hat{z}^* = \eta' \hat{z}\}. \tag{7}$$

Thus given $z \in B(X_k, \eta')$ and $\hat{z} \in B(X_k^*, \eta')$ the composite $z \cdot \hat{z}$ is an endomorphism of X_k^*, etc, and the bilinear form

$$\mathrm{tr}(z \cdot \hat{z}) = \mathrm{tr}(\hat{z} \cdot z)$$

exhibits the duality involved. Here tr denotes the reduced trace in the simple algebra $\mathrm{End}_D(X_k^*)$, etc.

Let $\widehat{ZN_k}$ be the Pontrjagin dual of ZN_k. The linear duality between $ZN_k = B(X_k, \eta')$ and $B(X_k^*, \eta')$ enables us to identify $B(X_k^*, \eta')$ with $\widehat{ZN_k}$ as follows. Let us fix a non-trivial character ψ of F. Then for $\beta \in B(X_k^*, \eta')$ we may define a character ψ_β by the formula

$$\psi_\beta(z) = \psi(\mathrm{tr}(z \cdot \beta)) \qquad (z \in ZN_k). \tag{8}$$

The map $\beta \mapsto \psi_\beta$ establishes the required isomorphism.

(2.3) Let G_1 be the isometry group of the form (,) in Definition 2.1. Suppose that G_1 is a member of a reductive dual pair (G_1, G') in the sense of Howe [6]. The groups G_1, G' will be mutual centralizers in some symplectic group Sp. In particular $G \subseteq G_1 \subseteq \mathrm{Sp}$. For $F \neq \mathbf{C}$ there will be a non-split short exact sequence

$$\mathbf{Z}_2 \to \widetilde{\mathrm{Sp}} \to \mathrm{Sp}, \tag{9}$$

where $\mathbf{Z}_2$ denotes the group with two elements and $\widetilde{\mathrm{Sp}}$ is of course the metaplectic group.

LEMMA 2.2. (*i*) *Suppose G is of type* (I_0) *so that* $G' = \mathrm{O}_m$ *is the orthogonal group. Then* (9) *splits over G if and only if m is even.*

(*ii*) *If G is of type* $(I_1), (I_3)$ *or* (I_4) *then* (9) *always splits over G.*

(*iii*) *Let G be of type* (I_2). *Then* (9) *does not split over G, but the obstruction is very mild. More precisely, let* $\tilde{G}$ *be the inverse image of G in* $\widetilde{\mathrm{Sp}}$. *Then there is a character* γ *of* $\tilde{G}$ *which does not factor through G. Consequently, any representation of* $\tilde{G}$ *either factors through G or is the tensor product of* γ *with a representation which factors through G.*

Proof. (i) is well know. We leave it to the reader to verify that (ii) and (iii) follow immediately from [10], Lemma 7.

NOTATION 2.3. (i) Suppose $F \neq \mathbf{C}$, and G is of type (I_0) so that $G = \mathrm{Sp}_{2n}$. We let $\bar{G}$ be the metaplectic two-fold cover of Sp_{2n}.

(ii) In all other cases we let $\bar{G} = G$.

Thus we have a canonical map

$$\bar{G} \to G.$$

In all cases (I_1)–(I_4), this is just the identity map. If E is a subgroup of G we let $\bar{E}$ be its inverse image in $\bar{G}$. Naturally, if it so happens that E has a unique lifting to a subgroup of $\bar{G}$ we shall use E to denote the lifted subgroup as well. Corresponding to (4) we have

$$\bar{P}_k = \overline{GL(X_k^*)} \cdot \overline{G(V_k^\perp)} \cdot N_k. \tag{10}$$

(2.4) Finally, we shall need to comment on the exact sequence (5). Write $W = \mathrm{Hom}_D(V_k^\perp, X_k^*)$. Since the restriction to $V_k^\perp$ of $(,)$ is non-degenerate, we may identify $V_k^\perp$ with its own linear dual via a standard procedure. Hence we have

$$W = \mathrm{Hom}_D(V_k^\perp, X_k^*) \simeq V_k^\perp \otimes_D X_k^*. \tag{11}$$

Suppose we are given a form $\beta \in B(X_k^*, \eta')$, and suppose that β is in fact non-degenerate on X_k^*. The formula

$$\langle , \rangle_\beta = \mathrm{tr}_{D/F}((,) \otimes_D \beta(,)^\#)$$

defines a non-degenerate symplectic form on W. (cf. [6]). Let $\mathrm{Sp}(W)$ be the corresponding symplectic group. Let G' be the stabilizer of β in $GL(X_k)$. The action by conjugation of $G(V_k^\perp)$. G' on N_l preserves ZN_l, and therefore gives rise to an action on $W = N_l/ZN_l$. Evidently this embeds $G(V_k^\perp)$. G' into $\mathrm{Sp}(W)$. It is not difficult to realize that $(G(V_k^\perp), G')$ is essentially a reductive dual pair in $\mathrm{Sp}(W)$. The word "essentially" is needed only because we have replaced O_n by SO_m in case (I_4), and thus when G is of type (I_4) the full centralizer of G' in $\mathrm{Sp}(W)$ is slightly bigger than $G(V_k^\perp)$. In similar occasions that we shall encounter later, we shall directly speak of $(G(V_k^\perp), G')$ as a reductive dual pair. In the context of this paper no harm will be caused by this slightly loose terminology.

Let N_β be the quotient group of N_k obtained by dividing ZN_k by the kernel of ψ_β. It is easy to see that N_β is a Heisenberg group with center

$$ZN_k/\mathrm{Ker}\,\psi_\beta \simeq F.$$

In fact N_β is naturally isomorphic to the Heisenberg group attached to the symplectic space W. Corresponding to (5) we have the exact sequence

$$1 \to F \to N_\beta \to W \to 1. \tag{12}$$

Section 3. The ZN_k-spectrum of a unitary representation

We place ourselves in the setting of Section 2.
Suppose that π is a unitary representation of $\bar{P}_k$. Consider the restriction $\pi|_{ZN_k}$. By the direct integral theory, this restriction is determined by a projection valued measure on $\widehat{ZN_k}$. We denote this measure by μ_π. Since μ_π comes from a representation of $\bar{P}_k$ it will satisfy a transformation law under the co-adjoint action of $\bar{P}_k$. Specifically, for a Borel set $S \subseteq \widehat{ZN_k}$ we have

$$\mu_\pi(\mathrm{Ad}^*\bar{p}(S)) = \pi(\bar{p})\mu_\pi(S)\pi(\bar{p})^{-1} \qquad (\bar{p} \in \bar{P}_k). \tag{13}$$

Suppose π is a unitary representation of $\bar{G}$. Then it restricts to one of $\bar{P}_k$, so the above applies to π.

Now the subgroup $G(\bar{V}_k^{\perp})\cdot N_k$ of $\bar{P}_k$ centralizes ZN_k, and hence acts trivially on $\widehat{ZN}_k$. The action of $\overline{GL(X_k^*)}$ will factor through $GL(X_k^*)$. Finally through the isomorphism $\widehat{ZN}_k \simeq B(X_k^*, \eta')$, the action of $GL(X_k^*)$ via Ad* is identified with its natural action on $B(X_k^*, \eta')$. Note that a $GL(X_k^*)$-orbit in $B(X_k^*, \eta')$ is nothing but an equivalence class of sesquilinear forms on X_k^* of the kind specified by η'. If $\beta \in B(X_k^*, \eta')$ we let $\mathcal{O}_\beta$ denote its orbit under $GL(X_k^*)$. By the rank of a form β we shall mean the maximal dimension of a subspace of X_k^*, upon which the restriction of β is non-degenerate. The rank of $\mathcal{O}_\beta$ is defined to be that of β.

Set

$$r_G = \max\{\operatorname{rank}\beta \mid \beta \in B(X_n^*, \eta')\}.$$

We clearly have

$$r_G = \begin{cases} n-1, & \text{if } G \text{ is of type}(I_4)\text{, and } 2 \nmid n \\ n, & \text{otherwise.} \end{cases} \tag{14}$$

THEOREM 3.1. *Let π be a unitary representation of $\bar{G}$ on a Hilbert space $\mathcal{H}$. Let μ_π be the ZN_n-spectrum of π. For each orbit $\mathcal{O}_\beta \subseteq B(X_n^* \eta')$, set*

$$\mathcal{H}_\beta = \mu_\pi(\mathcal{O}_\beta)\cdot\mathcal{H}. \tag{15}$$

Then for rank$\beta < r_G$, the subspace $\mathcal{H}_\beta$ is invariant under $\pi(\bar{G})$.

Proof. As in [5] we proceed by induction on r_G. Our proof however will be somewhat shorter than the one given in [5]. Set

$$r = \begin{cases} 2, & \text{if } G \text{ is of type } (I_4) \\ 1, & \text{otherwise.} \end{cases} \tag{16}$$

If $r_G = r$ then the only rank $< r_G$ is 0. Hence rank $\beta < r_G$ means $\beta = 0$. In this case the theorem follows from [8], which says $\mathcal{H}_0$ in fact decomposes over one dimensional representatives of $\bar{G}$.

Now assume $r_G > r$. Since $r < r_G \leqslant n$, $\bar{P}_r$ and $\bar{P}_n$ together generate $\bar{G}$. Since H_β is clearly invariant under $\pi(\bar{P}_n)$ it will be enough to show that H_β is invariant by $\pi(\bar{P}_r)$.

Let us write G^1 for $G(V_r^{\perp})$. We shall look at the subgroup $\bar{G}^1\cdot Nr$ of $\bar{P}_r$, and examine its representations by means of Mackey theory. Let

$$0 \neq \tau \in B(V_r^{\perp}, \eta').$$

Let $W = \mathrm{Hom}_D(V_r^\perp, X_r^*)$. Consider the character ψ_r and the Heisenberg group N_τ defined in Section 2.4. We may consider ψ_τ as a character of $ZN_r/\mathrm{Ker}\,\psi_\tau$. By the Stone-von Neumann theorem, there is a unique irreducible unitary representation ρ_τ of N_τ, whose central character is ψ_τ. Of course we may (and do) view ρ_τ as a representation of N_r.

According to Section 2.4, G^1 may be realized as a member of a reductive dual pair in $\mathrm{Sp}(W)$. In particular, we may extend our representation ψ_τ to a two fold cover of G^1 (i.e., the pre-image of G^1 in $\widetilde{\mathrm{Sp}(W)}$) by means of the oscillator representation. But Lemma 2.2 guarantees that we may in fact extend ρ_τ to $\bar{G}^1$. The extension is in general not unique, but any two such will differ by a character of $\bar{G}^1$. Let us fix one such extension, and denote the resulting representation of $\bar{G}^1 \cdot N_r$ by the same symbol ρ_τ. Then Mackey theory tells us that the correspondence

$$\nu \mapsto \nu \otimes \rho_\tau \tag{17}$$

is an equivalence between the category of unitary representations of $\bar{G}^1$ and the category of unitary representation of $\bar{G}_1 \cdot N_r$, whose restriction to ZN_r is a multiple of ψ_τ.

Now return to our representation π. Since the subspace of fixed vectors of $\pi(ZN_r)$ in $\mathscr{H}$ is invariant under $\pi(\bar{G})$ ([8] again), we may as well assume that $\mathscr{H}$ contains no fixed vectors for ZN_r. Then the analysis of the preceding paragraph shows that $\pi|_{\bar{G}^1 \cdot N_r}$ must decompose over representations of the form (17). Since π comes from a representation of $\bar{P}_r$ we can be more specific about this decomposition. Note that $GL(X_r^*)$ acts on $ZN_r = B(X_r^*, \eta')$. Under this action $B(X_r^*, \eta')$ breaks up into finitely many orbits. We have

$$\pi|_{\bar{G}^1 \cdot N_r} \simeq \sum_{\mathcal{O}'} \int_{\mathcal{O}'} \nu_\tau \otimes \rho_\tau \, d\tau. \tag{18}$$

Here the summation extends over all non-trivial orbits $\mathcal{O}'$ in $B(X_r^*, \eta')$, and ν_τ is a unitary representation of $\bar{G}^1 \simeq \bar{G}^1 \cdot N_r/N_r$ for each $\tau \in \mathcal{O}'$. Although it is irrelevant to our proof, we observe that the measure $d\tau$ in (18) can be described as follows: fix $\tau \in \mathcal{O}'$ and let G' be the stabilizer of τ in $GL(X_r^*)$. Then we may identify $\mathcal{O}'$ with $GL(X_r^*)/G'$. Under the action of $GL(X_r^*)$ the ZN_r-spectrum of π will obey the transformation law (13). It follows that $d\tau$ must come from the Haar measure on $GL(X_r^*)/G'$ by "transport of structure".

Set

$$X^1 = X_n \cap V_r^\perp, \qquad X^{1*} = X_n^* \cap V_r^\perp,$$

$$P^1 = G^1 \cap P_n, \qquad Q = GL(X_n^*) \cap N_r. \tag{19}$$

Then X^1, X^{1*} are maximal isotropic subspaces of $V_r^\perp$, and P^1 is the stabilizer of X^{1*} in G^1. Write the Levi decomposition (4) for P^1 as

$$P^1 = GL(X^{1*})\cdot G(V_n^\perp)\cdot N^1.$$

To further analyze (18) we must describe the action of $\rho_\tau(ZN_n)$. Observe that we have the direct sum decompositions

$$X_n = X_r \oplus X^1, \qquad X_n^* = X_r^* \oplus X^{1*}$$

$$V_r^\perp = X^1 \oplus V_n^\perp \oplus X^{1*}. \tag{20}$$

These enable us to write

$$ZN_n = ZN_r \oplus \mathrm{Hom}_D(X^1, X_r^*) \oplus ZN^1. \tag{21}$$

Write $X = \mathrm{Hom}_D(X^{1*}, X_r^*)$ (cf. (11)). We may view X as the isotropic subspace of W consisting of homomorphisms which vanish on $X^1 \oplus V_n^\perp$. According to [4], there will be a Hilbert space $\mathscr{F}$, such that ρ_τ can be realized on

$$\mathscr{Y}_\tau = L^2(X, \mathscr{F})$$

the space of $\mathscr{F}$-valued functions on X with square integrable norm. This is the so called mixed model realization of ρ_τ. In this model the action of ZN_n can be described as follows. First

$$\rho_\tau(z)(\phi)(x) = \psi_\tau(z)\cdot\phi(x), \qquad (z \in ZN_r, \phi \in \mathscr{Y}_\tau, x \in X).$$

If $z' \in ZN^1 = B(X^1, \eta')$ then the composite $x\cdot z'\cdot x^*$ lies in $B(X_r, \eta')$ (cf. (6), (7)). We have

$$\rho_\tau(z')(\phi)(x) = \psi_\tau(\tfrac{1}{2}(x\cdot z'\cdot x^*))\cdot\phi(x).$$

Finally if $y \in \mathrm{Hom}_D(X^1, X_r^*)$ then

$$\rho_\tau(y)(\phi)(x) = \psi(\mathrm{tr}(\tau\cdot x\cdot y^*))\cdot\phi(x).$$

The next lemma follows directly from these formulas.

LEMMA 3.2. *Define a form $\beta_\tau \in B(X_n^*, \eta')$ by requiring that X^{1*} is in its radical and the restriction of β_τ to X_r^* is τ. Then the ZN_n-spectrum of ρ_τ is concentrated on the Q-orbit of β_τ.*

Corresponding to (18) we have a decomposition

$$\mathscr{H} \simeq \sum_{\mathcal{O}'} \int_{\mathcal{O}'} \mathscr{H}_\tau \otimes \mathscr{Y}_\tau \cdot \mathrm{d}\tau, \tag{22}$$

where $\mathscr{H}_\tau$ denotes the Hilbert space on which v_τ acts. Our aim is to get a similar decomposition for $\mathscr{H}_\beta$. Suppose $\beta' \in B(X^{1*}, \eta')$ and $\mathcal{O}_{\beta'}$ is its orbit under $GL(X^{1*})$. Define the subspace $(\mathscr{H}_\tau)_{\beta'}$ of $\mathscr{H}_\tau$ is analogy to (15). We may extend β' to a form on X_n^* by requiring X_r^* to be in its radical. By Lemma 3.2 the ZN_n-spectrum of $(\mathscr{H}_\tau)_{\beta'} \otimes \mathscr{Y}_\tau$ will be supported on the set

$$\{\gamma + \gamma' \mid \gamma \in \mathcal{O}_{\beta'} \qquad \gamma' \in \mathrm{Ad}^*(Q)(\beta_\tau)\}.$$

Note that the group Q centralizes ZN^1. Let $\simeq$ denote equivalence of forms in $B(X_n^*, \eta')$. For $(\mathscr{H}_\tau)_{\beta'} \oplus \mathscr{Y}_\tau$ to make a contribution to $\mathscr{H}_\beta$ we must have

$$\beta' + \beta_\tau \simeq \beta. \tag{23}$$

By Witt cancellation (cf. [20]), we see that the orbit $\mathcal{O}_{\beta'}$ is uniquely determined by the $GL(X_r^*)$ orbit $\mathcal{O}'$ to which τ belongs. In conclusion we have

LEMMA 3.3. *For each* $\mathcal{O}'$ *in* (18) *let* $\mathcal{O}_{\beta'}$ *be the unique* $GL(X_r^*)$*-orbit with* β' *satisfying* (23) for all $\tau \in \mathcal{O}'$. *Then*

$$\mathscr{H}_\beta \simeq \sum_{\mathcal{O}'} \int_{\mathcal{O}'} (\mathscr{H}_\tau)_{\beta'} \otimes \mathscr{Y}_\tau \, \mathrm{d}r. \tag{24}$$

The group G^1 is a classical group (Definition 2.1) of the same type as G. We have $r_{G^1} = r_G - r$. Thus we may assume inductively that the theorem is valid for G^1. From (23) we get

$$\operatorname{rank} \beta' = \operatorname{rank} \beta - r < r_{G^1}.$$

The inductive hypothesis implies each $(\mathscr{H}_\tau)_{\beta'}$ in (24) is a $\bar{G}^1$-module, and hence (24) exhibits $\mathscr{H}_\beta$ as a module for $\bar{G}^1 \cdot N_r$. But $\bar{G}^1 \cdot N_r$ together with $\bar{P}_n$ already generate $\bar{G}$. This means $\mathscr{H}_\beta$ is a $\bar{G}$-module and concludes our proof of Theorem 3.1.

Section 4. Low rank representations

We keep the notations of Section 3.

DEFINITION 4.1. Let l be an integer. The representation π is said to be of rank $\leqslant l$ if the support of its ZN_n-spectrum μ_π is contained in the set of $\beta \in B(X_n^*, \eta')$ with rank $\beta \leqslant l$. It is said to be of rank l if it has rank less, but not strictly less, than l. Finally π is said to be of pure rank l if μ_π is supported on the orbits $\mathcal{O}_\beta \subseteq B(X_n^*, \eta')$ with rank $\mathcal{O}_\beta = l$.

For any integer $l \geqslant 0$ we let $\hat{\bar{G}}_l$ denote the subset of the unitary dual of $\bar{G}$ consisting of representations of pure rank l. For an orbit $\mathcal{O}_\beta \subseteq B(X_n^*, \eta')$, we let $\hat{\bar{G}}_\beta$ be the set of irreducible unitary representations whose ZN_n-spectrum is supported on $\mathcal{O}_\beta$. Theorem 3.1 implies that the whole unitary dual $\hat{\bar{G}}$ is a disjoint union

$$\hat{\bar{G}} = \hat{\bar{G}}_{r_G} \cup \left(\bigcup_{\mathrm{rank}\,\beta < r_G} \hat{\bar{G}}_\beta \right). \tag{25}$$

The purpose of this section is to give a description of $\hat{\bar{G}}_\beta$ when rank $\beta < r_G$.

We start with the description of $\hat{\bar{G}}_0$. By the theorem of Howe and Moore [8], this is precisely the set of characters of $\bar{G}$.

LEMMA 4.2. *Assume $n > 0$. Let $[\bar{G}, \bar{G}]$ be the commutator subgroup of $\bar{G}$*

(*a*) *In cases (I_0) and (I_1) the group $\bar{G}$ is perfect. That is $\bar{G} = [\bar{G}, \bar{G}]$.*

(*b*) *Suppose G is of type (I_2), so that D is a quadratic extension of F. Let D^1 be the group of norm one elements in D. Then taking determinants gives rise to an isomorphism*

$$G/[G,G] \xrightarrow{\sim} D^1. \tag{26}$$

In other words, $[G, G]$ is precisely the special unitary group.

(*c*) *Suppose G is of type (I_3). If $F = \mathbf{R}$ then G is a perfect group. Otherwise we have an isomorphism*

$$G/[G,G] \xrightarrow{\sim} F^\times/F^{\times 2}. \tag{27}$$

(*d*) *Let G be of type (I_4), i.e. $G = \mathrm{SO}_m$. Assume $m \geqslant 3$. Then taking the so called spinor norm gives us an isomorphism*

$$G/[G,G] \xrightarrow{\sim} F^\times/F^{\times 2}. \tag{28}$$

Proof. These statements are classical results. Here we merely remark that in case G is of type (I_3) the results of [1], [23] and [24] imply an isomorphism

$$G/[G,G] \xrightarrow{\sim} D^\times/F^\times.[D^\times, D^\times].$$

But it is well known that $[D^\times, D^\times]$ coincides with the norm one subgroup of $D^\times$.

So taking the standard norm N gives us an isomorphism

$$D^\times/F^\times\cdot[D^\times,D^\times] \xrightarrow{\sim} N(D^\times)/F^{\times 2}.$$

But

$$N(D^\times)=\begin{cases}F^{\times 2}, & \text{if } F=\mathbf{R}\\ F^\times, & \text{otherwise.}\end{cases}$$

Hence (27) follows.

THEOREM 4.3. *Let π be a unitary representation of $\bar{G}$ of pure* rank $l<r_G$, *then $\pi(\bar{P}_l)$ and $\pi(\bar{G})$ generate the same Von Neumann algebra. In particular, π is irreducible if and only if $\pi|_{P_l}$ is irreducible. If π' is another representation of pure* rank l *then π and π' are equivalent if and only if $\pi|_{\bar{P}_l}$ and $\pi'|_{\bar{P}_l}$ are equivalent.*

Proof. For $l=0$, we consider $\bar{P}_l$ to be $\bar{G}$ itself, so there is nothing to be proved. Assume $l>0$. Let r and other notations be as in the proof Theorem 3.1. For each $0\neq\tau\in B(X_r^*,\eta')$ we let ω_τ denote the restriction of ρ_τ to $\bar{G}^1$. According to (18) we have

$$\pi|_{\bar{G}^1}\simeq\sum_{\mathcal{O}'}\int_{\mathcal{O}'}v_\tau\otimes\omega_\tau\cdot d\tau, \tag{29}$$

where each v_τ is a representation of $\bar{G}^1$ of pure rank $l-r<r_{G^1}$.

First, let us take $l=r$. Consider $\overline{G(V_r)}$. Since $\overline{G(V_r)}$ and $\bar{P}_r$ together generate $\bar{G}$, it will be enough to show that the algebra generated by $\pi(\overline{G(V_r)})$ is contained in the one generated by $\pi(\bar{P}_r)$. Set

$$Q_r=G(V_r)\cap P_r.$$

It suffices to show that $\pi(\bar{Q}_r)$ and $\pi(\overline{G(V_r)})$ generate the same algebra.

Since $r=l<r_G$, $\overline{G(V_r)}$ will be conjugate to a subgroup of $\bar{G}^1$. Let $\bar{g}\in\bar{G}$ be an element realizing this conjugation, and let g be its image in G. Set

$$Y_r=g\cdot X_r,\qquad V_r^*=g\cdot X_r^*,$$

$$U_r=Y_r\oplus Y_r^*=g\cdot V_r\subseteq V_r^\perp.$$

$$Q_r'=g\cdot Q_r\cdot g^{-1}.$$

Then Q_r' is the parabolic subgroup of $G(U_r)=g\cdot G(V_r)g^{-1}$ preserving Y_r^*. It is enough to show that $\pi(\bar{Q}'_r)$ and $\pi(\overline{G(U_r)})$ generate the same algebra. But then in view of (29), it suffices to prove this statement with π replaced by the representations ν_τ and ω_τ in (29).

Since $l = r$, each ν_r is a rank 0 representation and therefore decomposes over characters of $\bar{G}^1$([8] again). The question for ν_τ then comes down to whether a character of $\overline{G(U_r)}$ is determined by its restriction to $\bar{Q}'_r$. That the answer is affirmative is an immediate consequence of Lemma 4.2 (applied to $\overline{G(U_r)}$). For example, suppose G is of type (I_4). Then the Levi component of Q'_r is isomorphic to $GL_2(F)$, and Q'_r may be realized as matrices of the form

$$\begin{pmatrix} A & * \\ 0 & {}^tA^{-1} \end{pmatrix}$$

with $A \in GL_2(F)$. The image of such an element of Q'_r under the map (28) is represented by det(A). (cf. [20], Chapter 9, Example 3.5). It follows that taking spinor norm maps Q'_r surjectively onto $F^\times/F^{\times 2}$. This verifies our assertion for type (I_4). The other cases are similar but easier, and are left to the reader.

Next we turn to the representation ω_τ. Let G' be the stabilizer of τ in $GL(X_r^*)$ (as in the proof of Theorem 3.1). According to Section 2, (G^1, G') is a reductive dual pair. We shall call ω_τ the oscillator representation associated to (G^1, G'). Similarly $(G(U_r), G')$ is a reductive dual pair. Let ω'_τ be the oscillator representation associated to the pair $(G(U_r), G')$. By the "functorial" property of the oscillator representation with respect to direct sums of symplectic spaces, the restriction $\omega_r|_{\overline{G(U_r)}}$ is a multiple of $\omega'_\tau|_{\overline{G(U_r)}}$ (cf. [5]). Consequently we only need to prove that $\omega'_\tau(\bar{Q}'_r)$ and $\omega'(\overline{G(U_r)})$ generate the same algebra. But the L^2-version of Howe's duality conjecture states (in our situation) that $\omega'_\tau(\overline{G(V_r)})$ generates precisely the centralizer of $\omega'_\tau(\tilde{G}')$. Howe [4] has proved a some what more precise version of his conjecture for stable dual pairs, which in the case that we are considering here says that $\omega'_\tau(\bar{Q}'_r)$ already generates the full centralizer of $\omega'_\tau(\tilde{G}')$. It follows in particular that $\omega'_\tau(\bar{Q}'_\tau)$ and $\omega'_\tau(\overline{G(U_r)})$ generate the same algebra. This finishes the proof of our theorem in the case $l = r$.

From now on we assume $l > r$. Then $\bar{P}_l$ and $\bar{P}_r$ together generate $\bar{G}$. In fact the group $\bar{G}^1$ together with $\bar{P}_l$ already generate $\bar{G}$. Set $Q = G^1 \cap P_l$. The representation ν_τ of $\bar{G}^1$ in (29) is of rank

$$l - r < r_G - r = r_{G^1}.$$

We may inductively assume that our theorem is true for $\bar{G}^1$. This implies $\nu_\tau(\bar{Q})$ and $\nu_\tau(\bar{G}^1)$ generate the same algebra. But $\bar{Q} \subseteq \bar{P}_l$, so the algebra generated by $\nu_\tau(\overline{G^1})$ is contained in the one generated by $\nu_\tau(\bar{P}_l)$. Now we have $N_r \subseteq \bar{P}_l$. The representation ρ_τ is already irreducible on N_r, and is simply extended to $\bar{G}^1$. Thus the algebra generated by $\omega_\tau(\bar{G}^1) = \rho_\tau(\bar{G}^1)$ is certainly contained in the algebra generated by $\rho_\tau(N_r)$. Putting things together, we see that (29) implies the algebra generated by $\pi(\bar{G}^1)$ is contained in the one generated by $\pi(\bar{P}_l)$. This concludes the proof of our theorem.

REMARK. By inspection of the above proof we see that the theorem remains valid if we replace $\bar{P}_l$ by any $\bar{P}_k$ with $k > l$. Indeed this is the version which was proved in [5] for $G = \mathrm{Sp}_{2n}$. The reason for this seemingly more general statement is as follows. Consider the subgroup

$$H_l = G(V_l^\perp)\cdot N_l \tag{30}$$

of $\bar{P}_l$. If π is a representation of $\bar{G}$ of pure rank $l < r_G$, then π is largely determined by the restriction π_{H_l} already. (It is in the case when $\bar{G}$ is a perfect group). But for any $k \geqslant l$ we clearly have $\bar{H}_l \subseteq \bar{P}_k$.

Fix a $GL(X_n^*)$-orbit $\mathcal{O}_\beta \subseteq B(X_n^*, \eta')$. Let $l = \operatorname{rank}\beta$ and assume $0 < l < r_G$. Theorem 4.3 implies that restriction gives us an injection

$$\hat{\hat{G}}_\beta \hookrightarrow \hat{\hat{P}}_l.$$

We may of course assume that the restriction of β to X_l^* is non-degenerate. Set

$$\gamma = \beta|_{X_l^*}.$$

LEMMA 4.4 *Let $\pi \in \hat{\hat{G}}_\beta$. Then the ZN_l-spectrum of π is supported on the $GL(X_l^*)$-orbit of γ.*

Proof. The set of $\beta' \in \mathcal{O}_\beta$ which restrict to degenerate forms on X_l^* is a sub-variety of $\mathcal{O}_\beta$ of positive codimension. On the other hand if $\beta' \in \mathcal{O}_\beta$ restricts to a non-degenerate form on X_l^* then we clearly have $\beta'|_{X_l^*} \simeq \gamma$. The lemma follows immediately from these two observations.

For $\pi \in \hat{\hat{G}}_\beta$ we may now describe the restriction $\pi|_{\bar{P}_l}$ according to Mackey theory. Let ψ_γ be the character of ZN_l defined by (8). Let J be the isotropy group of ψ_γ in $\bar{P}_l$ with respect to the co-adjoint action. Then Mackey theory tells us that we must have

$$\pi|_{\bar{P}_l} \simeq \mathrm{Ind}_J^{\bar{P}_l}\cdot\tau_1, \tag{31}$$

where τ_1 is an irreducible unitary representation of J which restricts to a multiple of ψ_γ on ZN_l. Let G' be the stabilizer of γ in $GL(X_l^*)$. It is clear that

$$J = \overline{G(V_l^\perp)}\cdot\bar{G}'\cdot N_l. \tag{32}$$

In complete analogy with the analysis in the proof of Theorem 3.1, there will be a unique irreducible unitary representation ρ_γ, of N_l, with central character ψ_γ. Let $W = \mathrm{Hom}_D(V_l^\perp, X_l^*)$. According to Section 2, W is a symplectic space and $(G(V_l^\perp), G')$ is a reductive dual pair in $\mathrm{Sp}(W)$. Let us first assume G is not of type (I_0), or it is of type (I_0) but $F = \mathbf{C}$. Let $\tilde{G}'$ be the pre-image of G' in $\widetilde{\mathrm{Sp}(W)}$. The group $G(V_l^\perp) \times \tilde{G}'$ acts on N_l via its projection to $G(V_l^\perp)\cdot G'$. Thus we can form the semi-direct product

$$\tilde{J} = (G(V_l^\perp) \times \tilde{G}') \times N_l.$$

There is an obvious projection $\tilde{J} \to J$. Let $\mathbf{Z}_2 \subseteq \tilde{J}$ be the kernel of this projection. It will be convenient to describe τ_1 as a representation of $\tilde{J}$ trivial on $\mathbf{Z}_2$. The restriction to $\tilde{G}'$ of the oscillator representation of $\widetilde{\mathrm{Sp}(W)}$ extends ρ_γ to $\tilde{G}'$. The extension is such that the subgroup $\mathbf{Z}_2$ will act according to (a multiple of) its unique non-trivial character, which we denote be ε. According to Lemma 2.2, we may further extend ρ_γ to $\tilde{J}$. Fix one such extension. From now on ρ_γ is considered a representation of $\tilde{J}$. Since the restriction of τ_1 to ZN_l is a multiple of ψ_γ, we must have

$$\tau_1 = \sigma_1 \otimes \rho_\gamma, \tag{33}$$

where σ_1 is an irreducible unitary representation of $G(V_l^\perp) \times \tilde{G}'$, and is extended to $\tilde{J}$ by making it trivial on N_l. We have of course

$$\sigma_1 = \chi \otimes \sigma, \tag{34}$$

where χ is a representation of $G(V_l^\perp)$ and σ is a representation of $\tilde{G}'$. Let $\hat{\tilde{G}}'(\varepsilon)$ be the subset of the unitary dual of $\tilde{G}'$ consisting of representations which restrict to ε on $\mathbf{Z}_2$. Since τ_1 comes from a representation of J, we must have $\sigma \in \hat{\tilde{G}}'(\varepsilon)$. Furthermore, an argument precisely analogous to the one leading to Lemma 3.3 shows that the representation χ of $G(V_l^\perp)$ is of rank 0, and hence must be a character. In conclusion we have obtained an injection

$$\hat{G}_\beta \hookrightarrow \widehat{G(V_l^\perp)}_0 \times \hat{\tilde{G}}'(\varepsilon), \tag{35}$$

which takes $\pi \in \hat{G}_\beta$ to the pair (χ, σ) as above.

REMARK. Except when $G = \mathrm{SO}_m$ with m odd, the set $\hat{\tilde{G}}'(\varepsilon)$ is identified with the unitary dual of G' in an obvious way.

Next assume G is of type (I_0). This case is of course spelled out in [5]. From [5] we know that $\hat{\hat{G}}_\beta$ consists of representations which factor through G if and only if l is even. The extension of ρ_γ to J can be chosen in such way that its restriction to $\mathbf{Z}_2 \subseteq J$ is trivial or not according as l is even or not. There is an injection

$$\hat{\hat{G}}_\beta \hookrightarrow \hat{G}', \tag{36}$$

which takes $\pi \in \hat{\hat{G}}_\beta$ to $\sigma \in \hat{G}'$, so that with τ_1 as in (31), we have

$$\tau_1 \simeq (1 \otimes \sigma) \otimes \rho_\gamma. \tag{37}$$

(Note that since $\overline{G(V_l^\perp)}$ is a perfect group, the character χ must be trivial here.)

The relations implied by (31), (34) and (37) show that the group $\overline{G(V_l^\perp)}$ plays a quite insignificant role in determining π. More precisely we have

THEOREM 4.5. *(a) Let $H_l = GL(X_l^*)\cdot N_l$. Suppose $\pi \in \hat{\hat{G}}_\beta$, then the restriction $\pi|_{\bar{H}_l}$ is already irreducible. (b) Suppose G is not of type (I_2). Let $\pi' \in \hat{\hat{G}}_\beta$ be another representation such that $\pi|_{\bar{H}_l} \simeq \pi'|_{\bar{H}_l}$. Then there is a character χ of $\bar{G}$, such that $\pi' \simeq \chi \oplus \pi$.*

Proof. According to the preceding analysis we have

$$\pi|_{\bar{P}_l} = \mathrm{Ind}_J^{\bar{P}_l}(\chi \otimes \sigma) \otimes \rho_\gamma. \tag{38}$$

The space of the induced representation on the right consists of functions from $\bar{P}_l$ to the space of $\sigma \otimes \rho_\gamma$, which transform on the left by J according to $(\chi \otimes \sigma) \otimes \rho_\gamma$. In particular, they transform on the left by $J_1 = \bar{G}' \cdot N_l$ according to the representation $\sigma \otimes \rho_\gamma$. It is immediately seen that restriction from $\bar{P}_l$ to $\bar{H}_l$ is an isometry onto the space of $\mathrm{Ind}_{J_1}^{\bar{H}_l}(\sigma \otimes \rho_\gamma)$, and intertwines the action of $\bar{H}_l$. Since $\mathrm{Ind}_{J_1}^{\bar{H}_l}(\sigma \otimes \rho_\gamma)$ is evidently irreducible for $\bar{H}_l$, the first statement of the theorem follows.

From Lemma 4.2 we see that χ can be extended to a character of $\bar{G}$, which we still denote by χ. Let $\chi^{-1} \cdot \sigma$ be the twist of σ by $\chi^{-1}|_{G'}$. From (38) we have

$$(\chi^{-1} \otimes \pi)|_{\bar{P}_l} \simeq \mathrm{Ind}_J^{\bar{P}_l}(1 \otimes \chi^{-1} \cdot \sigma) \otimes \rho_\gamma \tag{39}$$

If G is of type (I_0) or (I_1) the only character of $\bar{G}$ is the trivial one. If G is of type (I_3) or (I_4) then the restriction of χ to G' must be trivial. Hence the second statement of the theorem follows from (39).

COROLLARY 4.6. *Suppose G is of type (I_0) or (I_1); or of type (I_3) with $F = \mathbf{R}$; or of type (I_4) with $F = \mathbf{C}$. Let π be a unitary representation of $\bar{G}$ of pure* rank $l < r_G$. *Then $\pi(\bar{H}_l)$ and $\pi(\bar{G})$ generate the same Von Neumann algebra.*

Proof. For G as specified in the statement, the group $\bar{G}$ is perfect. (cf. Lemma 4.2) Hence the result follows from Theorem 4.5.

We are ready to show that the maps (35), (36) are in fact surjective. This will use results from [16]. Observe that the groups G and G' evidently form a reductive dual pair. In the statement of the next theorem, we do not assume $l < r_G$. The set $\hat{\bar{G}}_\beta$ and the group G', etc., are certainly defined even if $l = r_G$. However when $l = r_G$ the group N_l may sometimes be abelian, so that $N_l = ZN_l$. In such case we let ρ_γ denote the character ψ_γ. The relation $l \leqslant r_G$ implies that the dual pair (G, G') is in the so called stable range [4]. In [6] Howe conjectured a relationship between admissible representations of certain two-fold covers of G and G'. This is called Howe's duality correspondence in the literature. The covering groups involved in the duality correspondence are of course not necessarily our $\bar{G}$ and $\tilde{G}'$. But Lemma 2.2 shows that there exists a slight modification, so that Howe's correspondence may be thought of as expressing a relationship between representations of $\bar{G}$ and $\tilde{G}'$. With these understood, the following is then a simple reformulation of the main result of [16].

THEOREM 4.7. *Suppose G is not of type (I_0)* (resp. *of type (I_0)*). *To each $\sigma \in \hat{\tilde{G}}'(\varepsilon)$* (resp. $\sigma \in \hat{G}'$), *there is a representation $\pi(\sigma) \in \hat{\bar{G}}_\beta$, which is associated to σ by means of Howe's duality correspondence, such that*

$$\pi(\sigma)|_{P_l} \simeq \mathrm{Ind}_J^P(1 \otimes \sigma) \otimes \rho_\gamma \tag{40}$$

Comparing (40) with (39), we obtain

THEOREM 4.8. *Suppose* $\mathrm{rank}\, \beta = l < r_G$. *Then each $\pi \in \hat{\bar{G}}_\beta$ is of the form*

$$\pi \simeq \chi \otimes \pi(\sigma)$$

where χ is a character of $\bar{G}$ and $\sigma \in \hat{\tilde{G}}'(\varepsilon)$. ($\sigma \in \hat{G}'$ when G is of type (I_0)). In particular, the maps (35) *and* (36) *are bijective.*

Section 5. Distinguished representations

Having described the set $\hat{\bar{G}}_\beta$ for $\mathrm{rank}\, \beta < r_G$, it is now natural to ask whether $\hat{\bar{G}}_\beta$ admit a similar description when $\mathrm{rank}\, \beta = r_G$. Let us introduce the following.

DEFINITION 5.1. A unitary representation of $\bar{G}$ is called distinguished, if its ZN_n- spectrum is supported on a single $GL(X_n^*)$-orbit of rank r_G.

Suppose, for example, G is of type (I_4). Then a distinguished representation of G is simply a unitary representation of maximal pure rank r_G. Such representations are of course not so "distinguished" at all. One way to remedy the situation is perhaps to put more restrictive conditions in Definition 5.1, as was for example done in [17]. We shall not enter into this here. Instead we would like to discuss the validity of the following

STATEMENT 5.2. *Suppose π is a distinguished representation of $\bar{G}$. Then $\pi(\bar{P}_n)$ and $\pi(\bar{G})$ generate the same Von Neumann algebra.*

As we have indicated, this statement is certainly false in many cases. Nevertheless we have

PROPOSITION 5.3. *Let r be 2 or 1 according as G is of type (I_4) or not. If Statement* 5.2 is true for $r_G = r$, *then it is true in general.*

Proof. Let $\mathcal{O}_\beta$ be the $GL(X_n^*)$-orbit on which the ZN_n-spectrum of π is supported. Let $\mathcal{H}$ be the space of π. The assumption is that rank $\beta = r_G$, and $\mathcal{H}_\beta = \mathcal{H}$.

Since the statement is assumed to be true for $r_G = r$, we may suppose $r_G > r$.

We review the proof of Theorem 3.1. The argument leading to (18) certainly does not depend on the assumption that rank $\beta < r_G$. Thus, with notations as in the proof of Theorem 3.1, we have

$$\pi|_{G^1 \cdot N_r} \simeq \sum_{\mathcal{O}'} \int_{\mathcal{O}'} v_\tau \otimes \rho_\tau \, \mathrm{d}\tau \tag{41}$$

Since $\mathcal{H} = \mathcal{H}_\beta$ by our assumption, Lemma 3.3 implies that v_τ is a distinguished representation of $\bar{G}^1$. We may inductively assume that Statement 5.2 is true for G^1. That is to say $v_\tau(\bar{G}^1)$ and $v_\tau(\overline{P_n \cap G^1})$ generate the same Von Neumann algebra. But ρ_τ is an irreducible representation of N_r extended to $\bar{G}^1 \cdot N_r$. So the algebra generated by $\rho_\tau(\bar{G}^1 \cdot N_r)$ is certainly contained in the one generated by $\rho_\tau(N_r)$. It follows from (41) that the algebra generated by $\pi(\bar{G}^1)$ is contained in the one generated by $\pi(\bar{P}_n)$. Since $\bar{G}^1$ and $\bar{P}_n$ together generate the whole group $\bar{G}$, the proposition follows.

The next result shows that Proposition 5.3 is not always empty.

PROPOSITION 5.4. *Let F be a non-anchimedean local field and $G = \mathrm{Sp}_{2n}$. Let p be the residue characteristic of F.*

(a) Statement 5.2 is valid for $\bar{G}$ if $p \neq 2$.

(b) If $p = 2$ then a distinguished representation of $\bar{G}$ factors through G if and only if n is even.

Proof. Consider (a) first. By Proposition 5.3 it suffices to check the case when $n = 1$. And this comes down to an inspection of the known classification results for the unitary dual of $\bar{G} = \overline{SL(2)}$. From [2] [18] and [22] we see that for each non-zero β, the set $\hat{\bar{G}}_\beta$ consists of three representations. One of them factors to G and is supercuspidal. The other two are just the two irreducible components of the oscillator representation of $\bar{G}$ associated to the character ψ_β. (To put it another way, they are provided for by the procedure prescribed by Theorem 4.7). Since these three representations obviously have pair-wise inequivalent restrictions to $\bar{P}_1$, the assertion of part (a) follows.

For part (b) we use induction on n. Suppose $n = 1$. Then a result of [12] implies that no supercuspidal representation of SL_2 is distinguished, while the analysis of [2] on induced representations shows that no non-supercuspidal representation of SL_2 is distinguished. Thus there are no distinguished representation of SL_2 and this is precisely the content of (b) when $n = 1$.

Now let us assume $n > 1$, and look at the decomposition (41). We may assume the result is valid for $\bar{G}^1$. As we have seen before, each v_τ in (41) is a distinguished representation of $\bar{G}^1$. So it factors through G^1 if and only if $n - 1$ is even. But it is well known that the restriction of ρ_τ to the kernel of the projection $\bar{G}^1 \to G^1$ is the non-trivial character. It follows from (40) that $\pi|_{\bar{G}^1}$ factors through G^1 if and only if π factors through G. This proves part b).

Together with Theorem 4.7, the above proposition yields

COROLLARY 5.5. *Suppose F is a non-archimedean local field of odd residue characteristic, and $G = \mathrm{Sp}_{2n}$. Fix an orbit $\mathcal{O}_\beta \subseteq B(X_n^*, \eta')$ of full rank n. Let $\pi \in \hat{\bar{G}}_\beta$. If n is even we assume π factors through G; if n is odd we assume π does not factor through G. Then there is a $\sigma \in \hat{G}'$, such that $\pi = \pi(\sigma)$ as in Theorem* 4.7.

REMARK. (i) There is no reason to expect part (a) of Proposition 5.4 to fail when $p = 2$. (ii) Proposition 5.4 and Corollary 5.5 are the main local ingredients for [15].

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