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Introduction

In this paper we shall prove two results. The first one is of interest in number theory and automorphic forms, while the second is a result in harmonic analysis on *p*-adic reductive groups. The two results, even though seemingly different, are fairly related by a conjecture of Langlands [13].

To explain the first result let *F* be a number field and denote by $\mathbb{A}_F$ its ring of adeles. Given a place *v* of *F*, we let F_v denote its completion at *v*. Let π be a cusp form on $GL_2(\mathbb{A}_F)$. Write $\pi = \otimes_v \pi_v$. For an unramified *v*, let $\mathrm{diag}(\alpha_v, \beta_v)$ denote the diagonal element in $GL_2(\mathbb{C})$, the *L*-group of GL_2, attached to π_v. For a fixed positive integer *m*, let r_m denote the *m*-th symmetric power representation of the standard representation r_1 of $GL_2(\mathbb{C})$ which is an irreducible $(m + 1)$-dimensional representation. Then, for a complex number *s*, the local Langlands *L*-function [14] attached to π_v and r_m is

$$L(s, \pi_v, r_m) = \prod_{0 \leq j \leq m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1}. \tag{1}$$

Here q_v is the number of elements in the residue field of F_v.

Given a positive integer *m*, it is of great interest in number theory to define the local factors $L(s, \pi_v, r_m)$ at all other places in such a way that the global *L*-function

$$L(s, \pi, r_m) = \prod_v L(s, \pi_v, r_m)$$

extends to a meromorphic (holomorphic in many cases) function of *s* with a finite number of poles in $\mathbb{C}$, satisfying a standard functional equation (cf. [14] and [19]).

On the other hand Langlands functoriality, when applied to the homomorphism r_m: $GL_2(\mathbb{C}) \to GL_{m+1}(\mathbb{C})$, implies the existence of an automorphic form σ on $GL_{m+1}(\mathbb{A}_F)$, called the $m - th$ symmetric power of π, such that

$$L(s, \pi, r_m) = L(s, \sigma),$$

where the L-function on the right is the Godement–Jacquet [4] standard L-function of GL_{m+1}. (While the existence of such a σ for all m would lead to a proof of Ramanujan–Petersson's Conjecture for the corresponding cusp form π on $GL_2(\mathbb{A}_F)$, cf. [14], it is important to remark that the existence of σ even for $m = 3$ would provide us with the bound $q^{1/6}$ for the Fourier coefficients of π.) Consequently, the above mentioned properties of $L(s, \pi, r_m)$ would immediately follow from the known properties of $L(s, \sigma)$ (cf. [4]), and therefore an affirmative answer to the second problem implies one for the first one. While for $m = 1$ and 2 both problems have been affirmatively answered ($m = 1$ is due to Hecke and Jacquet–Langlands [5], while for $m = 2$ they were answered by Shimura [26] and Gelbart–Jacquet [3], respectively; for $m = 2$ the second problem has also been solved in [2] using the trace formula which does not seem to work for $m \geqslant 3$ at present), nothing as definitive is known for $m \geqslant 3$, and the first aim of this paper is to establish as definitive a result as possible for $m = 3$.

In view of the converse theorem for GL_4, it is best to consider another representation r_3^0 of $GL_2(\mathbb{C})$, closely related to r_3 and still of dimension 4, namely $r_3^0 = r_3 \otimes (\Lambda^2 r_1)^{-1}$. In fact if the automorphic representation $\sigma = \otimes_v \sigma_v$ of $GL_4(\mathbb{A}_F)$ is the image of π under the map defined by applying functoriality to r_3^0: $GL_2(\mathbb{C}) \to GL_4(\mathbb{C})$, and if v is an unramified place, then

$$L(s, \pi_v \otimes \varrho_v, r_3^0) = L(s, \sigma_v \otimes \varrho_v),$$

where $\varrho = \otimes_v \varrho_v$ is a character of $F^* \backslash \mathbb{I}_F$, $\mathbb{I}_F = \mathbb{A}_F^*$. Since r_3^0 generalizes the adjoint square representation $r_2^0 = r_2 \otimes (\Lambda^2 r_1)^{-1}$ of $GL_2(\mathbb{C})$ introduced in [3], in this paper we shall call it the adjoint cube even though r_3^0 does not factor through $PGL_2(\mathbb{C})$. We shall call σ the adjoint cube lift of π. It is then clear that $\sigma \otimes \omega$ will be the third symmetric power of π, where ω is the central character of π. To state our results we proceed as follows.

Fix a non-trivial character $\psi = \otimes_v \psi_v$ of $F \backslash \mathbb{A}_F$. After defining the local L-functions $L(s, \pi, r_3^0)$ and root numbers $\varepsilon(s, \pi_v, r_3^0, \psi_v)$ for every place v of F in Section 1, we let (for Re(s) > 1)

$$L(s, \pi, r_3^0, \varrho) = \prod_v L(s, \pi_v \otimes \varrho_v, r_3^0)$$

and

$$\varepsilon(s, \pi, r_3^0, \varrho) \;=\; \prod_v \varepsilon(s, \pi_v \otimes \varrho_v, r_3^0, \psi_v),$$

where $\varrho = \otimes_v \varrho_v$ is a character of $F^* \backslash \mathbb{I}_F$. Now, if ω is the central character of π, write $\omega = \omega_0 \otimes \alpha^{s_0}$ where ω_0 is a character of the compact group $F^* \backslash \mathbb{I}_F^1$ and s_0 is a pure imaginary complex number. Here α is the modulus character of $\mathbb{I}_F$ and $\mathbb{I}_F^1$ denotes its kernel. In view of the converse theorem for GL_4 we now state our first result, Theorem 4.1, as follows:

Let π be a non-monial cusp form on $GL_2(\mathbb{A}_F)$ (cf. Section 4 and [3]) with central character $\omega = \omega_0 \otimes \alpha^{s_0}$. Let ϱ be a character of $F^* \backslash \mathbb{I}_F^1$. Assume that the L-function $L(s, \pi \otimes \varrho\alpha^{-s_0/2})$ has no zero on the open interval $(1/2, 1)$ or the half open interval $[1/2, 1)$ according as $\omega_0 \varrho$ is trivial or not. Then $L(s, \pi, r_3^0, \varrho)$ extends to an entire function of s on $\mathbb{C}$. It satisfies

$$L(s, \pi, r_3^0, \varrho) \;=\; \tilde{\varepsilon}(s, \pi, r_3^0, \varrho)L(1 - s, \tilde{\pi}, r_3^0, \varrho^{-1}).$$

Corollary 4.2 then states our results on the first problem for $m = 3$ by restating Theorem 4.1 for the L-function $L(s, \pi, r_3)$.

As it is explained in Remark 1 of Section 4, if one believes in the Weak Riemann Hypothesis stated there, one would immediately see that Theorem 4.1 and Corollary 4.2 imply the holomorphy of both L-functions on the entire complex plane, except possibly at $s = \frac{1}{2}$ if ω_0 is non-trivial.

When $F = \mathbb{Q}$ and $\pi = \pi_f$ is a cusp form attached to a holomorphic modular newform [1] on a congruence group $\Gamma_0(N)$, $N \in \mathbb{Z}^+$, we prove a proposition (Proposition 5.1) which allows us to give examples for which the condition of Theorem 4.1 is satisfied. In particular when f is a rational newform of weight 2 on $\Gamma_0(N)$, Corollary 5.3 proves that for all but 16 values of $N \leqslant 423$, $L(s, \pi_f, r_3)$ is entire. This we have extracted from the tables in [17], using a theorem of Shimura [27]. Consequently we have avoided using any conjecture on elliptic curves in stating our results.

Existence of a global adjoint cube life σ for π requires twisting σ also by a cusp form on $GL_2(\mathbb{A}_F)$. Even though a meromorphic continuation and functional equation for such twisted L-functions can be proved, we prefer not to discuss it in this paper. Instead in Section 3 we prove the existence of a canonical local adjoint cube lift for every irreducible admissible representation of $GL_2(F)$, unless it is an extraordinary supercuspidal representation. Here F denotes any local field (Proposition 3.1). We refer the reader to Remarks 2 and 3 of Section 3 on the uniqueness of such a lift.

Proof of Theorem 4.1 is based on mixing the properties of Rankin–Selberg L-functions for $GL_2 \times GL_3$ applied to $\pi \times \Pi$ (Theorem 4.2) and those of Eisenstein series on a group of type G_2. Here Π is the adjoint square lift of π (cf. Sections 1 and 4; also see [3]).

There is a second set of results in this paper which is completely local, even though one of the proofs given is global. To explain, let **G** be a split group of type G_2 over a non-archimedean field F of characteristic zero. Let **P** be the parabolic subgroup of **G** generated by the long root of **G**. Write **G = MN**. Let **A** be the center of **M**. Then the short root α of **G** may be identified as the unique simple root of **A** in the Lie algebra of **N**. If ϱ is half the sum of positive roots in **N**, we let $\tilde{\alpha} = \langle \varrho, \alpha \rangle^{-1} \varrho$ as in [21]. We then identify $\mathbb{C}$ with a subspace of $\mathfrak{a}_\mathbb{C}^*$, the complex dual of the real Lie algebra of **A** by sending $s \in \mathbb{C}$ to $s\tilde{\alpha} \in \mathfrak{a}_\mathbb{C}^*$. For $s \in \mathbb{C} \subset \mathfrak{a}_\mathbb{C}^*$ and an irreducible admissible representation π of $M = \mathbf{M}(F)$, there is defined a complex number $\mu(s, \pi)$, the *Plancherel Constant* attached to s and π, which is of great importance in harmonic anlysis on $G = \mathbf{G}(F)$ (cf. [29]). The second result of this paper, Proposition 6.1, states that: *if π is infinite dimensional with central character ω, then*

$$\mu(s, \pi)\gamma(G/P)^{-2} = \gamma(2s, \omega, \psi)\gamma(-2s, \omega^{-1}, \bar{\psi})\gamma(s, \pi, r_3^0, \psi)\gamma(-s, \tilde{\pi}, r_3^0, \bar{\psi}).$$

Here

$$\gamma(s, \pi, r_3^0, \psi) = \varepsilon(s, \pi, r_3^0, \psi)L(1 - s, \tilde{\pi}, r_3^0)/L(s, \pi, r_3^0),$$

$\gamma(s, \omega, \psi)$ is the Tate γ-function attached to the character ω (cf. Section 2), and $\gamma(G/P)$ is a constant depending on the measures defining the γ-functions (cf. [29]). Moreover the additive character ψ is such that the defining measures are self dual with respect to ψ.

As consequences of this result we prove two propositions. The first one, Proposition 6.2, proves a conjecture of Langlands on normalization of intertwining operators defined by P and π [13], while the second one, Proposition 6.3, gives an exotic part of the tempered dual of G. More precisely, it determines when the representation $I(\pi)$ induced from a discrete series π is reducible. As it is remarked at the end of Section 6, we hope to study the unitary dual of all the rank 2 split p-adic groups in a future paper.

Proposition 6.1 is an easy consequence of Proposition 2.2 whose proof is local unless π is supercuspidal in which case we had to use global methods and frankly we do not see how it can be done locally at present. More precisely, we have imbedded our supercuspidal representation as the unique supercuspidal component of a cusp form on $GL_2(\mathbb{A}_F)$. We would like to

thank L. Clozel for explaining this to us. Thanks are also due to J. Rogawski for informing us of a reference to this in Langlands' book [15].

Finally, in an appendix, we prove that if r_5 is the six dimensional irreducible representation of $SL_2(\mathbb{C})$, the L-group of PGL_2, then the L-function $L_S(s, \pi, r_5)$ is non-zero (and holomorphic if π is attached to a holomorphic form; this was first observed by Serre) on the line Re $(s) = 1$, except possibly for a simple zero at $s = 1$ (simple pole at $s = 1$, respectively). As it has been pointed out by Serre [20], this will provide us with the best evidence for the validity of the Sato–Tate's conjecture [19] so for (see the remarks before Theorem A).

1. Adjoint cubes and local factors

We start by defining the following representation of $GL_2(\mathbb{C})$ which is more appropriate than the symmetric cube representation r_3 for formulation of our global results. The adjoint square of Gelbart–Jacquet [3], which we denote by r_2^0 is simply $r_2 \otimes (\Lambda^2 r_1)^{-1}$. We then define a similar representation attached to r_3 by $r_3^0 = r_3 \otimes (\Lambda^2 r_1)^{-1}$. In this paper we shall call r_3^0 the adjoint cube representation of $GL_2(\mathbb{C})$. But the reader must be careful with the term adjoint since r_3^0 does not factor through $PGL_2(\mathbb{C})$. We finally remark that for r_m, for example when m is odd, the appropriate substitute is $r_m^0 = r_m \otimes (\Lambda^2 r_1)^{-(m-1)/2}$.

We shall now define the local factors for both r_3 and r_3^0. Throughout this section we shall assume that F is a local field and π is an irreducible admissible representation of $GL_2(F)$.

We start with the archimedean places. If F is archimedean, let φ be the homomorphism from the Weil group $W_F = W(\bar{F}/F)$ into the L-group $GL_2(\mathbb{C})$ of GL_2, attached to π (cf. [16]). We then set

$$L(s, \pi, r_3^0) = L(s, r_3^0 \cdot \varphi),$$

where the L-function on the right is the Artin L-function attached to $r_3^0 \cdot \varphi$ (cf. [25]).

If $\pi = \pi(\mu, v)$ is an unramified principal series, it then immediately follows that

$$L(s, \pi, r_3^0) = \prod_{j=-1}^{2} L(s, \mu^j v^{1-j}). \tag{1.1}$$

From the standard relation

$$r_1 \otimes r_2^0 = r_3^0 \oplus r_1$$

it follows that for every unramified or archimedean place v,

$$L(s, \pi \times \Pi) \ = \ L(s, \pi, r_3^0)L(s, \pi), \tag{1.2}$$

where $L(s, \pi \times \Pi)$ is the Rankin–Selberg L-function for the pair π and Π (cf. [7]) and $L(s, \pi) = L(s, \pi, r_1)$ is the Hecke–Jacquet–Langlands L-function attached to π. Here Π is the Gelbart–Jacquet lift of π (cf. [3]).

We shall now use (1.2) to define $L(s, \pi, r_3^0)$ for the non-archimedean ramified places.

If $\pi = \pi(\mu, \nu)$ is a ramified principal series (not necessarily unitary), then it follows at once that (Proposition 9.4 of [7])

$$L(s, \pi, r_3^0) \ = \ \prod_{j=-1}^{2} L(s, \mu^j \nu^{1-j})$$

as expected.

Next assume π is special, i.e., $\pi = \sigma\,(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$, where α denotes the modulus character α_F. Then by [3] Π is the unique square integrable component of

$$\mathrm{Ind}(GL_3(F),\, B_3(F),\, \alpha,\, 1,\, \alpha^{-1}).$$

Now by Theorem 8.2 of [7], one has

$$L(s, \pi \times \Pi) \ = \ L(s, \mu\alpha^{3/2})L(s, \mu\alpha^{1/2})$$

and

$$L(s, \pi) \ = \ L(s, \mu\alpha^{1/2}).$$

We then set

$$L(s, \pi, r_3^0) \ = \ L(s, \mu\alpha^{3/2}).$$

It clearly satisfies (1.2).

It remains to study those π which are supercuspidal. Thus let π be an irreducible supercuspidal representation of $GL_2(F)$, where F is a non-archimedean local field. Then $L(s, \pi) = 1$ and $L(s, \pi \times \Pi) = 1$ unless

$$\Pi \ = \ \mathrm{Ind}(GL_3(F),\, P_{2,1}(F),\, \pi',\, \eta),$$

where π' is supercuspidal and η is a quasi-character of F^*. Here $P_{2,1}$ is the parabolic subgroup of type (2, 1).

In this case by Proposition 3.3 of [3], π is not extraordinary, and therefore by the discussion in page 488 of [3], there exists a quadratic extension K of F and a character χ of W_K, the Weil group of K such that if $\tau = \mathrm{Ind}(W_F, W_K, \chi)$, then $\pi = \pi(\tau)$. Observe that we may assume $\chi \in K^*$. It also follows from the same discussion that if χ' is the conjugate of χ, i.e., $\chi'(a) = \chi(\sigma(a))$, $\sigma \in \mathrm{Gal}(K/F)$, $\sigma \neq 1$, and $\mu = \mathrm{Ind}(W_F, W_K, \chi\chi'^{-1})$, then $\pi' = \pi(\mu)$ and η is the class field character of F^* defined by K/F, i.e., $\mathrm{Ker}\,(\eta) = N_{K/F}(K^*)$.

Now observe that $L(s, \pi \times \Pi) = L(s, \pi \times \pi')$ and therefore $L(s, \pi \times \Pi) \neq 1$ only if $\pi' \cong \tilde{\pi} \otimes \alpha^{s_0}$, where $\alpha = \alpha_F \cdot \det$ and $s_0 \in \mathbb{C}$ (Proposition 1.2 and Corollary 1.3 of [3]).

Consequently

$$\mathrm{Ind}(W_F, W_K, \chi\chi'^{-1}) \cong \mathrm{Ind}(W_F, W_K, \chi^{-1}) \otimes \alpha_F^{s_0}.$$

Given every quasi-character ϱ of F^*, let $\varrho_{K/F} \in \hat{K}^*$ be defined by $\varrho_{K/F} = \varrho \circ N_{K/F}$. Then $(\alpha_F)_{K/F} = \alpha_K$, the modulus character of K^*. Now the above isomorphism happens if and only if either

$$\chi\chi'^{-1} = \chi'^{-1} \cdot \alpha_K^{s_0}$$

or

$$\chi\chi'^{-1} = \chi^{-1} \cdot \alpha_K^{s_0}.$$

The first case implies that χ factors through the norm. This contradicts the fact that π is supercuspidal. The second equality implies $\chi^2\chi'^{-1} = \alpha_K^{s_0}$.

Consequently, if $\pi = \pi(\tau)$, where $\tau = \mathrm{Ind}(W_F, W_K, \chi)$ with χ a quasi-character of K^* which does not factor through the norm but $\chi^2\chi'^{-1} = \alpha_K^{s_0}$, then $L(s, \pi \times \Pi)$ may have a pole. More precisely, by Corollary 1.3 of [3]

$$L(s, \pi \times \Pi) = (1 - q^{-2(s+s_0)})^{-1}$$

if $\pi \cong \pi \otimes \eta_0$, where η_0 is the unramified quadratic character of F^*, and equals $(1 - q^{-(s+s_0)})^{-1}$ otherwise.

Finally, suppose $\pi \cong \pi \otimes \eta_0$ and assume $\eta_0 \neq \eta$. Then since $\eta_0 \cdot N_{K/F} \neq 1$, this implies

$$\chi' = \chi \cdot (\eta_0 \circ N_{K/F}).$$

Consequently

$$\chi = \alpha_K^{s_0} \cdot (\eta_0 \cdot N_{K/F})$$

which again implies that χ factors through the norm, a contradiction. Therefore, if $\pi \cong \pi \otimes \eta$, then $\eta = \eta_0$ and K/F is unramified.

Given every field E and a quasi-character ϱ of E^*, let $L_E(s, \varrho)$, $s \in \mathbb{C}$, be the corresponding Hecke L-function. More precisely, define

$$L_E(s, \varrho) = (1 - \varrho(\varpi_E)q_E^{-s})^{-1},$$

if ϱ is unramified, where ϖ_E is a uniformizing parameter for the ring of integers of E and $q_E = |\varpi_E|_E^{-1}$. Otherwise, we let $L_E(s, \varrho) = 1$. We now summerize what we have proved as:

PROPOSITION 1.1: *Let π be an irreducible supercuspidal representation of $GL_2(F)$, where F is a non-archimedean field. Denote by Π the Gelbart–Jacquet lift of π.*

(a) Suppose $\pi = \pi(\tau)$, where $\tau = \mathrm{Ind}(W_F, W_K, \chi)$ with $[K: F] = 2$. Regard χ as a character of K^*. Then

$$L(s, \pi \times \Pi) = L_K(s, \chi^2 \chi'^{-1})$$

(b) Otherwise, i.e., if π is extraordinary, then

$$L(s, \pi \times \Pi) = 1.$$

Now, let π be an irreducible supercuspidal representation of $GL_2(F)$. If $\pi = \pi(\tau)$, $\tau = \mathrm{Ind}(W_F, W_K, \chi)$, $\chi \in \hat{K}^*$, let

$$L(s, \pi, r_3^0) = L_K(s, \chi^2 \chi'^{-1}). \tag{1.3}$$

Otherwise, set

$$L(s, \pi, r_3^0) = 1. \tag{1.4}$$

Observe that by Proposition 1.1 it always satisfies (1.2).

Finally, if $\varepsilon(s, \pi \times \Pi, \psi)$ and $\varepsilon(s, \pi, \psi)$ are the corresponding root numbers, we let

$$\varepsilon(s, \pi, r_3^0, \psi) = \varepsilon(s, \pi \times \Pi, \psi)/\varepsilon(s, \pi, \psi). \tag{1.5}$$

Here ψ is a non-trivial character of F.

The factors for r_3 are then defined to be

$$L(s, \pi, r_3) \;=\; L(s, \pi \otimes \omega, r_3^0) \tag{1.6}$$

and

$$\varepsilon(s, \pi, r_3, \psi) \;=\; \varepsilon(s, \pi \otimes \omega, r_3^0, \psi). \tag{1.7}$$

Here ω is the central character of π. Observe that

$$L(s, \pi, r_3) \;=\; L(s, \pi \times (\Pi \otimes \omega))/L(s, \pi \otimes \omega).$$

Similarly for $\varepsilon(s, \pi, r_3, \psi)$.

Finally, if $\pi = \pi(\tau)$ is supercuspidal with $\tau = \mathrm{Ind}(W_F, W_K, \chi)$, $\chi \in \hat{K}^*$, then

$$L(s, \pi, r_3) \;=\; L_K(s, \chi^3). \tag{1.8}$$

This follows from the equality $\omega = \chi\eta$, where η is the character of F^* defined by K/F and χ denotes the restriction of χ to F^*.

2. Adjoint cubes and local coefficients

In this section we shall relate the local factors defined in the previous section to certain local coefficients defined in [22] (also cf. [23]). Later in Section 6 we shall use this to find a formula for certain Plancherel measures for a group of type G_2.

Let $\mathbf{G}$ be a split group of type G_2 defined over a local field F of characteristic zero. Fix a Cartan subgroup $\mathbf{T}$ in $\mathbf{G}$ and let α and β be the short and the long roots of $\mathbf{T}$ in $\mathbf{B}$, a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, respectively. Given a root γ of $\mathbf{T}$, let $H_\gamma \colon \bar{F}^* \to \mathbf{T}$ be the corresponding coroot. Let $\mathbf{M}$ be the centralizer of the image of $H_{2\alpha+\beta}$ in $\mathbf{G}$. It is a Levi factor for the maximal standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ generated by β. We start with the following lemma.

LEMMA 2.1: $\mathbf{M} \cong GL_2$.

Proof: Given a root γ of $\mathbf{T}$ in $\mathbf{B}$, let $X_\gamma \colon \bar{F} \to \mathbf{G}$ be the corresponding one-dimensional subgroup. The group generated by the images of X_β, $X_{-\beta}$, and $H_{2\alpha+\beta}$ is a closed subgroup of $\mathbf{G}$ contained in $\mathbf{M}$ and isomorphic to GL_2. On the other hand $\mathbf{M}$, being centralizer of a torus, is a connected reductive group of dimension 4, containing GL_2. By connectedness and equality of dimensions, it then must equal GL_2.

Now, let F be a non-archimedean field. Fix a non-trivial additive character ψ of F. It then naturally defines a non-degenerate character of the F-points U of the unipotent radical $\mathbf{U}$ of $\mathbf{B}$. Let $\mathbf{A}$ be the center of $\mathbf{M}$. Then α may be identified as the unique simple root of $\mathbf{A}$ in the Lie algebra of $\mathbf{N}$, $\mathbf{N} \subseteq \mathbf{U}$, the unipotent radical of $\mathbf{P}$. If ϱ is half the sum of positive roots in $\mathbf{N}$, we let $\tilde{\alpha} = \langle \varrho, \alpha \rangle^{-1} \varrho$ as in [21]. Then $\tilde{\alpha}$ belongs to $\mathfrak{a}_{\mathbb{C}}^{*}$, the complex dual of the real Lie algebra $\mathfrak{a}$ of $\mathbf{A}$. We now identify $\mathbb{C}$ with a subspace of $\mathfrak{a}_{\mathbb{C}}^{*}$ by sending $s \in \mathbb{C}$ to $s\tilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^{*}$.

Let π be an infinite dimensional irreducible admissible representation of $M = \mathbf{M}(F)$. Denote by $C_{\chi}(s, \pi)$ the local coefficient (cf. [22]) attached to $s\tilde{\alpha}$, π, χ, and the reflection $w_{2\alpha+\beta}$. If the notation of [22] is used, this will be denoted by $C_{\chi}(-s\tilde{\alpha}, \pi, \{\beta\}, w_{2\alpha+\beta})$. It is very important to compute $C_{\chi}(s, \pi)$. Beside its arithmetic importance (cf. Section 4) it can be used to compute certain Plancherel measures which we shall explain later.

For a quasi-character θ of F^{*}, let $L(s, \theta)$ and $\varepsilon(s, \theta, \psi)$ be the corresponding Hecke L-function and root number. Set

$$\gamma(s, \theta, \psi) = \varepsilon(s, \theta, \psi)L(1 - s, \theta^{-1})/L(s, \theta).$$

Also let

$$\gamma(s, \pi, r_{3}^{0}, \psi) = \varepsilon(s, \pi, r_{3}^{0}, \psi)L(1 - s, \tilde{\pi}, r_{3}^{0})/L(s, \pi, r_{3}^{0}).$$

Similarly define $\gamma(s, \pi, r_{3}, \psi)$. We shall now prove:

PROPOSITION 2.2: *Let π be an infinite dimensional irreducible admissible representation of $GL_{2}(F)$, where F is a local field of characteristic zero. If ω is the central character of π, then*

$$C_{\chi}(s, \pi) = \gamma(2s, \omega, \psi)\gamma(s, \pi, r_{3}^{0}, \psi), \tag{2.2.1}$$

provided that the measures defining $C_{\chi}(s, \pi)$ are self dual with respect to ψ.

Proof. Assume first that π is not supercuspidal. Then it is a subrepresentation of a principal series (not necessarily unitary) $\pi(\mu, v)$, $(\mu, v) \in (\hat{F}^{*})^{2}$. Moreover $C_{\chi}(s, \pi) = C_{\chi}(s, \pi(\mu, v))$. The identification $\mathbf{M} = GL_{2}(\bar{F})$ is such that $H_{\beta}(t) = \mathrm{diag}(t, t^{-1})$ and $H_{2\alpha+\beta}(t) = \mathrm{diag}(t, t)$, $t \in \bar{F}$. This then implies $H_{3\alpha+2\beta}(t) = \mathrm{diag}(t, 1)$, $H_{3\alpha+\beta}(t) = \mathrm{diag}(1, t)$, $H_{\alpha}(t) = \mathrm{diag}(t^{-1}, t^{2})$, and finally $H_{\alpha+\beta}(t) = \mathrm{diag}(t^{2}, t^{-1})$, $t \in \bar{F}$. Using Proposition 3.2.1 of [22] and Lemma 4.4 of [23], it can now be easily shown that

$$C_{\chi}(s, \pi(\mu, v)) = \prod_{j=-1}^{2} \gamma(s, \mu^{j}v^{1-j}, \psi) \cdot \gamma(2s, \omega, \psi),$$

where $\omega = \mu\nu$. The proposition in this case is now a consequence of Theorem 3.1 and Remark 3.3 of [7] which imply the equality of

$$\gamma(s, \pi \times \Pi, \psi)/\gamma(s, \pi, \psi)$$

with

$$\prod_{j=-1}^{2} \gamma(s, \mu^j\nu^{1-j}, \psi).$$

Next suppose π is supercuspidal. We may assume it is unitary. By the discussion in Pages 227–230 of [15], there exists a cuspidal automorphic representation $\sigma = \otimes_v\sigma_v$ of $GL_2(\mathbb{A}_K)$, K global, such that if $K_{v_0} = F$, then $\sigma_{v_0} = \pi$, while for every other finite place $v \neq v_0$ of K, σ_v is unramified.

Whether σ is monomial or not, let Σ be its lift as in [3] (cf. Page 491 of [3] for the lift of a monomial representation). When σ is monomial Σ is not cuspidal but still automorphic and always equals a full induced representation from either $P_{2,1}(\mathbb{A}_K)$ or $B_3(\mathbb{A}_K)$ (cf. [3]). At any rate (2.2.1) now follows from comparing the functional equation proved in [22] and those satisfied by $L(s, \sigma \times \Sigma)$ and $L(s, \sigma)$ (cf. Theorem 4.1 of [22], together with Theorems 5.1 of [24] and 3.1 of [25], respectively, for $L(s, \sigma \times \Sigma)$). This completes the proposition.

COROLLARY: *Replace π by $\pi \otimes \omega$. Then*

$$C_\chi(s, \pi \otimes \omega) = \gamma(2s, \omega^3, \psi)\gamma(s, \pi, r_3, \psi)$$

3. Local adjoint cube liftings

In this section F again will be a local field of arbitrary characteristic. Given an irreducible admissible representation σ of $GL_4(F)$, let $L(s, \sigma)$ and $\varepsilon(s, \sigma, \psi)$ denote its standard L-function and root number defined by Godement and Jacquet in [4], respectively.

Fix an irreducible admissible representation π of $GL_2(F)$. Given a quasi-character ϱ of F^*, let

$$L(s, \pi, r_3^0, \varrho) = L(s, \pi \otimes \varrho, r_3^0)$$

and

$$\varepsilon(s, \pi, r_3^0, \varrho; \psi) = \varepsilon(s, \pi \otimes \varrho, r_3^0, \psi).$$

DEFINITION: *An irreducible admissible representation σ of $GL_4(F)$ is called an adjoint cube lift of π if the following conditions are satisfied:*

(i) If ω is the central character of π, then the centeral character of σ is ω^2, and

(ii) for every quasi-character ϱ of F^*

$$L(s, \sigma \otimes \varrho) = L(s, \pi, r_3^0, \varrho)$$

and

$$\varepsilon(s, \sigma \otimes \varrho, \psi) = \varepsilon(s, \pi, r_3^0, \varrho, \psi).$$

REMARK 1: Suppose ψ is replaced by ψ_a defined by $\psi_a(x) = \psi(ax)$, where a and x are in F and $a \neq 0$. Then it can be easily shown (using the results of [7]) that

$$\varepsilon(s, \pi \times \Pi, \psi_a) = \omega^3(a)|a|^{6(s-1/2)}\varepsilon(s, \pi \times \Pi, \psi).$$

Moreover by relation (3.3.5) of [4]

$$\varepsilon(s, \pi, \psi_a) = \omega(a)|a|^{2(s-1/2)}\varepsilon(s, \pi, \psi)$$

and

$$\varepsilon(s, \sigma, \psi_a) = \omega_\sigma(a)|a|^{4(s-1/2)}\varepsilon(s, \sigma, \psi),$$

where ω_σ is the central character of σ. But now condition (i) implies that if σ is a lift for some ψ, then it is a lift for every ψ, and therefore the lifting does not depend on the choice of ψ.

REMARK 2: The uniqueness of local adjoint cube liftings will require twisting of σ with irreducible admissible representations of $GL_2(F)$ as well; and since that is not yet available, one cannot expect a uniqueness statement (cf. Remark 3 below).

In this section we shall prove:

PROPOSITION 3.1: *(a) Unless F is non-archimedean and π is extraordinary, every π admits a canonical adjoint cube lift σ which is explicitly given in the proof.*

(b) If F is non-archimedean and π is extraordinary and admits a lift σ, then σ is either supercuspidal or is a constituent of a representation induced from

a supercuspidal representation of $GL_2(F) \times GL_2(F)$, *regarded as Levi factor of* $GL_4(F)$.

Proof. First assume F is archimedean. Let $\varphi\colon W_F \to GL_2(\mathbb{C})$ be the homomorphism attached to π. Then $r_3^0 \cdot \varphi\colon W_F \to GL_4(\mathbb{C})$ defines an irreducible admissible representation of $GL_4(F)$. We contend that this is our σ. In fact

$$
\begin{aligned}
L(s, \sigma \otimes \varrho) &= L(s, r_3^0 \cdot (\varphi \otimes \varrho)) \\
&= L(s, \pi \otimes \varrho, r_3^0) \\
&= L(s, \pi, r_3^0, \varrho).
\end{aligned}
$$

Similarly for root numbers.

Now suppose F is non-archimedean. First assume $\pi = \pi(\mu, \nu)$. Write $\mu\nu^{-1} = \eta\alpha^t$ with η unitary and $t \in \mathbb{R}$. Here $\alpha = \alpha_F$ is the modulus character. Let

$$
\xi = \mathrm{Ind}(GL_4(F), B_4(F), \mu^2\nu^{-1}, \mu, \nu, \mu^{-1}\nu^2).
$$

If $t = 0$, then ξ is irreducible and quasi-tempered. We let $\sigma = \xi$. Now condition (i) is trivial while (ii) follows from Theorem 3.1(i) and Proposition 8.4 of [7]. Otherwise if $t > 0$, ξ may fail to be irreducible but it has a unique irreducible (Langlands) quotient. We then let σ be this quotient. Condition (ii) in this case is a consequence of Theorem 3.1 (i) and Proposition 9.4 of [7].

Next assume π is special, i.e., $\pi = \sigma\,(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$. The representation

$$
\mathrm{Ind}(GL_4(F), B_4(F), \mu\alpha^{3/2}, \mu\alpha^{1/2}, \mu\alpha^{-1/2}, \mu\alpha^{-3/2})
$$

has a unique quasi-square integrable constituent. We then let σ be this constituent. Its central character is $\mu^4 = \omega^2$. Given $\varrho \in \hat{F}^*$, it is easily seen that (cf. Theorem 7.1 of [4])

$$
\begin{aligned}
L(s, \sigma \otimes \varrho) &= L(s, \varrho\mu\alpha^{3/2}) \\
&= L(s, \pi, r_3^0, \varrho).
\end{aligned}
\tag{3.1.1}
$$

The equality of root numbers now follows from the identity

$$
\gamma(s, \pi \times (\Pi \otimes \varrho), \psi) = \gamma(s, \sigma \otimes \varrho, \psi)\gamma(s, \pi \otimes \varrho, \psi)
$$

and (3.1.1).

Finally assume π is a non-extraordinary supercuspidal representation. Then there exists a quasi-character χ of K^*, $[K:F] = 2$, such that $\pi = \pi(\tau)$, where $\tau = \mathrm{Ind}(W_F, W_K, \chi)$. For simplicity, let for every quasi-character ϱ of K^*, $\pi(\varrho) = \pi(\tau_\varrho)$, where $\tau_\varrho = \mathrm{Ind}(W_F, W_K, \varrho)$. The representations $\pi(\chi^2\chi'^{-1})$ and $\pi(\chi) = \pi$ have $\chi\eta$ as their common central character. Here χ denotes $\chi|_{F^*}$ and η is the quadratic character of F^* defined by K. Write $\chi|_{F^*} = \eta\alpha^t$, $\eta\bar\eta = 1$ and $t \in \mathbb{R}$. Then $\pi(\chi^2\chi'^{-1}) \otimes \alpha^{-t/2}$ and $\pi(\chi) \otimes \alpha^{-t/2}$ are both tempered and therefore the induced representation

$$\mathrm{Ind}(GL_4(F), P_{2,2}(F), \pi(\chi^2\chi'^{-1}), \pi(\chi) \otimes \eta)$$

being quasi-tempered is irreducible with central character $\chi^2|_{F^*}$. We contend that this is a σ which lifts $\pi = \pi(\chi)$.

Clearly for every quasi-character ϱ of F^*, relation (1.3) implies:

$$
\begin{aligned}
L(s, \sigma \otimes \varrho) &= L(s, \pi(\chi^2\chi'^{-1}) \otimes \varrho) \\[6pt]
&= L_K(s, \chi^2\chi'^{-1}\varrho_{K/F}) \\[6pt]
&= L(s, \pi \otimes \varrho, r_3^0).
\end{aligned}
$$

For root numbers, consider

$$\gamma(s, \sigma \otimes \varrho, \psi) = \gamma(s, \pi(\chi^2\chi'^{-1}) \otimes \varrho, \psi)\gamma(s, \pi(\chi) \otimes \eta\varrho, \psi)$$

$$= \gamma(s, (\tau_{\chi^2\chi'^{-1}} \oplus \tau_\chi \otimes \eta \oplus \tau_\chi) \otimes \varrho, \psi)\gamma(s, \pi \otimes \varrho, \psi)^{-1}. \qquad (3.1.2)$$

But

$$\tau_{\chi^2\chi'^{-1}} \oplus \tau_\chi \otimes \eta \oplus \tau_\chi = \tau_\chi \otimes \tau_{\chi\chi'^{-1}} \oplus \tau_\chi \otimes \eta$$

and thus (3.1.2) equals:

$$\gamma(s, \tau_\chi \otimes \tau_{\chi\chi'^{-1}} \otimes \varrho, \psi)\gamma(s, \tau_\chi \otimes \eta\varrho, \psi)/\gamma(s, \pi \otimes \varrho, \psi). \qquad (3.1.3)$$

But by equations (1.5.6) and (1.5.7), and paragraph (1.8) of [3]

$$\gamma(s, \tau_\chi \otimes \tau_{\chi\chi'^{-1}} \otimes \varrho, \psi) = \gamma(s, \pi(\chi) \times \pi(\chi\chi'^{-1}) \otimes \varrho, \psi).$$

Moreover

$$\gamma(s, \tau_\chi \otimes \eta\varrho, \psi) = \gamma(s, \pi(\chi) \otimes \eta\varrho, \psi).$$

Since the Gelbart–Jacquet lift of π is

$$\Pi = \mathrm{Ind}(GL_3(F), P_{2,1}(F), \pi(\chi\chi'^{-1}), \eta)$$

it follows immediately that (3.1.3) is equal to

$$\gamma(s, \pi \times (\Pi \otimes \varrho), \psi)/\gamma(s, \pi \otimes \varrho, \psi)$$

which by definition is $\gamma(s, \pi, r_3^0, \varrho; \psi)$. The equality of root numbers is now a consequence of the equality of L-functions. This completes part (a). We leave the trivial argument proving part (b) to the reader.

REMARK 3: To obtain a unique lift one would need to require more conditions (cf. Remark 7.5.4 of [6]). As global considerations dictate, these extra conditions must be related to the local factors attached to twists of σ with irreducible admissible representations of $GL_2(F)$. We are not able to state these extra conditions since a similar result in the side of π is not yet available. At any rate we believe that the lifts given in Proposition 3.1 are in fact the ones which will satisfy the extra conditions yet to be defined. That is why we call them canonical.

4. Global results

Throughout this section F will be a number field. We use $\mathbb{A}_F$ and $\mathbb{I}_F$ to denote the adeles and the ideles of F, respectively. Moreover if α denotes the modulus character of $\mathbb{I}_F$, and $\mathbb{I}_F^1$ is the kernel of α, then $\mathbb{I}_F = \mathbb{I}_F^1 \times \mathbb{R}_+^*$, where $\mathbb{R}_+^*$ is the multiplicative group of positive real numbers. Given a character ϱ of $F^*\backslash\mathbb{I}_F$, there exists a unique pure imaginary complex number s_0 and a unique character ϱ_0 of the compact group $F^*\backslash\mathbb{I}_F^1$ such that $\varrho = \varrho_0 \otimes \alpha^{s_0}$. In this way every character of $F^*\backslash\mathbb{I}_F^1$ may be considered as one of $F^*\backslash\mathbb{I}_F$, trivial on $\mathbb{R}_+^*$.

Let π be a cusp form on $GL_2(\mathbb{A}_F)$. We shall assume that π is not monomial (cf. [3]). This means that π is not of the form $\pi(\tau)$, where τ is a two dimensional irreducible representation of $W(\bar{F}/F)$, Weil group of $\bar{F}/F$, induced from $W(\bar{F}/K)$, where $[K:F] = 2$. This is equivalent to the fact that if η is the quadratic character of $\mathbb{I}_F$ defined by K, then $\pi \neq \pi \otimes \eta$ (cf. [12]). Then by [3] there exists a cusp form Π on $GL_3(\mathbb{A}_F)$ for which

$$L(s, \Pi) = L(s, \pi, r_2^0),$$

where the L-function on the left is the standard Godement–Jacquet [4] L-function of Π. This is the global Gelbart–Jacquet lift of π.

An automorphic form $\sigma = \otimes_v \sigma_v$ on $GL_4(\mathbb{A}_F)$ is called *an adjoint cube lift* of $\pi = \otimes_v \pi_v$ if for every v, σ_v is an adjoint cube lift of π_v. Observe that by strong multiplicity one theorem for $GL(4)$ (cf. [9]) and equation (1.1), if σ has a cuspidal adjoint cube lift, then the lift is unique. It is of great importance to see whether a cusp form π on $GL_2(\mathbb{A}_F)$ admits an adjoint cube lift on $GL_4(\mathbb{A}_F)$ which is cuspidal. (This, for example, would lead to the bound $q^{1/6}$ for the Fourier coefficients of non-monomial cusp forms on $GL_2(A_F)$ and in particular, when $F = \mathbb{Q}$, for Maass wave forms.) In view of the converse theorem for GL_4, it is in the direction of this existence that we shall prove Theorem 4.1 below. We need some preparation.

Fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}_F$ trivial on F. We shall now define our global L-functions and root numbers. More precisely, given a character $\varrho = \otimes_v \varrho_v$ of $F^* \backslash \mathbb{I}_F$ and for Re $(s) > 1$, let (cf. Theorem 5.3 of [9] for convergence):

$$L(s, \pi, r_3^0, \varrho) \;=\; \prod_v L(s, \pi_v, r_3^0, \varrho_v)$$

and

$$\varepsilon(s, \pi, r_3^0, \varrho) \;=\; \prod_v \varepsilon(s, \pi_v, r_3^0, \varrho_v; \psi_v),$$

where the local factors are defined as in Section 3.

We set

$$L(s, \pi, r_3^0) \;=\; L(s, \pi, r_3^0, 1)$$

and

$$L(s, \pi, r_3) \;=\; L(s, \pi, r_3^0, \omega),$$

where ω is the central character of π. Similarly for root numbers. We observe that the existence of a global adjoint cube σ implies the existence of a global *symmetric cube* $\sigma \otimes \omega$ which satisfies

$$L(s, \sigma \otimes \omega) \;=\; L(s, \pi, r_3)$$

and

$$\varepsilon(s, \sigma \otimes \omega) \;=\; \varepsilon(s, \pi, r_3).$$

Write $\omega = \omega_0 \otimes \alpha^{s_0}$, where $s_0 \in \mathbb{C}$ is pure imaginary and ω_0 is a character of the compact group $F^*\backslash \mathbb{I}_F^1$. They are both uniquely determined by ω. The following theorem may be considered as the first step towards the existence of global adjoint cubes. It is different from any previous result since we are now able to twist with any character of $F^*\backslash \mathbb{I}_F$. (In fact it is enough to twist with characters of $F^*\backslash \mathbb{I}_F^1$.)

THEOREM 4.1: *Let π be a non-monomial cusp form on $GL_2(\mathbb{A}_F)$ with central character $\omega = \omega_0 \otimes \alpha^{s_0}$. Let ϱ be a character of $F^*\backslash \mathbb{I}_F^1$. Assume that the Hecke–Jacquet–Langlands L-function $L(s, \pi \otimes \varrho\alpha^{-s_0/2})$, has no zeros on the open interval $(1/2, 1)$ or the half open interval $[1/2, 1)$ according as $\omega_0\varrho$ is trivial or not. Then $L(s, \pi, r_3^0, \varrho)$ extends to an entire function of s on $\mathbb{C}$. It satisfies*

$$L(s, \pi, r_3^0, \varrho) = \varepsilon(s, \pi, r_3^0, \varrho)L(1 - s, \tilde{\pi}, r_3^0, \varrho^{-1}). \tag{4.1.1}$$

COROLLARY 4.2: *With notation as in Theorem 4.1 assume that the Hecke–Jacquet–Langlands L-function $L(s, \pi \otimes \omega_0\alpha^{-s_0/2})$ has no zeros on the open interval $(1/2, 1)$ or half open interval $[1/2, 1)$ according as ω_0^3 is trivial or not. Then $L(s, \pi, r_3)$ extends to an entire function of s on $\mathbb{C}$. It satisfies*

$$L(s, \pi, r_3) = \varepsilon(s, \pi, r_3)L(1 - s, \tilde{\pi}, r_3). \tag{4.2.1}$$

REMARK 1: The Weak Riemann Hypothesis for the Hecke–Jacquet–Langlands L-function must state that $L(s, \pi \otimes \alpha^{-s_0/2})$ is non-zero for all real s unless $s = 1/2$. If one believes in this, one would immediately see that the theorem and its corollary imply the holomorphy of both L-functions on the entire complex plane, except possibly at $s = 1/2$ if ω_0 is non-trivial.

REMARK 2: One remarkable fact about the adjoint cube (and in fact odd degree adjoint cube L-functions $L(s, \pi, r_m^0)$, $r_m^0 = r_m \otimes r_1^{-(m-1)/2}$) L-functions is that

$$L(s, \pi \otimes \varrho, r_3^0) = L(s, \sigma \otimes \varrho)$$

for every character ϱ of $F^*\backslash \mathbb{I}_F$ and therefore the twisting with a character is attained by twisting the original form π. This is not the case with the even degree adjoint L-functions. In fact, defining $r_m^0 = r_m \otimes r_1^{-(m-2)/2}$ for even m,

it is clear that if π_v is unramified and σ_v is the class one representation of $GL_{m+1}(F_v)$ which satisfies

$$L(s, \pi_v, r_m^0) \;=\; L(s, \sigma_v),$$

then

$$L(s, \pi_v \otimes \varrho_v, r_m^0) \;=\; L(s, \sigma_v \otimes \varrho_v^2).$$

This may justify the use of the two sheet cover of $GL(2)$ in [3, 26] when $m = 2$.

To prove Theorem 4.1 we need some preparation.

Fix an integer $n \geqslant 1$. Let $\sigma = \otimes_v \sigma_v$ and $\varrho = \otimes_v \varrho_v$ be two cusp forms on $GL_{n-1}(\mathbb{A}_F)$ and $GL_n(\mathbb{A}_F)$, respectively. For Re $(s) > 1$, let

$$L(s, \sigma \times \varrho) \;=\; \prod_v L(s, \sigma_v \times \varrho_v),$$

where at each $v < \infty$, $L(s, \sigma_v \times \varrho_v)$ is the Rankin–Selberg L-function defined in [7]. Moreover, when $v = \infty$, we let $L(s, \sigma_v \times \varrho_v)$ be the Artin L-function attached to the tensor product of the representations of $W(\bar{F}_v/F_v)$ which correspond to σ_v and ϱ_v through local class field theory at v (cf. [8]). We shall now show how the following result can be deduced from the existing literature.

THEOREM 4.3: (*Jacquet–Piatetski-Shapiro-Shalika*) *The L-function* $L(s, \sigma \times \varrho)$ *extends to an entire function of s on* $\mathbb{C}$.

Proof: The discussion in Paragraph 3.5 of [10] (page 801) implies that the zeta function given by the left hand side of equation (3.5.1) of [10] (page 801) is in fact entire. Moreover for each v, the L-function $L(s, \sigma_v \times \varrho_v)$ has exactly the same poles as the local zeta function $\Psi(s, W_v, W_v')$ in the notation of [10] (cf. [7] for $v < \infty$, while for $v = \infty$ we refer to the discussion at the end of page 14 of [8] for the case $n \times (n - 1)$). But now the theorem is a consequence of the fact that for Re(s) sufficiently large $\Pi_v \Psi(s, W_v, W_v')$ is just the right hand side of equation (3.5.1) of [10].

Proof of Theorem 4.1: The functional equation (4.1.1) is just a special case of Theorem 4.1 of [22]. Next for every finite v, let P_v be the unique polynomial in q_v^{-s}, satisfying $P_v(0) = 1$ such that $P_v(q_v^{-s})$ is the numerator of $\gamma(s, \pi_v \otimes \varrho_v, r_3^0, \psi_v)$. It is easy to see that unless $\pi_v \otimes \varrho_v$ is a class one

complementary series representation of $GL_2(F_v)$

$$P_v(q_v^{-s})^{-1} = L(s, \pi_v, r_3^0, \varrho_v).$$

Consequently by Proposition 2.2, the L-function $L(s, \pi_v, r_3^0, \varrho_v)$ will appear in the numerator of $C_{\chi_v}(s, \pi_v \otimes \varrho_v)$ and therefore the poles of $L(s, \pi_v, r_3^0, \varrho_v)$ are among those intertwining operators (Part (b) of Proposition 3.3.1 of [22]). In particular if $F = \mathbb{Q}$, the L-functions given in Section 1 of this paper are the same as those defined in Section 6 of [21]. Moreover if π_v is a class one complementary series, then the polynomial $L(s, \pi_v, r_3^0, \varrho_v)^{-1}$ will have no multiple factors and therefore by Proposition 5.9 of [21] again the poles of $L(s, \pi_v, r_3^0, \varrho_v)$ are among those of intertwining operators. Finally let us consider those $v = \infty$ for which $\pi_v \otimes \varrho_v$ is ramified. Then it is either tempered or a non-class one (i.e. ramified) complementary series. If it is tempered then it is easily checked that $L(s, \pi_v, r_3^0, \varrho_v)$ has no poles for Re $(s) > 0$. Otherwise, the estimate that results from part (4) of Theorem 9.3 of [3] will be more than enough to assert the same fact, at least when Re $(s) \geqslant \frac{1}{2}$. We should remark that the fact that the representation is ramified is crucial.

Let S be a finite set of places containing all the ramified ones (archimedean or not), such that if $v = \infty$ and $\pi_v \otimes \varrho_v$ is of class one, then $v \notin S$. Now the proof of Theorem 6.2 of [21] implies that for Re $(s) \geqslant \frac{1}{2}$, the poles of

$$L_S(s, \pi, r_3^0, \varrho) = \prod_{v \notin S} L(s, \pi_v, r_3^0, \varrho_v)$$

are all on the interval $[1/2 - s_0, 1 - s_0)$.

Next the discussion at the beginning of this proof and an argument similar to Lemma 6.3 of [21] shows that for Re $(s) \geqslant \frac{1}{2}$, the poles of $\prod_{v \in S} L(s, \pi, r_3^0, \varrho_v)$ outside the line Im$(s) = -s_0/2$ are among the zeros of $L_S(s, \pi, r_3^0, \varrho)$ and therefore for Re $(s) \geqslant \frac{1}{2}$, all the poles of $L(s, \pi_v, r_3^0, \varrho_v)$ belong to the interval $[1/2 - s_0/2, 1 - (s_0/2))$. Observe that if $L(s, \pi \otimes \varrho\alpha^{-s_0/2})$ has no zeros on $[1/2, 1)$, then the same is true for $L(s, \pi \otimes \varrho)$ on $[1/2 - s_0/2, 1 - (s_0/2))$.

Finally, we appeal to Corollary 6.5 of [21] to conclude that $L(s, \pi, r_3^0, \varrho)$ has no pole at $1/2 - s_0/2$ if $\omega_0\varrho$ is trivial (assuming the Weak Riemann Hypothesis discussed above this is very important). Now the Theorem is a consequence of the functional equation (4.1.1) and the holomorphy of $L(s, \pi \times (\Pi \otimes \varrho))$ on this interval by Theorem 4.3.

REMARK: The introduction of the polynomial $P_v(q_v^{-s})$ is not necessary in the proof of Theorem 4.1. In fact unless $\pi_v \otimes \varrho_v$ is a class one complementary

series, $L(s, \pi, r_3^0, \varrho)$ will have no poles for Re $(s) > 0$. We have intro-
duced the polynomials $P(q^{-s})$ as to show that, at least when $F = \mathbb{Q}$, the
L-functions defined here agree with those defined quite generally in [21].

We now state the following Corollary of the proof of Theorem 4.1.

COROLLARY: *Let π be a cusp form on $PGL_2(\mathbb{A}_F)$. Then $L(s, \pi, r_3)$ is
holomorphic at the middle of the critical strip $s = 1/2$.*

5. Examples

Throughout this section we shall assume $F = \mathbb{Q}$. There is a wealth of
examples when the assumption of Theorem 4.1 holds. In fact as it is
remarked in [18], direct computations have shown that for every holo-
morphic cusp form (with respect to $SL_2(\mathbb{Z})$) of weight less than or equal
to 50, $L(s, \pi) \neq 0$ for $s \in (1/2, 1)$ and therefore for these forms $L(s, \pi, r_3^0) =
L(s, \pi, r_3)$ is entire.

Now, for a positive integer N let $\Gamma_0(N)$ be the congruence group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, c \equiv 0(N) \right\}.$$

If $\mathfrak{h}^*$ is the union of the upper half plane $\mathfrak{h}$ and the cusps of $\Gamma_0(N)$, let
$X_0(N) = \Gamma_0(N)\backslash\mathfrak{h}^*$. The image I of $[0, i\infty]$ in $X_0(N)$ is called the funda-
mental arc.

Let k be a positive integer. Fix a newform $f \in S_{2k}(\Gamma_0(N))$ in the sense of
Atkin and Lehner [1]. Here $S_{2k}(\Gamma_0(N))$ denotes the space of cusp forms of
weight $2k$ with respect to $\Gamma_0(N)$. Then f is an eigenfunction for all the Hecke
operators and therefore has real Fourier coefficients. Consequently
$f(iy) = \sum_{n=1}^{\infty} a_n e^{-2\pi ny}$ takes only real values. We now prove:

PROPOSITION 5.1: *Let $f \in S_{2k}(\Gamma_0(N))$ be a newform. Denote by π_f the corre-
sponding cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$. Assume that the
real valued function $f\,|\, I$ never changes sign. Then $L(s, \pi_f) \neq 0$ for $s \in (1/2,
1)$ and therefore $L(s, \pi_f, r_3)$ is entire.*

Proof: Let $s' = s + k - 1/2$. Then for Re $(s') > 1 + k$ (cf. Theorem 3.66
of [28])

$$L(s, \pi_f) = \int_0^\infty f(iy) y^{s'-1} \, dy$$

$$= \int_0^A f(iy) y^{s'-1} \, dy + \int_A^\infty f(iy) y^{s'-1} \, dy,$$

(5.1.1)

where $A = N^{-1/2}$ and $i = \sqrt{-1}$. Changing y to $1/Ny$ and using the equation

$$f(i/ny) = N^k (iy)^{2k} g(iy),$$

where

$$g = f \mid \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}_{2k}$$

(notation as in [28]), the first integral in (7.6.2) can be written

$$(-1)^k N^{k-s'} \int_A^\infty g(iy) y^{2k-1-s'}\, dy.$$

Since f is a newform, part (iii) of Theorem 3 of [1] implies that $g = \pm f$ and therefore (5.1.1) equals to

$$\int_A^\infty f(iy)[y^{s'-1} \pm (-1)^k N^{k-s'} y^{2k-1-s'}]\, dy. \tag{5.1.2}$$

But for $y > A = N^{-1/2}$ and $s' > k$, i.e., $s > 1/2$, (5.1.2) is non-zero since $f(iy)$ never changes sign. This completes the proposition.

Next, let **X** be a smooth projective variety over $\mathbb{Q}$ for which $\mathbf{X}(\mathbb{C}) = X_0(N)$. Assume $k = 1$, i.e., f is a newform in $S_2(\Gamma_0(N))$. Suppose f has rational Fourier coefficients. Then by Theorem 1 of [27], there exists an elliptic curve **E** over $\mathbb{Q}$ and a non-constant $\mathbb{Q}$-morphism $\varphi \colon \mathbf{X} \to \mathbf{E}$ such that $\varphi^* \omega = f(z)\, dz$, where ω is a differential of the first kind on $\mathbf{E}(\mathbb{C})$. We observe that since f is a newform this will not be the case for any $X_0(M)$ with $M < N$. As in [17], we shall call the ramification points of φ (as a map between $X_0(N)$ and $\mathbf{E}(\mathbb{C})$) which lie on the fundamental are I, the fundamental critical points of **E**. Finally we refer the reader to [17] for the definition of involuntary and semi-involuntary curves. Then by Corollaries 5 and 6 of §3 of [17], they have, at most, only one fundamental critical point. We now prove.

PROPOSITION 5.1: *Let f be a newform in $S_2(\Gamma_0(N))$ with rational Fourier coefficients. Denote by π_f and **E** the cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ and the $\mathbb{Q}$-rational elliptic curve attached to f, respectively. Let φ: $\mathbf{X} \to \mathbf{E}$ be the corresponding $\mathbb{Q}$-rational map. Assume that φ has at most one fundamental critical point (in particular if **E** is involuntary or semi-involuntary). Then $L(s, \pi_f, r_3)$ is entire.*

Proof: Let div($\varphi^*\omega$) denote the divisor of $\varphi^*\omega$, and denote by deg (div($\varphi^*\omega$)) its degree. If at every $P \in X_0(N)$, e_p is the corresponding ramification index, then

$$\deg (\varphi^*\omega) = \sum_P (e_P - 1)P. \tag{5.2.1}$$

Moreover if g is the genus of $X_0(N)$, then deg (div($\varphi^*\omega$)) $= 2g - 2$. Let P_0 be the only possible ramification point on I. Since the zeros of φ^* are symmetric with respect ot I, it then immediately follows from (5.2.1) that

$$2g - 2 = (e_{P_0} - 1) + \sum_{P \neq P_0} (e_P - 1)$$

and consequently $e_{P_0} - 1$ must be even. Thus the order of the zero of $f \, dz = \varphi^*\omega$ at P_0 must be even, and therefore $f(z)$ never changes sign on I. Now the proposition is a consequence of Proposition 5.1.

The following corollary is a consequence of the tables in §4 of [17].

COROLLARY: *Suppose $N \leqslant 423$. Let $f \in S_2(\Gamma_0(N))$ be a $\mathbb{Q}$-rational newform. Denote by π_f the corresponding cuspidal automorphic form on $GL_2(\mathbb{A}_\mathbb{Q})$. Then except possibly for 16 values of N, $L(s, \pi_f, r_3)$ is entire.*

Proof: These 16 values (which are explicitly given in Table 4 of [17]) are those where there are two or more (in fact three if $N \leqslant 423$) fundamental critical points.

6. Plancherel measures and *R*-groups

Throughout this section F dentoes a non-archimdean local field of characteristic zero. In what follows we shall obtain a formula for the Plancherel measure for G, a p-adic split group of exceptional type G_2, coming from the parabolic subgroup $P = \mathbf{P}(F)$ of G generated by its long root. A conjecture of Langlands in this case (Proposition 6.2) as well as the reducibility of certain induced representations (Proposition 6.3) will follow.

Let π be an infinite dimensional irreducible admissible representation of $\mathbf{M}(F) = GL_2(F)$. Using the notation of Section 2, given $s \in \mathbb{C}$, let $I(s, \pi)$ be the representation of G induced from

$$\pi \otimes q^{\langle s\tilde{\alpha}, H_P(\ )\rangle} \otimes 1$$

of *MN*. Here $H_P \colon M \to \mathrm{Hom}\,(X(\mathbf{M}),\, \mathbb{R})$ is defined by

$$q^{\langle \chi, H_p(m) \rangle} \;=\; |\chi(m)|,$$

$\chi \in X(\mathbf{M})$, where $X(\mathbf{M})$ is the group of F-rational characters of $\mathbf{M}$.

We use w to denote a representative for the reflection about the root $2\alpha + \beta$. Given f in the space of $I(s, \pi)$ and for $\mathrm{Re}(s)$ sufficiently large, we define:

$$A(s, \pi, w)f(g) \;=\; \int_N f(w^{-1}ng)\, \mathrm{d}n \quad (g \in G).$$

This is the standard intertwining operator attached to π. The Plancherel constant $\mu(s, \pi)$ is then defined through the relation (cf. [22], [29])

$$A(s, \pi, w)A(-s, w(\pi), w^{-1}) \;=\; \mu(s, \pi)^{-1}\gamma(G/P)^2,$$

where $\gamma(G/P)$ is a positive constant, depending on the measure $\mathrm{d}n$, defined in [29]. It does not depend upon the choice of w. We shall now prove:

PROPOSITION 6.1: *Let π be an infinite dimensional irreducible admissible representation of $M = GL_2(F)$ with central character ω. Then*

$$\mu(s, \pi)\gamma(G/P)^{-2} \;=\; \gamma(2s, \omega, \psi)\gamma(-2s, \omega^{-1}, \bar{\psi})$$

$$\cdot\, \gamma(s, \pi, r_3^0, \psi)\gamma(-s, \tilde{\pi}, r_3^0, \bar{\psi}),$$

where defining measures are self dual with respect to ψ.

Proof: By Proposition 3.1.1 of [22] we have

$$\mu(s, \pi)\gamma(G/P)^{-2} \;=\; C_\chi(s, \pi)C_\chi(-s, w(\pi))\omega(-1),$$

since $w^{-1} = w \cdot H_\beta(-1)$, where w is the standard representative for the reflection about $2\alpha + \beta$. Using Proposition 2.2 it is now enough to show $w(\pi) = \tilde{\pi}$. We shall show $w(\pi) = \pi \otimes \omega^{-1}$ which is isomorphic to $\tilde{\pi}$. It is easy to check that for every $m \in SL_2(F) \subset M$, $w(\pi)(m) = \pi(m)$. Now, using the identification $H_{3\alpha+2\beta}(t) = \mathrm{diag}\,(t, 1)$, $t \in F$, it can be shown that

$$w(\pi)\!\left(\!\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\!\right) \;=\; \pi\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\omega^{-1}(t)$$

proving $w(\pi) = \pi \otimes \omega^{-1}$. The proposition is now proved.

REMARK: This could also be proved using the equivalent identity

$$\mu(s, \pi)\gamma(G/P)^{-2} = C_\chi(s, \pi)C_{\tilde{\chi}}(-s, \tilde{\pi}).$$

Now set

$$\mathcal{A}(\pi, w) = \varepsilon(0, \tilde{\pi}, r_3^0 \oplus \Lambda^2 r_1, \psi)L(1, \tilde{\pi}, r_3^0 \oplus \Lambda^2 r_1)$$

$$\times L(0, \tilde{\pi}, r_3^0 \oplus \Lambda^2 r_1)^{-1} \cdot A(0, \pi, w),$$

where the right hand side is defined by taking limits as s approaches zero. Observe that $r_3^0 \oplus \Lambda^2 r_1$ is the adjoint action of $^L M = GL_2(\mathbb{C})$, the L-group of **M**, on the Lie algebra $^L \mathfrak{n}$ of $^L N$, the L-group of **N**. Thus we have the following result which proves a conjecture of Langlands [13] in this case.

PROPOSITION 6.2: *Let* π *be an infinite dimensional irreducible unitary representation of* $M = GL_2(F)$. *Denote by* $\mathcal{A}(\pi, w)^*$ *the adjoint of the operator* $\mathcal{A}(\pi, w)$ *with respect to the Hilbert space structures of* $I(0, \pi)$ *and* $I(0, w(\pi))$. *Then*

(a) $\mathcal{A}(\pi, w)\mathcal{A}(w(\pi), w^{-1}) = I,$

and

(b) $\mathcal{A}(\pi, w)^* = \mathcal{A}(w(\pi), w^{-1})$, *i.e.,* $\mathcal{A}(\pi, w)$ *is unitary.*

Proof: We only need to prove part (b) and that follows from the identities

$$\overline{L(s, \pi, r_3^0)} = L(\bar{s}, \tilde{\pi}, r_3^0)$$

and

$$\overline{\varepsilon(s, \pi, r_3^0, \psi)} = \varepsilon(\bar{s}, \tilde{\pi}, r_3^0, \psi)$$

the last of which is a consequence of the relations

$$\varepsilon(s, \pi \times \Pi, \bar{\psi}) = \omega(-1)\varepsilon(s, \pi \times \Pi, \psi)$$

and

$$\varepsilon(s, \pi, \bar{\psi}) = \omega(-1)\varepsilon(s, \pi, \psi).$$

COROLLARY: *Let π be an infinite dimensional irreducible unitary representation of $M = GL_2(F)$. Assume s is pure imaginary. Then*

$$\mu(s, \pi)\gamma(G/P)^{-2} = |\varepsilon(2s, \omega, \psi)\varepsilon(s, \pi, r_3^0, \psi)L(1 + 2s, \omega)$$

$$\times \ L(1 + s, \pi, r_3^0)|^2 \cdot |L(2s, \omega)L(s, \pi, r_3^0)|^{-2}$$

We now prove:

PROPOSITION 6.3: *Let π be a discrete series representation of $M = GL_2(F)$ with central character ω. Denote by $I(\pi)$ the representation of G induced from $\pi \otimes 1$ on MN. Then $I(\pi)$ is irreducible unless:*

(a) π is an extraordinary supercuspidal representation of $GL_2(F)$ and $\pi \otimes \omega \cong \pi$ (i.e., $\pi \cong \tilde{\pi}$) with $\omega \neq 1$, or

(b) π is a supercuspidal representation of $GL_2(F)$ of the form $\pi = \pi(\tau)$, where $\tau = \operatorname{Ind}(W_F, W_K, \chi)$, $[K:F] = 2$, $\chi \in \hat{K}^*$. Moreover either $\chi^3 \neq 1$ while $\chi|_{F^*} = 1$ or $\chi^2 = 1$ while $\chi|_{F^*} \neq \eta$, where η is the quadratic character of F^* defined by K.

In either case the corresponding R-group is $\mathbb{Z}_2$ and therefore $I(\pi)$ has two inequivalent tempered components.

Proof: By the general theory of R-groups if there is any reducibility, we must have $w(\pi) \cong \pi$ and therefore $\pi \cong \pi \otimes \omega$. To compute the R-groups we must study the zeros of $\mu(0, \pi)$ (cf. [11], [29], [30]). By the above corollary it would be enough to look at $L(0, \omega)^{-1}$. $L(0, \pi, r_3^0)^{-1}$. If $\pi = \sigma(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$, i.e. π is special, then $\pi \otimes \omega \cong \pi$ requires $\mu^2 = \omega = 1$ which provides a zero for $L(0, \omega)^{-1}$. Consequently $I(\pi)$ will be irreducible. It remains to study supercuspidal representations.

Assume first that π is extraordinary and $\pi \otimes \omega \cong \pi$. Then $L(0, \pi, r_3^0) = 1$, and $L(0, \omega)^{-1} = 0$ if and only if $\omega = 1$. The assumption $\omega \neq 1$ now proves part a.

Next let $\pi = \pi(\tau)$, $\tau = \operatorname{Ind}(W_F, W_K, \chi)$, $[K:F] = 2$. Then $\omega = \chi|_{F^*} \cdot \eta$ and $\pi \cong \tilde{\pi}$ if and only if

$$\operatorname{Ind}(W_F, W_K, \chi) \cong \operatorname{Ind}(W_F, W_K, \chi^{-1}).$$

But this happens if and only if either $\chi = \chi'^{-1}$ or $\chi = \chi^{-1}$. In the first case $\chi\chi' = 1$ which implies $\chi \cdot N_{K/F} = 1$. Consequently $\chi|_{F^*}$ is either η or trivial. The first case implies $\omega = 1$ and therefore to get reducibility we may assume $\chi|_{F^*} = 1$. It remains to check $L(0, \pi, r_3^0)$ which equals $L_K(0, \chi^2\chi'^{-1})$ by relation (1.3). Using $\chi\chi' = 1$, $L_K(0, \chi^2\chi'^{-1})^{-1}$ is zero if and only if $\chi^3 = 1$. This proves the first case in part b.

Finally assume $\chi^2 = 1$. Then $L_K(0, \chi^2\chi'^{-1})^{-1} = L_K(0, \chi'^{-1})^{-1}$ is zero if and only if $\chi = 1$. But this is a contradiction since π is supercuspidal. Consequently $\mu(0, \pi)$ is zero if and only if $\chi|_{F^*} = \eta$. This completes the proposition.

REMARK: The representations induced from the other maximal parabolic subgroup are much easier to treat since the Plancherel measures are much simpler. Having a formula for the Plancherel measure in hand must make it now possible to have a complete classification of non-supercuspidal tempered representations of a split p-adic group of type G_2 (cf. [11] for the minimal parabolic). It must also give us the formal degree of non-supercuspidal discrete series representations, as well as examples of non-tempered unitary representations. Since the cases of other rank two split groups are fairly similar, we hope to address the unitary dual of rank two split p-adic groups in a future paper.

Appendix

Let π be a cusp form on $PGL_2(\mathbb{A}_F)$ and let r_5 be the six dimensional irreducible representation of $SL_2(\mathbb{C}) = {}^L PGL_2$, the fifth symmetric power representation of $SL_2(\mathbb{C})$. Write $\pi = \otimes_v \pi_v$ and let S be a finite set of places of F containing the archimedean ones such that for every $v \notin S$, π_v is unramified. Finally, let

$$L_S(s, \pi, r_5) = \prod_{v \notin S} L(s, \pi_v, r_5),$$

where $L(s, \pi_v, r_5)$ is the inverse of a polynomial of degree 6 in q_v^{-s}, defined as in the introduction. In Theorem 4.1.2 of [22], it was proved that $L_S(s, \pi, r_5)$ extends to a meromorphic function of s on $\mathbb{C}$ which satisfies a standard functional equation.

As it has been pointed out by Serre [20], the following theorem can be used to obtain the best available evidence for the validity of the Sato–Tate Conjecture (cf. [19]) at present. More precisely, using this result, he showed that: *There exist infinitely many primes p such that $\tau(p)/p^{11/2} > 2\cos(2\pi/7) \cong 1.24697961$, where τ denotes the Ramanujan τ-function* (by the Ramanujan conjecture $-2 \leqslant \tau(p)/p^{11/2} \leqslant 2$). When π is attached to a $\mathbb{Q}$-rational newform f in $S_2(\Gamma_0(N))$ (cf. Section 5), this implies that there exist infinitely many rational primes p such that

$$N_p > 1 + p + 1.24697961p^{1/2},$$

where N_p is the number of rational points of **E** modulo p, **E** being the elliptic curve attached to f.

THEOREM A: (*a*) *Suppose π is not monomial. The $L_S(s, \pi, r_5) \neq 0$ for all s with Re $(s) = 1$ with the possible exception of a simple zero at $s = 1$.*

(*b*) *Assume further that π is an automorphic representation attached to a holomorphic cuspidal modular form. Then for Re $(s) \geqslant 1$, the partial L-function $L_S(s, \pi, r_5)$ is holomorphic and non-zero with the possible exception of either a simple zero or a simple pole at $s = 1$.*

Proof: Let Π be the Gelbart–Jacquet lift of π [3]. It is a cusp form on $PGL_3(\mathbb{A}_F)$. Applying Theorem 5.1 of [22] to the case $F_4 - 2$ of [21], we conclude that

$$L_S(1 + 3it, \pi, r_1)L_S(1 + 2it, \Pi \times \Pi)L_S(1 + 2it, \pi, r_2)^{-1}$$

$$\cdot L_S(1 + it, \pi, r_1)L_S(1 + it, \pi, r_3)L_S(1 + it, \pi, r_5) \neq 0 \qquad (\text{A.1})$$

for all $t \in \mathbb{R}$, where $i = \sqrt{-1}$. Here $L_S(s, \Pi \times \Pi)$ is the partial Rankin–Selberg L-function attached to (Π, Π). By Proposition 3.6 of [10], one knows that $L_S(1 + 2it, \Pi \times \Pi)$ is holomorphic (in fact by the proposition it is continuous, but then since it is meromorphic it must be holomorphic) for all $t \in \mathbb{R}$, except $t = 0$. Moreover, if $t = 0$, then $L_S(1, \Pi \times \Pi)$ has a simple pole. On the other hand every other partial L-function, except $L_S(1 + it, \pi, r_5)$, in (A.1) is holomorphic and non-zero (necessary for $L_S(1 + 2it, \pi, r_2)$) for $\forall t \in \mathbb{R}$. The first assertion then follows.

If π comes from a holomorphic modular cusp form, the $L_S(s, \pi, r_5)$ will be absolutely convergent for Re $(s) > 1$. This is a consequence of the validity of the Ramanujan-Petersson's conjecture which is due to Deligne. A standard use of equation (2.7) of [21] will then immediately imply that

$$L_S(1 + it, \pi, r_1)L_S(1 + it, \pi, r_3)L_S(1 + it, \pi, r_5)$$

is holomorphic for all $t \in \mathbb{R}$, except, possibly, for a simple pole at $t = 0$. Now the second statement is a consequence of the non-vanishing of the first two L-functions for all $t \in \mathbb{R}$ (cf. Theorem 5.3 of [22] for the second L-function).

REMARK: Part (*b*) of the theorem is also due to Serre [20].

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