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# On the Gauss maps of space curves in characteristic p

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## §0. Introduction

In this article, we discuss the extensions of function fields defined by Gauss maps of space curves in positive characteristic $p$. As is well-known, if $p = 0$, then the Gauss maps of space curves are always birational onto its image. However, if $p > 0$, then this is not true.

Precisely speaking, our purpose is to investigate the sets $\mathscr{K}'$, $\mathscr{K}'_{\text{un}}$ and $\mathscr{K}'_{\text{imm}}$ defined as follows: for a smooth connected complete curve $C$ over an algebraically closed field of positive characteristic $p$, we define $\mathscr{K}'$ as the set of subfields $K'$ of $K(C)$ such that there exists a morphism $\iota$ from $C$ to some projective space $\mathbb{P}$, birational onto its image, such that $\iota(C)$ is not a line in $\mathbb{P}$ and the extension of function fields defined by the Gauss map coincides with $K(C)/K'$; replacing the word "morphism" above by "unramified morphism" and "closed immersion", we moreover define $\mathscr{K}'_{\text{un}}$ and $\mathscr{K}'_{\text{imm}}$, respectively.

Our main results are

THEOREM 0.1:

(a) *If $C$ is an ordinary elliptic curve, then $\mathscr{K}'_{\text{imm}}$ contains any subfield $K'$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K'$ and the separable closure of $K'$ in $K(C)$ is a cyclic extension over $K'$ with degree indivisible by $p$.*

(b) *If $C$ is a supersingular elliptic curve, then*

$$\mathscr{K}'_{\text{imm}} = \begin{cases} \{K(C)^p, K(C)^{p^2}\} & \text{if } p = 2 \\ \{K(C), K(C)^p\} & \text{otherwise.} \end{cases}$$

(see Section 5)

THEOREM 0.2:

(a) *If $C$ is a curve of genus $g \geqslant 2$, then an arbitrary element $K'$ of $\mathcal{K}'_{\mathrm{imm}}$ is of the form $K(C)^{p^l}$ for some integer $l \geqslant 0$.*

(b) *Moreover, for an integer $l > 0$, let*

$$C \to C^{(p)} \to \cdots \to C^{(p^l)}$$

*be a sequence of Frobenius morphisms of $C$. Then, the following conditions are equivalent:*

(1) *$\mathcal{K}'_{\mathrm{imm}}$ contains $K(C)^{p^l}$;*

(2) *there exists a rank 2 vector bundle $\mathscr{E}$ on $C^{(p^l)}$ such that $\mathscr{E}_{C^{(p)}}$ is stable and $\mathscr{E}_C$ is isomorphic to the bundle $\mathscr{P}^1_C(\mathscr{L})$ of principal parts of $\mathscr{L}$ of first order for some line bundle $\mathscr{L}$ on $C$, where $\mathscr{E}_{C^{(p)}}$ and $\mathscr{E}_C$ are the pull-backs of $\mathscr{E}$ to $C^{(p)}$ and $C$, respectively.*

(see Corollaries 4.4 and 6.2)

We shall show also the following results: $\mathcal{K}'$ (respectively, $\mathcal{K}'_{\mathrm{imm}}$) contains $K(C)$ if and only if $p \neq 2$ (see Corollary 2.3); for a proper subfield $K'$ of $K(C)$, $\mathcal{K}'$ contains $K'$ if and only if $K(C)$ is finite, inseparable over $K'$ (see Corollaries 2.2 and 3.4); if $C$ is rational, then $\mathcal{K}' = \mathcal{K}'_{\mathrm{imm}}$ (see Corollary 3.6); and, $\mathcal{K}'_{\mathrm{un}} = \mathcal{K}'_{\mathrm{imm}}$ for any $C$ (see Corollary 4.3).

As a corollary to our results, one can show that a smooth plane curve has separable degree 1 over the dual curve via the Gauss map (see Corollary 4.5 and [16, p. 342]; compare with [24, Proposition 4.2]), and that if $C$ is a Tango–Raynaud curve (see, for example, [2] or [27]), then $\mathcal{K}'_{\mathrm{imm}}$ contains $K(C)^p$ (see Corollary 6.5).

The fundamental results in our study are Theorems 2.1, 3.1 and 4.1, from which we shall deduce all the results above.

## §1. Gauss maps

Throughout this article, we shall work over an algebraically closed field $k$ of positive characteristic, denoted by $p$.

Let $C$ be a smooth, connected, complete curve defined over $k$. For a morphism $\iota$ from $C$ to a projective space $\mathbb{P}$, birational onto its image, let $V$ be a vector space $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, and $\mathscr{P}^1_C(\iota^*\mathcal{O}_{\mathbb{P}}(1))$ the bundle of principal parts of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ of first order on $C$ (see, for example, [14, IV, A], [19, §1] or [25, §§2 and 6]). We have a natural map

$$a^1 \colon V \otimes_k \mathcal{O}_C \to \mathscr{P}^1_C(\iota^*\mathcal{O}_{\mathbb{P}}(1)),$$

which is surjective if and only if $\iota$ is unramified. The image of $a^1$ is a quotient bundle of $V \otimes_k \mathcal{O}_C$ of rank 2.

Let $\mathbb{G}$ be a Grassmann manifold consisting of 2-dimensional quotient spaces of $V$ with the universal quotient bundle

$$V \otimes_k \mathcal{O}_{\mathbb{G}} \to \mathcal{Q}.$$

DEFINITION: By virtue of the universality of $\mathbb{G}$, one obtains from $a^1$ a morphism

$$C \to \mathbb{G}.$$

We call this morphism the *Gauss map* of $C$ in $\mathbb{P}$ via $\iota$ (see, for example, [3, 2, §4], [4, 1, (e)] or [14, IV, B]).

In particular, the image of $a^1$ is isomorphic to the pull-back of $\mathcal{Q}$ to $C$, denoted by $\mathcal{Q}_C$. We identify these bundles.

If one considers $\mathbb{G}$ to be consisting of lines in $\mathbb{P}$, then, for a general point $x$ of $C$, the image of $x$ in $\mathbb{G}$ under the Gauss map is corresponding to the tangent line to $\iota(C)$ at $\iota(x)$.

We always assume that $\iota(C)$ is not a line in $\mathbb{P}$, and denote by $C^*$ the image of $C$ in $\mathbb{G}$ under the Gauss map, which is called the *dual curve* of $C$ if $\mathbb{P} = \mathbb{P}^2$ (see, for example, [5, I, Exercise 7.3] or [14, I, C]).

DEFINITION: For a curve $C$, we define $\mathcal{K}'$ as the set of subfields $K'$ of $K(C)$ as follows: there exists a morphism $\iota$ from $C$ to some projective space $\mathbb{P}$, birational onto its image, such that $\iota(C)$ is not a line in $\mathbb{P}$ and the extension $K(C)/K(C^*)$ defined by the Gauss map coincides with $K(C)/K'$. Replacing the word "morphism" above by "unramified morphism" and "closed immersion", we moreover define $\mathcal{K}'_{\text{un}}$ and $\mathcal{K}'_{\text{imm}}$, respectively.

REMARK: It is well-known that if $p = 0$, then $\mathcal{K}' = \{K(C)\}$.

## §2. Generic projections

Our first result is

THEOREM 2.1: *Let $\iota$: $C \to \mathbb{P}$ be a morphism birational onto its image as before, let $\Pi$: $\mathbb{P} \to \mathbb{P}_1$ be a projection of $\mathbb{P}$ from a general point in $\mathbb{P}$, and let $\iota_1$: $C \to \mathbb{P}_1$ be a composition of $\Pi$ with $\iota$. Let $C^*$ and $C_1^*$ be the images of $C$*

*under the Gauss maps via* $\iota$ *and* $\iota_1$, *respectively, and let* $K(C^*)_s$ *and* $K(C_1^*)_s$ *be the separable closures of* $K(C^*)$ *and* $K(C_1^*)$ *in* $K(C)$, *respectively. If* $\dim \mathbb{P} \geqslant 3$, *then*

$$K(C)/K(C^*)_s \;=\; K(C)/K(C_1^*)_s.$$

*Proof:* Denote by $P$ the centre of $\Pi$, let $\mathbb{G}$ and $\mathbb{G}_1$ be Grassmann manifolds of lines in $\mathbb{P}$ and $\mathbb{P}_1$, respectively, and let $\pi\colon \mathbb{G} \to \mathbb{G}_1$ be a rational map naturally induced by $\Pi$. Then, we have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{P} & \xleftarrow{\iota} & C & \longrightarrow & C^* & \subset & \mathbb{G} \\ {\scriptstyle \Pi}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}_1 & \xleftarrow{\iota_1} & C & \longrightarrow & C_1^* & \subset & \mathbb{G}_1. \end{array}$$

We note that, via the Plücker embeddings of $\mathbb{G}$ and $\mathbb{G}_1$ into some projective spaces, $\pi$ is compatible with a linear projection of the projective space, denoted by $\wedge\,\pi$. Writing $\sigma(P)$ for the subset of $\mathbb{G}$ consisting of lines in $\mathbb{P}$ passing through $P$, we see that the base locus of $\pi$ is equal to $\sigma(P)$, which coincides with the centre of $\wedge\,\pi$. Choose a smooth point $x$ of $C^*$, and consider the embedded tangent space $T_xC^*$ to $C^*$ at $x$. Counting the dimensions, one proves that $T_xC^*$ and $\sigma(P)$ do not meet for a general $P$, which implies that $C^*$ is separable over $C_1^*$ (see, for example, [28, §3]). This completes our proof.

Remark: Similarly, one can prove that if $\dim \mathbb{P} \geqslant 4$, then $K(C)/K(C^*) = K(C)/K(C_1^*)$. Moreover, it can be shown that when $\dim \mathbb{P} = 3$, this equality does not hold if and only if $\iota(C)$ is strange and not contained in any plane in $\mathbb{P}$, where $\iota(C)$ is called *strange* if there exists a point in $\mathbb{P}$ which lies on all the tangent lines at smooth points of $\iota(C)$ (see, for example, [5, IV, §3] or [29, II]).

Corollary 2.2: *For an element* $K'$ *of* $\mathscr{K}'$, *if* $K(C)$ *is separable over* $K'$, *then* $K(C) = K'$.

*Proof:* For a morphism $\iota\colon C \to \mathbb{P}$ associated to $K'$, using Theorem 2.1, one can reduce the problem to the case $\dim \mathbb{P} = 2$. In this case, the result is known (see, for example, [9, §9.4] or [14, p. 310]).

Corollary 2.3: *For a curve* $C$, *the following conditions are equivalent*:
(1) $\mathscr{K}'$ *contains* $K(C)$;
(2) $\mathscr{K}'_{\mathrm{un}}$ *contains* $K(C)$;
(3) $\mathscr{K}'_{\mathrm{imm}}$ *contains* $K(C)$;
(4) $p \neq 2$.

*Proof:* The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

For a morphism $\iota: C \to \mathbb{P}$, birational onto its image, we denote by $m$ the intersection multiplicity of $\iota(C)$ and a general tangent line to $\iota(C)$ at a general point. We note that $m + 1$ is equal to the third gap of the linear system defining $\iota$. It follows from Theorem 2.1 and [6, Proposition 4.4] that $K(C)/K(C^*)$ is separable if and only if $p \neq 2$ and $m = 2$. This proves the implication (1) $\Rightarrow$ (4).

Moreover, it follows from the Riemann–Roch theorem that if $\iota$ is a closed immersion defined by a complete linear system with sufficiently large degree, then $m = 2$. Thus, the implication (4) $\Rightarrow$ (3) follows from Corollary 2.2.

REMARK: Combining Theorem 2.1 with [6, Proposition 4.4], one can prove that if $K(C)/K(C^*)$ is not separable, then its inseparable degree is equal to $m$.

REMARK: Using Theorem 2.1, one can prove that if the Gauss map of a curve $C$ in $\mathbb{P}$ via $\iota$ gives a separable extension $K(C)/K(C^*)$, then $\iota(C)$ is not strange in $\mathbb{P}$.

## §3. Gauss maps and ruled surfaces

The following plays a key role throughout the rest of this article.

THEOREM 3.1: *For a subfield $K'$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K'$, the following conditions are equivalent:*

(1) *$\mathscr{K}'$ contains $K'$;*

(2) *there exist a rank 2 vector bundle $\mathscr{E}$ on a curve $C'$ with $K(C') = K'$ and a morphism $h: C \to \mathbb{P}(\mathscr{E})$, birational onto its image, such that the extension $K(h(C))/K(C')$ defined by the projection of $\mathbb{P}(\mathscr{E})$ over $C'$ coincides with $K(C)/K'$.*

*This is true for $\mathscr{K}'_{\mathrm{un}}$ (respectively, $\mathscr{K}'_{\mathrm{imm}}$) if one assumes moreover that $h$ is unramified (respectively, a closed immersion) in (2).*

We first verify the implication (1) $\Rightarrow$ (2).

LEMMA 3.2: *Consider a trivial extension $(\varepsilon_0)$ and a unique non-trivial extension $(\varepsilon_1)$ of $\mathscr{O}_C$ by $\Omega^1_C$. Then, for a line bundle $\mathscr{L}$ on $C$, we have a natural extension*

$$0 \to \Omega^1_C \otimes \mathscr{L} \to \mathscr{P}^1_C(\mathscr{L}) \to \mathscr{L} \to 0,$$

*which coincides with $(\varepsilon_0) \otimes \mathscr{L}$ if the degree of $\mathscr{L}$ is divisible by the characteristic $p$. Otherwise, the extension coincides with $(\varepsilon_1) \otimes \mathscr{L}$.*

*Proof:* See [11, §1].

For a morphism $\iota\colon C \to \mathbb{P}$, birational onto its image, combining the Euler sequence on $\mathbb{P}$ with the extension above, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \iota^*(\Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1)) & \to & V \otimes_k \mathcal{O}_C & \to & \iota^*\mathcal{O}_{\mathbb{P}}(1) & \to 0 \\ & \downarrow & & {\scriptstyle a^1}\downarrow & & \| & \\ 0 \to & \Omega^1_C \otimes \iota^*\mathcal{O}_{\mathbb{P}}(1) & \to & \mathscr{P}^1_C(\iota^*\mathcal{O}_{\mathbb{P}}(1)) & \to & \iota^*\mathcal{O}_{\mathbb{P}}(1) & \to 0 \end{array}$$

with exact rows. Let $C'$ be the normalization of the image $C^*$ of $C$ under the Gauss map, and let $C_0$ be a section of the ruled surface $\mathbb{P}(\mathcal{Q}_C)$ over $C$ associated to the quotient $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ of $\mathcal{Q}_C$, where $\mathcal{Q}_C$ is the image of $a^1$. We have a commutative diagram (see [11, §1])

$$\begin{array}{ccccccc} & \swarrow & C & \leftarrow & \mathbb{P}(\mathcal{Q}_C) & \searrow & \hookleftarrow C_0 \\ \mathbb{G} \hookleftarrow C^* & & \downarrow & & {\scriptstyle f}\downarrow & & X \hookrightarrow \mathbb{P}, \\ & \nwarrow & C' & \leftarrow & \mathbb{P}(\mathcal{Q}_{C'}) & \nearrow & \end{array}$$

where $X$ is an image of $\mathbb{P}(\mathcal{Q}_C)$ in $\mathbb{P}$, and $f$ is a natural morphism induced by the Gauss map. Intuitively, $C_0$ is consisting of points of contact on $\mathbb{P}(\mathcal{Q}_C)$. We see that the induced morphism $C_0 \to \mathbb{P}$ above coincides with $\iota$ via $C_0 \simeq C$.

Now, let $\mathscr{E}$ be the pull-back $\mathcal{Q}_{C'}$ and $h$ the composition of $f|_{C_0}$ with the isomorphism $C \to C_0$. Since $\iota$ is birational onto its image, so is $h$, and we have $K(h(C))/K(C') = K(C)/K(C^*)$. Moreover, if $\iota$ is unramified (respectively, a closed immersion), then so is $h$. This completes the proof of (1) $\Rightarrow$ (2).

REMARK: The curve $\iota(C)$ is strange if and only if the morphism $\mathbb{P}(\mathcal{Q}_{C'}) \to X$ above is not finite. It follows from the proof of [18, Proposition 3] that if $\iota$ is unramified, then $\iota(C)$ is not strange except for the case when $p = 2$ and $\iota(C)$ is a conic.

The converse (2) $\Rightarrow$ (1) follows from

LEMMA 3.3: *Let $\mathscr{E}$ be a rank 2 vector bundle on a curve $C'$, and let $h\colon C \to \mathbb{P}(\mathscr{E})$ be a morphism birational onto its image. If $h(C)$ is finite, inseparable over $C'$ via the projection of $\mathbb{P}(\mathscr{E})$, then there exists a morphism $\iota\colon C \to \mathbb{P}$, birational onto its image, such that the extension $K(C)/K(C^*)$ defined by the Gauss map coincides with $K(C)/K(C')$. Moreover, if $h$ is unramified (respectively, a closed immersion), then so is $\iota$.*

*Proof:* Take a sufficiently very ample line bundle $\mathscr{M}$ on $C'$, and let $\varrho$ be an embedding of $\mathbb{P}(\mathscr{E})$ into a projective space $\mathbb{P}$ as a scroll defined by a complete linear system associated to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{M}$, where $\pi$ is the projection of $\mathbb{P}(\mathscr{E})$. We define $\iota$ to be a composition of $\varrho$ with $h$. Then, the image of each fibre of $\mathbb{P}(\mathscr{E})$ under $\varrho$ is a line in $\mathbb{P}$, tangent to $\iota(C)$ since $h(C)$ is inseparable over $C'$. Denoting by $V$ the vector space $H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{M})$, we have a commutative diagram

$$\begin{array}{ccc} V_{\mathbb{P}(\mathscr{E})} & \to & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{M} \\ \downarrow & & \downarrow \\ V_{h(C)} & \to & \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{M}|_{h(C)}, \end{array}$$

where all the arrows are surjective, $\varrho$ is determined by the upper arrow, and $\iota$ is determined by the lower arrow. Let $\mathscr{L}$ be a quotient line bundle of $(\pi \circ h)^*\mathscr{E}$ associated to $h$ (see, for example, [5, II, Proposition 7.12]). Taking the inverse image by $h$ and a morphism $\mathbb{P}((\pi \circ h)^*\mathscr{E}) \to \mathbb{P}(\mathscr{E})$, taking the direct image by the projection of $\mathbb{P}((\pi \circ h)^*\mathscr{E})$ over $C$, we get

$$\begin{array}{ccc} V_C & \to & (\pi \circ h)^*(\mathscr{E} \otimes \mathscr{M}) \\ \| & & \downarrow \\ V_C & \to & \mathscr{L} \otimes (\pi \circ h)^*\mathscr{M}, \end{array}$$

where all the arrows are surjective. We find that this diagram coincides with

$$\begin{array}{ccc} V_C & \to & \mathscr{Q}_C \\ \| & & \downarrow \\ V_C & \to & \iota^*\mathscr{O}_{\mathbb{P}}(1) \end{array}$$

which is obtained from the commutative diagram under Lemma 3.2 for our $\iota$, and the extension of function fields defined by the Gauss map of $C$ via $\iota$ is equal to $K(C)/K(C')$.

This completes the proof of Theorem 3.1.

COROLLARY 3.4: *The set $\mathscr{K}'$ contains any subfield $K'$ of $K(C)$ such that $K(C)$ is finite, inseparable over $K'$.*

*Proof:* Let $K'$ be a subfield of $K(C)$ as above, and let $C'$ be a curve with $K(C') = K'$. According to Lemma 3.5 below, there is a primitive element $\psi$

of $K(C)$ over $K'$. From $\psi$ and a morphism $C \to C'$ defined by $K(C)/K'$, we obtain a morphism

$$h: C \to \mathbb{P}^1 \times C'.$$

Then, we find that $h$ is birational onto its image, and the result follows from Theorem 3.1.

LEMMA 3.5: *An arbitrary finite extension of function fields of dimension* 1 *over* $k$ *is simple.*

*Proof:* Let $K/K'$ be an extension as above, and let $K_s'$ be the separable closure of $K'$ in $K$. A finite, separable extension $K_s'/K'$ is simple. On the other hand, an arbitrary intermediate field of the purely inseparable extension $K/K_s'$ is of the form $K^{p^l}$ for some integer $l > 0$ (see, for example, [5, IV, Proposition 2.5]), whose number is finite. So, it follows from [10, Theorem 15, p. 55] that $K/K_s'$ is simple. Thus, according to [10, Theorem 14, p. 54], the extension $K/K'$ is simple.

From now on, we shall focus our attention on $\mathscr{K}_{\mathrm{un}}'$ and $\mathscr{K}_{\mathrm{imm}}'$.

COROLLARY 3.6: *If $C$ is a rational curve, then* $\mathscr{K}' = \mathscr{K}_{\mathrm{imm}}'$.

*Proof:* For a proper subfield $K'$ in $\mathscr{K}'$, let $C'$ be a curve with $K(C') = K'$, and let $h$ be a graph morphism of $C \to C'$ defined by $K(C)/K'$:

$$h: C \to C \times C',$$

which is a closed immersion. The result follows from Theorem 3.1 and Corollary 2.3.

REMARK: A similar argument to the above can be found in [26, Example 2.13].

COROLLARY 3.7: *For an element $K'$ of $\mathscr{K}_{\mathrm{un}}'$ (respectively, $\mathscr{K}_{\mathrm{imm}}'$), let $K_1'$ be an intermediate field of $K(C)/K'$. If $K(C)/K_1'$ is not separable, then $\mathscr{K}_{\mathrm{un}}'$ (respectively, $\mathscr{K}_{\mathrm{imm}}'$) contains $K_1'$.*

*Proof:* Let $C'$ be a curve with $K(C') = K'$. According to Theorem 3.1, we have a vector bundle $\mathscr{E}$ of rank 2 on $C'$ and an unramified morphism $C \to \mathbb{P}(\mathscr{E})$, birational onto its image, such that $K(h(C))/K(C') = K(C)/K'$.

Let $C_1'$ be a curve with $K(C_1') = K_1'$, and let $\mathscr{E}_1$ be a pull-back of $\mathscr{E}$ by a morphism $C_1' \to C'$ defined by $K_1'/K'$. Using $h$ and a morphism $C \to C_1'$ defined by $K(C)/K_1'$, we get an unramified morphism $C \to \mathbb{P}(\mathscr{E}_1)$, birational onto its image, such that $K(h_1(C))/K(C_1') = K(C)/K_1'$. Then, the result for $\mathscr{K}_{\text{un}}'$ follows from Theorem 3.1. If $h$ is a closed immersion, then so is $h_1$. This completes the proof.

COROLLARY 3.8: *For an element $K'$ of $\mathscr{K}_{\text{un}}'$ (respectively, $\mathscr{K}_{\text{imm}}'$), let $K_1'$ be a subfield of $K'$ such that $K'/K_1'$ is finite, purely inseparable, let $C'$ and $C_1'$ be curves with function fields $K'$ and $K_1'$, respectively, and let $f$: $C' \to C_1'$ be a morphism defined by $K'/K_1'$. Then, the following conditions are equivalent:*
(1) *$\mathscr{K}_{\text{un}}'$ (respectively, $\mathscr{K}_{\text{imm}}'$) contains $K_1'$;*
(2) *there exists a rank 2 vector bundle $\mathscr{E}_1$ on $C_1'$ such that*

$$f^*\mathscr{E}_1 \simeq \mathscr{E}$$

*for some vector bundle $\mathscr{E}$ on $C'$ associated to $K'$ as in* Theorem 3.1.

*Proof:* (1) $\Rightarrow$ (2). This follows from Theorem 3.1.
(2) $\Rightarrow$ (1). Let $h$ be the morphism $C \to \mathbb{P}(\mathscr{E})$ associated to $K'$ as in Theorem 3.1, and let $h_1$ be the morphism $h$ followed by a morphism $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}_1)$ induced from $f$. By virtue of Theorem 3.1, it suffices to show that $h_1$ is unramified, birational onto its image (respectively, a closed immersion). Since $f$ is purely inseparable, the morphism $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}_1)$ is injective. So, it is sufficient to verify that a tangent vector to $h(C)$ in $\mathbb{P}(\mathscr{E})$ at each point of $h(C)$ is not vanished under $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}_1)$. But, this is true since a fibre of $\mathbb{P}(\mathscr{E})$ over $C'$ is tangent to $h(C)$, and mapped isomorphically into $\mathbb{P}(\mathscr{E}_1)$ by $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}_1)$.

## §4. Calculation on ruled surfaces

The third of our fundamental results is

THEOREM 4.1: *Let $\iota$: $C \to \mathbb{P}$ be a morphism birational onto its image, let $g$ and $g'$ be the genera of $C$ and $C'$, respectively, and denote by $p_a(f(C_0))$ the arithmetic genus of $f(C_0)$, with the same notations as before. If $\iota$ is unramified, then we have*

$$p_a(f(C_0)) = g = g'.$$

We denote by $d$ the degree of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$, that is, the degree of $\iota(C)$ in $\mathbb{P}$, and by $d^*$ the degree of $C^*$ in $\mathbb{G}$ via the Plücker embedding, where $C^*$ is the image of $C$ in $\mathbb{G}$ under the Gauss map. We write $n$ for the degree of the extension $K(C)/K(C^*)$, that is, the degree of the Gauss map of $C$. We denote by $v$ the morphism $\mathbb{P}(\mathcal{Q}_{C'}) \to X$, and by $\gamma$ the degree of the cokernel of the map $a^1$.

One can easily prove the following (see, for example, [11, §1], [19, §3] or [25, §3]).

Lemma 4.2: *With the same notations as before, we have*

$$\begin{aligned} 2g - 2 + 2d - \gamma &= \deg \mathcal{Q}_C \\ &= nd^* \\ &= (\deg f)(\deg v)(\deg X). \end{aligned}$$

To prove Theorem 4.1, using a generic projection of $\mathbb{P}$, we may assume that $\mathbb{P} = \mathbb{P}^2$.

Writing $D$ for the image $f(C_0)$, we first study the numerical class of $D$ in the ruled surface $\mathbb{P}(\mathcal{Q}_{C'})$. For a general line $L$ in $\mathbb{P}^2$, set

$$H := v^*L$$

in $\mathbb{P}(\mathcal{Q}_{C'})$. Then, $H$ is a section of $\mathbb{P}(\mathcal{Q}_{C'})$ over $C'$. Since $X$ coincides with $\mathbb{P}^2$, in particular, it has degree 1, it follows from Lemma 4.2 that $v$ has degree $d^*$. Thus, we have $(H^2) = d^*$. Since $v|_{f(C_0)}$ is birational onto its image, we have $(D.H) = d$. On the other hand, denoting by $F$ the numerical class of a fibre of the projection of $\mathbb{P}(\mathcal{Q}_{C'})$ over $C'$, we obtain $(D.F) = n$ from the proof of (1) $\Rightarrow$ (2) in Theorem 3.1. Therefore, using Lemma 4.2, we find that

$$D \equiv nH - (2g - 2 + d - \gamma)F.$$

Denoting by $K$ the numerical class of a canonical divisor of $\mathbb{P}(\mathcal{Q}_{C'})$, we see

$$K \equiv -2H + (2g' - 2 + d^*)F.$$

Now, it follows from the adjunction formula that

$$2p_a(D) - 2 = n\{(2g' - 2) - (2g - 2)\} + (nd^* - 2d + n\gamma).$$

So, we obtain by Lemma 4.2 that

$$(2p_a(D) - 2) - (2g - 2)$$
$$= n\{(2g' - 2) - (2g - 2)\} + (n - 1)\gamma.$$

We here have $p_a(D) \geqslant g \geqslant g'$, $n \geqslant 1$, and $\gamma = 0$ since $\iota$ is unramified. Therefore, the result follows from the equality above. This completes the proof of Theorem 4.1.

COROLLARY 4.3: $\mathscr{K}'_{\mathrm{un}} = \mathscr{K}'_{\mathrm{imm}}$.

*Proof:* It follows from Theorem 4.1 that $f(C_0)$ is smooth, so that the unramified morphism $h$ in Theorem 3.1 is a closed immersion.

COROLLARY 4.4: *If $C$ has genus $g \geqslant 2$, then an arbitrary element $K'$ of $\mathscr{K}'_{\mathrm{un}}$ is of the form $K(C)^{p^l}$ for some integer $l \geqslant 0$.*

*Proof:* Use Hurwitz's formula (see, for example, [5, IV, §2]).

COROLLARY 4.5: *A smooth plane curve has separable degree* 1 *over the dual curve via the Gauss map.*

*Proof:* If $g \geqslant 2$, then the result follows directly from Corollary 4.4. If $g = 0$, then the result clearly follows. We hence assume that $g = 1$, so that $d = 3$. We obtain from Theorem 4.1 that $d^* \geqslant 3$, and the result follows from Corollary 2.2 and Lemma 4.2.

REMARK: At the last part of the proof above, we do not need Theorem 4.1 because we always have $n \leqslant d$.

REMARK: This result answers a question in [16, p. 342], and improves [24, Proposition 4.2].

## §5. Elliptic curves

This section is devoted to elliptic curves.

THEOREM 5.1: *Let $C$ be an ordinary elliptic curve in characteristic $p$, and let $K'$ be a subfield of $K(C)$ such that $K(C)$ is finite, inseparable over $K'$. If the separable closure of $K'$ in $K(C)$ is cyclic over $K'$ and the separable degree of $K(C)/K'$ is not divisible by $p$, then $\mathscr{K}'_{\mathrm{imm}}$ contains $K'$.*

*Proof:* Let $C'$ be a curve with $K(C') = K'$. By virtue of Theorem 3.1, it is sufficient to show that there exist a rank 2 vector bundle $\mathcal{E}$ on $C'$ and a closed immersion $h: C \to \mathbb{P}(\mathcal{E})$ such that $K(h(C))/K(C') = K(C)/K'$.

Writing $s$ and $q$ respectively for the separable and inseparable degree of $K(C)/K'$, we have an exact sequence of group schemes

$$0 \to \mathbb{Z}/s\mathbb{Z} \times \mu_q \to C \to C' \to 0.$$

Taking the dual, we get

$$0 \to \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to \hat{C}' \to \hat{C} \to 0.$$

Since $s$ and $q$ are coprime, one may choose a single element $\mathcal{L}$ of $\hat{C}'$ which generates the kernel above.

Now, we put

$$\mathcal{E} := \mathcal{O}_{C'} \oplus \mathcal{L}.$$

We note that there exist *only two* sections of $\mathbb{P}(\mathcal{E})$ over $C'$ such that the self-intersection number is equal to zero. It follows

$$\mathcal{E}_C \simeq \mathcal{O}_C^{\oplus 2},$$

and we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{P}^1 \times C \simeq & \mathbb{P}(\mathcal{E}_C) & \xrightarrow{f} & \mathbb{P}(\mathcal{E}) \\ & \downarrow & & \downarrow \\ & C & \longrightarrow & C', \end{array}$$

where $f$ is a morphism induced by $K(C)/K'$.

We consider a constant section $C_0$ of $\mathbb{P}(\mathcal{E}_C)$ over $C$ which comes from neither of the two sections of $\mathbb{P}(\mathcal{E})$ mentioned above, and let us prove that $f|_{C_0}$ is a closed immersion.

Let $C''$ be the normalization of the image $f(C_0)$. Using a base change by the naturally induced morphisms $C \to C'' \to C'$, we get a commutative diagram

$$\begin{array}{ccccc} C_0 & \to & C_0'' & \to & f(C_0) \\ \cap & & \cap & & \cap \\ \mathbb{P}(\mathcal{E}_C) & \to & \mathbb{P}(\mathcal{E}_{C''}) & \to & \mathbb{P}(\mathcal{E}) \\ \downarrow & & \downarrow & & \downarrow \\ C & \to & C'' & \to & C', \end{array}$$

where we denote by $C''_0$ a section of $\mathbb{P}(\mathscr{E}_{C''})$ over $C''$ naturally defined by the base change. Then, we find that $C_0$ is a pull-back of $C''_0$ and the self-intersection number of $C''_0$ is also equal to zero.

Now, suppose that $f|_{C_0}$ is not birational. It follows that $C \to C''$ should not be birational. Therefore, $\mathscr{E}_{C''}$ is not trivial and $C''_0$ comes from either of the two sections of $\mathbb{P}(\mathscr{E})$ specified above, so does $C_0$. This is a contradiction, and it follows that $f|_{C_0}$ is birational.

It remains to show that $f(C_0)$ is smooth in $\mathbb{P}(\mathscr{E})$. Computing its arithmetic genus, one can easily deduce this result.

THEOREM 5.2: *If C is a supersingular elliptic curve in characteristic p, then*

$$\mathscr{K}'_{\text{imm}} = \begin{cases} \{K(C)^p, K(C)^{p^2}\} & \text{if } p = 2 \\ \{K(C), K(C)^p\} & \text{otherwise.} \end{cases}$$

*Proof:* We show the inclusion $\subseteq$. By Theorem 3.1, we see that the converse follows from the existence of suitable vector bundles, which can be verified simultaneously below (use, for example, [1]).

Take an element $K'$ of $\mathscr{K}'_{\text{imm}}$, and let $\iota\colon C \to \mathbb{P}$ be a closed immersion associated to $K'$. We employ the same notations as before.

*Case: $p \neq 2$*

If $K(C) = K'$, then there is nothing to prove. We may assume that $K(C)$ is not separable over $K'$.

Let $C_1$, $C'$ and $C''$ be curves with function fields $K(C)^p$, $K'$ and $K'^{1/p}$, respectively. We get a commutative diagram

$$\begin{array}{ccc} C & \to & C_1 \\ \downarrow & & \downarrow \\ C'' & \to & C'. \end{array}$$

Since $\iota$ is unramified, using Lemmas 3.2 and 4.2, we have

$$\mathscr{Q}_C \simeq \iota^*\mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$$

and $C_0$ is a constant section of $\mathbb{P}(\mathscr{Q}_C)$ over $C$ via an isomorphism $\mathbb{P}(\mathscr{Q}_C) \simeq \mathbb{P}^1 \times C$.

I claim that $\mathscr{Q}_{C_1}$ is indecomposable. Suppose that $\mathscr{Q}_{C_1}$ is decomposed. Since $C_1$ is supersingular, a line bundle $\mathscr{L}$ on $C_1$ with $\mathscr{L}_C \simeq \iota^*\mathcal{O}_{\mathbb{P}}(1)$ is uniquely determined. Thus, $\mathcal{Q}_{C_1}$ should be of the form $\mathcal{L}^{\oplus 2}$ for some line bundle $\mathcal{L}$. This implies that the map

$$C_0 \hookrightarrow \mathbb{P}(\mathcal{Q}_C) \to \mathbb{P}(\mathcal{Q}_{C_1})$$

is not birational onto its image. Then, we find a contradiction because $f|_{C_0}$ is birational onto its image.

It follows that $\mathcal{Q}_{C'}$ is indecomposable. I claim moreover that $\mathcal{Q}_{C'}$ has even degree. Suppose that $\mathcal{Q}_{C'}$ has odd degree. It clearly follows that $\mathcal{Q}_{C''}$ should have also odd degree. Since $\mathcal{Q}_C$ is a direct sum of two line bundles with the same degree, $\mathcal{Q}_{C''}$ must be indecomposable. Therefore, $\mathcal{Q}_{C'}$ and $\mathcal{Q}_{C''}$ are both indecomposable if they have odd degree. Since $C \to C''$ is isomorphic to $C_1 \to C'$ as a finite cover of abstract curves, $\mathcal{Q}_C$ is indecomposable if and only if so is $\mathcal{Q}_{C_1}$ (see, for example, [1, II]). This is a contradiction.

Since $C''$ is supersingular, we have

$$\mathcal{Q}_{C''} \simeq \mathcal{M}^{\oplus 2}$$

for some line bundle $\mathcal{M}$ on $C''$.

If $K(C)/K'$ is not purely inseparable of degree $p$, again this contradicts the birationality of $f|_{C_0}$ because the map

$$\begin{array}{ccc} C_0 & \hookrightarrow & \mathbb{P}(\mathcal{Q}_C) \\ & & \downarrow \\ & & \mathbb{P}(\mathcal{Q}_{C''}) \end{array}$$

should not be birational onto its image. This completes the proof.

*Case:* $p = 2$

According to Corollaries 2.2 and 2.3, $K(C)$ is always inseparable over $K'$. Since $\iota$ is unramified, we have an exact sequence

$$(\varepsilon)\ 0 \to \iota^*\mathcal{O}_{\mathbb{P}}(1) \to \mathcal{Q}_C \to \iota^*\mathcal{O}_{\mathbb{P}}(1) \to 0.$$

It follows from Lemma 3.2 that $(\varepsilon)$ splits if and only if $d$ is even.

By the similar way to the former case, let us consider the curves $C_1$, $C'$, $C''$ and the commutative diagram above.

*Subcase:* $(\varepsilon)$ *splits*

I first claim that $\mathcal{Q}_{C_1}$ is indecomposable and has even degree. The indecomposability of $\mathcal{Q}_{C_1}$ follows by the similar way to the former case. If $\mathcal{Q}_{C_1}$ has odd degree, then, according to [23, Theorem 2.16], $\mathcal{Q}_C$ should be indecomposable. This is a contradiction.

Next, I claim that $K(C_1)/K(C')$ is either trivial or not separable. Suppose the contrary. Since $C_1$ is supersingular, the degree of $C_1 \to C'$ should not be divisible by $p$, that is, odd. Since $Q_{C_1}$ is indecomposable with even degree, so is $\mathcal{Q}_{C'}$. Thus, $\mathcal{Q}_{C''}$ should be of the form $\mathcal{M}^{\oplus 2}$ for some line bundle $\mathcal{M}$ on $C''$ because $C''$ is supersingular, and this contradicts the birationality of $f|_{C_0}$ as above.

Now, if $K(C_1) = K(C')$, then $K(C)$ is purely inseparable of degree $p$ over $K'$. We hence consider the case when $C_1 \to C'$ is not separable. Then, one can take a curve $C_2$ between $C_1$ and $C'$ with $K(C_2) = K(C)^{p^2}$, and $C'''$ between $C$ and $C''$ with $K(C''') = K'^{1/p^2}$. We obtain a commutative diagram

$$\begin{array}{ccccc} C & \to & C_1 & \to & C_2 \\ \downarrow & & & & \downarrow \\ C''' & \to & C'' & \to & C'. \end{array}$$

I claim that $\mathcal{Q}_{C_2}$ has odd degree. Suppose that $\mathcal{Q}_{C_2}$ has even degree. Then, $\mathcal{Q}_{C_1}$ should be decomposed since $C_1$ is supersingular. This is a contradiction.

Therefore, $\mathcal{Q}_{C'}$ is indecomposable with odd degree and $\mathcal{Q}_{C'''}$ is of the form $\mathrm{N}^{\oplus 2}$ for some line bundle $\mathcal{N}$ on $C'''$.

If $K(C)/K'$ is not purely inseparable of degree $p^2$, then we find a contradiction by the similar way to the last part of the former case. This completes our proof.

*Subcase:* ($\varepsilon$) *does not split*

We see that $C_0$ is a *unique* section of $\mathbb{P}(\mathcal{Q}_C)$ over $C$ such that the self-intersection number is equal to zero, and $\mathcal{Q}_C$ is indecomposable with even degree.

I claim that $\mathcal{Q}_{C_1}$ has odd degree. Suppose that $\mathcal{Q}_{C_1}$ has even degree. Then, $\mathcal{Q}_{C_1}$ should have a non-trivial extension

$$(\varepsilon_1)\ 0 \to \mathcal{L} \to \mathcal{Q}_{C_1} \to \mathcal{L} \to 0$$

with some line bundle $\mathcal{L}$ on $C_1$ because $\mathcal{Q}_{C_1}$ is indecomposable. We find that the pull-back of ($\varepsilon_1$) to $C$ must coincide with ($\varepsilon$). Therefore, similarly as above, this contradicts the birationality of $f|_{C_0}$.

Hence, $\mathcal{Q}_{C'}$ and $\mathcal{Q}_{C''}$ are indecomposable with odd and even degree, respectively, and we have $K(C_1) = K(C')$ since $f|_{C_0}$ is birational onto its image. Thus, $K(C)/K'$ is purely inseparable of degree $p$.

REMARK: In the latter subcase above, we have not used the assumption that $C$ is supersingular.

## §6. Curves of higher genus

The main purpose of this section is to prove

THEOREM 6.1: *Let $C$ be a curve in characteristic $p$ with genus $g \geqslant 2$, let*

$$f: C \to C'$$

*be a Frobenius morphism of $C$, and let $\mathscr{L}$ be a line bundle on $C$ such that the degree of $\mathscr{P}^1_C(\mathscr{L})$ is divisible by $p$. Then, the following conditions are equivalent:*
(1) *$\mathscr{K}'_{\mathrm{imm}}$ contains $K(C)^p$;*
(2) *there exists a stable vector bundle $\mathscr{E}$ of rank 2 on $C'$ such that*

$$\mathscr{P}^1_C(\mathscr{L}) \simeq f^*\mathscr{E}.$$

Combining Theorem 6.1 with Corollary 3.8, we get

COROLLARY 6.2: *Let $C$ be as above. For an integer $l > 0$, let*

$$C \to C^{(p)} \to \cdots \to C^{(p^l)}$$

*be a sequence of Frobenius morphisms of $C$. Then, the following conditions are equivalent:*
(1) *$\mathscr{K}'_{\mathrm{imm}}$ contains $K(C)^{p^l}$;*
(2) *there exists a vector bundle $\mathscr{E}$ of rank 2 on $C^{(p^l)}$ such that $\mathscr{E}_{C^{(p)}}$ is stable and $\mathscr{E}_C \simeq \mathscr{P}^1_C(\mathscr{L})$ for some line bundle $\mathscr{L}$ on $C$.*

To prove Theorem 6.1, let us use Theorem 3.1. For a rank 2 vector bundle $\mathscr{E}$ on $C'$, to give a morphism $h$: $C \to \mathbb{P}(\mathscr{E})$, it is equivalent to give a quotient line bundle $\mathscr{L}$ of $f^*\mathscr{E}$, and there is a commutative diagram

$$\begin{array}{ccccc} C & \longleftarrow & \mathbb{P}(f^*\mathscr{E}) & \supset & \mathbb{P}(\mathscr{L}) \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow & \quad \wr\searrow \\ C' & \longleftarrow & \mathbb{P}(\mathscr{E}) & \underset{h}{\longleftarrow} & C, \end{array}$$

where $\mathbb{P}(\mathscr{L})$ is a section of $\mathbb{P}(f^*\mathscr{E})$ over $C$. Then, we have

LEMMA 6.3: *The following conditions are equivalent:*
(1) *$h$ is birational onto its image;*
(2) *the quotient line bundle $\mathscr{L}$ of $f^*\mathscr{E}$ does not come from any quotient line bundle of $\mathscr{E}$.*

*Proof:* Since $\mathbb{P}(\mathscr{L})$ is purely inseparable of degree $p$ over $C'$, one of the two cases below occurs: $\mathbb{P}(\mathscr{L})$ is purely inseparable of degree $p$ over $h(C)$, and $K(h(C)) = K(C')$, so $h(C)$ is a section of $\mathbb{P}(\mathscr{E})$ over $C'$; $h$ is birational onto its image, and $K(h(C))/K(C')$ is purely inseparable of degree $p$. This completes the proof.

PROPOSITION 6.4: *Let $C$ be a curve in characteristic $p$, let $f: C \to C'$ be a Frobenius morphism of $C$, and let $\mathscr{L}$ be a line bundle on $C$ such that the degree of $\mathscr{P}^1_C(\mathscr{L})$ is divisible by $p$. Then, the following conditions are equivalent:*
(1) *$\mathscr{K}'_{\text{imm}}$ contains $K(C)^p$;*
(2) *there exists a vector bundle $\mathscr{E}$ of rank 2 on $C'$ such that*

$$\mathscr{P}^1_C(\mathscr{L}) \simeq f^*\mathscr{E}$$

*and the natural quotient line bundle $\mathscr{L}$ of $\mathscr{P}^1_C(\mathscr{L})$ does not come from any quotient line bundle of $\mathscr{E}$.*

*Proof:* We first show the implication (1) $\Rightarrow$ (2) in case of $p \neq 2$. For a closed immersion $\iota: C \to \mathbb{P}$ associated to $K(C)^p$ in $\mathscr{K}'_{\text{imm}}$, we have

$$\mathscr{P}^1_C(\iota^*\mathcal{O}_{\mathbb{P}}(1)) \simeq f^*\mathscr{Q}_{C'}.$$

In particular, $\mathscr{L}$ and $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ have the same degree modulo $p$ since $p \neq 2$. It follows from Lemma 3.2 that

$$\mathscr{P}^1_C(\mathscr{L}) \simeq \mathscr{P}^1_C(\iota^*\mathcal{O}_{\mathbb{P}}(1)) \otimes (\mathscr{L} \otimes \iota^*\mathcal{O}_{\mathbb{P}}(1)^\vee),$$

where $\iota^*\mathcal{O}_{\mathbb{P}}(1)^\vee$ is the dual of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$. Thus, there exists a vector bundle $\mathscr{E}$ on $C'$ such that $\mathscr{P}^1_C(\mathscr{L}) \simeq f^*\mathscr{E}$, and it follows from Lemma 6.3 (1) $\Rightarrow$ (2) that $\mathscr{E}$ has the required property.

Next, we consider the case $p = 2$. For a closed immersion $\iota: C \to \mathbb{P}$ such that the degree of $\iota(C)$ is even (respectively, odd), it follows from Corollaries 2.2 and 2.3 that the extension $K(C)/K(C^*)$ defined by the Gauss map via $\iota$ is not separable. Using Corollary 3.7, we obtain a closed immersion $\iota_1: C \to \mathbb{P}_1$ such that the degree of $\iota(C)$ is even (respectively, odd) and the extension defined by the Gauss map via $\iota_1$ is purely inseparable of degree $p$. In particular, this implies the condition (1) in case of $p = 2$. Now, apply the argument in case of $p \neq 2$ above to this $\iota_1$. We obtain the required $\mathscr{E}$ in (2) when the degree of $\mathscr{L}$ is even (respectively, odd). Thus, we have proved that both (1) and (2) are always satisfied when $p = 2$.

Finally, we show the implication (2) $\Rightarrow$ (1). For the natural quotient line bundle $\mathscr{L}$ of $f^*\mathscr{E}$ via $f^*\mathscr{E} \simeq \mathscr{P}^1_C(\mathscr{L})$, by virtue of Lemma 6.3 (2) $\Rightarrow$ (1), we have a morphism $h: C \to \mathbb{P}(\mathscr{E})$ such that $K(h(C))/K(C')$ is purely inseparable of degree $p$. Computing the arithmetic genus of $h(C)$, we see that $h(C)$ is smooth, so that $h$ is a closed immersion. Then, the result follows from Theorem 3.1.

We here consider a special case of this situation.

DEFINITION: For a curve $C$ with a Frobenius morphism $f: C \to C'$, if $C'$ has a line bundle $\mathscr{N}$ such that:
(1) $f^*\mathscr{N} \simeq \Omega^1_C$, and
(2) $f^*: H^1(C', \mathscr{N}^\vee) \to H^1(C, f^*\mathscr{N}^\vee)$ is not injective,
then $C$ is called a *Tango–Raynaud curve*.

COROLLARY 6.5: *If $C$ is a Tango-Raynaud curve in characteristic $p$, then $\mathscr{K}'_{\mathrm{imm}}$ contains $K(C)^p$.*

*Proof:* It follows from the assumption that there is a non-zero element $\xi$ of $H^1(C', \mathscr{N}^\vee)$ for some line bundle $\mathscr{N}$ on $C'$ such that $f^*(\xi) = 0$ in $H^1(C, f^*\mathscr{N}^\vee)$. Take $\mathscr{E}$ to be the extension of $\mathscr{N}$ by $\mathscr{O}_{C'}$ determined by $\xi$, and $\mathscr{L}$ to be $f^*\mathscr{O}_{C'}$ in the situation above. The result follows from Proposition 6.4.

REMARK: If a line bundle $\mathscr{N}$ on $C'$ with $\deg f^*\mathscr{N} = 2g - 2$ satisfies the condition (2), then we have $f^*\mathscr{N} \simeq \Omega^1_C$.

REMARK: Consider a curve $C$ of genus $g \geqslant 2$, and denote by $n(C)$ Tango's invariant of $C$ in [32, Definition 11]. Then, $C$ is a Tango-Raynaud curve in our sense if and only if

$$n(C) = (2g - 2)/p,$$

and our definition above is equivalent to the ordinary one (see, for example, [2] or [27]).

REMARK: Neither rational nor ordinary elliptic curve $C$ is a Tango–Raynaud curve. On the other hand, a supersingular elliptic curve $C$ is a Tango–Raynaud curve, because $\mathcal{O}_{C'}$ enjoys the required properties.

REMARK: Any curve $C$ in characteristic 2 is a Tango–Raynaud curve if $C$ is neither rational nor ordinary elliptic, because the cokernel of a natural map $\mathcal{O}_{C'} \to f_*\mathcal{O}_C$ is a line bundle having the required properties (see, for example, [2, Exemple i), p. 81] or [20, 2.2]).

Now, we go back to the proof of Theorem 6.1. For a rank 2 vector bundle $\mathscr{F}$ on a curve $C$ and a quotient line bundle $\mathscr{L}$ of $\mathscr{F}$, we put

$$s(\mathscr{F}, \mathscr{L}) := 2 \deg \mathscr{L} - \deg \mathscr{F}.$$

Moreover, we put

$$s(\mathscr{F}) := \inf_{\mathscr{L}} s(\mathscr{F}, \mathscr{L}),$$

where $\mathscr{L}$'s are quotient line bundles of $\mathscr{F}$. The value $s(\mathscr{F})$ is called the *stability degree* of $\mathscr{F}$ because of the fact that $\mathscr{F}$ is stable if and only if $s(\mathscr{F}) > 0$.

LEMMA 6.6: *For a rank* 2 *vector bundle* $\mathscr{F}$ *on a curve, if* $s(\mathscr{F}) < 0$, *then a quotient line bundle* $\mathscr{L}$ *of* $\mathscr{F}$ *with* $s(\mathscr{F}, \mathscr{L}) \leqslant 0$ *is uniquely determined.*

*Proof:* Taking a quotient line bundle $\mathscr{L}$ of $\mathscr{F}$ with $s(\mathscr{F}, \mathscr{L}) < 0$, we denote by $\mathscr{M}$ the kernel of $\mathscr{F} \to \mathscr{L}$. Let $\mathscr{L}'$ be an arbitrary quotient line bundle of $\mathscr{F}$ with $s(\mathscr{F}, \mathscr{L}') \leqslant 0$, and denote by $\mathscr{M}'$ the kernel of $\mathscr{F} \to \mathscr{L}'$. Then, we have that $\deg \mathscr{M} > \deg \mathscr{L}$, and $\deg \mathscr{M}' \geqslant \deg \mathscr{L}'$. It follows that $\deg \mathscr{M} > \deg \mathscr{L}'$, and $\deg \mathscr{M}' > \deg \mathscr{L}$. Thus, we see that $\mathscr{L} = \mathscr{L}'$ as a quotient line bundle of $\mathscr{F}$.

For a line bundle $\mathscr{L}$ on a curve $C$ with genus $g$, we have $s(\mathscr{P}^1_C(\mathscr{L}), \mathscr{L}) = -(2g - 2)$. Moreover, if $g \geqslant 2$, then it follows from Lemma 6.6 that

$$s(\mathscr{P}^1_C(\mathscr{L})) = -(2g - 2).$$

REMARK: One can say that $\mathscr{P}^1_C(\mathscr{L})$ is the farthest from a stable bundle on $C$ even if $\mathscr{P}^1_C(\mathscr{L})$ is indecomposable (see, for example, [5, V, Theorem 2.12]).

PROPOSITION 6.7: *Let $f: C \to C'$ be a finite morphism of curves, and let $\mathscr{F}$ be a rank 2 vector bundle on $C'$. If the pull-back $f^*\mathscr{F}$ has a quotient line bundle $\mathscr{L}$ with $s(f^*\mathscr{F}, \mathscr{L}) < 0$, then the following conditions are equivalent:*
(1) *the quotient line bundle $\mathscr{L}$ of $f^*\mathscr{F}$ does not come from any quotient line bundle of $\mathscr{F}$;*
(2) *$\mathscr{F}$ is stable.*

*Proof:* It suffices to show the implication (1) $\Rightarrow$ (2) since the converse is obvious. Assume that there exists a quotient line bundle $\mathscr{L}'$ of $\mathscr{F}$ such that $s(\mathscr{F}, \mathscr{L}') \leqslant 0$. It clearly follows that $s(f^*\mathscr{F}, f^*\mathscr{L}') \leqslant 0$. According to Lemma 6.6, the quotient line bundle $\mathscr{L}$ must coincide with the pull-back $f^*\mathscr{L}'$. This completes the proof.

Finally, Theorem 6.1 follows from Propositions 6.4 and 6.7.

REMARK: In Theorem 6.1, if the degree of $\mathscr{L}$ is not divisible by $p$ and there exists a vector bundle $\mathscr{E}$ on $C'$ such that $f^*\mathscr{E} \simeq \mathscr{P}^1_C(\mathscr{L})$, then $\mathscr{E}$ is stable.

REMARK: In characteristic 2, the line bundle $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ is not necessarily related to $\mathscr{L}$ in Theorem 6.1 and the condition (1) is always satisfied, as we have seen in the proof of Proposition 6.4.

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