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GERT-MARTIN GREUEL
ULRICH KARRAS

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Families of varieties with prescribed singularities

GERT-MARTIN GREUEL & ULRICH KARRAS
Universität Kaiserslautern, Fachbereich Mathematik, Erwin-Schrödinger-Strasse, D-6750 Kaiserslautern, FRG

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1. Introduction

The paper consists of two parts. In the first part, Sections 1–3, we consider deformations of holomorphic maps $f: X \to S$ between complex spaces, such that the induced deformation of X is locally trivial (i.e., does not change the singularities of X). In the second part, Sections 4–6, we apply this to families of reduced curves, in particular to embedded deformations of curves C lying on a smooth surface S (in which case $f: C \to S$ is the closed embedding). We are interested in questions concerning the family of all reduced projective plane curves with a fixed number and fixed analytic type of singularities: Does this "family" exist as a complex space, or even as an algebraic variety? What is its dimension? Under what conditions is it smooth or, in classical language, when is its characteristic linear series complete? Moreover, when do the singularities of C impose independent conditions, i.e., when is every deformation of the local multigerm $(C, \mathrm{Sing}(C))$ induced by an embedded deformation of the global curve C? An answer to the last question eventually allows even to construct curves with a given number and type of singularities (cf. 6.4 (4), (5)).

For families of plane curves with only ordinary nodes and cusps, these questions are classical and have been studied and answered by Severi and

Segre (cf. [Ta 1, 2] for a modern treatment). Actually, they never considered the problem of the existence of this family as an algebraic variety; this was first proved by J. Wahl [Wa]. Wahl also considered locally trivial embedded deformations of plane curves with arbitrary singularities, identified the infinitesimal deformations and obstructions as $H^0(C, \mathcal{N}'_{C/\mathbb{P}^2})$ and $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$ of a certain sheaf $\mathcal{N}'_{C/\mathbb{P}^2}$ on C and showed the existence of a formal versal deformation space in the sense of Schlessinger. We prove the existence of a *convergent* versal deformation space (in a much more general context) which is algebraic, if the curve has only simple singularities (2.4). Moreover, for an arbitrary smooth surface S containing C, we give sufficient conditions for $H^1(C, \mathcal{N}'_{C/S})$ to be zero, which implies the smoothness of the versal, locally trivial deformation space of C and also the independence of conditions imposed by the singularities. These sufficient conditions are given in terms of the genera, intersection numbers and Tjurina numbers of the irreducible components of C and are *very easy to compute* (cf. 6.1 (iii) and 6.3 (iii)). Although these conditions are in general not necessary, they are sharp in the sense, that for certain examples one cannot do better (cf. 6.4 (2), (3), (6)). For $S = \mathbb{P}^2$, the vanishing of $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$ is actually necessary and sufficient for the independence of conditions imposed by the singularities of C (6.3 (i)).

The difficulty for getting conditions which guarantee the vanishing of $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$ lies in the fact, that $\mathcal{N}'_{C/\mathbb{P}^2}$ is not invertible in the singular points of C. Usually one tries then to compare such a sheaf with an invertible sheaf on the normalization of C and to apply the usual vanishing theorem there. This is also possible in our case, but this requires somewhat complicated local computations and, in general, the results are weaker. For this reason we prove a vanishing theorem for arbitrary rank one sheaves on reduced curves in section 5. An argument like this should be well known although we could not find it in the literature. In addition, by introducing a local invariant (the index, cf. 5.1), we gain a little bit, which turns out to be essential in the applications.

In the first part we consider general locally trivial deformations of a holomorphic map $f\colon X \to S$, by which we mean deformations of f with fixed base S such that the induced deformation of X is locally trivial in each point of X. Let $\mathcal{D}'_{X/S}$ be the associated functor of isomorphism classes of such deformations. We show that for compact X with isolated singularities, there exists a convergent miniversal locally trivial deformation space and that "openness of versality" holds for $\mathcal{D}'_{X/S}$ (cf. 1.3). From this we deduce easily in Section 2 that the "locally trivial Hilbert functor" (of proper flat families of subspaces of a given space which are locally trivial) is representable by a complex space. The existence of a versal family is relatively easy to prove for isolated singularities, while for the openness result we have to show that $\mathcal{D}'_{X/S}$

has a "good" obstruction theory in order to apply Artin's criterion. This is done in Section 3, for X with arbitrary singularities. We would like to mention, that subsequently H. Flenner and S. Kosarew proved the existence of a miniversal deformation space for $\mathscr{D}'_{X/S}$ for compact X with arbitrary singularities. Hence our theorems 1.4 (iii), 2.2 and proposition 2.3 are valid for arbitrary singularities of X (for the smoothness of $\mathscr{H}_{X/S} \to \mathscr{D}_{X,\mathrm{Sing}}$ in 2.3 one has then to assume additionally that $H^1(X, \mathscr{T}_X^1) = 0$).

1. Locally trivial deformations

1.1. Let $f\colon X \to S$ be a holomorphic map of complex spaces. We are interested in deformations of X/S, i.e., deformations of f with fixed base S, such that the induced deformation of X is locally trivial. More precisely, a *deformation of X/S* over a complex space T with a distinguished point $t_0 \in T$ is a triple $(\mathscr{X}, F, i)$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] & \mathscr{X} \arrow[d, "F"] \arrow[dd, bend left=60, "\phi \text{ flat}"] \\
S \arrow[r, hook] \arrow[d] & S \times T \arrow[d, "\pi"] \\
t_0 \arrow[r, phantom, "\in"] & T
\end{tikzcd}$$

where i is a closed embedding and the composed morphism $\phi = \pi \circ F$: $\mathscr{X} \to T$ is flat ($S \hookrightarrow S \times T$ denotes the canonical embedding with image $S \times \{t_0\}$ and π the projection). If X is compact, we also require tht ϕ is proper. Two deformations $(\mathscr{X}, F, i)$ and $(\mathscr{X}', F', i')$ of X/S over T are isomorphic if there exists an isomorphism $\mathscr{X} \simeq \mathscr{X}'$ such that the obvious diagram (with the identity on $S \times T$) commutes. $\mathscr{D}_{X/S}$ denotes the functor from pointed complex spaces to sets defined by

$$\mathscr{D}_{X/S}(T) = \{\text{isomorphism classes of deformations of } X/S \text{ over } T\}.$$

Frequently we write $(\mathscr{X}, F)$ instead of $(\mathscr{X}, F, i)$, keeping in mind that the closed embedding $i\colon X \hookrightarrow \mathscr{X}$ is part of the data.

1.2. A deformation of X is a deformation of X/S with S the reduced point, and then we simply write $\mathscr{D}_X$. Note that a deformation of X/S, $(\mathscr{X}, F, i)$, induces via $(\mathscr{X}, \phi = F \circ \pi, i)$ a deformation of X and for any $x \in X$ this induces a deformation of the complex space germ (X, x). If this deformation of (X, x) happens to be trivial for all $x \in X$, we say that $(\mathscr{X}, F, i)$ induces a *locally trivial deformation* of X or by abuse of language that $(\mathscr{X}, F, i)$ is locally trivial (although we do not require that F is a locally trivial map). We are interested in the following subfunctor $\mathscr{D}'_{X/S}$ of $\mathscr{D}_{X/S}$:

$$\mathscr{D}'_{X/S}(T) = \{\text{elements of } \mathscr{D}_{X/S}(T) \text{ which induce a locally trivial deformation of } X\}.$$

Similarly one defines (locally trivial) deformations of X/S over complex space germs. Sometimes, in order to emphasize the difference to formal deformations, we speak of convergent deformations.

For $T = (\{t_0\}, A)$, A a complete local $\mathbb{C}$-algebra, there is the notion of a *formal deformation* (resp. *formal locally trivial deformation*) of X/S over T, meaning a sequence $\ldots \to (\mathscr{X}_n, F_n, i_n) \to (\mathscr{X}_{n-1}, F_{n-1}, i_{n-1}) \to \ldots$ with $(\mathscr{X}_n, F_n, i_n) \in \mathscr{D}_{X/S}(T_n)$ (resp. $\in \mathscr{D}'_{X/S}(T_n)$) where $T_n = (\{t_0\}, A/\mathfrak{m}^{n+1})$ is the n-th infinitesimal neighbourhood of $t_0 \in T$. For the notion of *(formal) versal* and *miniversal* (or *semi-universal*) deformations we refer to [Ar], [Bil] and [F12].

1.3. The aim of this section is to prove the following existence and openness result for $\mathscr{D}'_{X/S}$.

THEOREM: *Let f: $X \to S$ be a holomorphic map of complex spaces where X is compact. Then*

(*i*) *there exists a formal, formally miniversal locally trivial deformation of X/S.*

(*ii*) *for any convergent locally trivial deformation $(\mathscr{X}, F, i)$ of X/S over a complex space T, the set of points $t \in T$ where $(\mathscr{X}, F, i)$ is formally versal is Zariski-open in T.*

(*iii*) *If moreover X has only isolated singularities, then there exists a convergent miniversal locally trivial deformation of X/S and openness of versality holds for $\mathscr{D}'_{X/S}$, i.e., we may replace "formally versal" by "versal" in (ii).*

The infinitesimal deformation and obstruction theory for $\mathscr{D}'_{X/S}$ is studied in Section 3.

1.4. For the proof of 1.3 (iii) we need the following

LEMMA: *Let (X, x) be a complex germ with isolated singularity and $f\colon (\mathscr{X}, x) \to (S, s)$ any deformation of (X, x). Then the functor* Triv_f *from complex space germs to sets, given by*

$$\mathrm{Triv}_f(T, t) \quad = \quad \{\varphi\colon (T, t) \to (S, s) \,|\, \varphi^* f \text{ is a trivial deformation of } (X, x)\}$$

is representable by a closed subspace $(S', s) \in (S, s)$. Moreover, if (B, b) is the base space of the miniversal deformation of (X, x) and if $\psi\colon (S, s) \to (B, b)$ is any morphism which induces f via pull back, then $(S', s) = (\psi^{-1}(b), s)$.

Proof: We first have to check Schlessinger's conditions (cf. [Sch] or [Ar]) for Triv_f. But this follows easily from [Wa], Cor. 1.3.5.

Now, by Schlessinger's theorem, it follows that there exists a unique formal subspace $\hat{S}'$ of the completion $\hat{S} = (\{s\}, \hat{\mathcal{O}}_{S,s})$ such that for all Artinian complex space germs T and for any $\varphi \in \mathrm{Triv}_f(T)$, the completion $\hat{\varphi}$ factors through $\hat{S}' \subset \hat{S}$. Consider for a moment the case where $(S, s) = (B, b)$ is the base space of the miniversal deformation of (X, x). By the uniqueness property of miniversal deformations the Zariski tangent map of the closed embedding $\hat{S}' \subset \hat{S}$ is the zero-map, whence $\hat{S}'$ has to be the reduced point. The statement of the lemma now follows easily. □

1.5. Proof of Theorem 1.3: The main part of the proof, namely the study of infinitesimal deformations and obstructions, is done in Section 3. The statement in (ii) follows from proposition 3.8 and a criterion for openness of versality due to Artin ([Ar], theorem 4.4) in the algebraic category which was transferred by Bingener ([Bi], Satz 4.1) and Flenner ([F12]), Satz 4.3) to the analytic category.

Let us show 1.3 (iii): Since X is compact, there exists a miniversal deformation $(\mathscr{X}, F)$ of X/S over some complex germ (B, b), cf. [F11], Theorem 8.5. For each $x \in X$ let $f_x\colon (\mathscr{X}, x) \to (B, b)$ denote the deformation of (X, x) induced by $(\mathscr{X}, F)$. Because X has only isolated singularities, Triv_{f_x} is represented by a closed subspace $(B_x, b) \subset (B, b)$ by lemma 1.4. Let

$$(B', b) := \bigcap_{x \in X} (B_x, b)$$

and $(\mathscr{X}', F')$ the restriction of $(\mathscr{X}, F)$ to (B', b). It follows that $(\mathscr{X}', F')$ is a miniversal object for $\mathscr{D}'_{X/S}$. Since there exists a convergent miniversal deformation, formal versality is the same as versality by [F12], Satz 5.2, and openness of versality follows from (ii). □

2. Embedded deformations and the Hilbert functor

2.1. We now specialize to *locally trivial embedded deformations*, i.e., we consider the functor $\mathscr{D}'_{X/S}$ where $f\colon X \to S$ is a closed embedding. It is easy to see and follows from Nakayama's lemma that if $(\mathscr{X}, F)$ is a locally trivial deformation of X/S over (T, t_0) and X compact, then $F\colon \mathscr{X} \to S \times (T, t_0)$ is again a closed embedding. In this whole section we assume that X is compact and the induced deformations are proper. Then $\mathscr{D}'_{X/S}$ is just the local, locally trivial Hilbertfunctor and we write $\mathscr{H}'_{X/S}$ instead,

$$\mathscr{H}'_{X/S}(T, t_0) \quad = \quad \{\text{subspaces of } S \times (T, t_0)\text{, proper and locally trivial over } (T, t_0)\text{, inducing } f\colon X \hookrightarrow S \text{ over } t_0\}.$$

The openness and existence result (theorem 1.3) holds for $\mathscr{H}'_{X/S}$ if X has isolated singularities and we see from 3.2 that the tangent space of $\mathscr{H}'_{X/S}$ is equal to

$$\mathscr{H}'_{X/S}(\mathbb{D}) \quad = \quad H^0(X, \mathscr{N}'_{X/S}).$$

Here $\mathscr{N}'_{X/S} = \operatorname{Ker}(\mathscr{N}_{X/S} \to \mathscr{T}^1_X)$ and $\mathscr{N}_{X/S} = \mathscr{T}^1_{X/S} = (\mathscr{I}/\mathscr{I}^2)^*$ is the normal sheaf of $X \hookrightarrow S$ given by the ideal $\mathscr{I} \subset \mathscr{O}_S$; $\mathbb{D}$ denotes Spec of the dual numbers.

Concerning the obstructions, proposition 3.6 applies. In particular, if S is smooth then $\mathscr{H}'_{X/S}$ is formally unobstructed if $H^1(X, \mathscr{N}'_{X/S}) = 0$.

The same remarks apply if $f\colon X \to S$ is finite and generically a closed embedding.

2.2. Let S be a complex space. The Hilbert functor $\mathscr{H}_S$ on the category of complex spaces is defined by

$$\mathscr{H}_S(T) \quad = \quad \{\text{subspaces of } S \times T\text{, proper and flat over } T\}.$$

It is well known that $\mathscr{H}_S$ is representable by a complex space H_S (cf. [Bi2]). We define the *locally trivial Hilbertfunctor* $\mathscr{H}'_S$ to be the subfunctor of $\mathscr{H}_S$ by

$$\mathscr{H}'_S(T) \quad = \quad \{\text{subspaces of } S \times T\text{, proper, with finite singularities and locally trivial over } T\}$$

(finite singularities means that the fibres over T have only isolated singularities).

THEOREM: *The locally trivial Hilbertfunctor $\mathscr{H}'_S$ is representable by a complex space H'_S.*

This means that there exist a complex space H'_S and a universal family $\mathfrak{A} \subset S \times H'_S$, proper, with finite singularities and locally trivial over H'_S such that each element of $\mathscr{H}'_S(T)$, T a complex space, can be induced from $\mathfrak{A} \to H'_S$ via base change by a *unique* map $T \to H'_S$.

Proof: Use theorem 1.3, [Bi2], p. 339 (for the difference kernel) and apply the criterion [SV] Prop. 1.1 of Schuster and Vogt. □

2.3. Let X be a compact subspace of S with isolated singularities. X corresponds to a unique point of H'_S which we also denote by X. Then the germ of H'_S at X is the miniversal base space for the functor $\mathscr{H}'_{X/S}$. (Since $\mathscr{H}'_S$ is representable, it is even universal.) Therefore we can apply 2.1. Moreover we consider the functor $\mathscr{D}_{X,\mathrm{Sing}(X)}$ of deformations of the germ $(X, \mathrm{Sing}(X))$ and the natural forgetful map

$$\mathscr{H}_{X/S} \to \mathscr{D}_{X,\mathrm{Sing}(X)}.$$

PROPOSITION: *Let $f\colon X \hookrightarrow S$ be a closed embedding, X compact with isolated singularities. Then*
(*i*) $\mathrm{embdim}(H'_S, X) = \dim_{\mathbb{C}} H^0(X, \mathscr{N}'_{X/S})$.
(*ii*) *Assume S to be smooth. Then*

$$\dim H^0(X, \mathscr{N}'_{X/S}) - \dim H^1(X, \mathscr{N}'_{X/S})$$

$$\leqslant \dim(H'_S, X) \leqslant \dim H^0(X, \mathscr{N}'_{X/S}).$$

If $H^1(X, \mathscr{N}'_{X/S}) = 0$ then H'_S is smooth in X and $\mathscr{H}_{X/S} \to \mathscr{D}_{X,\mathrm{Sing}(X)}$ is smooth.

Remarks:
(1) The smoothness of $\mathscr{H}_{X/S} \to \mathscr{D}_{X,\mathrm{Sing}(X)}$ implies that any local deformation of the multigerm $(X, \mathrm{Sing}(X))$ is induced by a global embedded deformation of X in S. Moreover, the miniversal base spaces of $\mathscr{H}_{X/S}$ and $\mathscr{D}_{X,\mathrm{Sing}(X)}$ differ by a smooth factor of dimension $\dim H^0(X, \mathscr{N}'_{X/S})$.
(2) Of course, instead of assuming S smooth, we need only $f(X) \subset S - \mathrm{Sing}(S)$. Moreover the proposition is true in the same manner for $f\colon X \to S$ finite and generically a closed embedding ($\mathscr{N}'_{X/S}$ has to be replaced by $\mathscr{T}'^{1}_{X/S}$, cf. 3.2).

Proof: (ii) The assumptions imply $\mathcal{T}^0_{X/S} = 0$, $\mathrm{Ext}^i(Lf^*\mathcal{L}^{\cdot}_S, \mathcal{O}_X) = 0$ if $i > 0$, and hence reduce the exact sequences of 3.1 and 3.2 to

$$0 \to \mathcal{T}^0_X \to f^*\mathcal{T}^0_S \to \mathcal{T}^1_{X/S} \to \mathcal{T}^1_X \to 0,$$

$$\mathcal{T}^i_{X/S} \xrightarrow{\simeq} \mathcal{T}^i_X, \quad i \geqslant 2, \; T^1_{X/S} \simeq H^0(X, \mathcal{T}^1_{X/S}).$$

The local to global exact sequence reads

$$0 \to H^1(X, \mathcal{T}^1_{X/S}) \to T^2_{X/S}$$

$$\to H^0(X, \mathcal{T}^2_{X/S}) \to H^2(X, \mathcal{T}^1_{X/S}) \to T^3_{X/S}.$$

Since $\mathcal{N}'_{X/S} = \mathrm{Ker}(\mathcal{T}^1_{X/S} \to \mathcal{T}^1_X)$ and since $\mathcal{T}^1_X$ is concentrated on the finitely many singular points of X, the vanishing of $H^1(X, \mathcal{N}'_{X/S})$ implies $H^1(X, \mathcal{T}^1_{X/S}) = 0$. Therefore

$$T^1_{X/S} \cong H^0(X, \mathcal{T}^1_{X/S}) \to H^0(X, \mathcal{T}^1_X) \cong T^1_{X,\mathrm{Sing}(X)}$$

is surjective and

$$T^2_{X/S} \to H^0(X, \mathcal{T}^2_{X/S}) \cong H^0(X, \mathcal{T}^2_X) \cong T^2_{X,\mathrm{Sing}(X)}$$

is injective. This implies that $\mathcal{H}_{X/S} \to \mathcal{D}_{X,\mathrm{Sing}(X)}$ is formally smooth, but since both functors admit miniversal objects, this is equivalent to smoothness.

The second estimate for $\dim(H'_S, X)$ follows from (i). The first is due to the fact that (H'_S, X) is the fibre over the origin of a (non-linear) obstruction map $H^0(X, \mathcal{N}'_{X/S}) \to H^1(X, \mathcal{N}'_{X/S})$ (cf. [La] Theorem 4.2.4).

2.4. For hypersurfaces in $\mathbb{P}^n$ with only simple singularities we can show the algebraicity of the locally trivial Hilbert functor.

Let $\sigma = \{(X_1, 0), \ldots, (X_r, 0)\}$ be any finite set of complex space germs. We say that a complex space X is of *singularity type* σ if X has exactly r singular points $x_1, \ldots, x_r$ such that for all i (X, x_i) is analytically isomorphic to $(X_i, 0)$. X is called of *simple singularity type* if it is of singularity type σ where σ consists of finitely many simple hypersurface singularities. (The simple $(n-1)$-dimensional hypersurface singularities are given by the following local equations: A_k: $x_1^{k+1} + x_2^2 + q(x') = 0$ $(k \geqslant 1)$, D_k: $x_1(x_1^{k-2} + x_2^2) + q(x') = 0$ $(k \geqslant 4)$, E_6: $x_1^3 + x_2^4 + q(x') = 0$, E_7: $x_1(x_1^2 + x_2^3) + q(x') = 0$, E_8: $X_1^3 + x_2^5 + q(x') = 0$, where $q(x') = x_3^2 + \cdots + x_n^2$.)

We define $\mathscr{I}_{d,\sigma}$ to be the functor (on the category of complex algebraic varieties).

$\mathscr{I}_{d,\sigma}(T) :=$ {relative effective Cartier divisors $\mathscr{X} \subset \mathbb{P}^n \times T$, analytically locally trivial over T, such that each fibre of $\mathscr{X} \to T$ is a hypersurface in $\mathbb{P}^n$ of degree d and of singularity type σ}.

PROPOSITION: *For any simple singularity type σ, the functor $\mathscr{I}_{d,\sigma}$ is representable by an algebraic variety $\mathscr{I}_{d,\sigma}$ which is a disjoint union of quasiprojective subvarieties of $\mathbb{P}^N$, $N = [d(d + n + 1)]/2$.*

Proof. Same as in [Wa], Theorem 3.3.5, for plane curves with nodes and cusps, noting that a deformation $\mathscr{X} \to T$ of a hypersurface with only simple singularities is locally trivial at $t \in T$ iff $T^1_{\mathscr{X}/T}$ is flat over T at t and using the fact that simpleness is a Zariski-open condition. □

3. Infinitesimal deformations and obstructions

3.1. We now introduce infinitesimal deformation and obstruction spaces for locally trivial deformations. Let f: $X \to S$ be any holomorphic map of complex spaces. $\mathbb{D}$ denotes Spec of the dual numbers.

For any $\mathcal{O}_X$-module $\mathscr{F}$, let

$$T^i_{X/S}(\mathscr{F}) := \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{L}^{\cdot}_{X/S}, \mathscr{F}),$$

$$\mathscr{T}^i_{X/S}(\mathscr{F}) := \mathscr{E}xt^i_{\mathcal{O}_X}(\mathscr{L}^{\cdot}_{X/S}, \mathscr{F})$$

denote the global, resp. local cotangent cohomology of X/S with values in $\mathscr{F}$, where $\mathscr{L}^{\cdot}_{X/S}$ is the cotangent complex of X/S (cf. [Pa], [F11] and for a short summary also [Bi1]). Recall that

$$\mathscr{D}_{X/S}(\mathbb{D}) = T^1_{X/S}(\mathcal{O}_X) =: T^1_{X/S},$$

$$\mathscr{D}_{(X,x)/(S,f(x))}(\mathbb{D}) = \mathscr{T}^1_{X/S,x}(\mathcal{O}_{X,x}) =: \mathscr{T}^1_{X/S,x},$$

and that deformations of X/S resp. of the germ $(X, x)/(S, f(x))$ are obstructed by elements in $T^2_{X/S}$ resp. $\mathscr{T}^2_{X/S,x}$. Again, for S the reduced point we delete the subscript S and write T^i_X, $\mathscr{T}^i_{X,x}$ etc. There are natural morphisms

$$T^i_{X/S}(\mathscr{F}) \to T^i_X(\mathscr{F})$$

which fit into an exact sequence

$$0 \to T^0_{X/S}(\mathscr{F}) \to T^0_X(\mathscr{F}) \to \mathrm{Ext}^0_{\mathscr{O}_X}(Lf^*\mathscr{L}^{\cdot}_S, \mathscr{F}) \to T^1_{X/S}(\mathscr{F}) \to T^1_X(\mathscr{F}) \to \,.\,.$$

where $T^0_{X/S}(-) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, -)$ and $T^0_X(-) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_X, -)$. Similarly for the sheaves $\mathscr{T}^i_{X/S}(\mathscr{F}) \to \mathscr{T}^i_X(\mathscr{F})$. The local and the global cotangent cohomology is connected by a spectral sequence

$$E_2^{pq} = H^p(X, \mathscr{T}^q_{X/S}(\mathscr{F})) \Rightarrow T^{p+q}_{X/S}(\mathscr{F}).$$

3.2. We now define

$$\mathscr{T}'^1_{X/S}(\mathscr{F}) := \mathscr{K}er(\mathscr{T}^1_{X/S}(\mathscr{F}) \to \mathscr{T}^1_X(\mathscr{F})),$$

$$T'^1_{X/S}(\mathscr{F}) := \mathrm{Ker}(T^1_{X/S}(\mathscr{F}) \to T^1_X(\mathscr{F}) \to H^0(X, \mathscr{T}^1_X(\mathscr{F}))).$$

The morphism $T^1_X(\mathscr{F}) \to H^0(X, \mathscr{T}^1_X(\mathscr{F}))$ occurs in the following commutative diagram with exact rows, which results from the first terms of the local to global spectral sequence:

$$\begin{array}{ccccccccccc} 0 \to & H^1(X,\mathscr{T}^0_{X/S}(\mathscr{F})) & \to & T^1_{X/S}(\mathscr{F}) & \to & H^0(X,\mathscr{T}^1_{X/S}(\mathscr{F})) & \to & H^2(X,\mathscr{T}^0_{X/S}(\mathscr{F})) & \to & T^2_{X/S}(\mathscr{F})) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^1(X,\mathscr{T}^0_X(\mathscr{F})) & \to & T^1_X(\mathscr{F}) & \to & H^0(X,\mathscr{T}^1_X(\mathscr{F})) & \to & H^2(X,\mathscr{T}^0_X(\mathscr{F})) & \to & T^2_X(\mathscr{F}). \end{array}$$

From this we deduce the exact sequence

$$0 \to H^1(X, \mathscr{T}^0_{X/S}(\mathscr{F})) \to T'^1_{X/S}(\mathscr{F}) \to H^0(X, \mathscr{T}'^1_{X/S}(\mathscr{F})) \to K \qquad (*)$$

where $K = \mathrm{Ker}(H^2(X, \mathscr{T}^0_{X/S}(\mathscr{F})) \to H^2(X, \mathscr{T}^0_X(F)))$.
Since the $\mathscr{T}^i_{X/S}$ are coherent functors, $\mathscr{T}'^1_{X/S}$ is a coherent functor and $\mathscr{T}'^1_{X/S}(\mathscr{F})$ is the sheafification of $T'^1_{X/S}$ for coherent $\mathscr{F}$. Moreover, it follows immediately from the definitions and the corresponding properties of the $\mathscr{T}^i$ that $T'^1_{X/S}$ resp. $\mathscr{T}'^1_{X/S,x}$ are the vector spaces of first order locally trivial deformations of X/S resp. of $(X, x)/(S, f(x))$, i.e.,

$$\mathscr{D}'_{X/S}(\mathbb{D}) = T'^1_{X/S}(\mathscr{O}_X) =: T'^1_{X/S}$$

$$\mathscr{D}'_{(X,x)/(S,f(x))}(\mathbb{D}) = \mathscr{T}'^1_{X/S,x}(\mathscr{O}_{X,x}) =: \mathscr{T}'^1_{X/S,x}.$$

3.3. More generally, for any morphism $F\colon \mathscr{X} \to S \times T$ over T and any $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{F}$ we define

$$\mathscr{T}'^{1}_{\mathscr{X}/S\times T/T}(\mathscr{F}) := \mathscr{K}er(\mathscr{T}^{1}_{\mathscr{X}/S\times T}(\mathscr{F}) \to \mathscr{T}^{1}_{\mathscr{X}/T}(\mathscr{F})),$$

$$T'^{1}_{\mathscr{X}/S\times T/T}(\mathscr{F}) := \operatorname{Ker}(T^{1}_{\mathscr{X}/S\times T}(\mathscr{F}) \to H^0(\mathscr{X}, \mathscr{T}^{1}_{\mathscr{X}/T}(\mathscr{F})).$$

For T the reduced point, these definitions reduce to those of 3.2. Moreover, the diagram and the exact sequence (*) of 3.2 generalize. In particular we obtain the exact sequence

$$0 \to H^1(\mathscr{X}, \mathscr{T}^{0}_{\mathscr{X}/S\times T}(\mathscr{F})) \to T'^{1}_{\mathscr{X}/S\times T/T}(\mathscr{F}) \to H^0(\mathscr{X}, \mathscr{T}'^{1}_{\mathscr{X}/S\times T/T}(\mathscr{F})) \qquad (*)$$

$$\to H^2(\mathscr{X}/\mathscr{T}^{0}_{\mathscr{X}/S\times T}(\mathscr{F})) \to H^2(\mathscr{X}, \mathscr{T}^{0}_{\mathscr{X}/T}(\mathscr{F})).$$

Consider now a diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "F"] \arrow[dr, "\phi"'] & S \times T \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

which is a locally trivial deformation of X/S over T and which we abbreviate by $a = (\mathscr{X}, F)$. For $\mathscr{M}$ a coherent $\mathcal{O}_T$-module, let $T[\mathscr{M}]$ denote the trivial extension of T by $\mathscr{M}$, i.e., $\mathcal{O}_{T[\mathscr{M}]} = \mathcal{O}_T \oplus \mathscr{M}$ with $\mathscr{M}^2 = 0$. Following Artin [Ar] we let

$D'_a(\mathscr{M})$ = {locally trivial deformations of X/S over $T[\mathscr{M}]$ which extend a}
modulo isomorphisms which leave a fixed.

In particular, if $a_0 = (X, f, id)$ denotes the constant "deformation" over the reduced point, then $D'_{a_0}(\mathbb{C}) = \mathscr{D}'_{\mathscr{X}/S}(\mathbb{D})$. Likewise $D_{a_0}(\mathbb{C}) = \mathscr{D}_{X/S}(\mathbb{D})$, where

$D_a(\mathscr{M})$ = {deformations of $\mathscr{X}/S$ over $T[\mathscr{M}]$ which extend a}
modulo isomorphism which leave a fixed.

By [F11], 3.20 or [Bi1], 5.17,

$$D_a(\mathscr{M}) \simeq T^{1}_{\mathscr{X}/S\times T}(\phi^*\mathscr{M})$$

as $\Gamma(T, \mathcal{O}_T)$-module, where, as above, $\phi = \pi \circ F\colon \mathcal{X} \to T$. We have to consider the sheafification of $D'_a(\mathcal{M})$ which is the sheaf $\mathcal{D}'_a(\mathcal{M})$ on T associated to the presheaf

$$T \supset U \to D'_{a|U}(\mathcal{M}|U)$$

and similarly for $\mathcal{D}_a(\mathcal{M})$ (which is isomorphic to the relative Ext-sheaf $\mathcal{E}xt^1(\phi;\ \mathcal{L}^{\cdot}_{\mathcal{X}/S\times T}, \phi^*\mathcal{M})$).

3.4. LEMMA:
(i) $D'_a(\mathcal{M}) \cong T'^1_{\mathcal{X}/S\times T/T}(\phi^*\mathcal{M})$,
(ii) $\mathcal{D}'_a(\mathcal{M})$ *is a coherent* $\mathcal{O}_T$*-sheaf if* ϕ *is proper.*

Proof: (i) Let $\bar{a} = (\mathcal{X}, \phi)$ be the deformation of X induced by a and for $x \in \mathcal{X}$, $(\bar{a}, x)$ denotes the induced deformation of $(\phi^{-1}(\phi(x)), x)$ (which is trivial, since a was assumed to be locally trivial). The groups $D_{\bar{a}}(\mathcal{M})$ and $D_{(\bar{a},x)}(\mathcal{M})$ are defined in the same manner as above. Note that $D_{(\bar{a},x)}(\mathcal{M})$ is the stalk at x of the sheaf $\tilde{\mathcal{D}}_{\bar{a}}(\mathcal{M})$ on $\mathcal{X}$ associated to the presheaf

$$\mathcal{X} \supset U \to D_{\bar{a}|U}(\mathcal{M}).$$

$\bar{a}/U$ is the restriction $\phi/U\colon U \to T$. Note that $\tilde{\mathcal{D}}_{\bar{a}}(\mathcal{M})$ is just a $\phi^{-1}\mathcal{O}_T$-module and as such it is isomorphic to $\mathcal{T}^1_{\mathcal{X}/T}(\phi^*\mathcal{M})$ by 3.3.

The associations $a \to \bar{a} \to (\bar{a}, x)$ induce canonical homomorphisms

$$D_a(\mathcal{M}) \to \tilde{D}_{\bar{a}}(\mathcal{M}) \to H^0(\mathcal{X}, \tilde{\mathcal{D}}_{\bar{a}}(\mathcal{M}))$$

and a deformation b of X/S over $T[\mathcal{M}]$ which extends a is locally trivial iff it is in the kernel of the composed map. Using the identifications of 3.3, the sequence reads

$$T^1_{\mathcal{X}/S\times T}(\phi^*\mathcal{M}) \to T^1_{\mathcal{X}/T}(\phi^*\mathcal{M}) \to H^0(\mathcal{X}, \mathcal{T}^1_{\mathcal{X}/T}(\phi^*\mathcal{M})).$$

Since $T'^1_{\mathcal{X}/S\times T/T}(\phi^*\mathcal{M})$ is by definition the kernel of this composition, (i) follows.

(ii) $\mathcal{D}'_a(\mathcal{M})$ is the sheaf associated to $U \to T'^1_{\mathcal{X}/S\times U/U}(\phi^*\mathcal{M}|_U)$. Therefore, sheafifying the exact sequence 3.2 (*) over T, we obtain an exact sequence

$$0 \to R^1\phi_*(\mathcal{T}^0_{\mathcal{X}/S\times T}(\phi^*\mathcal{M})) \to \mathcal{D}'_a(\mathcal{M})$$

$$\to \phi_*\mathcal{T}'^1_{\mathcal{X}/S\times T/T}(\phi^*\mathcal{M}) \to R^2\phi_*(\mathcal{T}^0_{\mathcal{X}/S\times T}(\phi^*\mathcal{M})).$$

Since $\mathcal{T}^0_{\mathcal{X}/S\times T}$ and $\mathcal{T}'^1_{\mathcal{X}/S\times T/T}$ are coherent functors and since ϕ is proper, $\mathcal{D}'_a(\mathcal{M})$ is coherent. □

3.5. In order to describe the obstructions for the functor $\mathcal{D}'_{X/S}$ we have not only to consider isomorphism classes of deformations but also deformations itself. Let S be a fixed complex space and $G = G^S$ the *groupoid of locally trivial deformations* over S which is defined as follows: It is the fibered groupoid over the category of complex spaces whose objects over T are holomorphic maps $F\colon \mathcal{X} \to S \times T$ over T such that $\phi = \pi \circ F\colon \mathcal{X} \to T$ is locally trivial ($\pi\colon S \times T \to T$ the projection). A morphism between $F\colon \mathcal{X} \to S \times T$ and $F'\colon \mathcal{X}' \to S \times T'$ is a cartesian diagram, where the base map $S \times T \to S \times T'$ is of the form $id \times \phi$, $\phi\colon T \to T'$. It follows from [Wa], Cor. 1.3.5 that G satisfies Schlessingers conditions, even the stronger condition (S_1') (cf. [Ar], [Bi1], [F1]) which we shall need. For $a \in G$ we shall use the usual notation G_a for the groupoid of morphisms $a \to a'$ in G, while $\bar{G}$ resp. $\bar{G}_a$ etc. denotes the corresponding set of isomorphism classes.

Now let $T_0 \subset T$ be a closed subspace such that $(T_0)_{\mathrm{red}} = T_{\mathrm{red}}$ and $\mathcal{M}$ a coherent $\mathcal{O}_{T_0}$-module. For an object $a = (\mathcal{X}, \mathcal{F}) \in G(T)$ let $\mathrm{Ex}(a, \mathcal{M})$ denote the set of isomorphism classes of locally trivial extensions of a by $\mathcal{M}$. Such an extension is a pair (a', T') where T' is an extension of T by $\mathcal{M}$ and $a' \in G(T')$ is an extension $F'\colon \mathcal{X}' \to S \times T'$ of a. There is a natural map $D'_a(\mathcal{M}) \xrightarrow{i} \mathrm{Ex}(a, \mathcal{M})$ such that the image of i is just the kernel of the projection $\mathrm{Ex}(a, \mathcal{M}) \to T^1_T(\mathcal{M})$, $(a', T') \mapsto T'$ (cf. [F12], (2.3)). The cokernel of this map is called the module of obstructions $\mathrm{Ob}_a(\mathcal{M})$. Altogether we have an exact sequence of $H^0(T, \mathcal{O}_T)$-modules

$$D'_a(\mathcal{M}) \xrightarrow{i} \mathrm{Ex}(a, \mathcal{M}) \longrightarrow T^1_T(\mathcal{M}) \longrightarrow \mathrm{Ob}_a(\mathcal{M}) \longrightarrow 0$$

([F12], loc. cit.).

For the explicit description of the image of $\mathrm{Ex}(a, \mathcal{M})$ in $T^1_T(\mathcal{M})$ we define the following sheaves on $\mathcal{X}$:
$\widetilde{\mathrm{Ex}}(a, \mathcal{M})$, $\widetilde{\mathrm{Ob}}_a(\mathcal{M})$ and $\tilde{D}'_a(\mathcal{M})$ are the sheaves associated to the presheaves.

$$U \mapsto \mathrm{Ex}(a|_U, \mathcal{M}),\ \mathrm{Ob}_{a|U}(\mathcal{M}) \quad \text{and} \quad D'_{a|U}(\mathcal{M}),$$

respectively, where $U \subset \mathcal{X}$ is open and $a|_U$ is the restriction $F|_U\colon U \to S \times T$. Note that by 3.4

$$\tilde{D}'_a(\mathcal{M}) \cong \mathcal{T}'^1_{\mathcal{X}/S\times T/T}(\phi^*\mathcal{M}).$$

It follows from [I1], III.2.2 that

$$\widetilde{\mathrm{Ob}}_{\mathscr{a}}(\mathscr{M}) \hookrightarrow \mathscr{E}xt^1_{\mathscr{O}_{T0}}(L\psi_0^*\mathscr{L}^{\cdot}_S, \phi_0^*\mathscr{M})$$

where $\psi: \mathscr{X} \to S$ is the composition of $F: \mathscr{X} \to S \times T$ with the projection on S. The index 0 indicates restriction to T_0. We have to consider also $A_{\mathscr{a}}(\mathscr{M})$, the $H^0(T, \mathscr{O}_T)$-module of $T[\mathscr{M}]$-automorphisms of the trivial extension $\mathscr{a}[\mathscr{M}]$ of $\mathscr{a}$ over $T[\mathscr{M}]$. Taking derivatives, we get an isomorphism

$$A_{\mathscr{a}}(\mathscr{M}) \cong T^0_{\mathscr{X}/S\times T}(\phi^*\mathscr{M}).$$

The presheaf $\mathscr{X} \supset U \mapsto A_{\mathscr{a}|U}(\mathscr{M})$ generates a sheaf on $\mathscr{X}$ which is isomorphic to $\mathscr{T}^0_{\mathscr{X}/S\times T}(\phi^*\mathscr{M})$.

3.6. Consider first the canonical map

$$\mathrm{ob}_1\colon T^1_T(\mathscr{M}) \to H^0(\widetilde{\mathrm{Ob}}_{\mathscr{a}}(\mathscr{M})) \hookrightarrow H^0(\mathscr{X}_0, \mathscr{E}xt^1_{\mathscr{O}_{T0}}(L\psi_0^*\mathscr{L}^{\cdot}_S, \phi_0^*\mathscr{M}).$$

Let T' be an infinitesimal extension of T by $\mathscr{M}$ (hence $\mathscr{M}^2 = 0$). We shall find necessary and sufficient conditions for $[T']$ to be in the image of $\mathrm{Ex}(\mathscr{a}, \mathscr{M})$. Here and in the following [ ] denotes equivalence classes. We have $\mathrm{ob}_1[T'] = 0$ iff there is an open Stein covering $\{U_i\}$ of $\mathscr{X}$ such that $\mathscr{a}|U_i$ can be lifted to an object $\mathscr{a}'_i \in G_{\mathscr{a}|U_i}(T')$ for all i.

Since the groupoid G satisfies (S'_1), the additive group of $D'_{\mathscr{a}0}(\mathscr{M})$ acts effectively and transitively on $\bar{G}_{\mathscr{a}}(T')$ (cf. [Sch], p. 213). Thus, the elements $\mathscr{a}'_i \in G_{\mathscr{a}|U_i}(T')$ determine a Cech-cocycle (d_{ij}) of $\tilde{D}'_{\mathscr{a}0}(\mathscr{M})$ such that $[\mathscr{a}'_i|U_i \cap U_j] = d_{ij} \cdot [\mathscr{a}'_j|U_i \cap U_j]$ in $\bar{G}_{\mathscr{a}|U_i\cap U_j}(T')$.
(Note that $H^0(U, \tilde{D}'_{\mathscr{a}0}(\mathscr{M})) = D'_{\mathscr{a}0|U}(\mathscr{M})$ for U Stein by 3.4). Since the class $[(d_{ij})]$ of (d_{ij}) in $H^1(\mathscr{X}_0, \tilde{D}'_{\mathscr{a}0}(\mathscr{M}))$ is independent of the local liftings $\mathscr{a}'_i$, we obtain a map

$$\mathrm{ob}_2\colon \ker(\mathrm{ob}_1) \to H^1(\mathscr{X}_0, \mathscr{T}'^1_{X_0/S\times T_0/T_0}(\phi_0^*(\mathscr{M})),$$

$$[T'] \mapsto [(d_{ij})].$$

$\mathrm{ob}_1[T'] = \mathrm{ob}_2[T'] = 0$, implies the existence of an open Stein covering $\{U_i\}$ of $\mathscr{X}$ and local liftings $a'_i \in G_{a|U_i}(T')$ such that $[a'_i|U_i \cap U_j] = [a'_j|U_i \cap U_j]$. Hence there are isomorphisms

$$\varphi_{ij}\colon a'_j|U_i \cap U_j \to a'_i|U_i \cap U_j.$$

Since $A_{a0|U_i \cap U_j}$ acts transitively and effectively on such isomorphisms, the φ_{ij} determine a 2-Cech-cocycle (α_{ijk}) of $\mathscr{T}^0_{\mathscr{X}/S\times T_0}(\phi_0^*\mathscr{M})$. Note that the class of (α_{ijk}) in $H^2(\mathscr{X}, \mathscr{T}^0_{X/S\times T_0}(\phi_0^*\mathscr{M}))$ depends on the local liftings a'_j. (We are grateful to J. Bingener for pointing this out to us.) Different liftings a''_i differ from a'_i by an element $[d_i]$ of $D'_{a0|U_i} = H^0(U_i, \tilde{\mathscr{D}}'_{a0}(\mathscr{M}))$, $d_i \in G_{a0|U_i}(T_0[\mathscr{M}])$. As above, these d_i are isomorphic on double intersections and determine therefore on triple intersections a 2-Cech-cocycle of $\mathscr{T}^0_{\mathscr{X}/S\times T}(\phi^*\mathscr{M})$ whose cohomology class is determined by the $[d_i]$. This defines a map

$$\mathrm{o}\colon H^0(\mathscr{X}, \tilde{D}'_{a0}(\mathscr{M})) \to H^2(\mathscr{X}, \mathscr{T}^0_{\mathscr{X}/S\times T_0}(\phi_0^*\mathscr{M})),$$

and

$$\mathrm{ob}_3\colon \ker(\mathrm{ob}_2) \to H^2(\mathscr{X}, \mathscr{T}^0_{\mathscr{X}/S\times T_0}(\phi_0^*\mathscr{M}))/\mathrm{o}(H^0(\mathscr{X}, \mathscr{T}'^1_{\mathscr{X}/S\times T_0/T_0}(\phi_0^*\mathscr{M}))$$

$[T'] \mapsto [(\alpha_{ijk})]$, is well defined. Moreover $\mathrm{ob}_3([T']) = 0$ iff the local liftings $[a'_i] \in \mathrm{Ex}(a|U_i, \mathscr{M})$ glue to a global lifting $[a'] \in \mathrm{Ex}(a, \mathscr{M})$.

3.7. We say that $D'_{X/S}$ is *formally unobstructed*, if any locally trivial deformation over an Artinian base T is locally trivial liftable over infinitesimal extensions T' of T. Since every infinitesimal extension factors through extensions with $\mathscr{M}^2 = 0$, and since $\mathscr{T}^i_{X/S}(f^*\mathscr{M}) \simeq \mathscr{T}^i_{X/S} \otimes_{\mathbb{C}} \mathscr{M}$ for finite dimensional vector-spaces $\mathscr{M}$ (and similar for the other sheaves above), we have shown

PROPOSITION: *For any morphism $f\colon X \to S$ of complex spaces, $D'_{X/S}$ is formally unobstructed, if the following holds*:

(*i*) $H^0(X, \mathscr{E}xt^1_{\mathscr{O}_X}(Lf^*L^{\cdot}_S, \mathscr{O}_X)) = 0$

(*ii*) $H^1(X, \mathscr{T}'^1_{X/S}) = 0$

(*iii*) $H^2(X, \mathscr{T}^0_{X/S}) = 0.$

Remarks: (1) If $D'_{X/S}$ admits a miniversal deformation (e.g. X compact with isolated singularities by 1.3), then $D'_{X/S}$ is formally unobstructed iff the base space of the miniversal deformation is smooth.

(2) (i) holds if S is smooth since $\mathscr{E}xt^i_{\mathcal{O}_X}(Lf^*L^{\cdot}_S, \mathcal{O}_X)$ is concentrated on $F^{-1}(\mathrm{Sing}(S))$ for $i \geqslant 1$. (iii) holds if $f: X \to S$ is finite and generically smooth or a closed embedding since in both cases the sheaf of relative vectorfields $\theta_{X/S} = \mathscr{T}^0_{X/S}$ vanishes.

3.8. Note that the obstruction morphisms defined in 3.6 yield $\mathcal{O}_T$-module homomorphisms

$$ob_1: \mathscr{T}^1_T(\mathscr{M}) \to \phi_* \mathscr{E}xt^1_{\mathcal{O}_{\mathscr{X}_0}}(L\psi_0^* \mathscr{L}^{\cdot}_S, \phi_0^* \mathscr{M}),$$

$$ob_2: \mathscr{K}er(ob_1) \to R^1\phi_*(\mathscr{T}'^1_{\mathscr{X}_0/S \times T_0/T_0}(\phi_0^* \mathscr{M})),$$

$$ob_3: \mathscr{K}er(ob_2) \to R^2\phi_*(\mathscr{T}^0_{\mathscr{X}/S \times T_0}(\phi_0 \mathscr{M}))/o\phi_* \mathscr{T}'^1_{\mathscr{X}/S \times T_0/T_0}(\phi_0^* \mathscr{M}).$$

Since ϕ is proper, it follows from the coherence of $\mathscr{T}'$, $\mathscr{E}xt^1$ and $\mathscr{T}'^1$ (cf. 3.4) that all sheaves are coherent. In particular

$$\mathcal{O}b_a(\mathscr{M}) := \mathscr{T}^1_T(\mathscr{M})/\mathscr{K}er(ob_3)$$

is coherent. Sheafification on T of the exact sequences of 3.5. yields therefore an exact sequence of coherent $\mathcal{O}_T$-modules

$$\mathscr{T}^0_T(\mathscr{M}) \longrightarrow \mathscr{D}'_a(\mathscr{M}) \longrightarrow \mathscr{E}x(a, \mathscr{M}) \longrightarrow \mathscr{T}^1_T(\mathscr{M}) \xrightarrow{ob} \mathcal{O}b_a(\mathscr{M}) \longrightarrow 0.$$

PROPOSITION: *$\mathcal{O}b_a(-)$ and ob define a good obstruction theory for the groupoid of locally trivial deformations $\mathscr{D}'$.*

An obstruction theory is to be understood in the sense of [Ar], [Bi] or [F12]. We use here its most convenient form as formulated in [F12], Section 4. "Good" means that it satisfies Flenner's condition (S3) (coherence) and (S4) (cnstructibility).

Proof: Everything except the constructibility has already been shown. For the functor $\mathcal{O}b_a(-)$ this means that the canonical morphisms

$$\mathcal{O}b_a(\mathscr{M})_t \otimes \mathbb{C}_t \to \mathcal{O}b_a(\mathscr{M} \otimes \mathbb{C}_t)_t$$

is generically, on the support of $\mathscr{M}$, an isomorphism. But this is true since an analogous statement for $\mathscr{K}er(ob_3)$ and $\mathscr{T}^1_T(\mathscr{M})$ holds. The constructibility for $\mathscr{D}'_a(-)$ follows from the last exact sequence in the proof of Lemma 3.4.

4. Various deformation functors for curves

4.1. By a curve we always mean a reduced, compact and pure 1-dimensional complex space. Let C be a curve on a complex manifold S, $j: C \hookrightarrow S$ the inclusion, $n: \tilde{C} \to C$ the normalization of C and $f = j \circ n$ such that the following diagram commutes

$$\begin{tikzcd} \tilde{C} \arrow[d, "n"'] \arrow[dr, "f"] & \\ C \arrow[r, hook, "j"'] & S \end{tikzcd} .$$

We consider the functors $\mathcal{D}_{C/S}$, $\mathcal{D}'_{C/S}$, $\mathcal{D}_{\tilde{C}/S}$ and $\mathcal{D}_{\tilde{C}/C}$. Note that $\mathcal{D}_{\tilde{C}/S}$ corresponds to deformations of $\tilde{C}$ and of f, with varying image in S, while the deformations corresponding to $\mathcal{D}_{\tilde{C}/C}$ leave C fixed. $\mathcal{D}_{C/S}$ resp. $\mathcal{D}'_{C/S}$ correspond to embedded, resp. locally trivial embedded deformations of $C \subset S$.

4.2. LEMMA: *There are natural transformations*

$$\begin{tikzcd} & \mathcal{D}'_{C/S} \arrow[d, "L"] \\ \mathcal{D}_{\tilde{C}/C} \arrow[r, "G"'] & \mathcal{D}_{\tilde{C}/S} \end{tikzcd}$$

such that for any locally trivial deformation a of C/S, $L(a) \in \mathrm{Im}(G)$ if and only if a is globally trivial.

Proof: Let $\tilde{\mathcal{C}} \to C \times T$ be a deformation of $\tilde{C}/C$. Then the composition $\tilde{\mathcal{C}} \to C \times T \hookrightarrow S \times T$ defines an element of $\mathcal{D}_{\tilde{C}/S}(T)$. This gives G.

For the definition of L consider a locally trivial deformation a of C/S given by $F: \mathscr{C} \hookrightarrow S \times T$. For each singular point $x_i \in C$, $i = 1, \ldots, r$, choose small open Stein neighbourhoods U_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Let V_i be an open neighbourhood of x_i s.t. $\bar{V}_i \subset U_i$ and define $U_0 := C - \bigcup_{i=1}^{r} \bar{V}_i$ such that $U_0, \ldots, U_r$ cover C. Since the deformation is locally trivial, the family $\mathscr{C} \to T$ arises by patching the $U_i \times T$ together via transition functions, given by a cocycle $(g_{ij}) \in Z^1(\{U_i\}, \mathscr{G}_T)$ where $\mathscr{G}_T$ is the sheaf of relative automorphisms of C over (T, t_0). Note that $U_i \cap U_j \cap U_k = \emptyset$ for $i \neq j \neq k$.
Then $\{\tilde{U}_i := n^{-1}(U_i)\}$ defines an open Stein covering and satisfies $\tilde{U}_i \cap \tilde{U}_j \cap \tilde{U}_k = \emptyset$ for $i \neq j \neq k$. Since the restriction $n_{ij} = n: \tilde{U}_i \cap \tilde{U}_j \to U_i \cap U_j$ is biholomorphic, the

$$\tilde{g}_{ij} := (n_{ij} \times id_T)^{-1} \circ g_{ij} \circ (n_{ij} \times id_T)$$

define a cocycle $(\tilde{g}_{ij}) \in Z^1(\{\tilde{U}_i\}, \tilde{\mathscr{G}}_T)$ and hence a flat family $\tilde{C} \to T$. Let $\tilde{n}_i := n|_{\tilde{U}_i} \times id_T$. These local maps fit together to a map $\tilde{n}: \tilde{\mathscr{C}} \to \mathscr{C}$ over T. $L(\alpha)$ is then represented by $F \circ \tilde{n}: \tilde{\mathscr{C}} \to S \times T$.

The second assertion is an immediate consequence of the construction.

□

4.3. The long exact sequence of cotangent complexes associated to $\tilde{C} \xrightarrow{n} C \xhookrightarrow{j} S$ reduces to

$$0 \to \mathscr{T}^1_{\tilde{C}/C} \to \mathscr{T}^1_{\tilde{C}/S} \to \mathrm{Ext}^1_{\mathcal{O}_{\tilde{C}}}(Ln^*\mathscr{L}^{\cdot}_{C/S}, \mathcal{O}_{\tilde{C}}) \to \mathscr{T}^2_{\tilde{C}/C} \to 0$$

$$0 \to \mathscr{T}^i_{C/S}(n_*\mathcal{O}_{\tilde{C}}) \to n_*\mathscr{T}^{i+1}_{\tilde{C}/C} \to 0, \quad i \geqslant 2.$$

Namely, $\Omega^1_{C/S} = 0$, $\Omega^1_{\tilde{C}/S} = \Omega^1_{\tilde{C}/C}$ is generically zero and $\mathscr{L}^{\cdot}_{C/S} \to \Omega^1_{C/S}$ is generically quasiisomorphic. Hence $\mathscr{T}^0_{\tilde{C}/C} = \mathscr{T}^0_{\tilde{C}/S} = \mathrm{Hom}_{\mathcal{O}_{\tilde{C}}}(Ln^*\mathscr{L}^{\cdot}_{C/S}, \mathcal{O}_{\tilde{C}}) = 0$. Since S is smooth, $\mathscr{T}^i_{\tilde{C}/S} \cong \mathscr{T}^i_{\tilde{C}}$ (cf. proof of prop. 2.3) if $i \geqslant 2$, but $\mathscr{T}^i_{\tilde{C}} = 0$ for $i \geqslant 1$ since $\tilde{C}$ is smooth. By the projection formula we obtain for $i \geqslant 1$, $n_*\,\mathrm{Ext}^i_{\mathcal{O}_{\tilde{C}}}(Ln^*\mathscr{L}^{\cdot}_{C/S}, \mathcal{O}_{\tilde{C}}) \cong \mathscr{T}^i_{C/S}(n_*\mathcal{O}_{\tilde{C}})$ since n_* is right exact.

We define $\mathscr{N}_f'$ by the exact sequence

$$0 \to \mathscr{T}^1_{\tilde{C}/C} \to \mathscr{T}^1_{\tilde{C}/S} \to \mathscr{N}_f' \to 0.$$

Clearly, the $\mathcal{O}_{\tilde{C}}$-morphism $\mathscr{T}^1_{\tilde{C}/C} \to \mathscr{T}^1_{\tilde{C}/S}$ is induced by the tangent map of $G: \mathscr{D}_{\tilde{C}/C} \to \mathscr{D}_{\tilde{C}/S}$. The tangent map of $L: \mathscr{D}'_{C/S} \to \mathscr{D}_{\tilde{C}/S}$ induces an $\mathcal{O}_C$-morphism $\mathscr{N}'_{C/S} \to n_*\mathscr{T}^1_{\tilde{C}/S} \to n_*\mathscr{N}_f'$. Recall $\mathscr{N}'_{C/S} = \mathrm{Ker}(\mathscr{N}_{C/S} \to \mathscr{T}^1_C)$ and $\mathscr{N}_{C/S}$ is the normal sheaf of $C \subset S$.

By Prop. 4.2 this composition is injective. Calling the cokernel $\mathscr{T}$ we obtain:

Corollary: *There exists an exact sequence*

$$0 \to \mathscr{N}'_{C/S} \to n_*\mathscr{N}_f' \to \mathscr{T} \to 0,$$

where $\mathscr{N}'_{C/S} \to n_*\mathscr{N}_f'$ is induced by the tangent map of L, where $\mathscr{T}$ is a torsion sheaf concentrated on Sing(C).

One can show that this is the same sequence obtained by Tannenbaum [Ta2], 1.5, in a completely different manner. We wanted to point out its geometric meaning.

4.4. Let R denote the ramification divisor of $\tilde{C} \to C$ on $\tilde{C}$ defined by the 0-th Fitting ideal $F^0(\Omega^1_{\tilde{C}/C})$, i.e., $F^0(\Omega^1_{\tilde{C}/C}) = \mathcal{O}_{\tilde{C}}(-R)$. Moreover let $\mathcal{K}_{\tilde{C}} = \operatorname{Ker}(f^*\Omega^1_S \to \Omega^1_{\tilde{C}})$. The following lemma describes the sheaves $\mathcal{T}^1_{\tilde{C}/C}$, $\mathcal{T}^1_{\tilde{C}/S}$ and $\mathcal{N}'_f$. We write θ instead of $\mathcal{T}^0$ for the tangent sheaves.

LEMMA: *The following diagram commutes and is exact*

$$\begin{tikzcd}
0 \arrow[r] & \theta_{\tilde{C}} \arrow[r] \arrow[d, equal] & \theta_{\tilde{C}}(R) \arrow[r] \arrow[d, hook] & \mathcal{T}^1_{\tilde{C}/C} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \theta_{\tilde{C}} \arrow[r] & f^*\theta_S \arrow[r] \arrow[d] & \mathcal{T}^1_{\tilde{C}/S} \arrow[r] \arrow[d] & 0 \\
 & & \operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(\mathcal{K}_{\tilde{C}}, \mathcal{O}_{\tilde{C}}) \arrow[r, phantom, "\cong"] & \mathcal{N}'_f &
\end{tikzcd}$$

Moreover, $\mathcal{T}^1_{\tilde{C}/C} \cong \operatorname{Ext}^1_{\mathcal{O}_{\tilde{C}}}(\Omega^1_{\tilde{C}/C}, \mathcal{O}_{\tilde{C}})$. *In particular,* $\mathcal{N}'_f$ *is locally free on* $\tilde{C}$ *(of rank* $\dim_{f(x)} S - 1$ *for* $x \in \tilde{C}$*).*

Proof: The exact sequences of cotagent sheaves associated to $\tilde{C} \to C$ and $\tilde{C} \to S$ give rise to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \theta_{\tilde{C}} \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(n_*\Omega^1_C, \mathcal{O}_{\tilde{C}}) \arrow[r] \arrow[d, hook] & \mathcal{T}^1_{\tilde{C}/C} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \theta_{\tilde{C}} \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(f^*\Omega^1_S, \mathcal{O}_{\tilde{C}}) \arrow[r] & \mathcal{T}^1_{\tilde{C}/S} \arrow[r] \arrow[d] & 0 \\
 & & & \mathcal{N}'_f &
\end{tikzcd}$$

On the other hand, we can apply $\operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(-, \mathcal{O}_{\tilde{C}})$ to the diagram

$$\begin{tikzcd}
 & n^*\Omega^1_C \arrow[r] \arrow[d] & \Omega^1_{\tilde{C}} \arrow[r] \arrow[d, equal] & \Omega^1_{\tilde{C}/C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K}_{\tilde{C}} \arrow[r] & f^*\Omega^1_S \arrow[r] & \Omega^1_{\tilde{C}} \arrow[r] & \Omega^1_{\tilde{C}/S} \arrow[r] & 0.
\end{tikzcd}$$

It is then not difficult to see that $\mathcal{T}^1_{\tilde{C}/C} \cong \operatorname{Ext}^1(\Omega^1_{\tilde{C}/C}, \mathcal{O}_{\tilde{C}})$. On the other hand, since $\mathcal{O}_{\tilde{C}}$ is locally principal, it follows that $F^0(\Omega^1_{\tilde{C}/C}) = \operatorname{Ann}_{\mathcal{O}_{\tilde{C}}}(\Omega^1_{\tilde{C}/C})$, and the last exact sequence splits into the two following sequences

$$0 \to \mathcal{K}_{\tilde{C}} \to f^*\Omega^1_S \to \Omega^1_{\tilde{C}}(-R) \to 0,$$

$$0 \to \Omega^1_{\tilde{C}}(-R) \to \Omega^1_{\tilde{C}} \to \Omega^1_{\tilde{C}/S} \to 0.$$

Dualizing again, we obtain the desired result.

4.5. We keep the notations of above. Moreover $T'^2_{C/S}$ denotes the obstruction space of $\mathscr{D}'_{C/S}$.

PROPOSITION:

$$
\begin{aligned}
(i)\ & T^1_{C/S} \cong H^0(C, \mathscr{N}_{C/S}),\\
& T^2_{C/S} \cong H^1(C, \mathscr{N}_{C/S}) \oplus H^0(C, \mathscr{T}^2_C).\\
(ii)\ & T'^1_{C/S} \cong H^0(C, \mathscr{N}'_{C/S}),\\
& T'^2_{C/S} \cong H^1(C, \mathscr{N}'_{C/S}).\\
(iii)\ & T^1_{\tilde{C}/S} \cong H^0(\tilde{C}, f^*\theta_S/\theta_{\tilde{C}}),\\
& T^2_{\tilde{C}/S} \cong H^1(\tilde{C}, f^*\theta_S/\theta_{\tilde{C}}).\\
(iv)\ & \mathscr{T}^1_{\tilde{C}/C} \cong \mathscr{E}xt^1_{\mathscr{O}_{\tilde{C}}}(\Omega^1_{\tilde{C}/C}, \mathscr{O}_{\tilde{C}}) \cong \theta_{\tilde{C}}(R)/\theta_{\tilde{C}},\\
& n_*\mathscr{T}^2_{\tilde{C}/C} \cong \mathscr{E}xt^1_{\mathscr{O}_C}(\Omega^1_C, n_*\mathscr{O}_{\tilde{C}}),\\
& T^1_{\tilde{C}/C} \cong H^0(C, n_*\mathscr{T}^1_{\tilde{C}/C}),\\
& T^2_{\tilde{C}/C} \cong H^0(C, n_*\mathscr{T}^2_{\tilde{C}/C}).
\end{aligned}
$$

Moreover, $\mathscr{T}^1_{\tilde{C}/C}$ *and* $\mathscr{T}^2_{\tilde{C}/C}$ *are torsion sheaves and we have* $\dim_{\mathbb{C}}(n_*\mathscr{T}^1_{\tilde{C}/C})_x = \operatorname{mult}(C, x) - r(C, x)$.

Here $\operatorname{mult}(C, x)$ is the multiplicity of the local ring $\mathscr{O}_{C,x}$ and $r(C, x)$ the number of branches of (C, x).

Proof: Nearly everything follows from the previous discussion. Applying the long exact cotangent sequence of $\tilde{C} \to C$ to $\mathscr{O}_{\tilde{C}}$ we see as in 4.3 that $\mathscr{T}^i_C(n_*\mathscr{O}_{\tilde{C}}) \cong n_*\mathscr{T}^{i+1}_{\tilde{C}/C}$ for $i \geqslant 1$. But since C is reduced and $n_*\mathscr{O}_{\tilde{C}}$ torsion follows easily that $\mathscr{T}^1_C(n_*\mathscr{O}_{\tilde{C}}) \cong \operatorname{Ext}^1(\Omega^1_C, n_*\mathscr{O}_{\tilde{C}})$. The dimension statement follows from (cf. [B–G], 6.1.2). □

5. A vanishing theorem for rank one sheaves on curves

5.1. Let C be an arbitrary reduced curve, $\mathscr{F}$ and $\mathscr{G}$ coherent sheaves on C and $x \in C$. For any local homomorphism $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$ such that $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ have finite dimension over $\mathbb{C}$ (call φ *admissible* in this case), we define

$$\operatorname{ind}_x(\varphi; \mathscr{F}, \mathscr{G}) := \dim_{\mathbb{C}} \ker(\varphi) - \dim_{\mathbb{C}} \operatorname{coker}(\varphi).$$

It is not difficult to see that such φ exists iff on each irreducible component of (C, x), $\mathscr{F}_x$ and $\mathscr{G}_x$ have the same rank. If this is the case, we define

$$\operatorname{ind}_x(\mathscr{F}, \mathscr{G}) = \sup \operatorname{ind}_x(\bar{\varphi}; \bar{\mathscr{F}}, \bar{\mathscr{G}})$$

where the supremum is taken over all admissible $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$. The $\bar{}$ denotes reduction modulo torsion. We call $\mathrm{ind}_x(\mathscr{F}, \mathscr{G})$ the *local index* of $\mathscr{F}$ in $\mathscr{G}$ in x. If $t_x(\mathscr{F}) := \dim_{\mathbb{C}} \mathrm{Tors}(\mathscr{F})_x$, where $\mathrm{Tors}(\mathscr{F})$ is the torsion subsheaf of $\mathscr{F}$, then

$$\begin{aligned} \mathrm{ind}_x(\varphi; \mathscr{F}, \mathscr{G}) &= t_x(\mathscr{F}) - t_x(\mathscr{G}) + \mathrm{ind}_x(\bar{\varphi}; \bar{\mathscr{F}}, \bar{\mathscr{G}}) \text{ and} \\ \mathrm{ind}_x(\bar{\varphi}; \bar{\mathscr{F}}, \bar{\mathscr{G}}) &= -\dim_{\mathbb{C}} \mathrm{coker}(\bar{\varphi}) \end{aligned}$$

for any admissible φ. We have preferred to define the index on the reduction modulo torsion, since it is usually easier to compute. In particular, $\mathrm{ind}_x(\mathscr{F}, \mathscr{G})$ is a non positive integer and on a compact curve this number is not zero only at finitely many points. So, for a compact, reduced curve and for coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ which have the same rank on each reducible component of C, we define

$$\mathrm{ind}(\mathscr{F}, \mathscr{G}) := \sum_{x \in C} \mathrm{ind}_x(\mathscr{F}, \mathscr{G})$$

and call it the *total* (*local*) *index* of $\mathscr{F}$ in $\mathscr{G}$. If D is an irreducible component of C and $x \in D$, we set

$$\mathscr{F}_D := \overline{\mathscr{F} \otimes \mathcal{O}_D},$$

$$\mathrm{ind}_{D,x}(\mathscr{F}, \mathscr{G}) := \sup \mathrm{ind}_x(\varphi_D; \mathscr{F}_D, \mathscr{G}_D)$$

(here the supremum is taken over all morphisms $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$, such that the induced map φ_D is admissible) and $\mathrm{ind}_D(\mathscr{F}, \mathscr{G}) := \Sigma_{x \in D}\, \mathrm{ind}_x(\mathscr{F}, \mathscr{G})$. Note that $\mathrm{ind}_D(\mathscr{F}, \mathscr{G}) \leqslant 0$.

Example: Let $\mathscr{C} = \mathscr{A}nn(n_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C)$ be the conductor sheaf on C, D a component of C, then it is easy to see that

$$\mathrm{ind}_{C,x}(\mathscr{C}, \omega_C) = -\dim_{\mathbb{C}}(n_* \mathcal{O}_{\tilde{C},x}/\mathcal{O}_{C,x}) = -\delta(C, x)$$

where ω_C is the dualizing sheaf.
If (C, x) is a plane curve singularity, then one can also show that

$$\mathrm{ind}_{D,x}(\mathscr{C}, \omega_C) = -\delta(D, x) - (C' \cdot D, x)$$

where $C = C' \cup D$ and $(C' \cdot D, x)$ denotes the intersection number of C' and D in x.

5.2. PROPOSITION: *Let C be a compact, reduced curve and $\mathscr{F}$ a coherent $\mathcal{O}_C$-module which has rank 1 on each irreducible component C_i of C, $i = 1, \ldots, s$. Then*
(*i*) $H^1(C, \mathscr{F}) = 0$ *if for* $i = 1, \ldots, s$

$$\chi(\mathscr{F}_{C_i}) > \chi(\omega_{C,C_i}) + \mathrm{ind}_{C_i}(\mathscr{F}, \omega_C).$$

(*ii*) *Let $\mathscr{F}$ be torsion free, then* $H^0(C, \mathscr{F}) = 0$ *if for* $i = 1, \ldots, s$

$$\chi(\mathscr{F}_{C_i}) < 1 - p_a(C_i) - \mathrm{ind}_{C_i}(\mathcal{O}_C, \mathscr{F}).$$

Here ω_C denotes the dualizing sheaf, $\omega_{C,C_i} = \overline{\omega_c \otimes \mathcal{O}_{C_i}}$, $\chi(\mathscr{M}) = \dim_{\mathbb{C}} H^0(C, \mathscr{M}) - \dim_{\mathbb{C}} H^1(C, \mathscr{M})$ for a coherent sheaf $\mathscr{M}$ on C and $p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 + \chi(\omega_C)$ is the arithmetic genus.

Remarks:
(1) If $\mathscr{F}$ is locally free, (ii) gives just the usual estimate while in (i) we have

$$\mathrm{ind}_C(\mathscr{F}, \omega_C) \leqslant -\sum_{x \in C} (\dim_{\mathbb{C}}(\omega_{C,x}/m_x\omega_{C,x}) - 1)$$

which is negative, if C is not Gorenstein.
(2) It is convenient (cf. [vS], 3.3) to define the degree of $\mathscr{F}$ by

$$\deg(\mathscr{F}) := \chi(\mathscr{F}) - \mathrm{rank}(\mathscr{F}) \cdot \chi(\mathcal{O}_C)$$

for any coherent sheaf $\mathscr{F}$ on C which has the same rank on every irreducible component C_i of C. Using Riemann-Roch, 5.2 reads for C irreducible,
(i) $H^1(C, \mathscr{F}) = 0$ if $\deg(\mathscr{F}) > 2p_a(C) - 2 + \mathrm{ind}(\mathscr{F}, \omega_C)$,
(ii) $H^0(C, \mathscr{F}) = 0$ if $\deg(\mathscr{F}) < -\mathrm{ind}(\mathcal{O}_C, \mathscr{F})$.
Moreover, if C lies on a smooth surface S, the adjunction formula tells us that $\deg(\omega_C \otimes \mathcal{O}_{C_i}) = \deg(\omega_{C_i}) + C' \cdot C_i$, where $C' \cdot C_i = \deg(\mathcal{O}_S(C') \otimes \mathcal{O}_{C_i})$ is the intersection number in S of $C' = \Sigma_{j \neq i} C_j$ and C_i. Since $(K_S + C)C_i = 2p_a(C_i) - 2 + C'C_i$ we have that $H^1(C, \mathscr{F}) = 0$ if

$$\deg(\mathscr{F}_{C_i}) > (K_S + C)C_i + \mathrm{ind}_{C_i}(\mathscr{F}, \omega_C)$$

for $i = 1, \ldots, s$.

5.3. Proof: By Serre duality $H^1(C, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{F}, \omega_C)^*$, and if this is not 0, there is a $\varphi: \mathscr{F} \to \omega_C$ which is not the zero-map. Since ω_C is torsion free,

$\mathrm{im}(\varphi)$ must have rank 1 on at least one component C_i of C. Therefore, the restriction of φ to C_i,

$$\varphi_i\colon \mathscr{F} \otimes \mathcal{O}_{C_i} \to \omega_C \otimes \mathcal{O}_{C_i}$$

has image of rank 1 and this implies that $\ker(\varphi_i)$ and $\mathrm{im}(\varphi_i)$ are torsion sheaves. Hence φ_i is admissible and

$$\begin{aligned}\chi(\mathscr{F}_{C_i}) &= \chi(\omega_{C,C_i}) + \sum_{x \in C_i} \mathrm{ind}_x(\varphi_{C_i}; \mathscr{F}_{C_i}, \omega_{C,C_i}) \\ &\leqslant \chi(\omega_{C,C_i}) + \mathrm{ind}_{C_i}(\mathscr{F}, \omega_C),\end{aligned}$$

which implies the result.
For (ii) we argue similarly, noting that $H^0(C, \mathscr{F}) \neq 0$ implies the existence of a non-zero map $\mathcal{O}_C \to \mathscr{F}$. □

EXAMPLE: Let C lie on the smooth surface S and let D be any divisor on S. Using example 5.1 we see that $H^1(\mathscr{C} \otimes \mathcal{O}_C(D)) = 0$ if for each component C_i of C, $C_i \cdot D - C_i \cdot C' > 2p_a(C_i) - 2$. In particular, if $S = \mathbb{P}^2$, we obtain the well known fact that $H^1(\mathscr{C} \otimes \mathcal{O}_C(C)) = 0$.

6. On the completeness of the characteristic linear series

6.1. We now consider curves on a smooth surface S. Let $H := H_S$ resp. $H' := H'_S$ be the representing spaces for the Hilbert functor resp. for the locally trivial Hilbert functor (cf. Section 3).

$C \subset S$ denotes a compact reduced curve and $C = C_1 \cup \ldots C_s$ its decomposition into irreducible components. We pose

$$\delta(C, x) := \dim_{\mathbb{C}} (n_* \mathcal{O}_{\tilde{C}}/\mathcal{O}_C)_x$$

$$\tau(C, x) := \dim_{\mathbb{C}} T^1_{C,x} \qquad \text{(Tjurina number of } (C, x))$$

and

$$\alpha(C) := \sum_{x \in C} \alpha(C, x) \quad \text{for} \quad \alpha = \delta \quad \text{or} \quad \alpha = \tau.$$

THEOREM:

(*i*) $\dim(H, C) \geqslant C^2 + 1 - p_a(C)$,

$$\dim(H', C) \geqslant C^2 + 1 - p_a(C) - \tau(C).$$

(*ii*) *If* $H^1(C, \mathcal{N}'_{C/S}) = 0$ *then* $\mathcal{H}_{C/S}$, $\mathcal{H}'_{C/S}$, $\mathcal{D}_{\tilde{C}/C}$ *are unobstructed and* (H, C), (H', C) *are smooth space germs of dimensions* $\dim(H, C) = C^2 + 1 - p_a(C)$ *and* $\dim(H', C) = \dim(H, C) - \tau(C)$. *Moreover, the miniversal base of* $\mathcal{D}_{\tilde{C}/S}$ *is smooth of dimension* $\dim(H, C) - \delta(C)$ *and the morphism of functors* $\mathcal{H}_{C/S} \to \mathcal{D}_{C,\mathrm{Sing}(C)}$ *is smooth.*

(*iii*) $H^1(C, \mathcal{N}'_{C/S}) = 0$ *if for* $i = 1, \ldots, s$

$$-K_S \cdot C_i > C' \cdot C_i + \tau(C_i) + \mathrm{ind}_{C_i}(\mathcal{N}'_{C/S}, \omega_C)$$

where $C' = \bigcup_{j \neq i} C_j$ and K_S *denotes the canonical sheaf on S. Moreover,* $\mathrm{ind}_{C_i}(\mathcal{N}'_{C/S}, \omega_C) \leqslant -\# C_i \cap \mathrm{Sing}(C)$.

Remarks:

(1) From the adjunction formula we obtain $C_i^2 - 2p_a(C_i) + 2 = -K_S \cdot C_i$.

(2) Let u, v be local coordinates of S in $x \in C_i$ and $f(u, v) = 0$ resp. $f_i(u, v) = 0$ local equations of (C, x) resp. (C_i, x). We define $j(C, x)$, the *Jacobian ideal* of (C, x), to be the ideal generated by $(f, \partial f/\partial u, \partial f/\partial v)$ in $\mathbb{C}\{u, v\}/(f) \cong \mathcal{O}_{C,x}$ and similar for (C_i, x). Then

$$\tau(C, x) = \dim_{\mathbb{C}} \mathcal{O}_{C,x}/j(C, x) \text{ and}$$

$$\mathrm{ind}_{C_i,x}(\mathcal{N}'_{C/S}, \omega_C) = -\inf \dim_{\mathbb{C}}(\mathrm{coker}\ \varphi: j(C_i, x) \to \mathcal{O}_{C_i,x})$$

where the infimum is taken over all admissible φ_i which are restrictions of $\varphi: j(C, x) \to \mathcal{O}_{C,x}$. It follows that $\mathrm{ind}_{C_i,x}(\mathcal{N}'_{C/S}, \omega_C) \leqslant -1$ if $x \in \mathrm{Sing}(C)$ with equality if (C, x) is quasihomogeneous.

6.2. Proof: We argue as in the proof of proposition 2.3.

Since $\mathcal{T}^2_C = 0$ we see from proposition 4.5 $\dim(H, C) \geqslant \chi(\mathcal{N}_{C/S})$ and $\dim(H', C) \geqslant \chi(\mathcal{N}'_{C/S})$. Moreover, by definition of $\mathcal{N}'_{C/S}$, $\chi(\mathcal{N}_{C/S}) = \chi(\mathcal{N}'_{C/S}) + \tau(C)$ and from Riemann-Roch follows $\chi(\mathcal{N}_{C/S}) = 1 - p_a(C) + C^2$.

(ii) The unobstructedness of $\mathcal{H}_{C/S}$ and $\mathcal{H}'_{C/S}$ follows immediately from prop. 4.5. From corollary 4.3 we see that $H^1(\tilde{C}, \mathcal{N}_f) = 0$ and from the definition of $\mathcal{N}'_f$ in 4.3, we get $H^1(\tilde{C}, \mathcal{T}^1_{\tilde{C}/S}) = 0$ since $\mathcal{T}^1_{\tilde{C}/C}$ is a torsion sheaf. Since $T^2_{\tilde{C}/S} = H^1(C, \mathcal{T}^1_{\tilde{C}/S})$, the unobstructedness of $\mathcal{D}_{\tilde{C}/S}$ follows.

Hence, by 4.5 (iii), the miniversal base of $\mathcal{D}_{\tilde{C}/S}$ has dimension $\dim_{\mathbb{C}} H^0(\tilde{C}, \mathcal{T}^1_{\tilde{C}/S}) = \dim_{\mathbb{C}} H^0(C, \mathcal{N}'_f) + \dim_{\mathbb{C}} H^0(C, \mathcal{T}^1_{\tilde{C}/C}) = \dim_{\mathbb{C}} H^0(C, \mathcal{N}'_{C/S}) + \dim_{\mathbb{C}} H^0(C, \mathcal{T}) + \dim_{\mathbb{C}} H^0(C, n_* \mathcal{T}^1_{\tilde{C}/C})$ by 4.3. $\mathcal{T}$ and $n_* \mathcal{T}^1_{\tilde{C}/C}$ are torsion sheaves and we have

$$\dim_{\mathbb{C}} \mathscr{T}_x = \tau(C, x) - \delta(C, x) + r(C, x) - \operatorname{mult}(C, x)$$

(cf. [Gr]) and by 4.5 (i) $\dim_{\mathbb{C}}(\mathscr{T}^1_{\tilde{C}/C,x}) = \operatorname{mult}(C, x) - r(C, x)$. This implies the dimension formula for $\mathscr{D}_{\tilde{C}/S}$.

The smoothness of $\mathscr{H}_{C/S} \to \mathscr{D}_{C,\operatorname{Sing}(C)}$ is proved in Prop. 2.3.

(iii) By 5.2, $H^1(C, \mathscr{N}'_{C/S}) = 0$ if for $i = 1, \dots, s$

$$\chi(\overline{\mathscr{N}'_{C/S} \otimes \mathscr{O}_{C_i}}) > p_a(C_i) - 1 + C'C_i + \operatorname{ind}_{C_i}(\mathscr{N}'_{C/S}, \omega_C)$$

where $\overline{}$ denotes the reduction modulo torsion and $C' = \bigcup_{j \neq i} C_j$. From the exact sequence

$$0 \to \overline{\mathscr{N}'_{C/S} \otimes \mathscr{O}_{C_i}} \to \mathscr{N}_{C/S} \otimes \mathscr{O}_{C_i} \to \mathscr{T}^1_C \otimes \mathscr{O}_{C_i} \to 0$$

we obtain

$$\chi(\overline{\mathscr{N}'_{C/S} \otimes \mathscr{O}_{C_i}}) = \chi(\mathscr{N}_{C/S} \otimes \mathscr{O}_{C_i}) - \dim_{\mathbb{C}} H^0(\mathscr{T}^1_C \otimes \mathscr{O}_{C_i}).$$

Moreover, by Riemann-Roch,

$$\chi(\mathscr{N}_{C/S} \otimes \mathscr{O}_{C_i}) = 1 - p_a(C_i) + C \cdot C_i.$$

This implies that $H^1(C, \mathscr{N}'_{C/S}) = 0$ if

$$C \cdot C_i - 2p_a(C_i) + 2 > C' \cdot C_i + \dim_{\mathbb{C}} H^0(\mathscr{T}^1_C \otimes \mathscr{O}_{C_i}) + \operatorname{ind}_{C_i}(\mathscr{N}'_{C/S}, \omega_C).$$

If $f_i(u, v) = 0$ resp. $f(u, v) = 0$ s.t. $f = f_i \cdot g$ are local equations of (C_i, x) resp. (C, x), then

$$\dim_{\mathbb{C}} \mathscr{T}^1_{C,x} \otimes \mathscr{O}_{C_i,x} = \dim_{\mathbb{C}} \mathscr{O}_{C_i,x}/g \cdot j(C_i, x).$$

The exact sequence

$$0 \to \mathscr{O}_{C_i,x}/j(C_i, x) \xrightarrow{\cdot g} \mathscr{O}_{C_i,x}/g \cdot j(C_i, x) \to \mathscr{O}_{C_i,x}/g\mathscr{O}_{C_i,x} \to 0$$

shows that $\dim_{\mathbb{C}} \mathscr{T}^1_{C,x} \otimes \mathscr{O}_{C_i,x} = \tau(C_i, x) + (C_i \cdot C', x)$, hence

$$H^0(\mathscr{T}^1_C \otimes \mathscr{O}_{C_i}) = \tau(C_i) + C_i \cdot C'.$$

This implies the result. □

6.3. COROLLARY: *Let $C \subset \mathbb{P}^2$ be a reduced projective plane curve of degree d and let d_i be the degree of the irreducible component C_i of C, $i = 1, \ldots, s$.*

(*i*) $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 0$ *iff* $\mathcal{H}_{C/\mathbb{P}^2} \to \mathcal{D}_{C,\mathrm{Sing}(C)}$ *is surjective.*

(*ii*) *If* $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 0$ *then* (H', C) *is smooth of dimension* $\dim(H', C) = \frac{1}{2}d(d + 3) - \tau(C)$.

(*iii*) $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 0$ *if for* $i = 1, \ldots, s$

$$3d_i > d_i(d - d_i) + \tau(C_i) - \# \,\mathrm{Sing}(C) \cap C_i.$$

If C is irreducible, the last condition reads $3d > \tau(C) - \# \,\mathrm{Sing}(C)$.

Proof: (i) Since $T^2_{C/\mathbb{P}^2} = H^1(C, \mathcal{N}_{C/\mathbb{P}^2}) = 0$ and $T^2_{C,\mathrm{Sing}(C)} = 0$, $\mathcal{H}_{C/\mathbb{P}^2}$ and $\mathcal{D}_{C/\mathrm{Sing}(C)}$ are unobstructed. Therefore any deformation of $(C, \mathrm{Sing}(C))$ is induced by an embedded deformation of $C \subset \mathbb{P}^2$ iff this is infinitesimally valid, i.e., iff $T^1_{C/\mathbb{P}^2} \to T^1_{C,\mathrm{Sing}(C)}$ is surjective, which is equivalent to $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 0$.

(ii) and (iii) follow immediately from theorem 6.1 and remark 6.1. (2). □

6.4. EXAMPLES:

(1) Assume, in the situation of 6.1 (iii), that $x \in C_i$ is a node of C. Then either x is a node of C_i or C_i is smooth in x. In any case, $(C' \cdot C_i, x) + \tau(C_i, x) = 1$ and $\mathrm{ind}_{C_i,x}(\mathcal{N}'_{C/S}, \omega_C) = -1$. Therefore, nodes do not count to the right hand side of 6.1 (iii) and 6.3 (iii). In particular we obtain the well known result of Severi, that every irreducible component of the space of nodal curves of degree d in $\mathbb{P}^2$ is smooth of dimension $[d(d + 3)]/2$-# nodes and that each node can be smoothed independently. An ordinary cusp counts 1 to the r.h.s. of 6.1 (iii) and 6.3 (iii). More generally if $x \in C_i$ is a singularity of type A_k (i.e., local equation of (C, x) is $x^2 + y^{k+1} = 0$) then x counts $k - 1$ if C_i is the unique component of C which contains x; if k is odd and (C_i, x) is one of the two branches of (C, x) then x counts only $(k + 1)/2 - 1$.

(2) Let $C \subset \mathbb{P}^2$ be given by $f^2 + g^3 = 0$, where f and g are generic homogeneous forms of degree $3n$ resp. $2n$. Then C is of degree $d = 6n$ with exactly $6n^2$ ordinary cusps. Tannenbaum [Ta2] has shown that $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$ has dimension $(n - 1)(n - 2)/2$ but that H' is smooth at C. Our numerical condition of 6.3. (iii) is fulfilled for $n = 1, 2$ hence is sharp for the vanishing of $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$. On the other hand, the vanishing of this group is not at all necessary for the smoothness of H'.

(3) Consider the family of curves C, where C is the union of 3 smooth cubics C_1, C_2, C_3 which meet in 9 ordinary triple points (D_4). For C_3 to go

through the 9 intersection points of C_1 and C_2 we have only to demand that C_3 meets 8 of these points, the ninth is met automatically. Hence the 9 singularities of C impose dependent conditions and indeed it is easy to see, that this family has dimension 19 while the "expected dimension" is $\chi(\mathcal{N}'_{C/\mathbb{P}^2}) = 18$. For each C_i both sides of our numerical condition give 9, hence this condition can in general not be improved.

(4) Let $C = C_1 \cup C_2$ where C_1 is the cuspidal cubic $y^2 - x^3 = 0$ and C_2 the nonsingular quadric $y^2 = 6(x-1)^2 + 2$. C_1 and C_2 meet in $(2, \sqrt{2})$ and $(2, -2\sqrt{2})$ with intersection number 3. Hence C has in both points a singularity of type A_5. The numerical condition of 6.3 (iii) is fulfilled, hence each singular point can be deformed into two ordinary cusps. This gives an irreducible curve of degree 5 with 5 cusps, which is the maximal possible number (by the Plücker-formulas). This example was found by Koelman [Ko], who showed the existence by computing an explicit deformation. Moreover, our method shows that for each pair of integers $\delta, \varkappa$ such that $0 \leqslant \delta + \varkappa \leqslant 5$ there exists an irreducible quintic with δ nodes and $\varkappa$ cusps.

(5) In general it is a difficult problem to determine for a given degree d those k such that there exists a projective plane curve of degree d which has a singularity of type A_k. Of course, the irreducible curve C given by the affine equation $x^2 + x^{k+1} + y^{k+1} = 0$ has 1 singular point which is of type A_k and fulfills the condition of 6.3 (iii). A better example, communicated by the referee, to whom we are grateful, is the degree d curve $(1 + y^{d-6})(x + y^3)^2 + x^d = 0$, $d \geqslant 6$, with one A_{3d-1}-singularity. 6.3 (iii) still applies and we conclude that there exist plane curves of degree $d\,(\geqslant 6)$ with r singular points $p_1, \ldots, p_r$ of type $A_{\ell_1}, \ldots, A_{\ell_r}$ such that the disjoint union of the Dynkin diagrams of the A_{ℓ_i} is an adjacent subdiagram of A_{3d-1} (i.e., obtained from A_{3d-1} by deleting points and the corresponding lines meeting these points). Moreover, at these curves H' is smooth.

(6) Wahl [Wa] was the first to show the existence of a curve C with nodes and cusps such that (H', C) is not smooth, but in his example H'_{red} is smooth at C. Up to now, no example of a curve with only nodes and cusps and singular H'_{red} is known. Luengo [Lu] constructed the first examples of curves C with higher order singularities such that (H'_{red}, C) is singular: e.g., $y(xy^3 + z^4)^2 + x^9 = 0$ defines an irreducible projective plane curve which has only one singular point (at $(0, 1, 0)$) which is of type A_{35}. Luengo showed that $\dim(H', C) = 19$ (which is the "expected dimension" by 6.1. (iii)), $\dim_{\mathbb{C}} H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 1$, $\dim_{\mathbb{C}} H^0(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 20$. So in this case, the obstructions against smoothness of (H', C) are all of $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2})$.

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