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Finite-dimensional categorial complement theorems in shape theory

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Introduction

The aim of a categorial complement theorem in shape theory is to describe
a class $\mathfrak{C}$ of compacta,
"admissible embeddings" of compacta $X \in \mathfrak{C}$ into "ambient spaces" M,
a "complementary category" **K** of spaces,

such that the following holds: When $X, Y \in \mathfrak{C}$ are admissibly embedded into ambient spaces M, N, there is a "canonical 1-1 correspondence" between shape morphisms $X \to Y$ and morphisms $M - X \to N - Y$ in the complementary category **K**.

We shall make this more precise. The admissible embeddings will be specified by
a class $\mathfrak{A}$ of spaces (occurring as ambient spaces),
a topological embedding condition (E) for compacta $X \subset M \in \mathfrak{A}$.

The category **K** is required to have the property
$M - X \in \mathrm{Ob}\mathbf{K}$ whenever $M \in \mathfrak{A}$ and $X \subset M$ is a compactum.

Let $\mathbf{Sh}(\mathfrak{C})$ denote the full subcategory of the shape category with $\mathrm{Ob}\mathbf{Sh}(\mathfrak{C}) = \mathfrak{C}$, and let $\mathbf{K}(\mathfrak{C}, \mathfrak{A}, (E))$ denote the category having as objects the pairs (M, X) with $M \in \mathfrak{A}$, $X \in \mathfrak{C}$ and $X \subset M$ satisfying (E), and having as morphisms $(M, X) \to (N, Y)$ the morphisms $M - X \to N - Y$ in **K**. We call $(\mathfrak{A}, (E), \mathbf{K})$ *data of a categorial complement theorem for* $\mathbf{Sh}(\mathfrak{C})$ (briefly: data for $\mathfrak{C}$) if there is an equivalence of categories T: $\mathbf{K}(\mathfrak{C}, \mathfrak{A}, (E)) \to \mathbf{Sh}(\mathfrak{C})$ such that $T(M, X) = X$.

Chapman's category isomorphism theorem [2] says that $(\{Q\}, (Z\text{-set}), \mathbf{wHP})$, where Q is the Hilbert cube and **wHP** is the weak proper homotopy category, are data for the class $\mathfrak{CM}$ of *all* compacta. In [7] we obtained the following

Extended Chapman Theorem. ($\mathfrak{AR}$, (*Z-set*), **wHC**), *where* $\mathfrak{AR}$ *is the class of* ARs *with a complete uniform structure and* **wHC** *is the weak complete homotopy category, are data for* $\mathfrak{CM}$.

With these data for $\mathfrak{CM}$, however, there are *no* admissible embeddings of nonempty compacta into $\mathbb{R}^n$ (or S^n). This seems to be a remarkable insufficiency, especially since the finite-dimensional complement theorems in shape theory [3, 8, 10], guarantee the existence of embeddings of compacta X into $\mathbb{R}^n$ (or S^n) such that the shape type of X and the *homeomorphism type of the complement* of X determine each other.

The purpose of this paper is to present finite-dimensional versions of the Extended Chapman Theorem *without* the above mentioned insufficiency. "Finite-dimensional" means that data are given no longer for the whole class $\mathfrak{CM}$, but only for the classes $\mathfrak{CM}_m$ of compacta X with fundamental dimension $\mathrm{Fd}X \leqslant m$. In the m-dimensional case, the *complementary category* is the weak complete m-homotopy category $\mathbf{wH_mC}$ (introduced in §1; we point out that the concept of weak m-homotopy for complete maps resembles the concept of m-homotopy for ordinary maps used e.g. in [6]). *Examples for admissible embeddings* of a compactum X are Z-set embeddings into arbitrary m-connected ANRs, or ILC embeddings into the interior of m-connected piecewise-linear manifolds M *provided* $\mathrm{Fd}X \leqslant \dim M - 2 - m$. For the details see §3,4 (in particular, Main Theorem 4.2).

1. Weak homotopy in categories of uniform spaces

We shall assume that the reader is familiar with weak proper homotopy theory as developed in [4]. Let $\mathbf{P}$ be the category of σ-compact spaces (= locally compact Hausdorff spaces that can be written as the countable union of compact subsets) and proper maps, and let $\mathbf{P}_\infty$ be the quotient category $\mathbf{P}\backslash$ (cofinal inclusions in $\mathbf{P}$) as in [4] §6.2. Associated to $\mathbf{P}$ and $\mathbf{P}_\infty$ are the weak proper homotopy category, $\mathbf{wHP}$, and the weak proper homotopy category at ∞, $\mathbf{wHP}_\infty$, respectively (see again [4]). We shall now introduce "uniform" analogues of these categories. Let us call a uniform space A *σ-complete* if there exists a map h: $A \to [0, \infty)$ satisfying the following two conditions:

(C1) For each complete $S \subset A$, $\mathrm{cl}(h(S))$ is compact (where "cl" denotes closure).

(C2) For each n, $h^{-1}([0, n])$ is complete.

For example, let M be a complete uniform space whose underlying topological space is metrizable (N.B.: This does not imply that the uniform structure on

M is induced by a metric). Then, for each compact $X \subset M$, the uniform subspace $M - X \subset M$ is σ-complete: Choose a metric d on the topological space M and define $h\colon M - X \to [0, \infty)$ by $h(a) = d(a, X)^{-1}$.

It is easy to verify

1.1. PROPOSITION. *Let A be a σ-complete uniform space.*
a) *Each closed $A' \subset A$ is σ-complete.*
b) *For each compact K, $A \times K$ is σ-complete.*

1.2. REMARK. Recall that each compact topological space K admits a unique uniform structure compatible with its topology, i.e., K can be regarded as a uniform space in a natural way.

The symbol $\mathbf{C}$ will denote the category of σ-complete uniform spaces and complete maps (a map $f\colon A \to B$ between uniform spaces is *complete* if preimages of complete subsets are complete; cf. [7]). The restriction to σ-complete uniform spaces is motivated by the nice behaviour of their "ends" defined in the next section. In the present section, however, everything could also be done with arbitrary uniform spaces.

A uniform subspace A' of a uniform space A is called *cofinal* in A if $A' \subset A$ is closed and $\mathrm{cl}(A - A')$ is complete. In this case also the inclusion $i\colon A' \to A$ is said to be *cofinal*. Let Σ denote the class of all cofinal inclusions in $\mathbf{C}$ (observe that a cofinal inclusion $i\colon A' \to A$ *is in* $\mathbf{C}$ iff A is σ-complete). The quotient category $\mathbf{C}\backslash\Sigma$ will be denoted by $\mathbf{C}_\infty$. As in [4] one can show that $\mathbf{C}_\infty$ admits a calculus of right fractions, i.e., each morphism $\mathfrak{f}\colon A \to B$ in $\mathbf{C}_\infty$ can be regarded as an equivalence class of complete maps $f'\colon A' \to B$ defined on cofinal subspaces A' of A, two such maps $f'\colon A' \to B$ and $f''\colon A'' \to B$ being equivalent if they agree on some cofinal subspace of A. The composition of morphisms in $\mathbf{C}_\infty$ is obvious since for each $\mathfrak{f} \in \mathbf{C}_\infty(A, B)$ and each cofinal $B' \subset B$ there exists a representative $f'\colon A' \to B$ of $\mathfrak{f}$ with $f'(A') \subset B'$.

1.3. LEMMA. *There exist full embeddings $e\colon \mathbf{P} \to \mathbf{C}$ and $e_\infty\colon \mathbf{P}_\infty \to \mathbf{C}_\infty$ such that $U_{\mathbf{C}}e = e_\infty U_{\mathbf{P}}$, where $U_{\mathbf{P}}\colon \mathbf{P} \to \mathbf{P}_\infty$ and $U_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}_\infty$ are the canonical functors into the quotient categories.*

Proof. For each σ-compact X, let X^* be the one-point compactification. Let $\tilde{X}$ denote the uniform subspace of X^* (cf. 1.2) whose underlying set is X; $\tilde{X}$ is totally bounded. Hence, the complete subsets of $\tilde{X}$ are precisely the compact subsets of X, and therefore any proper map $h\colon X \to [0, \infty)$ (see [4] 6.3.5 for existence) satisfies (C1), (C2). Moreover, since a map $f\colon A \to B$ between totally bounded uniform spaces is complete iff it is proper, we can define e: $\mathbf{P} \to \mathbf{C}$ by $e(X) = \tilde{X}$ and $e(f) = f$. Since i: $X' \to X$ is a cofinal inclusion in $\mathbf{P}$ iff $e(i)$ is a cofinal inclusion in $\mathbf{C}$, we see that e induces the desired e_∞: $\mathbf{P}_\infty \to \mathbf{C}_\infty$.

1.4. REMARK. By 1.3 we can regard $\mathbf{P}$ and $\mathbf{P}_\infty$ as full subcategories $\mathbf{P} \subset \mathbf{C}$ and $\mathbf{P}_\infty \subset \mathbf{C}_\infty$.

We shall need the notion of *weak homotopy* in $\mathbf{C}$, $\mathbf{C}_\infty$.

Weak homotopy in $\mathbf{C}$: $f_0, f_1 \in \mathbf{C}(A, B)$ are called *weakly completely homotopic*, $f_0 \equiv f_1$, if for every complete $S \subset B$ there exists a map H: $A \times I \to B$ such that $H^{-1}(S) \subset A \times I$ is complete and $H_i = f_i$, $i = 0, 1$ (H_i is given by $H_i(x) = H(x, i)$). It is not hard to verify that $f_0 \equiv f_1$ iff for every cofinal $B' \subset B$ there exist a map H: $A \times I \to B$ and a cofinal $A' \subset A$ such that $H(A' \times I) \subset B'$ and $H_i = f_i$, $i = 0, 1$.

Weak homotopy in $\mathbf{C}_\infty$: $\mathfrak{f}_0$, $\mathfrak{f}_1 \in \mathbf{C}_\infty(A, B)$ are called *weakly completely homotopic at* ∞, $\mathfrak{f}_0 \equiv \mathfrak{f}_1$, if for every cofinal $B' \subset B$ and for all representatives f_i: $A_i \to B$ of $\mathfrak{f}_i$ with $f_i(A_i) \subset B'$, $i = 0, 1$, there exists a cofinal $A' \subset A$ such that $A' \subset A_0 \cap A_1$ and $f_0|A' \simeq f_1|A'$ in B'.

We are now able to define, for each $m = 0, 1, \ldots, \infty$, the concept of weak m-homotopy in $\mathbf{D} = \mathbf{C}$, $\mathbf{C}_\infty$: α_0, $\alpha_1 \in \mathbf{D}(A, B)$ are called *weakly* m*-homotopic in* $\mathbf{D}$, $\alpha_0 \equiv_m \alpha_1$, if $\alpha_0\varphi \equiv \alpha_1\varphi$ for each σ-compact polyhedron P with $\dim P \leqslant m$ and each $\varphi \in \mathbf{D}(P, A)$. Note that $P \in \mathrm{Ob}\mathbf{D}$ via 1.4.

It is easy to verify that "weak homotopy" and "weak m-homotopy" are equivalence relations for morphisms in $\mathbf{C}$, $\mathbf{C}_\infty$ which are compatible with composition. This yields *weak homotopy categories* $\mathbf{wHC}$, $\mathbf{wHC}_\infty$ and *weak* m*-homotopy categories* $\mathbf{wH_mC}$, $\mathbf{wH_mC}_\infty$. One readily sees that $U_\mathbf{C}$: $\mathbf{C} \to \mathbf{C}_\infty$ induces a functor $\tilde{U}_\mathbf{C}$: $\mathbf{wH_mC} \to \mathbf{wH_mC}_\infty$.

1.5. REMARK. The inclusions $\mathbf{P} \subset \mathbf{C}$ and $\mathbf{P}_\infty \subset \mathbf{C}_\infty$ induce inclusions $\mathbf{wHP} \subset \mathbf{wHC}$ and $\mathbf{wHP}_\infty \subset \mathbf{wHC}_\infty$. Moreover, the notion of weak m-homotopy in $\mathbf{C}$, $\mathbf{C}_\infty$ restricts to an intrinsic notion of weak m-homotopy in $\mathbf{P}$, $\mathbf{P}_\infty$. This gives weak m-homotopy categories $\mathbf{wH_mP} \subset \mathbf{wH_mC}$ and $\mathbf{wH_mP}_\infty \subset \mathbf{wH_mC}_\infty$.

2. The end of a σ-complete uniform space

We introduce the end of a uniform space A following the lines of [4]. Let **Top** denote the category of topological spaces and continuous maps, **HTop** its homotopy category, and **pro-Top** resp. **pro-HTop** their pro-categories. The *end* of A is the inverse system in **pro-Top**

$$\varepsilon(A) = \{A_V = \mathrm{cl}(A - V)\}_{V \in c(A)}$$

indexed by $c(A) = \{V \subset A \text{ complete}\}$ and bonded by inclusion maps. Of course, $c(A)$ is ordered by set inclusion.

2.1. REMARK. Let M be a locally compact non-compact space equipped with a complete uniform structure, and let $X \subset M$ be compact. Then, for each compact neighbourhood M_0 of X, we can identify the topological end $\varepsilon(M_0 - X)$ defined in [4] with a cofinal subsystem of our uniform end $\varepsilon(M - X)$. Thus, the uniform end $\varepsilon(M - X)$ may be interpreted as the "end of $M - X$ around X" which obviously cannot be defined in terms of the topological structure of $M - X$.

We can extend the above end-construction to a functor ε: $\mathbf{C}_\infty \to \mathbf{pro\text{-}Top}$: For each $\mathfrak{f} \in \mathbf{C}_\infty(A, B)$ we have a canonical $\varepsilon(\mathfrak{f})$: $\varepsilon(A) \to \varepsilon(B)$ in **pro-Top** (let $f: A' \to B$ represent $\mathfrak{f}$; an index function $\chi: c(B) \to c(A)$ is defined by $\chi(V) = f^{-1}(V) \cup \mathrm{cl}(A - A')$; for each $V \in c(B)$, $f_V: A_{\chi(V)} \to B_V$ is the restriction of f). Note that the restriction of ε to $\mathbf{P}_\infty \subset \mathbf{C}_\infty$ is the topological end-functor as defined in [4]. The same arguments as in §6.3 of [4] yield

2.2. THEOREM. *ε induces a full embedding*

$$\varepsilon^*: \mathbf{wHC}_\infty \to \mathbf{pro\text{-}HTop}.$$

2.3. REMARK. What is needed to copy the arguments in [4] are the following facts. Let us call a filtered topological space $\tilde{A} = (A_0 \supset A_1 \supset \ldots)$ a *filtered model* of a σ-complete uniform space A if $A_n = h^{-1}([n, \infty))$ for some $h: A \to [0, \infty)$ satisfying (C1), (C2).

(*1*) If $\tilde{A}$ is a filtered model of A, then $\varepsilon'(\tilde{A}) = \{A_n\}$ is a cofinal subtower of $\varepsilon(A)$. Moreover, the projection $\mathrm{Tel}(\varepsilon'(\tilde{A})) \to \tilde{A}$ is a filtered homotopy equivalence (cf. [4] 6.3.5).

(*2*) If $\tilde{A}$ and $\tilde{B}$ are filtered models of σ-complete uniform spaces A and B, then a map $f: A \to B$ is complete iff $f: \tilde{A} \to \tilde{B}$ is filtered.

We now explore the behaviour of $\bar{\varepsilon}$: $\mathbf{C}_\infty \to \mathbf{pro\text{-}HTop}$ with respect to weak m-homotopy in $\mathbf{C}_\infty$. For this purpose we need some general nonsense from category theory.

Let $\mathbf{K}$ be a category and $\mathbf{K}_0 \subset \mathbf{K}$ a full subcategory. Morphisms f_0, $f_1: X \to Y$ in $\mathbf{K}$ are called $\mathbf{K}_0$-*equal* if for each $X_0 \in \mathrm{Ob}\mathbf{K}_0$ and each $g \in \mathbf{K}(X_0, X)$, $f_0 g = f_1 g$. This yields a category $\mathbf{K}/\mathbf{K}_0$ having the same objects as $\mathbf{K}$ and having as morphisms $\mathbf{K}_0$-equality classes of morphisms of $\mathbf{K}$. There is canonical functor $\varrho: \mathbf{K} \to \mathbf{K}/\mathbf{K}_0$.

2.4. PROPOSITION. *For each* $X_0 \in \mathrm{Ob}\mathbf{K_0}$ *and each* $Y \in \mathrm{Ob}\mathbf{K}$, ϱ: $\mathbf{K}(X_0, Y) \to \mathbf{K/K_0}(X_0, Y)$ *is a bijection. In particular, the restriction* $\varrho|\mathbf{K_0}$: $\mathbf{K_0} \to \mathbf{K/K_0}$ *is a full embedding.*

The *saturation* of $\mathbf{K_0}$ is the full subcategory $\mathbf{Sat(K_0)} \subset \mathbf{K}$ defined as follows: $X' \in \mathrm{Ob}\mathbf{Sat(K_0)}$ iff for any $\mathbf{K_0}$-equal morphisms $f_0, f_1 \in \mathbf{K}(X, Y)$ and any $g \in \mathbf{K}(X', X)$, $f_0 g = f_1 g$.

2.5. PROPOSITION. *Let* $\mathbf{K_0'} \subset \mathbf{K}$ *be a full subcategory with* $\mathbf{K_0} \subset \mathbf{K_0'} \subset \mathbf{Sat(K_0)}$. *Then morphisms* $f_0, f_1 \in \mathbf{K}(X, Y)$ *are* $\mathbf{K_0}$*-equal iff they are* $\mathbf{K_0'}$*-equal; i.e.,* $\mathbf{K/K_0} = \mathbf{K/K_0'}$.

Now, let $\mathbf{L}$ be another category and F: $\mathbf{K} \to \mathbf{L}$ be a full embedding. By $F(\mathbf{K_0}) \subset \mathbf{L}$ we mean the full subcategory of all $F(X_0)$, $X_0 \in \mathrm{Ob}\mathbf{K_0}$.

2.6. PROPOSITION. F: $\mathbf{K} \to \mathbf{L}$ *induces a full embedding* $F/\mathbf{K_0}$: $\mathbf{K/K_0} \to \mathbf{L/F(K_0)}$.

Finally, a morphism f: $X \to Y$ in $\mathbf{K}$ is called a $\mathbf{K_0}$*-equivalence* if for each $X_0 \in \mathrm{Ob}\mathbf{K_0}$, $f_\#$: $\mathbf{K}(X_0, X) \to \mathbf{K}(X_0, Y)$ is a bijection (where $f_\#(g) = fg$).

2.7. PROPOSITION. *Let* f: $X \to Y$ *be a morphism of* $\mathbf{K}$. *If* $\varrho(f)$ *is an isomorphism in* $\mathbf{K/K_0}$, *then* f *is a* $\mathbf{K_0}$*-equivalence. The converse holds provided* $Y \in \mathrm{Ob}\mathbf{K_0}$.

2.8. REMARK. In general $\mathbf{K/K_0}$ is not the quotient category obtained from $\mathbf{K}$ by inverting the $\mathbf{K_0}$-equivalences.

2.9. EXAMPLE. Let Π_m and $\Pi_{m,\infty}$ be the full subcategories of $\mathbf{wHC}$ and $\mathbf{wHC}_\infty$ whose objects are the σ-compact polyhedra of dimension $\leq m$. Then $\mathbf{wH_mC} = \mathbf{wHC}/\Pi_m$ and $\mathbf{wH_mC}_\infty = \mathbf{wHC}_\infty/\Pi_{m,\infty}$.

Let Ω_m^* be the full subcategory of **pro-HTop** whose objects are all inverse systems which are isomorphic in **pro-HTop** to some inverse system $\underline{P} = \{P_\lambda\}$ such that the P_λ are polyhedra with dimension $\leq m$. We now define

$$\Omega_m = \Omega_m^* \cap \mathbf{Sat}(\varepsilon^*(\Pi_{m,\infty})) \subset \mathbf{pro\text{-}HTop}.$$

Clearly, $\varepsilon^*(\Pi_{m,\infty}) \subset \Omega_m \subset \mathbf{Sat}(\varepsilon^*(\Pi_{m,\infty}))$. From 2.5, 2.6, 2.9 we obtain

2.10. PROPOSITION. ε^*: $\mathbf{wHC}_\infty \to$ **pro-HTop** *induces a full embedding* ε_m: $\mathbf{wH_mC}_\infty \to \mathbf{pro\text{-}HTop}/\Omega_m$.

An important property of Ω_m is

2.11. LEMMA. Let $\underline{P} = \{P_n, \gamma_n\}$ *be an inverse sequence in* **pro-HTop**, *where each* P_n *is a compact polyhedron with* $\dim P_n \leqq m$. *Then* $\underline{P} \in \mathrm{Ob}\Omega_m$.

2.12. REMARK. Elementary examples show that $\underline{P}$ need not be isomorphic to an object of $\varepsilon^*(\Pi_{m,\infty})$. One can only say that $\underline{P}$ is isomorphic to an object of $\varepsilon^*(\Pi_{m+1,\infty})$, e.g., to $\varepsilon^*(\mathrm{Tel}(\{P_n, p_n\}))$ with PL representatives p_n of γ_n.

Proof of 2.11. Let $\underline{\varphi_0}, \underline{\varphi_1}: \underline{X} \to \underline{Y}$ be $\varepsilon^*(\Pi_{m,\infty})$-equal morphisms and $\underline{\psi}: \underline{P} \to \underline{X}$. Letting $\underline{\mu_i} = \underline{\varphi_i \psi}$, we have to show $\underline{\mu_0} = \underline{\mu_1}$. Set $\mathrm{P}^* = \bigcup_r P_r \times \{r\}$. This is a σ-compact polyhedron with dimension $\leqq m$. Letting $P_n^* = \bigcup_{r \geqq n} P_r \times \{r\}$, we see that the inclusion-bonded inverse sequence $\underline{P}^* = \{P_n^*\}$ is cofinal in $\varepsilon^*(P^*)$. A level morphism $\underline{\gamma}^*: \underline{P}^* \to \underline{P}$ in **pro-HTop** is defined by $\gamma_n^*: P_n^* \to P_n, \gamma_n^* | P_r \times \{r\} = \gamma_n^r = \gamma_n \cdots \gamma_{r-1}$. The morphisms $\underline{\mu_i}: \underline{P} \to \underline{Y} = \{Y_\alpha\}_{\alpha \in A}$ can be represented by maps of inverse systems $\{\mu_{i,\alpha}: P_{\chi(\alpha)} \to Y_\alpha\}$ with a common index function $\chi: A \to \mathbb{N}$. Hence the morphisms $\underline{\mu_i \gamma^*}$ can be represented by $\{\mu_{i,\alpha}\gamma^*_{\chi(\alpha)}: P^*_{\chi(\alpha)} \to Y_\alpha\}$. By assumption, $\underline{\mu_0 \gamma^*} = \underline{\varphi_0 \psi \gamma^*} = \underline{\varphi_1 \psi \gamma^*} = \underline{\mu_1 \gamma^*}$; thus, each $\alpha \in A$ admits $n \geqslant \chi(\alpha)$ such that $\mu_{0,\alpha}\gamma^*_{\chi(\alpha)} | P_n^* = \mu_{1,\alpha}\gamma^*_{\chi(\alpha)} | P_n^*$. But this implies $\mu_{0,\alpha}\gamma^n_{\chi(\alpha)} = \mu_{1,\alpha}\gamma^n_{\chi(\alpha)}$, which completes the proof.

2.13. COROLLARY. *Let X be a compactum with* $\mathrm{Fd}X \leqslant m$ *and* $\underline{p}: X \to \underline{X}$ *be an* **HPol**-*expansion* (*cf.* [5]). *Then* $\underline{X} \in \mathrm{Ob}\Omega_m$.

Proof. $\underline{X}$ is isomorphic in **pro-HTop** to an inverse sequence $\underline{P}$ as in 2.11; cf. [5] Ch. II §1.

2.14. THEOREM. *Let A, B be σ-complete and assume*

(1) *A is metrizable, and there exist* $\underline{A} \in \mathrm{Ob}\Omega_m$ *and an* Ω_m-*equivalence* $\underline{\varphi}: \bar{\varepsilon}(A) \to \underline{A}$.

(2) *B is an m-connected ANR.*

Then $\tilde{U}_{\mathrm{C}}$: $\mathbf{wH_mC}(A, B) \to \mathbf{wH_mC_\infty}(A, B)$ *is a bijection.*

Proof. 1) Surjectivity. We may assume that $\underline{A} = \{P_\lambda\}$, where the P_λ are polyhedra with dimension $\leqslant m$. By 2.7, $\varrho_m(\underline{\varphi})$ is an isomorphism (where ϱ_m: **pro-HTop** $\to$ **pro-HTop**$/\Omega_m$). Hence, there is $\underline{\psi}: \underline{A} \to \bar{\varepsilon}(A)$ with $\varrho_m(\underline{\psi}) = \varrho_m(\underline{\varphi})^{-1}$. Consider an equivalence class $[\mathfrak{f}] \in \mathbf{wH_mC_\infty}(A, B)$. By (2.2), there exists $\mathfrak{g} \in \mathbf{C}_\infty(A, B)$ with $\bar{\varepsilon}(\mathfrak{g}) = \bar{\varepsilon}(\mathfrak{f})\underline{\psi\varphi}$. We have $\varepsilon_m([\mathfrak{g}]) = \varrho_m\bar{\varepsilon}(\mathfrak{g}) = \varrho_m\bar{\varepsilon}(\mathfrak{f}) = \varepsilon_m([\mathfrak{f}])$, i.e., $[\mathfrak{g}] = [\mathfrak{f}]$ by 2.10. Let $\mathfrak{g}$ be represented by $g: A' \to B$ with a cofinal $A' \subset A$. Since $\bar{\varepsilon}(\mathfrak{g})$ factors through $\underline{\varphi}$, there is cofinal $A_0 \subset A'$ such that $g | A_0$ factors up to homotopy through a map into a polyhedron P with $\dim P \leqslant m$. Since B is m-connected, $g | A_0$ must be inessential. Borsuk's homotopy extension theorem [1] yields an extension $G: A \to B$ of $g|A_0$. It is clear that G is a complete map with $U_C(G) = \mathfrak{g}$. Thus we found $[G] \in \mathbf{wH_m C}(A, B)$ with $\tilde{U}_C([G]) = [U_C(G)] = [\mathfrak{g}] = [\mathfrak{f}]$.

2) Injectivity. Consider $F_0, F_1 \in \mathbf{C}(A, B)$ such that $U_C(F_0) \equiv_m U_C(F_1)$. Let P be a σ-compact polyhedron with dim $P \leqslant m$ and $G: P \to A$ be a complete map. Then $U_C(F_0 G) \equiv U_C(F_1 G)$. Let $B' \subset B$ be cofinal. There exist cofinal $A' \subset A$ and $P' \subset P$ such that $F_i(A') \subset B'$ and $G(P') \subset A'$; $U_C(F_i G)$ is represented by $f_i = F_i G|_{P'}$. There exist a cofinal $P'' \subset P'$ and a homotopy $h: P'' \times I \to B$ such that $h_i = f_i|_{P''}$ and $h(P'' \times I) \subset B'$. Because $\mathrm{cl}(P - P'')$ is compact, there is a compact subpolyhedron R of P with $P - P'' \subset R$. Note that $P_0 = \mathrm{cl}(P - R) \subset P''$ is cofinal in P, and that the topological boundary $\mathrm{Bd}(R \times I) = R \times \{0, 1\} \cup (\mathrm{Bd}\, R) \times I$ of $R \times I$ in $P \times I$ is a compact polyhedron of dimension $\leqslant m$. The maps $F_i G$ and the homotopy h determine a map h': $\mathrm{Bd}(R \times I) \to B$ which must be inessential. Thus there is an extension $h'': R \times I \to B$ of h'. We can put h and h'' together to obtain a homotopy $H: P \times I \to B$ satisfying $H_i = F_i G$ and $H(P_0 \times I) \subset B'$. This implies $F_0 \equiv_m F_1$.

3. Admissible embeddings

Throughout this section, let M denote an ANR and $X \subset M$ a compactum. We shall need the following conditions (where $k \geqslant 0$):

(A_{-1}) X is not open in M.

(A_k) Each neighbourhood U of X in M admits a neighbourhood $V \subset U$ of X in M such that the inclusion-induced $\pi_k(V, V - X, *) \to \pi_k(U, U - X, *)$ is trivial for each $* \in V - X$.

3.1. Remark. In (A_0) we adopted the convention $\pi_0(Y, B, *) = \pi_0(Y, *)/_{IM(i_\#)}$ with $i_\#: \pi_0(B, *) \to \pi_0(Y, *)$. Also observe that (A_2) clearly implies the inessential loops condition ILC (cf. [10]).

For each $m \geqslant -1$, we say that $X \subset M$ is *m-admissible*, or satisfies the embedding condition (Ad_m), if the $m + 3$ conditions $(A_{-1}), \ldots, (A_{m+1})$ are fulfilled.

3.2. Example. Let $X \subset M$ be a compact Z-set. Then $X \subset M$ is m-admissible for each $m \geqslant -1$. This is so because the inclusion $U - X \to U$ is a homotopy equivalence for each neighbourhood U of X in M.

3.3. Lemma. *$X \subset M$ is (-1)-admissible iff no component of X is open in M.*

Proof. (*1*) Let $X \subset M$ be (-1)-admissible. Assume that there is a component K of X which is open in M. Then K must be a component of M. Since M is an ANR, K is also a path-component of M. By (A_0) there is a neighbourhood V of X in M such that $\pi_0(V, V - X, *) \to \pi_0(M, M - X, *)$ is trivial for each $* \in V - X$ (N.B.: By (A_{-1}) there *exists* $* \in V - X$). Clearly, K is a path-component of V, i.e., $K \in \pi_0(V, *)$. From above we infer the existence of a path in M from K to $M - X$, whence K cannot be a path-component of M; a contradiction.

(*2*) Let $X \subset M$ have no open component. Then X satisfies (A_{-1}): Otherwise X would be an ANR, and each component of X would be open in X and therefore open in M. To prove (A_0) it suffices to show that for any open neighbourhood U of X in M and each $* \in U - X$, $\pi_0(U, U - X, *) = 0$, i.e., that $i_\#: \pi_0(U - X, *) \to \pi_0(U, *)$ is onto. Assume that $i_\#$ is not onto. This means that there exists a path-component P of U such that $P \subset X$. Let K be the component of X with $P \subset K$. Since U is an ANR, P is a component of U; hence $P = K$. We infer that P is not open in M and therefore not open in U. This is a contradiction because path-components of ANRs are always open.

3.4. Corollary. *Let M be connected. Then $X \subset M$ is (-1)-admissible iff $X \neq M$.*

3.5. Example. Let M be a piecewise-linear manifold and X a compactum in the interior $\mathring{M}$ of M.
a) If $\mathrm{Fd}X \leqslant \dim M - 1$, then $X \subset M$ is (-1)-admissible (apply 3.3).
b) If $\mathrm{Fd}X \leqslant \dim M - 2$, then $X \subset M$ is 0-admissible (apply *a*) and [11] Lemma 1.11).
c) If $\mathrm{Fd}X \leqslant \dim M - 3$ and X satisfies ILC, then $X \subset M$ is $(\dim M - 2 - \mathrm{Fd}X)$-admissible (apply *b*) and [11] Lemma 1.12).

In particular, if $\mathrm{Fd}X \leqslant d(M) = \max\{k \mid 2k + 2 \leqslant \dim M\}$ and X satisfies ILC, then $X \subset M$ is $d(M)$-admissible; but if $\mathrm{Fd}X \geqslant d(M) + 1$, then $X \subset M$ is (in general) not $d(M)$-admissible. Note that the ILC requirement can be dropped if $\dim M \leqslant 3$.

We shall now give some conditions equivalent to (Ad_m). By $U(X) = U(M, X) = \{U_\lambda, j_{\lambda\mu}\}_{\lambda \in \Lambda}$ we denote the inverse system of all neighbourhoods of X in M, bonded by inclusions $j_{\lambda\mu}: U_\mu \to U_\lambda$. Letting $U^*(X) = U^*(M, X) = \{U_\lambda - X, j^*_{\lambda\mu}\}_{\lambda \in \Lambda}$, bonded by inclusions $j^*_{\lambda\mu}: U_\mu - X \to U_\lambda - X$, we obtain the inclusion level morphism $i_X = i_{(M,X)} = \{i_\lambda: U_\lambda - X \to U_\lambda\}: U^*(X) \to U(X)$ in **pro-Top**.

3.6. LEMMA. *The following are equivalent.*

(i) *$X \subset M$ is m-admissible.*

(ii) *Each λ admits a $\mu \geqslant \lambda$ such that for each polyhedral pair (P, R) with* dim $(P - R) \leqslant m + 1$, *any map α: $(P, R) \to (U_\mu, U_\mu - X)$ is homotopic in U_λ* rel *R to a map into $U_\lambda - X$.*

(iii) *i_X: $U^*(X) \to U(X)$ is an unpointed $(m + 1)$-equivalence in pro-Top, i.e., $U^*(X)$ is not isomorphic to the trivial rudimentary system $(\emptyset)$, and each λ admits a $\mu \geqslant \lambda$ such that for each $* \in U_\mu - X$*

$$\ker((i_\mu)_\#: \pi_k(U_\mu - X, *) \to \pi_k(U_\mu, *)) \subset$$

$$\ker((j^*_{\lambda\mu})_\#: \pi_k(U_\mu - X, *) \to \pi_k(U_\lambda - X, *)) \quad \textit{for } k = 0, \ldots, m$$

and

$$im((j_{\lambda\mu})_\#: \pi_k(U_\mu, *) \to \pi_k(U_\lambda, *)) \subset$$

$$im((i_\lambda)_\#: \pi_k(U_\lambda - X, *) \to \pi_k(U_\lambda, *)) \quad \textit{for } k = 0, \ldots, m + 1.$$

(iv) *X is not open in M, and for each base ray ω: $[0, \infty) \to M - X$ the induced $(i_X)_\#$:* pro $-$ *$\pi_k(U^*(X, \omega))$* $\to$ pro $-$ *$\pi_k(U(X, \omega))$ is an isomorphism for $k = 0, \ldots, m$ and an epimorphism for $k = m + 1$ (cf.* [4] 6.4.1).

The proof follows the classical pattern (see e.g., [9] p. 404) and is left to the reader.

3.7. REMARK. Our definition of an "unpointed $(m + 1)$-equivalence" in (iii) is somewhat complicated because $U^*(X)$ has no "basepoints" to form an inverse system of pointed spaces; but in contrast to the formulation in (iv) it applies – mutatis mutandis – to more general cases.

3.8. COROLLARY. *If $X \subset M$ is m-admissible, then $[i_X]$: $U^*(X) \to U(X)$ is an Ω^*_m-equivalence in* **pro-HTop** *(here, $[i_X]$ denotes the image of i_X in* **pro-HTop**).

The proof is based on 3.6(ii) and follows again the classical pattern (cf. [9] p. 405).

3.9. COROLLARY. *If $X \subset M$ is m-admissible and M is m-connected, then $M - X$ is m-connected.*

Proof. Let f: $S^k \to M - X$ be any map, $k = 0, \ldots, m$. There is an extension F: $D^{k+1} \to M$ of f. Choose a neighbourhood $V = U_\mu$ of X in M

as in 3.6(ii) (for $U_\lambda = M$). There is a compact PL manifold neighbourhood N of $F^{-1}(X)$ in D^{k+1} such that $N \subset F^{-1}(V)$. The topological boundary $\mathrm{Bd}N$ of N is a polyhedron which does not intersect $F^{-1}(X)$. Then F restricts to a map F': $(N, \mathrm{Bd}N) \to (V, V - X)$. By assumption, F' is homotopic in M rel $\mathrm{Bd}N$ to a map g: $N \to M - X$. Extend this homotopy by the identity off N to obtain a homotopy of F rel S^k to a map G into $M - X$. This proves that f is inessential.

4. Data of a categorial complement theorem for $\mathbf{Sh}(\mathfrak{CM}_m)$

Let A, B be σ-complete, $\underline{A}$, $\underline{B} \in \mathrm{Ob}\Omega_m$ and $\underline{\varphi}$: $\bar{\varepsilon}(A) \to \underline{A}$, $\underline{\psi}$: $\bar{\varepsilon}(B) \to \underline{B}$ be Ω_m-equivalences. The following commutative diagram describes the basic situation (cf. 2.2, 2.4, 2.7, 2.10).

$$\begin{array}{ccccccc}
\mathbf{wHC}_\infty(A, B) & \xrightarrow[\approx]{\varepsilon^*} & \mathbf{pro\text{-}HTop}(\bar{\varepsilon}(A), \bar{\varepsilon}(B)) & \xrightarrow{\underline{\psi}_\#} & \mathbf{pro\text{-}HTop}(\bar{\varepsilon}(A), \underline{B}) & \xleftarrow{\underline{\varphi}^\#} & \mathbf{pro\text{-}HTop}(\underline{A}, \underline{B}) \\
\big\downarrow & & \varrho_m \big\downarrow & & \varrho_m \big\downarrow & & \varrho_m \big\downarrow \approx \\
\mathbf{wH_mC}_\infty(A, B) & \xrightarrow[\approx]{\varepsilon_m} & \mathbf{pro\text{-}HTop}/\Omega_m(\bar{\varepsilon}(A), \bar{\varepsilon}(B)) & \xrightarrow[\approx]{\underline{\psi}_\#} & \mathbf{pro\text{-}HTop}/\Omega_m(\bar{\varepsilon}(A), \underline{B}) & \xleftarrow[\approx]{\underline{\varphi}^\#} & \mathbf{pro\text{-}HTop}/\Omega_m(\underline{A}, \underline{B})
\end{array}$$

Note that if $\underline{\varphi}$, $\underline{\psi}$ are isomorphisms in **pro-HTop**, then *each* arrow is a bijection.

4.1. THEOREM. *Let* $\mathfrak{ANR}$ *be the class of* ANRs *with a complete uniform structure. Then* ($\mathfrak{ANR}$, (Ad$_m$), $\mathbf{wH_mC}_\infty$) *are data of a categorial complement theorem for* $\mathbf{Sh}(\mathfrak{CM}_m)$.

Proof. We have to construct an equivalence of categories T_∞: $\mathbf{wH_mC}_\infty(\mathfrak{CM}_m, \mathfrak{ANR}, (\mathrm{Ad}_m)) \to \mathbf{Sh}(\mathfrak{CM}_m)$ (cf. Introduction). Let (M, X) be a pair, where $M \in \mathfrak{ANR}$, $\mathrm{Fd}X \leqslant m$, and $X \subset M$ is m-admissible. There is a canonical isomorphism $\varphi_{(M,X)}$: $\varepsilon(M - X) \to U^*(M, X)$ in **pro-HTop**($\varepsilon(M - X)$ can be identified with a cofinal subsystem of $U^*(M, X)$). By 3.8, $\psi_{(M,X)} = [i_{(M,X)}]\varphi_{(M,X)}$: $\varepsilon(M - X) \to U(M, X)$ is an Ω_m-equivalence in **pro-HTop**. Moreover, $U(M, X) \in \mathrm{Ob}\Omega_m$ by 2.13 (note that $U(M, X)$ is isomorphic in **pro-HTop** to the cofinal subsystem of *open* neighbourhoods of X in M). We are now in the basic situation considered above: Given $\alpha \in \mathbf{wH_mC}_\infty(M - X, N - Y)$, we can define $T_\infty(\alpha) = \varrho_m^{-1}(\varrho_m(\psi_{(N,Y)})\,\varepsilon_m(\alpha)\,\varrho_m(\psi_{(M,X)})^{-1}) \in \mathbf{pro\text{-}HTop}(U(M, X), U(N, Y)) = \mathbf{Sh}(X, Y)$.

4.2. MAIN THEOREM. *Let* $\mathfrak{ANR}_m$ *be the class of m-connected* ANRs *with a complete uniform structure. Then* ($\mathfrak{ANR}_m$, (Ad$_m$), $\mathbf{wH_mC}$) *are data of a categorial complement theorem for* $\mathbf{Sh}(\mathfrak{CM}_m)$.

Proof. Let $T = T_\infty \tilde{U}_C$: $\mathbf{wH_m C}(\mathfrak{CM}_m, \mathfrak{ANR}_m, (\mathrm{Ad}_m)) \to \mathbf{Sh}(\mathfrak{CM}_m)$. It follows from 2.14, 3.9 and 4.1 that T is an equivalence of categories.

4.3. Corollary. *Let X, Y be Z-sets in compact* ARs *M, N such that* $\mathrm{Fd}\,X$, $\mathrm{Fd}\,Y < \infty$. *Then $M - X$ and $N - Y$ have the same weak proper homotopy type iff they have the same weak proper m-homotopy type for some/any $m \geqslant$* $\mathrm{Fd}\,X$, $\mathrm{Fd}\,Y$.

For each fixed m-connected ambient ANR M, we obtain the following finite-dimensional version of Chapman's category isomorphism [2].

4.4. Corollary. *The shape category of m-admissible compacta $X \subset M$ with* $\mathrm{Fd}\,X \leqslant m$ *is isomorphic to the weak complete m-homotopy category of their complements $M - X$. If M is compact, one can replace the weak complete m-homotopy category by the weak proper m-homotopy category.*

4.5. Corollary. *Let M be a $d(M)$-connected piecewise-linear manifold. Then the shape category of ILC compacta $X \subset \mathring{M}$ with* $\mathrm{Fd}\,X \leqslant d(M)$ *is isomorphic to the weak complete $d(M)$-homotopy category of their complements $M - X$ (the* ILC *requirement can be dropped if* $\dim M \leqslant 3$).

The next corollary is well-known for $n \geqslant 5$, but seems to be new for $n = 4$ (note that $d(S^4) = 1$).

4.6. Corollary. *Let X, Y be* ILC *compacta in S^n such that* $\mathrm{Fd}\,X$, $\mathrm{Fd}\,Y \leqslant d(S^n)$. *If $S^n - X$ and $S^n - Y$ are homeomorphic, then X and Y have the same shape.*

4.7. Corollary. *Let X, Y be compacta in S^n, $n \geqslant 2$, such that* $\mathrm{Fd}\,X = \mathrm{Fd}\,Y = 0$. *If $S^n - X$ and $S^n - Y$ are homeomorphic, then X and Y have the same shape.*

4.8. Remark. In 4.6 and 4.7 one can also replace S^n by $\mathbb{R}^n$ (identify $\mathbb{R}^n$ with $S^n - \{*\}$).

We close with the following property of the equivalence of categories T constructed in 4.2.

4.9. Proposition. *Let (M, X), (N, Y) be objects of* $\mathbf{wH_m C}(\mathfrak{CM}_m, \mathfrak{ANR}_m, (\mathrm{Ad}_m))$ *and F: $M \to N$ be a map such that $F(X) \subset Y$ and $F(M - X) \subset N - Y$. If F': $X \to Y$ and F'': $M - X \to N - Y$ are defined as restrictions of F,* then

(a) *F'' is a complete map.*
(b) $T([F'']) = \check{S}([F'])$.
Here $\check{S}$: **HTop** $\to$ **Sh** *denotes the shape functor, $[F'']$ the weak complete m-homotopy class of F'' and $[F']$ the homotopy class of F'.*

Proof. a) is obvious, since a subset of $M - X$ (resp. $N - Y$) is complete iff it is closed in M (resp. N).

b) F induces a morphism $\underline{F}$: $U(M, X) \to U(N, Y)$ in **pro-Top**. Obviously, $[\underline{F}]\psi_{(M,X)} = \psi_{(N,Y)}\varepsilon^*([U_C(F'')])$. But then it is clear from the construction of T_∞ in 4.1 that $T([F'']) = \check{S}([F'])$.

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