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On the degree of the L-function associated with an exponential sum

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0. Introduction

The purpose of the present paper is to derive estimates for the degree of the L-function associated with a certain type of exponential sum defined over a finite field $\mathbb{F}_q$ of characteristic p. Let V be an affine variety defined by the vanishing of polynomials $\{\bar{h}_i(X)\}_{i=1}^r \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$; let f and $\{g_i\}_{i=1}^s$ be regular functions on V induced by polynomials $\bar{f}(X)$, $\{\bar{g}_i(X)\}_{i=1}^s \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$. Let ψ be a non-trivial additive character of $\mathbb{F}_q$, and $\{\chi_i\}_{i=1}^s$ a collection of non-trivial multiplicative characters of $\mathbb{F}_q^*$, extended to functions on $\mathbb{F}_q$ by setting $\chi_i(0) = 0$. Then the exponential sums of interest in this paper are

$$K_m(V, \underline{f}, \psi, \{\underline{g}_i, \chi_i\}) = \sum_{\underline{x} \in V(\mathbb{F}_{q^m})} \psi^{(m)}(\underline{f}(\underline{x})) \prod_{i=1}^{s} \chi_i^{(m)}(\underline{g}_i(\underline{x}))$$

where $V(\mathbb{F}_{q^m})$ denotes the $\mathbb{F}_{q^m}$-rational points on V, and $\psi^{(m)} = \psi \circ Tr_{\mathbb{F}_{q^m}/\mathbb{F}_q}$, $\chi_i^{(m)} = \chi_i \circ \mathbb{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$. Associated to this collection of exponential sums is an L-function

$$L(\{K_m(V, \underline{f}, \psi, \{\underline{g}_i, \chi_i\})\}, T) = \exp\left(\sum_{m=1}^{\infty} K_m T^m/m\right)$$

known from the work of Dwork and Grothendieck to be a rational function of T with coefficients in the field $\mathbb{Q}(\zeta_p, \zeta_{q-1})$ (where the symbol ζ_m denotes an arbitrary choice of primitive mth root of 1 for all $m \geqslant 1$).

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It has been known for some time that the pre-cohomological part of Dwork's theory can be used to estimate the degree of $L(T)$ as rational function (= Euler characteristic of L = degree numerator − degree denominator) and the "total" degree of $L(T)$ (= degree numerator + degree denominator). The Euler characteristic appears in the functional equation for L, and when L is a polynomial gives the actual degree of L. The total degree of L, when combined with information concerning the archimedean size of the reciprocal zeros and poles of L gives estimates for the absolute value of the exponential sum K_m, often an important ingredient in calculations in analytic number theory. These invariants also appear in the recent work of Fried and Jarden [9, 10]. The basic work in estimating these two types of degrees of L-functions is due to Bombieri [4, 5], in the case of an exponential sum involving only an additive character. (Dwork has an alternative approach involving the use of cohomology in simple non-singular cases, and then "deforming" to handle the general case.) In a series of papers [1, 2, 13], the authors have exploited Bombieri's approach to give improved estimates in many cases, and to extend the applicability of the results to more general character sums, in particular, allowing multiplicative characters. In the present paper, we continue this approach, utilizing an idea of Dwork on how to reduce character sums such as K_m above to additive character sums (see (1.2) below). We then study the particular character sum that so arises first by using Adolphson's trace formula [3] to obtain estimates for the Frobenius matrix; then using Bombieri's approach in this particular case, we derive the desired estimates.

One of the main results in this paper is Theorem (5.21) which gives a sharp estimate for the degree as rational function of the L-function associated with a certain type of character sum. From this result, we can extract at once the following estimate for degree of L as rational function (Theorem (5.23)):

$$0 \leqslant \deg L(\{K_m(\mathbb{A}^n_{\mathbb{F}_q} - H_0; \underline{f}, \psi, \{\underline{g}_i, \chi_i\}_{i=1}^s)\}; T)^{(-1)^{n+1}}$$
$$\leqslant D_n(d_0, d_1, \ldots, d_s)$$

where H_0 is the coordinate hypersurface defined by equation $X_1X_2 \ldots X_n = 0$, where $d_0 = \deg \bar{f}(X)$, $d_i = \deg \bar{g}_i(X)$ and where $D_n(d_0, d_1, \ldots, d_s)$ denotes the sum of all monomials in $d_0, d_1, \ldots, d_s$ of degree n.

Another result that follows from (5.23) is the more general estimate (Theorem 5.27):

$$|\deg L(\{K_m(V; \underline{f}, \psi; \{\underline{g}_i; \chi_i\}_{i=t+1}^k)\}, T)|$$
$$\leqslant 2^t \cdot D_n(d_0 + 1, d_1 + 1, \ldots, d_k + 1)$$

where here V is defined over $\mathbb{F}_q$ by the simultaneous vanishing of $\{\bar{h}_j(X)\}_{j=1}^{t} \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$ of respective degrees given by $\{\deg \bar{h}_j(X) = d_j\}_{j=1}^{t}$ and where as before $\underline{f}$ and $\{\underline{g}_i\}_{i=t+1}^{k}$ are regular functions on V induced by polynomials $\bar{f}(X)$ and $\bar{g}_i(X)$ of respective degrees d_0 and $\{d_i\}_{i=t+1}^{k}$. In terms of total degree, we derive (Theorem (6.12)) (where the given multiplicative characters χ_i have the form $\chi_i^{(0)} \circ \mathbb{N}_{\mathbb{F}_q/\mathbb{F}_p}$, for suitable multiplicative characters $\chi_i^{(0)}$ of $\mathbb{F}_p^*$)

$$\text{total deg } L(\{K_m(V, \underline{f}, \psi, \{\underline{g}_i, \chi_i\}_{i=t+1}^{k})\}, T)$$

$$\leqslant (2e^3)^k (2e^3 + 1)^n \left[\left(2 + \frac{3k}{n+1}\right) D + 1\right]^n$$

where $D = \max \{d_i\}_{i=0}^{k}$.

In the case when the multiplicative characters have exponent $p^{a'} - 1$, the same estimate holds with D replaced by $a'D$. In the case, where no multiplicative characters are present at all, we derive the estimate (6.13):

$$\text{total deg } L(\{K_m(V, \underline{f}, \psi)\}, T) \leqslant (2e^3)^n (2e^3 + 1)^n (5D + 1)^n.$$

This result may be compared with [5, Theorem 2] where the exponent that appears (when V is a closed subset of $\mathbb{A}^n_{\mathbb{F}_q}$ which is not equal to $\mathbb{A}^n_{\mathbb{F}_q}$ itself) is $2n + 1$ rather than n.

We thank B. Dwork for his encouragement and helpful comments.

1. Definitions

Let p be a prime number, $q = p^a$, and let $\mathbb{F}_{q^m}$ denote the finite field of q^m elements. Let $\mathbb{Q}_p$ denote the p-adic number field, and let Ω be the completion of an algebraic closure of $\mathbb{Q}_p$. Let K_a denote the unique unramified extension of $\mathbb{Q}_p$ in Ω of degree a over $\mathbb{Q}_p$. The residue class field of K_a is $\mathbb{F}_q$. The Frobenius automorphism $x \to x^p$, the canonical generator of Gal $(\mathbb{F}_q/\mathbb{F}_p)$, lifts to a generator τ of Gal $(K_a/\mathbb{Q}_p)$. If ζ_{q-1} is a $(q - 1)$st root of 1 in K_a, a so-called Teichmuller unit, then $\tau\zeta_{q-1} = \zeta_{q-1}^p$. Let $\Omega_0 = K_a(\zeta_p)$, $\Omega_1 = Q_p(\zeta_p)$, and let $\mathcal{O}_0$ and $\mathcal{O}_1$ be their respective ring of integers. Denote by "ord" the additive valuation on Ω normalized so that ord $p = 1$ and denote by "ord_q" the additive valuation normalized so that $\text{ord}_q\, q = 1$.

Let V be an algebraic variety defined over $\mathbb{F}_q$. Let $\{\chi_i\}_{i=1}^{s}$ be a collection of non-trivial multiplicative characters of $\mathbb{F}_q^*$ with values in Ω (all such in fact have values in K_a). We extend χ_i to all of $\mathbb{F}_q$ by setting $\chi_i(0) = 0$. For the trivial multiplicative character χ_0, we define $\chi_0(0) = 1$. Let ψ be a non-trivial additive character of $\mathbb{F}_q$. Let $\underline{f}$, $\{\underline{g}_i\}_{i=1}^s$ be regular functions on V. Associated with this data is the family of mixed or twisted exponential sums

$$K_m(V, \underline{f}, \psi; \{\underline{g}_i, \chi_i\}) = \sum_{\underline{x} \in V(\mathbb{F}_{q^m})} \prod_{i=1}^{s} \chi_i^{(m)}(\underline{g}_i(\underline{x}))\psi^{(m)}(\underline{f}(\underline{x})) \tag{1.1}$$

where $V(\mathbb{F}_{q^m})$ denotes the $\mathbb{F}_{q^m}$ rational points of V, the characters $\psi^{(m)}$ (resp $\chi_i^{(m)}$) of $\mathbb{F}_{q^m}$ (resp $\mathbb{F}_{q^m}^*$) are obtained from ψ (resp χ_i) by composition with the trace (resp norm) so that $\psi^{(m)} = \psi \cdot Tr_{\mathbb{F}_{q^m}/\mathbb{F}_q}$, $\chi_i^{(m)} = \chi_i \cdot N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$.

Our aproach to the study of (1.1) will involve exponential sums of the following type. Let ω: $\mathbb{F}_q^* \to \Omega$ be the Teichmüller character, the canonical generator of the cyclic group $\hat{\mathbb{F}}_q^*$. Let $\bar{F}(X)$, $\{\bar{H}_i(X)\}_{i=1}^b \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$. Let $\{j_i\}_{i=1}^b \subseteq \mathbb{Z}$, $0 \leqslant j_i \leqslant q-2$. Set

$$S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\}_{i=1}^k) = \sum_{(\bar{x}, \bar{y}) \in (\mathbb{F}_{q^m})^{n+k}} \prod_{i=1}^{k} \omega^{(m)}(\bar{y}_i)^{j_i}\psi^{(m)}\left(\bar{F}(\bar{x}) + \sum_{i=1}^{k} \bar{y}_i \bar{H}_i(\bar{x})\right). \tag{1.2}$$

We now assume V is affine and defined by the simultaneous vanishing of polynomials $\{\bar{h}_i(X)\}_{i=1}^t \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$. We also assume that $\underline{f}$ and $\{\underline{g}_i\}_{i=1}^s$ are induced by polynomials $\bar{f}(X)$, $\{\bar{g}_i(X)\}_{i=1}^s \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$. Finally let $\chi_i^{-1} = \omega^{j_i}$. Then the relation of (1.1) and (1.2) is given by the following lemma.

(1.3) LEMMA. *If V is affine as above and the notation above is used then*

$$S_m(\bar{f}, \psi; \{0, \bar{h}_i\}_{i=1}^t; \{j_i, \bar{g}_i\}_{i=1}^s)$$

$$= (-1)^s \left(q^t \prod_{i=1}^{s} G_q(\chi_i^{-1}, \psi)\right)^m K_m(V, \underline{f}, \psi, \{\underline{g}_i, \chi_i\}_{i=1}^s).$$

Here the notation $G_q(\chi, \psi)$ denotes the negative of the Gauss sum determined by a non-trivial multiplicative and additive character of $\mathbb{F}_q$ (χ and ψ respectively):

$$G_q(\chi, \psi) = -\sum_{\bar{x} \in \mathbb{F}_q^*} \chi(\bar{x})\psi(\bar{x}).$$

Proof. By (1.2), we may write

$$S_m(\bar{f}, \psi; \{0, \bar{h}_i\}_{i=1}^t, \{j_i, \bar{g}_i\}_{i=1}^s)$$

$$= \sum_{(\bar{x},\bar{y},\bar{z}) \in \mathbb{F}_{q^m}^{n+s+t}} \prod_{i=1}^{s} \chi_i^{(m)^{-1}}(\bar{y}_i) \cdot \psi^{(m)}\left(\bar{f}(\bar{x}) + \sum_{i=1}^{s} \bar{y}_i \bar{g}_i(\bar{x}) + \sum_{i=1}^{t} \bar{z}_i \bar{h}_i(\bar{x})\right)$$

and since $\chi_i^{(m)^{-1}}(0) = 0$ for all $i = 1, \ldots, s$ we obtain same result from the sum on the right if we sum only over $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{F}_{q^m}^n \times (\mathbb{F}_{q^m}^*)^s \times \mathbb{F}_{q^m}^t$. Then by orthogonality of characters

$$S_m(\bar{f}, \psi; \{0, \bar{h}_i\}_{i=1}^t, \{j_i, \bar{g}_i\}_{i=1}^s)$$

$$= q^{mt} \sum_{\underline{x} \in V(\mathbb{F}_{q^m})} \psi^{(m)}(\underline{f}(\underline{x})) \cdot \prod_{i=1}^{s} \left(\sum_{\bar{y}_i \in \mathbb{F}_{q^m}^*} \chi_i^{(m)^{-1}}(\bar{y}_i) \psi^{(m)}(\bar{y}_i \underline{g}_i(\underline{x}))\right)$$

Note that if $\underline{x} \in V(\mathbb{F}_{q^m})$ such that $\underline{g}_i(\underline{x}) = 0$ for some i, then

$$\sum_{\bar{y}_i \in \mathbb{F}_{q^m}^*} \chi_i^{(m)^{-1}}(\bar{y}_i) \psi^{(m)}(\bar{y}_i \underline{g}_i(\underline{x})) = \sum_{\bar{y}_i \in \mathbb{F}_{q^m}^*} \chi_i^{(m)^{-1}}(\bar{y}_i) = 0$$

since $\chi_i^{(m)^{-1}}$ is a non-trivial character of $\mathbb{F}_{q^m}^*$. On the other hand, if $\underline{x} \in V(\mathbb{F}_{q^m})$ is fixed such that $\underline{g}_i(\underline{x}) \neq 0$, then the change of variables $\bar{y}_i = \bar{w}_i \underline{g}_i(\underline{x})^{-1}$ yields

$$\sum_{\bar{y}_i \in \mathbb{F}_{q^m}^*} \chi_i^{(m)^{-1}}(\bar{y}_i) \cdot \psi^{(m)}(\bar{y}_i \underline{g}_i(\underline{x})) = (-1) \chi_i^{(m)}(\underline{g}_i(\underline{x})) \cdot G_{q^m}(\chi_i^{(m)^{-1}}, \psi^{(m)})$$

from which the desired conclusion now follows using the Hasse–Davenport relation;

$$G_{q^m}(\chi^{(m)}; \psi^{(m)}) = G_q(\chi, \psi)^m.$$

Associated with the collections of elements in Ω_0 given by $\lambda_m = K_m(V; \underline{f}, \psi; \{\chi_i, \underline{g}_i\}_{i=1}^s)$ as in (1.1) or by $\lambda_m = S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\}_{i=1}^b)$ as in (1.2) is an L-function

$$L(\{\lambda_m\}, T) = \exp\left(\sum_{m=1}^{\infty} \lambda_m T^m / m\right) \tag{1.4}$$

which is known to be a rational function of T with coefficients in $\mathbb{Q}(\zeta_p, \zeta_{q-1})$. Similarly if $\lambda_m = S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\}_{i=1}^b)$ denotes the right-side of (1.2) but where the sum runs only over $(\bar{x}, \bar{y}) \in (\mathbb{F}_{q^m}^*)^{n+b}$, then (1.4) defines an associated L-function which again belongs to $\mathbb{Q}(\zeta_p, \zeta_{q-1})(T)$. Finally, if V is affine/$\mathbb{F}_q$ with coordinate ring $\mathbb{F}_q[X_1, \ldots, X_n]/I$ (where I is an ideal in $\mathbb{F}_q[X_1, \ldots, X_n]$), we can define V^*, the complement in V of the coordinate hypersurface H_0 having equation $X_1 X_2 \ldots X_n = 0$. Thus the collection $\lambda_m = K_m(V^*; \underline{f}, \psi; \{\underline{g}_i, \chi_i\}_{i=1}^b)$ also defines via (1.4) an element of $\mathbb{Q}(\zeta_p, \zeta_{q-1})(T)$, the associated L-function.

The proof of the following remark is identical with the proof of Lemma (1.3).

Remark (1.5). Let the hypotheses and notations be the same as those in Lemma (1.3). Then

$$S_m^*(\bar{f}, \psi; \{j_i, \bar{g}_i\}_{i=1}^s)$$

$$= (-1)^s \left(\prod_{i=1}^s G_q(\chi_i^{-1}, \psi) \right)^m K_m(\mathbb{A}_{\mathbb{F}_q}^n - H_0; \underline{f}, \psi; \{\underline{g}_i, \chi_i\}_{i=1}^s)$$

Finally, (1.3) and (1.5) admit the following L-function formulations:

Corollary (1.6). *Let the hypotheses and notations be the same as those in Lemma (1.3). Then*

$$L(\{S_m(\bar{f}, \psi; \{0, \bar{h}_i\}_{i=1}^t, \{j_i, \bar{g}_i\}_{i=1}^s)\}, T)$$

$$= L(\{K_m(V; \underline{f}, \psi; \{\chi_i, \underline{g}_i\}_{i=1}^s)\}, q^t \prod_{i=1}^s G_q(\chi_i^{-1}, \psi) T)^{(-1)^s}$$

and

$$L(\{S_m^*(\bar{f}, \psi; \{j_i, \bar{g}_i\}_{i=1}^s)\}, T)$$

$$= L(\{K_m(\mathbb{A}_{\mathbb{F}_q}^n - H_0; \underline{f}, \psi; \{\chi_i, \underline{g}_i\}_{i=1}^s)\}, q^t \prod_{i=1}^s G_q(\chi_i^{-1}, \psi) T)^{(-1)^s}.$$

The importance of this result for the present study is that properties of the L-functions associated with exponential sums (1.1) may be established by studying the L-functions associated with sums of the type (1.2).

2. Trace formula

Our approach will utilize the basic framework of Dwork's theory; in particular we will need Adolphson's form of the trace formula [3]. We review here the major features of the theory that will be used in subsequent sections.

Let $j = (j_1, \ldots, j_k)$ be an ordered k-tuple of integers satisfying $0 \leqslant j_i \leqslant q - 2$. We define a p-adic Banach space of formal series in integral powers of one set of variables $\{X_i\}_{i=1}^n$ and fractional powers of another set of variables $\{Y_i\}_{i=1}^k$. More precisely let

$$\Gamma_{(j)} = \left\{ (\mathbf{R}, \mathbf{M}) \in (\mathbb{Z}_{\geqslant 0})^n \times \left(\frac{1}{q-1} \mathbb{Z}_{\geqslant 0} \right)^k \middle| M_i \equiv j_i/(q-1) \bmod \mathbb{Z} \right\}.$$

We will write $X^{\mathbf{R}} Y^{\mathbf{M}}$ to denote $\Pi_{i=1}^n X_i^{R_i} \cdot \Pi_{j=1}^k Y_j^{M_j}$. In the notation of (1.2), let $d_0 = \deg \bar{F}(X)$, $d_i = \deg \bar{H}_i(X)$. In terms of these quantities, we can define a weight function on $\Gamma_{(j)}$

$$w(\mathbf{R}, \mathbf{M}) = \max\{|\mathbf{R}| - \mathbf{M} \cdot \mathbf{d}, 0\} + |\mathbf{M}| d_0 \tag{2.2}$$

where we employ for brevity the usual dot product and the notation $|\mathbf{R}| = \Sigma_{i=1}^n R_i$ for ordered tuples. This weight function takes values in $(q-1)^{-1}\mathbb{Z}_{\geqslant 0}$ and satisfies the following properties:

(i) $w(c\mathbf{R}, c\mathbf{M}) = cw(\mathbf{R}, \mathbf{M})$, (provided $c \in \mathbb{Z}_{\geqslant 0}$, $c \equiv 1 \bmod q - 1$)

(ii) $w(\mathbf{R}^{(1)}, \mathbf{M}^{(1)}) + w(\mathbf{R}^{(2)}, \mathbf{M}^{(2)}) \geqslant w(\mathbf{R}^{(1)} + \mathbf{R}^{(2)}, \mathbf{M}^{(1)} + \mathbf{M}^{(2)})$. (2.3)

(iii) there exists positive constants ε_1 and ε_2 such that

$$\varepsilon_1(|\mathbf{R}| + |\mathbf{M}|) \leqslant w(\mathbf{R}, \mathbf{M}) \leqslant \varepsilon_2(|\mathbf{R}| + |\mathbf{M}|).$$

We define an $\mathcal{O}_0$-module of formal series for $b, c \in \mathbb{R}$, $b > 0$.

$$L_{(j)}(b, c) = \left\{ \sum_{(\mathbf{R},\mathbf{M}) \in \Gamma_{(j)}} C(\mathbf{R}, \mathbf{M}) X^{\mathbf{R}} Y^{\mathbf{M}} \,\middle|\, C(\mathbf{R}, \mathbf{M}) \in \Omega_0, \right.$$

$$\left. \operatorname{ord} C(\mathbf{R}, \mathbf{M}) \geqslant c + b d_0^{-1} w(\mathbf{R}, \mathbf{M}) \right\}, \tag{2.4}$$

and an Ω_0-vector space

$$L_{(j)}(b) = \bigcup_{c \in \mathbb{R}} L_{(j)}(b, c).$$

In fact $L_{(j)}(b)$ is a p-adic Banach space of $b(\Gamma_{(j)})$-type in Serre's notation [12].

We now define a completely continuous Ω_0-linear operator α acting on $L_{(j)}(b)$. Let ψ_q act by

$$\psi_q\left(\sum_{(\mathbf{R},\mathbf{M})\in\Gamma_{(j)}} C(\mathbf{R},\mathbf{M})X^{\mathbf{R}}Y^{\mathbf{M}}\right) = \sum_{(\mathbf{R},\mathbf{M})\in\Gamma_{(j)}} C(q\mathbf{R}, q\mathbf{M})X^{\mathbf{R}}Y^{\mathbf{M}}.$$

Observe that $(\mathbf{R}, \mathbf{M}) \in \Gamma_{(j)}$ if and only if $(q\mathbf{R}, q\mathbf{M}) \in \Gamma_{(j)}$. Clearly, $\psi_q(L_{(j)}(b, c)) \subseteq L_{(j)}(qb, c)$.

Let $E(X) = \exp\,(\Sigma_{i=0}^{\infty} X^{p^i}/p^i)$ be the Artin–Hasse exponential series; let $\gamma \in \Omega$ be a root of $\Sigma_{i=0}^{\infty} X^{p^i}/p^i = 0$ with $\operatorname{ord}\gamma = 1/(p-1)$. Then $\theta_\infty(x) = E(\gamma x)$ is a splitting function in the terminology of [7]; if we write $\theta_\infty(X) = \Sigma_{m=0}^{\infty} B_m X^m$, it is a consequence of the p-adic integrality of the coefficients of the Artin–Hasse series that

$$\operatorname{ord} B_m \geqslant m/(p-1). \tag{2.5}$$

Suppose now that we write

$$\bar{F}(X) + \sum_{i=1}^{k} Y_i \bar{H}_i(X) = \sum_{w(0)\in\Lambda^{(0)}} \bar{A}^{(0)}_{w(0)} X^{w(0)} + \sum_{i=1}^{k} \sum_{w(i)\in\Lambda^{(i)}} \bar{A}^{(i)}_{w(i)} Y_i X^{w(i)}$$

where $\Lambda^{(i)}$ is a finite set of ordered n-tuples of non-negative integers for each $i \in \{0, 1, \ldots, k\}$ and where $\{\bar{A}^{(i)}_{w(i)}\}_{i\in\{0,\ldots,n\};w(i)\in\Lambda^{(i)}} \subseteq \mathbb{F}_q$. Let $A^{(i)}_{w(i)}$ denote the Teichmüller lifting of $\bar{A}^{(i)}_{w(i)}$, so that $(A^{(i)}_{w(i)})^q = A^{(i)}_{w(i)}$, and $\tau(A^{(i)}_{w(i)}) = (A^{(i)}_{w(i)})^p$. Consider

$$F(X, Y) = \prod_{w(0)\in\Lambda^{(0)}} \theta_\infty(A^{(0)}_{w(0)} X^{w(0)}) \prod_{i=1}^{k} \prod_{w(i)\in\Lambda^{(i)}} \theta_\infty(A^{(i)}_{w(i)} X^{w(i)} Y_i). \tag{2.6}$$

If we write $F(X, Y) = \Sigma_{(R,M)\in\Gamma_{(0)}} F(\mathbf{R}, \mathbf{M})X^{\mathbf{R}}Y^{\mathbf{M}}$ then from (2.6)

$$F(\mathbf{R}, \mathbf{M}) = \sum \prod_{w(0)\in\Lambda^{(0)}} B_{m_{w(0)}} \prod_{i=1}^{k} \prod_{w(i)\in\Lambda^{(i)}} B_{m_{w(i)}}$$

where the sum runs over ordered l-tuples ($l = \Sigma_{i=0}^{k} \operatorname{card}\Lambda^{(i)}$), $(\{m_{w(0)}\}_{w(0)\in\Lambda^{(0)}}, \ldots, \{m_{w(k)}\}_{w(k)\in\Lambda^{(k)}})$, of non-negative integers satisfying

$$\begin{aligned} \mathbf{R} &= \sum_{i=0}^{k} \sum_{w(i)\in\Lambda^{(i)}} m_{w(i)} \cdot w(i) \\ M_i &= \sum_{w(i)\in\Lambda^{(i)}} m_{w(i)}. \end{aligned} \tag{2.7}$$

Thus using (2.5)

$$\operatorname{ord} F(\mathbf{R}, \mathbf{M}) \geqslant (p-1)^{-1} \inf \left\{ \sum_{i=0}^{k} \sum_{w(i) \in \Lambda^{(i)}} m_{w(i)} \right\}$$

where the infimum runs over all l-tuples of non-negative integers satisfying (2.7).

Hence

$$\operatorname{ord} F(\mathbf{R}, \mathbf{M}) \geqslant (p-1)^{-1} \left(\sum_{i=1}^{k} M_i + \inf \left\{ \sum_{w(0) \in \Lambda^{(0)}} m_{w(0)} \right\} \right)$$

where the infimum is taken over the same set. Combining the equalities in (2.7) with the inequalities $|w(i)| \leqslant d_i$, we obtain

$$|\mathbf{R}| \leqslant \sum_{i=0}^{k} d_i \sum_{w(i) \in \Lambda^{(i)}} m_{w(i)} = d_0 \sum_{w(0) \in \Lambda^{(0)}} m_{w(0)} + \sum_{i=1}^{k} M_i d_i .$$

Thus

$$\sum_{w(0) \in \Lambda^{(0)}} m_{w(0)} \geqslant \max \left\{ 0, \frac{1}{d_0} (|\mathbf{R}| - \mathbf{M} \cdot \mathbf{d}) \right\}$$

which yields the estimate

$$\operatorname{ord} F(\mathbf{R}, \mathbf{M}) \geqslant \frac{w(\mathbf{R}, \mathbf{M})}{d_0(p-1)}$$

so that

$$F(X, Y) \in L_{(0)}\left(\frac{1}{p-1}, 0 \right).$$

Let

$$F_0(X, Y) = \prod_{j=0}^{a-1} \tau^j F(X^{p^j}, Y^{p^j}) \tag{2.8}$$

then

$$F_0(X, Y) \in L_{(0)}\left(\frac{p}{q(p-1)}, 0 \right).$$

In fact multiplication by $F_0(X, Y)$ defines via (2.3) an endomorphism of $L_{(j)}(p/q(p-1))$. If

$$i\colon L_{(j)}\left(\frac{p}{p-1}\right) \to L_{(j)}\left(\frac{p}{q(p-1)}\right)$$

denotes inclusion, then i and $\alpha = \psi_q \circ F_0(X, Y) \circ i$ are completely continuous endomorphisms of $L_{(j)}(p/(p-1))$ in the sense of Serre [12]. Furthermore the trace formula of Adolphson [3] yields

$$(q^m - 1)^{n+k}\, Tr\,(\alpha^m) = S_m^*\,(\bar{F}, \psi; \{j_i, \bar{H}_i\}_{i=1}^k). \tag{2.9}$$

For a completely continuous endomorphism α, $Tr\,(\alpha^m)$ and the Fredholm determinant $\det\,(I - T\alpha)$ are well-defined, independent of choice of orthonormal basis, and are related by

$$\det\,(I - T\alpha) = \exp\left(-\sum_{m=1}^{\infty} Tr\,(\alpha^m)T^m/m\right). \tag{2.10}$$

Let δ denote the operator

$$\delta\colon g(T) \to \frac{g(T)}{g(qT)}.$$

Then (2.9) is equivalent by (2.10) to

$$\det\,(I - T\alpha)^{\delta^{n+k}} = L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\}_{i=1}^k)\}, T)^{(-1)^{n+k+1}}. \tag{2.11}$$

3. Reduction step

Our method gives significantly better estimates for the Newton polygon of $\det\,(I - T\alpha)$ in the case when all the χ_i have exponent $p-1$, i.e. all the j_i are divisible by $(q-1)/(p-1) = 1 + p + \cdots + p^{a-1}$. If

$$j_i = (q-1)v_i/(p-1) \tag{3.1}$$

with $0 \leqslant v_i \leqslant p-2$, we may define

$$\alpha_0 = \psi_p \circ \tau^{-1} \circ F(X, Y)$$

an $\Omega_1(=\mathbb{Q}_p(\zeta_p))$-linear completely continuous endomorphism of $L_{(j)}(p/(p-1))$. We have the relationship

$$\alpha = \alpha_0^a \tag{3.2}$$

and the relationship of Fredholm determinants given by

$$\det(I - T^a\alpha)^a = \prod \det_{\Omega_1}(I - \zeta T\alpha_0) \tag{3.3}$$

where the product is taken over all roots ζ of $\zeta^a = 1$. This establishes the following result.

THEOREM (3.4). *Under the hypothesis* (3.1), *a point* $(x, y) \in \mathbb{R}^2$ *is a vertex of the Newton polygon of* $\det(I - T\alpha)$ *computed with respect to the valuation* "ord_q" *if and only if* (ax, ay) *is a vertex of the Newton polygon of* $\det_{\Omega_1}(I - T\alpha_0)$ *computed with respect to the valuation* "ord".

We are thus reduced in the case (3.1) to estimating the Newton polygon of $\det_{\Omega_1}(I - T\alpha_0)$. In the other cases we estimate the Newton polygon of $\det(I - T\alpha)$ directly and use the somewhat weaker results.

4. Estimates for Frobenius; Newton polygon

Let $\{\xi_1, \ldots, \xi_a\}$ be an integral basis for Ω_0 over Ω_1 that has the property of p-adic directness [7, §3c], i.e., for any $\{\beta_1, \ldots, \beta_a\} \subseteq \Omega_1$,

$$\text{ord} \sum_{j=1}^{a} \beta_j \xi_j = \min_j \{\text{ord}\, \beta_j\}.$$

An orthonormal basis for $L_{(j)}(p/p-1)$ as an Ω_1-linear space can then be obtained from the set

$$I_{(j)} = \{\xi_l X^{\mathbf{R}} Y^{\mathbf{M}} \mid 1 \leqslant l \leqslant a, (\mathbf{R}, \mathbf{M}) \in \Gamma_{(j)}\}$$

by multiplying each $i \in I$ by a suitable constant $\gamma_i \in \Omega_0$. We obtain first estimates for the Frobenius matrix with respect to $I_{(j)}$. In order to do so, it is convenient to rewrite $F(X, Y)$ in terms of $I_{(j)}$; in particular, if

$$F(\mathbf{R}, \mathbf{M}) = \sum_{l=1}^{a} F(\mathbf{R}, \mathbf{M}, l)\xi_l$$

then ord $(F(\mathbf{R}, \mathbf{M}, l)) \geqslant w(\mathbf{R}, \mathbf{M})/(p-1)d_0$, and

$$F(X, Y) = \sum_{\xi_l X^{\mathbf{R}} Y^{\mathbf{M}} \in I_{(j)}} F(\mathbf{R}, \mathbf{M}, l)\xi_l X^{\mathbf{R}} Y^{\mathbf{M}}.$$

If we write

$$\tau^{-1}(\xi_l \xi_{\tilde{l}}) = \sum_{l'=1}^{a} u(l, \tilde{l}; l')\xi_{l'}, \text{ with } u(l, \tilde{l}; l') \in \Omega_1,$$

then ord $u(l, \tilde{l}; l') \geqslant 0$. Furthermore, if $i = \xi_l X^{\mathbf{R}} Y^{\mathbf{M}}$, $i' = \xi_{l'} X^{\mathbf{R}'} Y^{\mathbf{M}'}$ belong to $I_{(j)}$ and we write

$$\alpha_0(i) = \sum_{i' \in I_{(j)}} A(i, i')i',$$

then

$$A(i, i') = \sum_{\tilde{l}=1}^{a} \tau^{-1}(F(p\mathbf{R}' - \mathbf{R}, p\mathbf{M}' - \mathbf{M}, \tilde{l})u(l, \tilde{l}; l')$$

and with the given notation we obtain

$$\begin{aligned} \text{ord } A(i, i') &\geqslant \frac{w(p\mathbf{R}' - \mathbf{R}, p\mathbf{M}' - \mathbf{M})}{(p-1)d_0} \\ &\geqslant \frac{pw(\mathbf{R}', \mathbf{M}') - w(\mathbf{R}, \mathbf{M})}{(p-1)d_0} \end{aligned} \tag{4.1}$$

using the properties (2.3).

If we write

$$\det_{\Omega_1}(I - T\alpha_0) = 1 + \sum_{m \geqslant 1} c_m T^m$$

then $(-1)^m c_m$ is the sum of all $m \times m$ principal minors of the matrix of α_0 with respect to an orthonormal basis. Since $I_{(j)}$ differs from an orthonormal basis only by scalar multiples, it is easy to see that $(-1)^m c_m$ is also the sum of the $m \times m$ principal minors of the matrix $(A(i, i'))_{i,i' \in I_{(j)}}$. Thus

$$\text{ord } c_m \geqslant \inf\left\{\sum_{r=1}^{m} \text{ord } A((\mathbf{R}^{(r)}, \mathbf{M}^{(r)}, l^{(r)}), (\mathbf{R}^{\sigma(r)}, \mathbf{M}^{\sigma(r)}, l^{\sigma(r)}))\right\} \tag{4.2}$$

where the infimum is taken over all collections $\{i^{(r)} = (\mathbf{R}^{(r)}, \mathbf{M}^{(r)}, l^{(r)})\}_{r=1,2,\ldots m}$ of m distinct elements in $I_{(j)}$, and over all $\sigma \in$ permutations on m letters. As a consequence of (4.1), we obtain from (4.2)

$$\operatorname{ord} c_m \geqslant d_0^{-1} \cdot \inf \sum_{i=1}^{m} w(i^{(r)}) \tag{4.3}$$

in which the infimum is taken over all collections of m distinct elements from $I_{(j)}$.

For each $K \in \mathbb{Z}_{\geqslant 0}$, we define

$$\begin{aligned} W(K) &= a^{-1} \operatorname{card} \{ i = \xi_l X^{\mathbf{R}} Y^{\mathbf{M}} \in I_{(j)} \mid (p-1) w(\mathbf{R}, \mathbf{M}) = K \} \\ &= \operatorname{card} \{ X^{\mathbf{R}} Y^{\mathbf{M}} \in \Gamma_{(j)} \mid (p-1) w(\mathbf{R}, \mathbf{M}) = K \}. \end{aligned} \tag{4.4}$$

Summarizing our above result (4.3) we have

THEOREM (4.5). *Suppose the integers $\{j_i\}_{i=1}^k$ are all divisible by $(q-1)/(p-1)$. Then the Newton polygon of* $\det(I - T\alpha)$ *is contained in the convex closure in $\mathbb{R}^2$ of the points* $(0, 0)$ *and*

$$\left(\sum_{K=0}^{N} W(K), (p-1)^{-1} d_0^{-1} \sum_{K=0}^{N} K W(K) \right)_{N=0,1,2,\ldots}.$$

It remains to compute $W(K)$. This will be done in the next theorem. We employ the following notation:

$$c_n(m) = \binom{n+m-1}{m}, \quad \mathbf{j} = \frac{q-1}{p-1} \mathbf{v}.$$

THEOREM (4.6). *If $K < |\mathbf{v}| d_0$, then $W(K) = 0$. Assume $K \geqslant |\mathbf{v}| d_0$. Let $K_0 = K - |\mathbf{v}| d_0 = Q d_0 (p-1) + r$ with $0 \leqslant Q$, $0 \leqslant r < d_0(p-1)$.*

Case (i). $r \neq 0$: If, in addition, $K_0 + \mathbf{v} \cdot \mathbf{d} \not\equiv 0 \bmod (p-1)$, then $W(K) = 0$. If $K_0 + \mathbf{v} \cdot \mathbf{d} = Q' \cdot (p-1)$ then

$$W(K) = \sum c_n(Q' + \mathbf{N} \cdot \mathbf{d} - |\mathbf{N}| d_0)$$

where $\mathbf{N} = (N_1, \ldots, N_k) \in (\mathbb{Z}_{\geqslant 0})^k$ and the sum is taken over all such $\mathbf{N}$ satisfying $0 \leqslant |\mathbf{N}| \leqslant Q$.

Case (ii). $r = 0$: *If* $\mathbf{v} \cdot \mathbf{d} = E(p - 1) + r'$ *with* $0 < r' < p - 1$, *then*

$$W(K) = \sum_{|\mathbf{N}|=Q} c_{n+1}(E + \mathbf{N} \cdot \mathbf{d}).$$

If $\mathbf{v} \cdot \mathbf{d} = E(p - 1)$ *then*

$$W(K) = \sum_{|\mathbf{N}|=Q} c_{n+1}(E + \mathbf{N} \cdot \mathbf{d}) + \sum_{|\mathbf{N}|=0}^{Q-1} c_n(Qd_0 + E + \mathbf{N} \cdot \mathbf{d} - |\mathbf{N}|d_0).$$

Proof. Let

$$\mathbf{M} = \tilde{\mathbf{M}} + \frac{1}{q-1}\mathbf{j} = \tilde{\mathbf{M}} + \frac{1}{p-1}\mathbf{v}.$$

Then

$$(p - 1)w(\mathbf{R}, \mathbf{M}) = \max\{(p - 1)|\mathbf{R}| - (p - 1)\tilde{\mathbf{M}} \cdot \mathbf{d} - \mathbf{v} \cdot \mathbf{d}, 0\} + |\tilde{\mathbf{M}}|d_0(p - 1) + |\mathbf{v}|d_0.$$

So that $(p - 1)w(\mathbf{R}, \mathbf{M}) = K_0 + |\mathbf{v}|d_0$ if and only if

$$\max\{(p - 1)|\mathbf{R}| - (p - 1)\tilde{\mathbf{M}} \cdot \mathbf{d} - \mathbf{v} \cdot \mathbf{d}, 0\} + |\tilde{\mathbf{M}}|d_0(p - 1) = K_0 \tag{4.7}$$

and all terms involved are rational integers. Clearly if K_0 is not a multiple of $d_0(p - 1)$ a solution $(\mathbf{R}, \tilde{\mathbf{M}})$ of (4.7) must have positive max term on the left. But then $K_0 + \mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod (p - 1)$. On the other hand, if $K_0 + \mathbf{v} \cdot \mathbf{d} = Q'(p - 1)$, the solutions $(\mathbf{R}, \tilde{\mathbf{M}})$ of (4.7) are those for which

$$|\mathbf{R}| + \sum_{i=1}^{k} \tilde{M}_i(d_0 - d_i) = Q'$$

$$|\tilde{\mathbf{M}}|d_0(p - 1) \leqslant K_0$$

which completes the case (i).

If $r = 0$ and $(\mathbf{R}, \tilde{\mathbf{M}})$ is a solution of (4.7) then $|\tilde{\mathbf{M}}| \leqslant Q$ and

$$\max\{(p - 1)|\mathbf{R}| - (p - 1)\tilde{\mathbf{M}} \cdot \mathbf{d} - \mathbf{v} \cdot \mathbf{d}, 0\} = (Q - |\tilde{\mathbf{M}}|)d_0(p - 1). \tag{4.8}$$

If $\mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod p - 1$, then the only solutions of (4.8) arise when $Q = |\tilde{\mathbf{M}}|$ and

$$|\mathbf{R}| \leqslant \tilde{\mathbf{M}} \cdot \mathbf{d} + \frac{\mathbf{v} \cdot \mathbf{d}}{p - 1}. \tag{4.9}$$

On the other hand if $\mathbf{v} \cdot \mathbf{d} = E(p - 1)$ one obtains other solutions to (4.8) in addition to the solutions of the inequality (4.9) when $|\tilde{\mathbf{M}}| = Q$. In particular $|\tilde{\mathbf{M}}| < Q$ and

$$|\mathbf{R}| = (Q - |\tilde{\mathbf{M}}|)d_0 + E + \tilde{\mathbf{M}} \cdot \mathbf{d}$$

yield the additional solutions.

This completes the proof of case (ii) and the theorem.

5. Degree of the L-function

It is known that $L(\{S_m^*(\bar{F}, \psi, \{j_i, \bar{H}_i\}_{i=1}^k)\}, T)$ is a rational function of T with coefficients in $Q(\zeta_p, \zeta_{q-1})$. (It follows again from (2.11) and the Dwork rationality criterion [6].) We write

$$L(\{S_m^*(\bar{F}, \psi, \{j_i, \bar{H}_i\}_{i=1}^k)\}, T)^{(-1)^{n+k+1}} = \prod_{i=1}^{r_1} (1 - \varrho_i T) \Big/ \prod_{j=1}^{r_2} (1 - \eta_j T) \tag{5.1}$$

so that the degree of $L(\{S_m^*\}, T)^{(-1)^{n+k+1}}$ as a rational function is $r_1 - r_2$. Inverting (2.11) and solving for the Fredholm determinant of α yields

$$\det (I - T\alpha) = D_1(T)/D_2(T)$$

where

$$D_1(T) = \prod_{i=1}^{r_1} \prod_{m=0}^{\infty} (1 - q^m \varrho_i T)^{c_{n+k}(m)}$$

$$D_2(T) = \prod_{j=1}^{r_2} \prod_{m=0}^{\infty} (1 - q^m \eta_j T)^{c_{n+k}(m)}.$$

LEMMA (5.2) [4, Corollary to Lemma 3]. If $L(\{S_m^*(\bar{F},\psi,\{j_i,\bar{H}_i\}_{i=1}^k)\},T)^{(-1)^{n+k}}$ is written as in (5.1), then

$$\sum_{i=1}^{r_1}\sum{}'(x-\operatorname{ord}_q(q^m\varrho_i))c_{n+k}(m)-\sum_{j=1}^{r_2}\sum{}'(x-\operatorname{ord}_q(q^m\eta_j))c_{n+k}(m)$$

$$\leqslant d_0^{-1}(p-1)^{-1}\sum_{K\leqslant d_0(p-1)x}(d_0(p-1)x-K)W(K) \tag{5.3}$$

where the sums Σ' are over all m such that the summands are positive. □

By a well-known formula (Knopp, Infinite Series, Ch. 14), for $l \geqslant 1$

$$\sum_{n=1}^{K} n^l = \frac{1}{l+1}K^{l+1}+\tfrac{1}{2}K^l+\frac{l}{12}K^{l-1}+O(K^{l-3}). \tag{5.4}$$

This yields at once that

$$\sum_{m\leqslant x}(x-m)c_n(m) = \frac{x^{n+1}}{(n+1)!}+\frac{1}{2}\frac{x^n}{(n-1)!}+O(x^{n-1}) \tag{5.5}$$

as $x\to+\infty$. Since $\operatorname{ord}_q(q^m\varrho) = m+\operatorname{ord}_q(\varrho)$, (5.5) implies that the left side of (5.3) equals

$$(r_1-r_2)\frac{x^{n+k+1}}{(n+k+1)!}+O(x^{n+k}). \tag{5.6}$$

It is our intention next to estimate the asymptotic behavior of the right-side of (5.3). It will be useful to establish a lemma first. Let $\{d_i\}_{i=0}^k$ and r be for the present arbitrary positive integers. Let $D_r(d_0,\ldots,d_k)$ denote the sum of all monomials in the $\{d_i\}_{i=0}^k$ of degree r

$$D_r(d_0,\ldots,d_k) = \sum_{|\mathbf{i}|=r}\prod_{l=0}^{k} d_l^{i_l}. \tag{5.7}$$

Let

$$\varphi_{r,\{d_i\}}(\lambda) = \sum\left(\sum_{i=0}^{k}N_id_i\right)^r\Big/ r! \tag{5.8}$$

where the outer sum runs over all $k+1$ tuples $(N_0,\ldots,N_k)$ of non-negative integers satisfying $\Sigma_{i=0}^k N_i = \lambda$.

LEMMA (5.9). *Assume* $\{d_i\}_{i=0}^k$ *and* r *are arbitrary positive integers. Then*

$$\varphi_{r,\{d_i\}}(\lambda) = \frac{\lambda^{r+k}}{(r+k)!} D_r(d_0, \ldots, d_k) + O(\lambda^{r+k-1}).$$

Proof. We proceed by induction on k. The case $k = 0$ is immediate. We assume the result now for k and consider

$$\frac{1}{r!} \sum_{\Sigma_{i=0}^{k+1} N_i = \lambda} \left(\sum_{i=0}^{k+1} N_i d_i \right)^r = \frac{1}{r!} \sum_{j=0}^{\lambda} \sum_{\Sigma_{i=0}^{k} N_i = j} \left(\sum_{i=0}^{k} N_i d_i + (\lambda - j) d_{k+1} \right)^r$$

$$= \frac{1}{r!} \sum_{j=0}^{\lambda} \sum_{\Sigma_{i=0}^{k} N_i = j} \sum_{l=0}^{r} \binom{r}{l} \left(\sum_{i=0}^{k} N_i d_i \right)^l ((\lambda - j) d_{k+1})^{r-l}$$

$$= \frac{1}{r!} \sum_{j=0}^{\lambda} \sum_{l=0}^{r} \binom{r}{l} \frac{l!}{(l+k)!} j^{l+k} \cdot D_l(d_0, d_1, \ldots, d_k) \cdot ((\lambda - j) d_{k+1})^{r-l}$$

$$+ O(\lambda^{r+k}) \quad \text{(using the induction hypothesis)}$$

$$= \frac{1}{r!} \sum_{j=0}^{\lambda} \sum_{l=0}^{r} \binom{r}{l} \frac{l!}{(l+k)!} \sum_{m=0}^{r-l} \binom{r-1}{m} (-1)^{r-l-m} \lambda^m j^{r+k-m}$$

$$\times (d_{k+1}^{r-l} D_l(d_0, d_1, \ldots, d_k)) + O(\lambda^{r+k})$$

$$= \frac{1}{r!} \sum_{j=0}^{r} \binom{r}{l} \frac{l!}{(l+k)!} \sum_{m=0}^{r-l} \binom{r-l}{m} (-1)^{r-l-m} \frac{\lambda^{r+k+1}}{r+k+1-m}$$

$$\times (d_{k+1}^{r-l} D_l(d_0, d_1, \ldots, d_k)) + O(\lambda^{r+k}) \quad \text{(summing over } j \text{ and using (5.4))}. \tag{5.10}$$

Now we assert

$$\sum_{m=0}^{r-l} \binom{r-l}{m} (-1)^{r-l-m} \frac{1}{r+k+1-m} = \frac{(k+l)!(r-l)!}{(r+k+1)!}, \tag{5.11}$$

for the left side is equal to

$$\int_0^1 \sum_{m=0}^{r-l} \binom{r-l}{m} (-1)^{r-l-m} z^{r+k-m}\, dz = \int_0^1 z^{r+k-1} (z^{-1} - 1)^{r-l}\, dz$$

$$= \int_0^1 z^{k+l} (1 - z)^{r-l}\, dz = \frac{\Gamma(k + l + 1)\Gamma(r - l + 1)}{\Gamma(k + r + 2)}$$

by the usual beta function evaluation.

Substituting (5.11) into the last expression in (5.10) yields finally

$$\frac{1}{r!} \sum_{\Sigma_{i=0}^{k+1} N_i = \lambda} \left(\sum_{i=0}^{k+1} N_i d_i \right)^r = \frac{\lambda^{r+k+1}}{(r + k + 1)!} \sum_{l=0}^{r} d_{k+1}^{r-l} D_l(d_0, d_1, \ldots, d_k)$$

$$+ O(\lambda^{r+k}).$$

Of course,

$$\sum_{l=0}^{r} d_{k+1}^{r-l} D_l(d_0, \ldots, d_k) = D_r(d_0, \ldots, d_{k+1})$$

which completes the induction proof of Lemma 5.9. □

Note that in the notation of Theorem (4.6), if we view Q, Q', and E as functions of K, then

$$Q = \frac{K}{d_0(p - 1)} + O(1)$$

$$Q' = \frac{K}{(p - 1)} + O(1) \tag{5.12}$$

$$E = O(1).$$

LEMMA (5.13). *Consider K such that $d_0(p - 1)$ does not divide $K - |\mathbf{v}| d_0$ (case (i) in the language of Theorem (4.6)). Then*

(i) $W(K) = 0$, *if $p - 1$ does not divide $K - |\mathbf{v}| d_0 + \mathbf{v} \cdot \mathbf{d}$.*

(ii) *if* $K - |\mathbf{v}|d_0 + \mathbf{v}\cdot\mathbf{d} \equiv 0 \bmod p - 1$, *then*

$$W(K) = \frac{K^{n+k-1}}{(d_0(p-1))^{n+k-1}(n+k-1)!} D_{n-1}(d_0, d_1, \ldots, d_k)$$

$$+ O(K^{n+k-2}).$$

Proof of (ii). Clearly (5.12) implies

$$Q' = d_0 Q + O(1)$$

If we set $N_0 = Q - |\mathbf{N}|$, then

$$Q' = N_0 d_0 + |\mathbf{N}| d_0 + O(1)$$

so that

$$W(K) = \sum c_n \left(\sum_{i=0}^{k} N_i d_i + O(1) \right)$$

where the sum runs over ordered $k + 1$-tuples of non-negative integers $(N_0, \ldots, N_k)$ satisfying $\Sigma_{i=0}^{k} N_i = Q$. Using (5.4) and lemma (5.9) with $r = n - 1$, $\lambda = Q$ we obtain immediately the desired conclusion. □

In precisely the same way we prove the following result.

LEMMA (5.14). *Consider K such that* $d_0(p - 1)$ *does divide* $K - |\mathbf{v}|d_0$ *(case (ii) in the language of Theorem* (4.6)).

(i) *If* $p - 1$ *does not divide* $\mathbf{v}\cdot\mathbf{d}$, *then*

$$W(K) = \frac{K^{n+k-1}}{(d_0(p-1))^{n+k-1}(n+k-1)!} D_n(d_1, \ldots, d_k) + O(K^{n+k-2})$$

(ii) *If* $(p - 1)$ *does divide* $\mathbf{v}\cdot\mathbf{d}$, *then*

$$W(K) = \frac{K^{n+k-1}}{(d_0(p-1))^{n+k-1}(n+k-1)!} [D_n(d_1, \ldots, d_k)$$

$$+ D_{n-1}(d_0, \ldots, d_k)] + O(K^{n+k-2}).$$

□

THEOREM (5.15). *The right side of* (5.3) *equals*

$$\frac{x^{n+k+1}}{(n+k+1)!} D_n(d_0, d_1, \ldots, d_k) + O(x^{n+k}).$$

Proof. The calculation will depend on whether or not $\mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod (p-1)$. Assume first that $\mathbf{v} \cdot \mathbf{d} \not\equiv 0 \bmod (p-1)$. Consider the contribution to the right-side of (5.3) coming from those K for which $K_0 \not\equiv 0 \bmod (d_0(p-1)$; more particularly, those K for which $K_0 + \mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod (p-1)$. Using the notation of (4.6) we may write $K = Q'(p-1) + |\mathbf{v}| d_0 - \mathbf{v} \cdot \mathbf{d}$ and re-express the right-side of (5.3) in terms of Q', namely

$$d_0^{-1} \sum_{Q' \leqslant d_0 x + c_0} (d_0 x - Q' + c_0) \left(\frac{(Q' d_0^{-1})^{n+k-1}}{(n+k-1)!} D_{n-1}(d_0, d_1, \ldots, d_k) \right.$$
$$\left. + \, O(Q'^{n+k-2}) \right)$$

where c_0 is a fixed constant ($c_0 = (\mathbf{v} \cdot \mathbf{d} - |\mathbf{v}| d_0)/(p-1)$) independent of Q', and where we have used the estimates of (5.13). This yields a contribution to the right side of (5.3) equal to

$$\frac{x^{n+k+1}}{(n+k+1)!} d_0 D_{n-1}(d_0, d_1, \ldots, d_k) + O(x^{n+k}).$$

The contribution from those K for which $d_0(p-1)$ divides K_0 is obtained in similar fashion writing $K = Qd_0(p-1) + |\mathbf{v}| d_0$ and re-expressing the right-side of (5.3) in terms of Q. This yields a contribution to the right side of (5.3) equal to

$$\frac{x^{n+k+1}}{(n+k+1)!} D_n(d_1, d_2, \ldots, d_k) + O(x^{n+k}).$$

Since

$$D_n(d_0, d_1, \ldots, d_k) = D_n(d_1, \ldots, d_k) + d_0 D_{n-1}(d_0, d_1, \ldots, d_k),$$

this completes the proof of the theorem when $\mathbf{v} \cdot \mathbf{d} \not\equiv 0 \bmod (p-1)$.

On the other hand, if $\mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod (p-1)$, then the two cases $K_0 + \mathbf{v} \cdot \mathbf{d} \equiv 0 \bmod p - 1$ (i.e. $K_0 \equiv 0 \bmod p - 1$) and $K_0 \equiv 0 \bmod (d_0(p-1))$

are not independent. When $K_0 \equiv 0 \pmod{p-1}$ in *all* cases (whether or not $K_0 \equiv 0 \bmod d_0(p-1)$) there is a contribution to the right side of (5.3) equal to

$$\frac{x^{n+k+1}}{(n+k+1)!} d_0 D_{n-1}(d_0, d_1, \ldots, d_k) + O(x^{n+k}).$$

When $K_0 \equiv 0 \bmod d_0(p-1)$, there is an *additional* contribution to the right-side of (5.3) equal to

$$\frac{x^{n+k+1}}{(n+k+1)!} D_n(d_1, \ldots, d_k) + O(x^{n+k})$$

This completes the proof of the theorem.

THEOREM (5.16). *If*

$$\mathbf{j} = \frac{(q-1)}{(p-1)} \mathbf{v} \quad \text{where } \mathbf{v} = (v_1, \ldots, v_k) \in (\mathbb{Z}_{\geqslant 0})^k,$$

then

$$0 \leqslant \deg L(\{S_m^*(\bar{F}, \psi, \{j_i, \bar{H}_i\})\}, T)^{(-1)^{n+k-1}} \leqslant D_n(d_0, d_1, \ldots, d_k)$$

Proof. The right-most inequality follows from a comparison of (5.6) and (5.15) as $x \to +\infty$. The inequality on the left is a consequence via (2.11) of the following lemma provided by the referee.

LEMMA (5.11). *Let f be a rational function, $f(0) = 1$ such that for some $m \in \mathbb{N}$, $f^{(1-\varphi)^{-m}}$ is an entire function (where $f(t)^{\varphi} = f(qt)$ so that $\delta = 1 - \varphi$). Then degree $f \geqslant 0$.*

Proof. Write $f = \prod_{i=1}^{s} (1 - \lambda_i t)^{\varepsilon_i}$, $\varepsilon_i = \pm 1$. We let the cyclic multiplicative group $\langle q \rangle$ act on Ω by multiplication. The intersection of the orbits under this action with $\{\lambda_1, \ldots, \lambda_s\}$ partitions this set. For each equivalence class, we choose τ so that $\lambda = q^m \tau$ (for some $m \geqslant 0$) for every λ in the class. Then $f = \prod_{j=1}^{l} (1 - \tau_j t)^{h_j(\varphi)}$ where each $h_j(\varphi) \in \mathbb{Z}[\varphi]$ and $j \neq j'$ implies $\tau_j / \tau_{j'} \notin \langle q \rangle$. Thus $(1 - \tau_j t)^{h_j(\varphi)(1-\varphi)^{-m}}$ has no factor in common with $(1 - \tau_{j'} t)^{h_{j'}(\varphi)(1-\varphi)^{-m}}$. As a consequence, $(1 - \tau_j t)^{h_j(\varphi)(1-\varphi)^{-m}}$ is entire, and we may assume $f = (1 - \tau t)^{h(\varphi)}$. Now

$$h(\varphi)(1-\varphi)^{-m} = \sum_{n \geqslant 0} a_n \varphi^n \quad (a_n \geqslant 0)$$

so we may write $h(x) = (1 - x)^m \sum_{n\geqslant 0} a_n x^n$. Thus $h(x) \geqslant 0$ for x in the interval $[0, 1)$, so that $h(1) \geqslant 0$. But $h(1) = \text{degree} f$.

REMARK 1. We also wish to treat the case in which $\bar{F}(x) = 0$ identically. (Or what is almost the same thing, for our purposes, $d_0 = 0$). We revise the discussion in preceding sections as follows. Let

$$\Gamma'_{(j)} = \left\{(\mathbf{R}, \mathbf{M}) \in (\mathbb{Z}_{\geqslant 0})^n \times \left(\frac{1}{q-1}\mathbb{Z}_{\geqslant 0}\right)^k \middle| M_i \equiv j_i/(q-1) \bmod \mathbb{Z}, \right.$$

$$\left. |\mathbf{R}| \leqslant \mathbf{M}\cdot\mathbf{d}\right\}, \tag{2.1$'$}$$

$$w'(\mathbf{R}, \mathbf{M}) = |\mathbf{M}|. \tag{2.2$'$}$$

In terms of these, we define

$$L'_{(j)}(b, c) = \left\{ \sum_{(\mathbf{R},\mathbf{M})\in\Gamma'_{(j)}} C(\mathbf{R}, \mathbf{M}) X^{\mathbf{R}} Y^{\mathbf{M}} \right| \tag{2.4$'$}$$

$$\left. C(\mathbf{R}, \mathbf{M}) \in \Omega_0, \text{ord } C(\mathbf{R}, \mathbf{M}) \geqslant c + bw'(\mathbf{R}, \mathbf{M})\right\},$$

$$L'_{(j)}(b) = \bigcup_{c\in\mathbb{R}} L'_j(b, c),$$

$$F'(X, Y) = \prod_{i=1}^{k} \prod_{w(i)\in\Lambda^{(i)}} \theta_\infty(A^i_{w(i)} X^{w(i)} Y_i) \in L'_{(0)}\left(\frac{1}{p-1}, 0\right) \tag{2.6$'$}$$

where we have preserved the notation for $\bar{H}_i$ from Section 2. Furthermore, let

$$\begin{aligned} F'_0(X, Y) &= \prod_{j=0}^{a-1} \tau^j F'(X^{p^j}, Y^{p^j}) \in L'_{(0)}\left(\frac{p}{q(p-1)}, 0\right), \\ \alpha' &= \psi_q \circ F'_0(X, Y) \circ i \colon L'_{(j)}\left(\frac{p}{p-1}\right) \to L'_{(j)}\left(\frac{p}{p-1}\right) \end{aligned} \tag{2.8$'$}$$

so that

$$(q^m - 1)^{n+k}\, Tr\, (\alpha')^m = S^*_m\, (0, \psi; \{j_i, \bar{H}_i\}_{i=1}^k) \tag{2.9$'$}$$

and

$$\det (I - \alpha' T)^{\delta^{n+k}} = L(\{S_m^* (0, \psi; \{j_i, \bar{H}_i)\}_{i=1}^k)\}, T)^{(-1)^{n+k+1}}. \qquad (2.11)'$$

If $\mathbf{j} = (q - 1)\mathbf{v}/(p - 1)$, we define as in §3, $\alpha_0' = \psi_p \circ \tau^{-1} \circ F'(X, Y)$ and obtain the analogous result to Theorem (3.4) for α_0' and α'.

The argument is now clear: The Newton polygon of $\det (I - T\alpha')$ may be estimated as follows:

THEOREM (4.5)′. *Suppose* $\bar{F}(x) = 0$ *identically. Suppose* $\mathbf{j} = (q - 1)\mathbf{v}/(p - 1)$. *Then the Newton polygon of* $\det (I - T\alpha')$ *is contained in the convex closure in* $\mathbb{R}^2$ *of the points* $(0, 0)$ *and*

$$\left(\sum_{K'=0}^{N} W'(K'), (p - 1)^{-1} \sum_{K=0}^{N} K' W'(K')\right)_{N=0,1,2\ldots}$$

where

$$W'(K) = \operatorname{card} \{X^{\mathbf{R}} Y^{\mathbf{M}} \in \Gamma'_{(j)} | (p - 1)w'(\mathbf{R}, \mathbf{M}) = K\}.$$

We prove the following in the same way as (4.6).

THEOREM (4.6)′. *If* $K' < |\mathbf{v}|$, *then* $W'(K') = 0$. *Assume* $K' \geqslant |\mathbf{v}|$, *let* $K_0' = K' - |\mathbf{v}| = \tilde{Q}'(p - 1) + r$, $0 \leqslant r < p - 1$. Then

$$W'(K') = \begin{cases} 0 & \text{if } r \neq 0 \\ \Sigma\, c_{n+1}\left(\mathbf{N} \cdot \mathbf{d} + \left[\dfrac{\mathbf{v} \cdot \mathbf{d}}{p - 1}\right]\right), & \text{if } r = 0, \end{cases}$$

where the sum on the right is taken over k-tuples $\mathbf{N} = (N_1, \ldots, N_k)$ *of non-negative integers satisfying* $|\mathbf{N}| = \tilde{Q}'$.

This result leads to the following theorem in the same way as (5.16) was proved.

THEOREM (5.16)′. *If* $\mathbf{j} = (q - 1)\mathbf{v}/(p - 1)$ *where* $\mathbf{v} = (v_1, \ldots, v_k) \in (\mathbb{Z}_{\geqslant 0})^k$, *then*

$$0 \leqslant \deg L (\{S_m^* (0, \psi; \{j_i, \bar{H}_i\})\}, T)^{(-1)^{n+k+1}} \leqslant D_n(d_1, \ldots, d_k).$$

REMARK 2. In the case $j_i \neq 0$ for all $i = 1, \ldots, k$ then as we have shown in Section 1

$$\text{degree } L(\{S_m^*(0, \psi; \{j_i, \bar{H}_i\})\}, T)^{(-1)^{n+k+1}}$$

$$= \text{degree } L^*(\bar{H}_1, \ldots, \bar{H}_k; \chi_1, \ldots, \chi_k; T)^{(-1)^{n+1}} \tag{5.18}$$

where the latter uses the notation of [2]. However, since $D_n(d_1, \ldots, d_k) \leqslant (\Sigma_{i=1}^k d_i)^n$ (and this is a strict inequality unless $n = 1$ or $k = 1$) the upper estimate of (5.16)′ is an improvement on the upper estimate of [2, Theorem 4], which we believed then to be best possible and generically attained. The argument we used in [2] involved estimating degree $Z(X, T)$ where X is the complement in $\mathbb{A}^n$ of the hypersurface defined by the vanishing of the polynomial $X_1X_2 \ldots X_n\bar{H}_1\bar{H}_2 \ldots \bar{H}_k$ and then utilizing the known relationship

$$\text{degree } Z(X, T) = \text{degree } L^*(\bar{H}_1, \ldots, \bar{H}_k; \chi_1, \ldots, \chi_k; T). \tag{5.19}$$

However our method of estimating degree $Z(X, T)$ treated $\bar{H} = \bar{H}_1\bar{H}_2 \ldots \bar{H}_k$ as it would a generic polynomial of degree $\Sigma_{i=1}^k d_i$ and did not make use of the special feature of $\bar{H}$ namely its *reducibility* to improve the estimate to $D_n(d_1, \ldots, d_k)$.

REMARK 3. We believe the estimates of (5.16) and (5.16)′ are generically attained. We note that in the case of (5.16)′ if

$$\bar{H}_i(X) = \bar{a}_{i1}X_1 + \bar{a}_{i2}X_2 + \bar{b}_i; \quad 1 \leqslant i \leqslant k$$

and if

$$\{X_1, X_2, \bar{H}_1(X), \ldots, \bar{H}_k(X)\}$$

are in general position in the sense that no three of them intersect in $\mathbb{P}^2_{\mathbb{F}_q}$ then

$$\text{card } X(\mathbb{F}_{q^m}) = q^{2m} - (s + 2)q^m + \binom{s+2}{2} \tag{5.20}$$

where here again X is the complement in $\mathbb{A}^2_{\mathbb{F}_q}$ of the hypersurface defined by the vanishing of the polynomial $X_1X_2 \Pi_{i=1}^k \bar{H}_i$. Using (5.18), (5.19) and

(5.20), we obtain

$$\text{degree } L(\{S_m^*(0, \psi; \{j_i, \bar{H}_i\})\}, T)^{(-1)^{k+1}} = \binom{s+1}{2} \tag{5.21}$$

which is the upper estimate in (5.16)′ when $n = 2$ and $d_1 = \cdots = d_k = 1$. It does not seem difficult to extend this example to the case of linear hypersurfaces in $\mathbb{A}^n_{\mathbb{F}_q}$ when $n > 2$.

REMARK 4. (cf. Remark following Theorem 4 in [2].) By the result of Deligne [11] on Euler–Poincaré characteristics, degree $L(\{S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$ and degree $L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$ are independent of the choice of the $\{j_i\}_{i=1}^k$. In particular, we may conclude the following.

THEOREM (5.22). *For arbitrary choice of*

$$\mathbf{j} = (j_1, \ldots, j_k) \in (\mathbb{Z}_{\geqslant 0})^k,$$

$$0 \leqslant \text{degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)^{(-1)^{n+k+1}} \leqslant D_n(d_0, d_1, \ldots, d_k).$$

Using (1.5) and (5.22) we find in the particular case when none of the j_i's are divisible by $q - 1$ the following.

COROLLARY (5.23). *Let H_0 be the union of the coordinate hyperplanes. Then*

$$0 \leqslant \text{degree } L(\{K_m(\mathbb{A}^n_{\mathbb{F}_q} - H_0; \underline{f}, \psi; \{\underline{g}_i, \chi_i\}_{i=1}^s)\}, T)^{(-1)^{n+1}}$$

$$\leqslant D_n(d_0, d_1, \ldots, d_s)$$

where $d_0 = \deg \bar{f}(X)$, $d_i = \deg \bar{g}_i(X), \bar{f}(X)$ and $\{\bar{g}_i(X)\}_{i=1}^s \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$ inducing $\underline{f}$ and $\{\underline{g}_i\}_{i=1}^s$ respectively on V.

REMARK 5. Let us renumber if necessary so that $j_i = 0$, if $i = 1, \ldots, t$ and $j_i \neq 0$, if $t < i \leqslant k$. Let $\mathscr{S} = \{X_1, \ldots, X_n\}$, $\mathscr{T} = \{1, \ldots, t\}$, $\mathscr{U} = \{t+1, \ldots, k\}$, $\mathscr{V} = \mathscr{T} \cup \mathscr{U}$. For each subset B of $\mathscr{T}$, we define $B' = B \cup \mathscr{U}$, a subset of $\mathscr{V}$. Let $A \subseteq \mathscr{S}$, $B \subseteq \mathscr{T}$ denote arbitrary subsets. We define

$$S_{m,A \cup B} = S_m(\bar{F}_A, \psi; \{j_i, \bar{H}_{i,A}\}_{B'})$$

where if $\bar{f}(X) \in \mathbb{F}_q[\mathscr{S}]$ then $\bar{f}_A(X) \in \mathbb{F}_q[A]$ is the result of specializing the variables X_i, $i \notin A$, to equal zero, and where if $\mathscr{C}$ is a collection indexed by $\mathscr{V}$, $\mathscr{C}_{B'}$ is the subcollection indexed by B'. A similar definition is immediate for $S^*_{m,A\cup B}$. Then

$$S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\}) = \sum_{\substack{A\subseteq\mathscr{S};\\ B\subseteq\mathscr{T}}} S^*_{m,A\cup B}.$$

In terms of L-functions

$$L(\{S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T) = \prod_{\substack{A\subseteq\mathscr{S};\\ B\subseteq\mathscr{T}}} L(\{S^*_{m,A\cup B}\}, T). \tag{5.24}$$

From (5.16) we obtain

$$0 \leqslant \deg L(\{S^*_{m,A\cup B}\}, T)^{(-1)^{|A|+|B|+k-t}} \leqslant D_{|A|}(d_0, \{d_i\}_{B'}) \tag{5.25}$$

where $|A|$ (respectively $|B|$) denotes the cardinality of the given set. Utilizing the weaker estimate

$$|\deg L(\{S^*_{m,A\cup B}\}, T)| \leqslant D_{|A|}(d_0, \{d_i\}_{B'}\},$$

we obtain via (1.3) the following result

THEOREM (5.26). *If V is defined over $\mathbb{F}_q$ by the simultaneous vanishing of $\{\bar{H}_i(X)\}_{i=1}^t \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$ then*

$$|\deg L(\{K_m(V, \bar{F}, \psi; \{\bar{H}_i, \chi_i\}_{i=t+1}^k)\}, T)| \leqslant \sum_{\substack{A\subseteq\mathscr{S}\\ B\subseteq\mathscr{T}}} D_{|A|}(d_0, \{d_i\}_{B'}).$$

Greater precision may of course be obtained if the upper estimate in (5.25) is attained in this case for all $A \subseteq \mathscr{S}$ $B \subseteq \mathscr{T}$. Finally we note that the following estimate follows directly from (5.26).

THEOREM (5.27). *Assume V is defined over $\mathbb{F}_q$ by the simultaneous vanishing of $\{\bar{h}_i(X)\}_{i=1}^t \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$ where degree $\bar{h}_i(X) = d_i$. Let $\underline{F}$ and $\{\underline{g}_i\}_{i=t+1}^k$ be regular functions on V induced by polynomials $\bar{F}(X)$, $\{\bar{g}_i(X)\}_{i=t+1}^k \subseteq \mathbb{F}_q[X_1, \ldots, X_n]$ of respective degrees d_0, $\{d_i\}_{i=t+1}^k$. Then*

$$|\deg L(\{K_m(V; \underline{F}, \psi; \{\underline{g}_i, \chi_i\}_{i=t+1}^k)\}, T)|$$

$$\leqslant 2^t D_n(d_0 + 1, d_1 + 1, \ldots, d_k + 1).$$

Proof. Fixing A and using

$$D_{|A|}(d_0, \{d_i\}_{B'}) \leqslant D_{|A|}(d_0, d_1, \ldots, d_k)$$

we get immediately from (5.24) that the upper estimate is majorized by

$$2^t \sum_{A \subseteq \mathscr{S}} D_{|A|}\,(d_0, d_1, \ldots, d_k).$$

A calculation with binonical coefficients then yields

$$\sum_{r=0}^{n} \binom{n}{r} D_r\,(d_0, d_1, \ldots, d_k) \leqslant D_n(d_0 + 1, \ldots, d_k + 1)$$

which concludes the proof of (5.27).

6. Total degree of the L-function

In this section we estimate the total degree of $L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$. Suppose first that the integers $\{j_i\}_{i=1}^k$ are all divisible by $(q - 1)/(p - 1)$. We follow the method of [5] to deduce this estimate from the lower bound for the Newton polygon of det $(I - T\alpha)$ (Theorems (4.5) and (4.6)). Recall the basic idea: From (5.1), (2.11), and the definition of δ we have

$$\frac{\Pi(1 - \varrho_i T)}{\Pi(1 - \eta_j T)} = \prod_{m=0}^{n+k} \det\,(I - q^m T\alpha)^{(-1)^m \binom{n+k}{m}} \tag{6.1}$$

But [15, Exp. XXI, Cor. 5.5.3(iii)] says that $0 \leqslant \operatorname{ord}_q \varrho_i, \operatorname{ord}_q \eta_j \leqslant n + k$. Hence the reciprocal zeros and poles on the left hand side of (6.1) all occur among the reciprocal zeros of $\Pi_{m=0}^{n+k} \det\,(I - q^m T\alpha)^{\binom{n+k}{m}}$ of $\operatorname{ord}_q \leqslant n + k$. Let N_m be the number of reciprocal zeros of det $(I - q^m T\alpha)$ of $\operatorname{ord}_q \leqslant n + k$ (i.e., the number of reciprocal zeros of det $(I - T\alpha)$ of $\operatorname{ord}_q \leqslant n + k - m$). Then

$$\text{total degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T) \leqslant \sum_{m=0}^{n+k} \binom{n+k}{m} N_m. \tag{6.2}$$

To estimate N_m, we use Theorems (4.5), (4.6), and the fact that N_m is the total length of the projections on the x-axis of the sides of slope $\leqslant n + k - m$ of the Newton polygon of det $(I - T\alpha)$. Suppose we can find points $(x(r), y(r)) \in \mathbb{R}^2$, $r = 0, 1, 2, \ldots$ lying on or below the Newton polygon of det $(I - T\alpha)$ such that

$$\frac{y(r)}{x(r)} \geqslant r. \tag{6.3}$$

It is then clear that the total length of the projections on the x-axis of the sides of slope $\leqslant r$ of the Newton polygon of det $(I - T\alpha)$ is $\leqslant x(r)$. Hence $N_m \leqslant x(n + k - m)$ and by (6.2)

$$\text{total degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T) \leqslant \sum_{m=0}^{n+k} \binom{n+k}{m} x(n + k - m). \tag{6.4}$$

Theorem (4.5) gives us a sequence of points lying on or below the boundary of the Newton polygon of det $(I - T\alpha)$. The next step is to determine which of these points satisfy (6.3) for a given r. We begin with a simple lemma.

LEMMA 6.5. *Suppose $\tilde{W}(K)$ is a real-valued function of K and δ is a positive real number such that for $N = 0, 1, 2, \ldots$*

$$\delta N\tilde{W}(N) \geqslant \sum_{K=0}^{N-1} \tilde{W}(K).$$

Then for $N = 0, 1, 2, \ldots,$

$$\sum_{K=0}^{N} K\tilde{W}(K) \geqslant N(1 + \delta)^{-1} \sum_{K=0}^{N} \tilde{W}(K).$$

Proof. By induction on N, the case $N = 0$ being trivial. Suppose the inequality holds with $N - 1$ in place of N:

$$\sum_{K=0}^{N-1} K\tilde{W}(K) \geqslant N - 1(1 + \delta)^{-1} \sum_{K=0}^{N-1} \tilde{W}(K).$$

Adding $N\tilde{W}(N)$ to both sides:

$$\sum_{K=0}^{N} K\tilde{W}(K) \geqslant N(1 + \delta)^{-1} \sum_{K=0}^{N} \tilde{W}(K) + N(1 - (1 + \delta)^{-1})\tilde{W}(N)$$

$$- (1 + \delta)^{-1} \sum_{K=0}^{N-1} \tilde{W}(K).$$

But

$$N(1 - (1 + \delta)^{-1})\tilde{W}(N) - (1 + \delta)^{-1} \sum_{K=0}^{N-1} \tilde{W}(K)$$

$$= (1 + \delta)^{-1}\left(\delta N\tilde{W}(N) - \sum_{K=0}^{N-1} \tilde{W}(K)\right),$$

which is $\geqslant 0$ by hypothesis.

COROLLARY (6.6). *Suppose in addition that $\tilde{W}(K)$ is a non-decreasing function of K. Then for $N = 0, 1, 2, \ldots,$*

$$\sum_{K=0}^{N} K\tilde{W}(K) \geqslant \frac{N}{2} \sum_{K=0}^{N} \tilde{W}(K).$$

Proof. The hypothesis that $\tilde{W}(K)$ is non-decreasing implies that we may take $\delta = 1$ in Lemma (6.5). □

The function $W(K)$ that appears in Theorem (4.5) is not non-decreasing, as is easily seen from Theorem (4.6). However, we shall see that on a certain subsequence of the sequence of points given in Theorem (4.5) $W(K)$ behaves on average as though it were non-decreasing.

Let $E = [\mathbf{v} \cdot \mathbf{d}/(p - 1)]$ be as in Theorem (4.6) and put

$$\tilde{W}_1(Q) = \sum_{|\mathbf{N}|=Q} c_{n+1}(E + \mathbf{N} \cdot \mathbf{d})$$

where $\mathbf{N} = (N_1, \ldots, N_k) \in (\mathbb{Z}_{\geqslant 0})^k$ and $\mathbf{d} = (d_1, \ldots, d_k)$. For $0 \leqslant r < d_0(p - 1)$, if $r \equiv -\mathbf{v} \cdot \mathbf{d} (\operatorname{mod} p - 1)$ put

$$\hat{W}_r(Q) = \begin{cases} \displaystyle\sum_{|\hat{\mathbf{N}}|=Q} c_n\left(\hat{\mathbf{N}} \cdot \hat{\mathbf{d}} + \frac{r + \mathbf{v} \cdot \mathbf{d}}{p - 1}\right) & \text{if } r > 0 \\ \displaystyle\sum_{|\hat{\mathbf{N}}|=Q-1} c_n\left(\hat{\mathbf{N}} \cdot \hat{\mathbf{d}} + \frac{\mathbf{v} \cdot \mathbf{d}}{p - 1}\right) & \text{if } r = 0 \end{cases}$$

where $\hat{\mathbf{N}} = (N_0, N_1, \ldots, N_k) \in (\mathbb{Z}_{\geqslant 0})^{k+1}$ and $\hat{\mathbf{d}} = (d_0, d_1, \ldots, d_k)$; otherwise set $\hat{W}_r(Q) = 0$. It is easily seen that $\tilde{W}_1$ and $\hat{W}_r$ are non-decreasing functions of Q. We may reformulate Theorem (4.6) as follows. Put $K_0 = K - |\mathbf{v}| d_0$ and write

$$K_0 = Qd_0(p - 1) + r,$$

where $0 \leqslant r < d_0(p-1)$. Then $W(K) = 0$ if $K < |\mathbf{v}|d_0$ and for $K \geqslant |\mathbf{v}|d_0$ we have

$$W(K) = \begin{cases} \tilde{W}_1(Q) + \hat{W}_0(Q) & \text{if } r = 0 \\ \hat{W}_r(Q) & \text{if } r > 0. \end{cases} \tag{6.7}$$

Now consider N such that $N - |\mathbf{v}|d_0 \equiv -1 (\bmod d_0(p-1))$ and let $\lambda(N) = [(N - |\mathbf{v}|d_0)/(p-1)]$. Using (6.7) and the trivial estimate $K \geqslant Qd_0(p-1)$ we get

$$\sum_{K=0}^{N} KW(K) \geqslant \sum_{Q=0}^{\lambda(N)} Qd_0(p-1)\tilde{W}_1(Q) + \sum_{r=0}^{d_0(p-1)-1} \sum_{Q=0}^{\lambda(N)} Qd_0(p-1)\hat{W}_r(Q).$$

Now apply Corollary (6.6) to $\tilde{W}_1$ and $\hat{W}_r$ to conclude

$$\begin{aligned} \sum_{K=0}^{N} KW(K) &\geqslant \frac{d_0(p-1)\lambda(N)}{2}\left(\sum_{Q=0}^{\lambda(N)} \tilde{W}_1(Q) + \sum_{r=0}^{d_0(p-1)-1} \sum_{Q=0}^{\lambda(N)} \hat{W}_r(Q)\right) \\ &= \frac{d_0(p-1)\lambda(N)}{2} \sum_{K=0}^{N} W(K). \end{aligned}$$

It follows that for such N, the line from the origin through

$$\left(\sum_{K=0}^{N} W(K), (p-1)^{-1}d_0^{-1} \sum_{K=0}^{N} KW(K)\right)$$

has slope $\geqslant \lambda(N)/2$. So to get slope $\geqslant r$, we must choose $N = N_r$ such that $\lambda(N_r) \geqslant 2r$. From the definition of λ, it is clear that it suffices to take

$$N_r = 2rd_0(p-1) + d_0(p-1) - 1 + |\mathbf{v}|d_0.$$

Thus the point $(x(r), y(r))$, where

$$x(r) = \sum_{K=0}^{N_r} W(K)$$

$$y(r) = (p-1)^{-1}d_0^{-1} \sum_{K=0}^{N_r} KW(K),$$

satisfies (6.3). So from (6.4)

$$\text{total degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T) \leqslant \sum_{m=0}^{n+k} \binom{n+k}{m} \sum_{K=0}^{N_{n+k-m}} W(K). \quad (6.8)$$

Next, we estimate $\Sigma_{K=0}^{N_{n+k-m}} W(K)$. We have from (6.7) and the definition of N_r

$$\sum_{K=0}^{N_{n+k-m}} W(K) = \sum_{Q=0}^{2(n+k-m)} \tilde{W}_1(Q) + \sum_{r=0}^{d_0(p-1)-1} \sum_{Q=0}^{2(n+k-m)} \hat{W}_r(Q).$$

Set $D = \max \{d_i\}_{i=0}^{k}$. We have from the definitions of $\tilde{W}_1$, $\hat{W}_r$

$$\tilde{W}_1(Q) \leqslant c_k(Q)c_{n+1}\left(QD + \sum_{i=1}^{k} d_i\right) \leqslant c_k(Q)c_{n+1}((Q+k)D)$$

$$\hat{W}_r(Q) \leqslant c_{k+1}(Q)c_n\left(QD + \sum_{i=0}^{k} d_i\right) \leqslant c_{k+1}(Q)c_n((Q+k+1)D)$$

$$\text{if } r \equiv -\mathbf{v}\cdot\mathbf{d} (\mathrm{mod}\, p-1)$$

$$\hat{W}_r(Q) = 0 \quad \text{if } r \not\equiv -\mathbf{v}\cdot\mathbf{d} (\mathrm{mod}\, p-1).$$

Thus

$$\sum_{Q=0}^{2(n+k-m)} \tilde{W}_1(Q) \leqslant c_{k+1}(2(n+k-m))c_{n+1}((2n+3k-2m)D)$$

$$\sum_{Q=0}^{2(n+k-m)} \hat{W}_r(Q) \leqslant c_{k+2}(2(n+k-m))c_n((2n+3k-2m+1)D)$$

$$\text{if } r \equiv -\mathbf{v}\cdot\mathbf{d} (\mathrm{mod}\, p-1)$$

$$= 0 \quad \text{if } r \not\equiv -\mathbf{v}\cdot\mathbf{d} (\mathrm{mod}\, p-1).$$

Hence

$$\sum_{K=0}^{N_{n+k-m}} W(K) \leqslant c_{k+1}(2(n+k-m))c_{n+1}(2n+3k-2m)D)$$

$$+ d_0 c_{k+2}(2(n+k-m))c_n((2n+3k-2m+1)D).$$

From (6.8) and the elementary estimate

$$\binom{n+k}{m} \leqslant 2^{n+k-1}$$

we get

$$\text{total degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$$

$$\leqslant 2^{n+k-1}(n+k+1)(c_{k+1}(2(n+k))c_{n+1}((2n+3k)D)$$

$$+ d_0 c_{k+2}(2(n+k))c_n((2n+3k+1)D). \tag{6.9}$$

Using the inequalities

$$\binom{r+s}{r} \leqslant (r+s)^r/r!$$

and $r^s/s! \leqslant e^r/(|r-s|+1)$ one sees that this last expression is less than or equal to

$$2^{n+k-1}(n+k+1)\left[\frac{e^{2n+3k}}{2n+2k+1}((2n+3k)D+n)^n/n!\right.$$

$$\left.+ d_0\frac{e^{2n+3k+1}}{2n+2k+1}((2n+3k+1)D+n-1)^{n-1}/(n-1)!\right].$$

Using $(n+1)^n/n! \leqslant e^n$ and $(n+1)^{n-1}/(n-1)! \leqslant 2e^{n-1}$, this is less than or equal to

$$2^{n+k-1}\frac{n+k+1}{2n+2k+1}e^{3(n+k)}\left[\left(\left(2+\frac{3k-2}{n+1}\right)D+\frac{n}{n+1}\right)^n\right.$$

$$\left.+ 2d_0\left(\left(2+\frac{3k-1}{n+1}\right)D+\frac{n-1}{n+1}\right)^{n-1}\right]. \tag{6.10}$$

Weakening this estimate slightly to produce a more compact formula, we have the following result.

THEOREM (6.11).

$$\text{total degree } L(\{S_m^*(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$$

$$\leqslant (2e^3)^{n+k}\left(\left(2 + \frac{3k}{n+1}\right)D + 1\right)^n.$$

From this we can also derive an estimate for the total degree of the L-function when the coordinate hyperplanes are included in the exponential sum. Apply Theorem (6.11) to estimate total degree of each $L(\{S^*_{m,A\cup B}\}, T)$ (see (5.24)):

$$\text{total degree } L(\{S^*_{m,A\cup B}\}, T) \leqslant (2e^3)^{a+k}\left(\left(2 + \frac{3k}{a+1}\right) + 1\right)^a.$$

where a = cardinality of A. From (5.24),

$$\text{total degree } L(\{S_m(\bar{F}, \psi; \{j_i, \bar{H}_i\})\}, T)$$

$$\leqslant \sum_{j=0}^{n}\binom{n}{j}(2e^3)^{j+k}\left(\left(2 + \frac{3k}{j+1}\right)D + 1\right)^j.$$

The estimate

$$\left(\left(2 + \frac{3k}{j+1}\right)D + 1\right)^j \leqslant \left(\left(2 + \frac{3k}{n+1}\right)D + 1\right)^n$$

leads immediately to

(6.12) THEOREM. *Assume that each character χ_i has the form $\chi_i = \chi_i^{(0)} \circ \mathbb{N}_{\mathbb{F}_q/\mathbb{F}_p}$ where $\chi_i^{(0)}$ is a non-trivial multiplicative character of $\mathbb{F}_p^*$.*

$$\text{total deg } L(\{K_m(V, \underline{f}, \psi, \{\underline{g}_i, \chi_i\}_{i=t+1}^k)\}, T)$$

$$\leqslant (2e^3)^k(2e^3 + 1)^n\left(\left(2 + \frac{3k}{n+1}\right)D + 1\right)^n$$

where $\bar{f}(X)$ and $\{\bar{g}_i(X)\}_{i=t+1}^k$ of respective degrees $d_0 = \deg \bar{f}(X)$, $d_i = \deg \bar{g}_i(X)$ induce the given regular functions $\underline{f}$ and $\{\underline{g}_i\}_{i=t+1}^k$ on V;

and where V itself is defined by the simultaneous vanishing of $\{\bar{h}_j(X)\}_{j=1}^{\ell}$ of respective degrees, $\deg \bar{h}_j = d_j$.

Consider now the special case where all the multiplicative characters are trivial, i.e., $j_i = 0$, for $i = 1, 2, \ldots, k$. Then by Lemma (1.3),

$$\text{total deg } L(\{S_m(\bar{F}, \psi; \{0, \bar{H}_i\})\}, T) = \text{total deg } L(\{K_m(V, f, \psi)\}, T),$$

where V is the affine variety defined by the vanishing of the $\bar{H}_i$'s. It is known [8] and [14] that over some extension field of $\mathbb{F}_q$, V can be defined by n equations. Furthermore, this extension of scalars does not change the total degree of $L(\{K_m(V, f, \psi)\}, T)$ (see [5]). Hence for this particular L-function, we may assume that $k \leqslant n$ in Theorem (6.12).

Corollary (6.13).

$$\text{total degree } L(\{K_m(V, \underline{f}, \psi)\}, T) \leqslant (2e^3)^n(2e^3 + 1)^n(5D + 1)^n$$

This may be compared with [5, Theorem 2], where the exponent that appears is $2n + 1$ rather than n.

We can prove somewhat weaker results without the assumption that the j_i's are divisible by $(q - 1)/(p - 1)$. Let a' be chosen so that the characters $\chi_1, \ldots, \chi_s$ have orders dividing $p^{a'} - 1$. The conclusion of Theorem (4.5) is still valid provided the Newton polygon in question is the Newton polygon of $\det(I - T\alpha)$ computed with respect to $\text{ord}_{p^{a'}}$, rather than ord_q. The subsequent arguments may then be repeated without change to establish the following.

Theorem (6.14). *If in Theorem* (6.12) *the* χ_i's *have exponent* $p^{a'} - 1$, *then the conclusion of the Theorem is valid when D is replaced by $a'D$.*

We do not know whether the conclusions of Theorem (6.12) are valid under the weaker hypothesis of Theorem (6.14).

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