COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 68, nº 1 (1988), p. 31-40

<http://www.numdam.org/item?id=CM_1988__68_1_31_0>

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Compositio Mathematica 68: 31–40 (1988)
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Lax equation representation of certain completely integrable systems

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Received 28 August 1987; accepted in revised form 24 March 1988

0. Introduction

Most of the known examples of algebraically completely integrable (ACI) systems admit a 1-parameter family of Lax representations. These representations can be used to show complete integrability and to express the solutions in terms of theta functions ([1], [6]). At present there is no general procedure for producing a 1-parameter family for a given ACI. In this paper we present a 1-parameter family $(A(z), B(z))$ of 2×2 Lax representations for the case of an ACI system whose Liouville tori are isogenous to Jacobians of hyperelliptic curves in Weierstrass form. We then apply the technique to the Kovalevskaya top [9].

Section 1 contains a brief review of the algebro-geometric method for linearizing a flow with a 1-parameter family of Lax pairs. The construction of the matrix $A(z)$ corresponding to a single hyperelliptic curve is given in Section 2. In Section 3 we consider a family of hyperelliptic curves and prove the main result. Finally, in Section 4 we present a family of Lax pairs for the Kovalevskaya top.

Shortly after this work had been completed, other Lax representations (of higher dimensions) for Kovalevskaya's top were found by Adler–van Moerbeke [2] and Haine–Horozov [8]. Both works depend on converting the Kovalevskaya system to other differential equations for which Lax representations were previously known. The referee informed us of two other solutions to the problem, [3] and [12]. The matrix $A(z)$ also appeared, in a somewhat different context, in [5] and [11].

1. ACI systems, Lax representations

Several definitions of an ACI system exist in the literature, e.g., [1], [10]. We offer the following definition.

DEFINITION 1. An algebraically completely integrable system (X, v) consists of a 2n-dimensional algebraic variety X, an n-dimensional base variety B, and an algebraic vector field v such that there is a map $\pi: X \to B$ with the following properties:

(a) the fibers $\pi^{-1}(b)$ are Zariski open in abelian varieties A_b,

(b) v is vertical and constant on $\pi^{-1}(b)$.

DEFINITION 2. A Lax representation of a differential equation (X, v) is a pair (A, B) of endomorphisms of a vector space V (or a pair of matrices) whose entries are meromorphic functions on X such that the vector field v is represented by the equation

$$\dot{A} = [B, A].$$

The algebro-geometric approach to understanding the flow associated with an ACI system (X, v) is as follows. First we associate with each point $x \in X$ a representation of the system in the form of a Lax pair $(A(z), B(z))$ with complex parameter z. The form of the Lax equation implies that as time t changes the matrix $A(z)$ remains in its coadjoint orbit:

$$A(z)(t) = (G(z)(t))A(z)(0)(G(z)(t))^{-1}.$$

Hence the spectral curve

$$C_x = \{(z, w) | \det(wI - A(z)) = 0\},$$

is time invariant. We also have an eigenvector map associated with $A(z)$:

$$\phi_x : C_x \to \mathbf{P}(V)$$

sending $(z, w) \in C_x$ to $\ker(wI - A(z))$. We assume that, generically, this is one-dimensional. The eigenvector map induces a line bundle

$$L_x = \phi_x^{-1}(\mathcal{O}_{\mathbf{P}(V)}(1)) \in \mathrm{Pic}^d(C_x),$$

where $d = \deg \phi_x(C_x)$. In certain cases as $x(t)$ flows along integral curves for v, $L_{x(t)}$ varies linearly in $\mathrm{Pic}^d(C)$. After we choose a base point in the curve C_x this is quivalent to $L_{x(t)}$ flowing linearly on the Jacobian, $\mathrm{J}(C_x)$. The solutions of the differential equation can then be expressed in terms of the theta function of C. For more details, c.f. [6].

2. The matrix

Consider a hyperelliptic curve C of genus g whose equation is given in Weierstrass form by

$$w^2 = f(z)$$

where f is a polynomial of degree $2g + 1$. In this form, the point at infinity is a distinguished Weierstrass point. Let $p_1, \ldots, p_g$ be points of C and denote the corresponding point $(p_1, \ldots, p_g)$ of $S^g C$ by $\bar{p}$. Consider the Jacobi polynomials

$$U(z, \bar{p}) = \prod_{i=1}^{g} (z - z(p_i))$$

$$V(z, \bar{p}) = \sum_{i=1}^{g} w(p_i)$$

$$\times \left[\frac{(z - z(p_1)) \ldots (z - \widehat{z(p_i)}) \ldots (z - z(p_g))}{(z(p_i) - z(p_1)) \ldots \widehat{(z(p_i) - z(p_i))} \ldots (z(p_i) - z(p_g))} \right]$$

$$W(z, \bar{p}) = \frac{f(z) - V^2(z, \bar{p})}{U(z, \bar{p})}.$$

they satisfy the identity

$$UW + V^2 = f(z),$$

cf. [10]. We consider the matrix

$$A(z, \bar{p}) = \begin{pmatrix} -V(z, \bar{p}) & U(z, \bar{p}) \\ W(z, \bar{p}) & V(z, \bar{p}) \end{pmatrix}.$$

Its characteristic polynomial

$$\det(wI - A(z)) = w^2 - f(z)$$

is just the equation of C. The eigenvector corresponding to $(z, w) \in C$ is

$$\phi_{\bar{p}}(z, w) = \begin{pmatrix} 1 \\ m(z, w, \bar{p}) \end{pmatrix}$$

where

$$m(z, w, \bar{p}) = \frac{w + V(z, \bar{p})}{U(z, \bar{p})}.$$

The entries of A are polynomials in z which depend rationally on $\bar{p} \in S^g C$. By Jacobi's inversion theorem [7], the symmetric product $S^g C$ is birationally isomorphic to $\mathrm{Pic}^g(C)$, so we may consider A as a matrix-valued rational function on $\mathrm{Pic}^g(C)$: for a generic line bundle $x \in \mathrm{Pic}^g(C)$, $A(z)(x)$ is $A(z, \bar{p})$ where $\bar{p} \in S^g C$ is the unique effective divisor satisfying

$$x = \mathcal{O}_C(\bar{p}).$$

(more precisely, this is defined when x is non-special.) In particular, we think of the entries of A as rational functions $U(z, x)$, $V(z, x)$, $W(z, x)$ of $x \in \mathrm{Pic}^g(C)$. For generic $x \in \mathrm{Pic}^g(C)$ we have the eigenvector map

$$\phi_x : C \to \mathbf{P}^1,$$

hence a linearization map

$$L : \mathrm{Pic}^g(C) \to \mathrm{Pic}^d(C) \tag{1}$$

which a-priori is defined only on a Zariski-open subset.

LEMMA 1. *The linearization map (1) is*

$$L : \mathrm{Pic}^g(C) \to \mathrm{Pic}^{g+1}(C) \tag{2}$$

$$x \mapsto x \otimes \mathcal{O}_C(\infty).$$

and is hence defined everywhere.

Proof. It suffices to show that (2) holds generically, so we assume that x is non-special and that

$$h^0(x(-\infty)) = 0,$$

i.e. that the divisor $\bar{p}$ is uniquely defined and does not contain ∞. We need to show that $m(z, w, \bar{p})$, as a function on C, has polar divisor $\bar{p} + \infty$. The divisor of $w + V(z, \bar{p})$ is

$$\sum_{i=1}^{g} j(p_i) + \sum_{k=1}^{g+1} q_k - (2g + 1) \cdot \infty$$

where j is the hyperelliptic involution, since at $j(p_i) = (-w(p_i), z(p_i))$ the value of $V(x, \bar{p})$ is $w(p_i)$ (the points q_k in the above formula have not been determined). The divisor of $U(z, \bar{p})$ is

$$\sum_{i=1}^{g} (p_i + j(p_i)) - 2g \cdot \infty.$$

Hence the divisor of $m(z, w, \bar{p})$ is

$$\sum_{j=1}^{g+1} q_j - \sum_{i=1}^{g} p_i - \infty.$$

■

LEMMA 2. *Let $x \in \mathrm{Pic}^g(C)$ be a differentiable function of t, and assume $A(z, x)$ is well-defined at $x = x(t)$. Then*

$$(\mathrm{d}/\mathrm{d}t)A(z, x) = [B(z, x), A(z, x)]$$

where

$$B(z, x) = \frac{1}{U(z, x)} \begin{pmatrix} 0 & 0 \\ (\mathrm{d}/\mathrm{d}t)V(z, x) & -(\mathrm{d}/\mathrm{d}t)U(z, x) \end{pmatrix}$$

Proof. Let

$$\Lambda = \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ m(z, w, x) & m(z, -w, x) \end{pmatrix}$$

then

$$A = P\Lambda P^{-1}.$$

Now

$$(\mathrm{d}/\mathrm{d}t)(P\Lambda P^{-1}) = [B, P\Lambda P^{-1}]$$

where

$$B = \dot{P}P^{-1}$$

$$= \begin{pmatrix} 0 & 0 \\ (\mathrm{d}/\mathrm{d}t)m(z, w, x) & (\mathrm{d}/\mathrm{d}t)m(z, -w, x) \end{pmatrix}$$

$$\times \begin{pmatrix} m(z, -w, x) & -1 \\ -m(z, w, x) & 1 \end{pmatrix} \left(\frac{U(z, x)}{-2w} \right).$$

■

3. Families of hyperelliptic Jacobians

DEFINITION 3. A family $\mathscr{C} \to B$ of hyperelliptic curves is said to be a Weierstrass family if it is given in $B \times \mathbf{P}^2$, by an equation of the form

$$w^2 = f(z, b)$$

where $b \in B$, (z, w) are affine coordinates of a point in $\mathbf{P}^2\backslash$(line at ∞), and f is a polynomial of odd degree $2g + 1$ in z.

DEFINITION 4. Given an ACI system (X, v) where X fibres

$$\pi : X \to B$$

over a base B, and the fibers $\pi^{-1}(b)$, $b \in B$ are Zariski open in Abelian varieties A_b, we say (X, v) is hyperelliptic if each A_b is isogenous to a hyperelliptic Jacobian. We say that (X, v) is Weierstrass if there is a Weierstrass family $\mathscr{C} \to B$ of hyperelliptic curves such that A_b is isogenous to $J(C_b)$, where C_b is the fiber of $\mathscr{C}$ over $b \in B$.

Our main result is:

THEOREM 1. *Any Weierstrass ACI system (X, v) admits a one-parameter family of 2×2 Lax representations $(A(z), B(z))$ whose linearization map*

$$L : X \to J(\mathscr{C}/B)$$

is an isogeny on fibers.

REMARK. On a hyperelliptic curve C in Weierstrass form, one of the Weierstrass points, denoted ∞, is distinguished. We thus have a natural isomorphism

$$\mathrm{Pic}^d(C) \to J(C),$$

for any d. The linearization map L can therefore be taken to be a map into $J(\mathscr{C}/B)$.

Proof. Since there are only countably many isogeny types, one of them must hold for the generic $b \in B$, hence for all $b \in B$, in other words, there is a morphism

$$l: X \to J(\mathscr{C}/B)$$

inducing isogenies on fibers. Since the vector field v is constant on each A_b, it descends to a (constant) vector field l_*v on $J(C_b)$. Replacing (X, v) by the quotient system $(J(\mathscr{C}/B), l_*v)$, we may assume that each fiber A_b of $\pi: X \to B$ is isomorphic to the Jacobian $J(C_b)$. By the remark, we have isomorphisms

$$X \cong \mathrm{Pic}^g(\mathscr{C}/B) \cong \mathrm{Pic}^{g+1}(\mathscr{C}/B).$$

We can thus interpret $A(z, x)$, $B(z, x)$ of Section 2 as matrix-valued functions depending polynomially on $z \in \mathrm{C}$ and rationally on $x \in X$:

$$X \cong \mathrm{Pic}^g(\mathscr{C}/B) \overset{A(z),\, B(z)}{\underset{\longrightarrow}{\longrightarrow}} sl(2, \mathrm{C}).$$

They form a Lax representation by Lemma 2, and the linearization map

$$L: \mathrm{Pic}^g(\mathscr{C}/B) \to \mathrm{Pic}^{g+1}(\mathscr{C}/B),$$

when translated to X, becomes the map described in Lemma 1. ■

4. The Kovalevskaya equations

The Kovalevskaya equations for heavy rigid body motion are

$$\begin{aligned} \dot{\Gamma} &= [\Gamma, \Omega] \\ \dot{M} &= [M, \Omega] + [\Gamma, \Xi] \end{aligned} \tag{3}$$

where

$$M = \begin{pmatrix} 0 & -r & 2q \\ r & 0 & -2p \\ -2q & 2p & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & -u & 0 \end{pmatrix}.$$

M represents angular momentum, Ω angular velocity, Γ position, and Ξ center of gravity. The first integrals

$$1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

$$2l = 2(p\gamma_1 + q\gamma_2) + \gamma_3 r$$

define the coadjoint orbit where the system is completely integrable (see [14]). Two additional integrals define the Liouville tori where the flow occurs:

$$6h = 2(p^2 + q^2) + r^2 - 2u\gamma_1$$

$$k^2 = (p^2 - q^2 + u\gamma_1)^2 + (2pq + u\gamma_2)^2.$$

S. Kovalevskaya constructed the map (using her notation)

$$s_{1,2} = 3h + \frac{R(x_1, x_2) \mp (R(x_1)R(x_2))^{1/2}}{(x_1 - x_2)^2}$$

where

$$x_{1,2} = p \pm iq$$

$$R(z) = -z^4 + 6hz^2 + 4ulz + u^2 - k^2$$

$$R(x_1, x_2) = -x_1^2 x_2^2 + 6hx_1 x_2 + 2ul(x_1 + x_2) + u^2 - k^2.$$

She showed the above map is an isogeny and that the equations transform to

$$\dot{s}_1 = \frac{2i(f(s_2))^{1/2}}{(s_1 - s_2)}, \quad \dot{s}_2 = \frac{-2i(f(s_2))^{1/2}}{(s_1 - s_2)} \tag{4}$$

where $f(z) = \{z((z - 3h)^2 + u^2 - k^2) - 2u^2l^2\}((z - 3h)^2 - k^2)$. Here our curve C is hyperelliptic of genus 2. Our point $\bar{p}$ on S^gC is

$$(p_1, p_2) = ((s_1, (f(s_1))^{1/2}), \quad (s_2, (f(s_2))^{1/2})).$$

The function $V(z, \bar{p})$ can be written

$$\begin{aligned} V(z, \bar{p}) &= \frac{w_1(z - s_2) - w_2(z - s_1)}{s_1 - s_2} \\ &= (1/2i)[(\dot{s}_1)(z - s_2) + (\dot{s}_2)(z - s_1)] \\ &= (1/2i)[(\mathrm{d}/\mathrm{d}t)(s_1 + s_2)z - (\mathrm{d}/\mathrm{d}t)(s_1 s_2)] \\ &= -(1/2i)(\mathrm{d}/\mathrm{d}t)U(z, \bar{p}). \end{aligned}$$

the matrices A and B are

$$A(z, \bar{p}) = \begin{pmatrix} (1/2i)(\mathrm{d}/\mathrm{d}t)U(z, \bar{p}) & U(z, \bar{p}) \\ \{f(z) + (1/4)((\mathrm{d}/\mathrm{d}t)U(z, \bar{p}))^2\}/U(z, \bar{p}) & -(1/2i)(\mathrm{d}/\mathrm{d}t)U(z, \bar{p}) \end{pmatrix},$$

$$B(z, \bar{p}) = (1/U(z, \bar{p}))\begin{pmatrix} 0 & 0 \\ -(1/2i)(\mathrm{d}/\mathrm{d}t)^2 U(z, \bar{p}) & -(\mathrm{d}/\mathrm{d}t)U(z, \bar{p}) \end{pmatrix}$$

where

$$\begin{aligned} U(z, \bar{p}) &= z^2 - (s_1 + s_2)z + s_1 s_2 \\ &= z^2 + (1/2q^2)(4ulp + 6h(p^2 - q^2) - (p^2 + q^2)^2 + u^2 - k^2)z \\ &\quad -(1/q^2)(2ulp(p^2 + q^2 + 3h) + p^2(u^2 - k^2 + 9h^2) + u^2l^2), \end{aligned}$$

$$(\mathrm{d}/\mathrm{d}t)U(z,\bar{p}) = -(1/2q^3)\{u^2\gamma_3^2(3pr + u) - (pr + \gamma_3 u)(2q^2(12h + u\gamma_1)$$

$$- r^2(q^2 + 3p^2)) - 2r(uq^3\gamma_2 + p^3r^2)\}z - (1/q^3)$$

$$\times \{p((p^2 + q^2)r + up\gamma_3)(9h^2 - k^2 + u^2) + ul(r(2p^2 + q^2)$$

$$+ 2\gamma_3 pu)(3h + p^2 + q^2) + l^2u^2(rp + u\gamma_3) - 2\gamma_3 lu^2pq^2\}$$

The derivatives are resolved by using (3) (u, l, h, k are all constants; $2\dot{p} = qr$, $2\dot{q} = -(pr + u\gamma_3)$, etc.).

Acknowledgement

The author gladly thanks his advisor Dr. Ron Donagi for his guidance and suggestions without which this paper would not be.

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