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## Invariant theory for linear algebraic groups II (char $k$ arbitrary)

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Given a linear algebraic group  $G$  and an action of it on a quasi-projective variety  $X$ , all defined over an algebraically closed field  $k$ , there will in general be no quasi-projective orbit space. When the group  $G$  is reductive, Mumford in GIT gave a reasonable criteria for the existence of a quasi-projective quotient using his notion of stable points. In order to generalize his concepts to arbitrary linear groups it is necessary to treat the case of unipotent group actions. If  $H$  is the unipotent radical of  $G$ , then one first must construct  $Y = X/H$  and assuming  $Y$  is well behaved, apply the technique of Mumford to the action of the reductive group  $G/H$  on  $Y$ .

There are two technical problems involved in this program. The first is to find reasonable conditions which guarantee that  $Y$  exists and is quasi-projective. The second is to insure that the pair  $(Y, G/H)$  satisfies the hypothesis required in order to apply the methods of GIT.

The results of [1] give a method for handling these problems when the ground field  $k$  has characteristic zero. The purpose of this note is to extend the key results on unipotent actions given in [1; Section 1] to the case of arbitrary characteristics. It is then a straightforward matter to extend to arbitrary algebraic groups  $G$  over arbitrary fields  $k$  the notions of stability given in [1].

Let  $X$  be a quasi-factorial variety over  $k$ , i.e.,  $B = \Gamma(X, \mathcal{O}_X)$  is a unique factorization domain and the canonical map:  $X \rightarrow \text{Spec } B$  is an open immersion. Let  $H$  be a connected unipotent algebraic group defined over  $k$ . We assume throughout that  $H$  acts on  $X$  and that the isotropy group in  $H$  of each point of  $X$  is finite. We recall here some definitions from [1].

(1) A point  $x \in X$  is *semi-stable* if  $\dim c^{-1}(cx) = \dim H$  where  $c: X \rightarrow \text{Spec } A$ ,  $A = B^H$ , is the natural map. If  $X^{ss}$  denotes the set of semi-stable points of  $X$  then  $X^{ss}$  is open and  $H$ -stable (c.f. [1]). Moreover there exists a quasi-factorial variety  $Q$  and an  $H$ -equivariant map  $\pi: X \rightarrow Q$  making  $Q$  an  $s$ -categorical quotient of  $X$  by  $H$ . This means that for any morphism  $f$ :

$X \rightarrow Y$ ,  $Y$  a separated algebraic scheme, with  $f$  constant on  $H$ -orbits, there is a unique map  $g: Q \rightarrow Y$  with  $f = g\pi$ . Further  $Q = c(X^{ss})$  is open in  $\text{Spec } A$ .

(2) A point  $x \in X$  is *stable* if there is an open neighborhood  $U$  of  $x$  with  $HU = U$  and such that  $U/H$  exists and is affine. We denote by  $X^s$  the open set of stable points. It is evidently invariant under the action of  $H$  and a geometric quotient  $X^s/H$  exists as an algebraic scheme. A point  $x$  is *properly stable* if it is stable and there exists an open  $H$ -invariant neighborhood  $V$  of  $x$  with  $V \subseteq X^s$  such that the action of  $H$  on  $V$  is proper. The set of properly stable points  $X^{ps}$  is evidently open and  $H$ -stable.

**PROPOSITION 1.1.** *Let  $U \subseteq X^s$  be an open  $H$  stable subset and suppose  $Y = U/H$ . If the quotient morphism  $U \rightarrow Y$  is affine then  $U \subseteq X^{ss}$ . If further,  $Y$  is separated then the natural map  $Y \rightarrow Q$  is an open immersion so  $Y$  is quasi-factorial.*

*Proof.* First assume that  $Y$  and  $U$  are affine. Let  $A = B^H$ . Then  $A$  is factorial and  $k[U] = B[a^{-1}]$ ,  $k[Y] = A[a^{-1}]$  for some  $a \in A$ . The triangle below clearly commutes

$$\begin{array}{ccc} U & \xrightarrow{c} & \text{Spec } A \\ & \searrow q & \nearrow \\ & & Y \end{array}$$

The non empty fibres of  $c|_U$  are of dimension  $l = \dim H$ . If  $x \in U \subset X$  then  $c(x) = q(x) = y$  so  $a(y) \neq 0$  and hence  $a(x) \neq 0$ . Thus each point of the fibre  $c^{-1}(c(x))$  lies in  $U$  and it follows that the dimension of each fiber is  $l$  so  $U \subset X^{ss}$ . Now in the general case  $U$  is covered by  $H$ -stable open affine subsets with affine quotients so  $U \subseteq X^{ss}$ . If  $Y$  is separated, then  $Y \rightarrow Q$  is a birational quasi-finite map, hence an open immersion by Zariski's Main Theorem.

In [1] it was shown that when  $\text{char } k = 0$ ,  $X^{ps}$  is the set of points in  $X$  for which the action of  $H$  is locally trivial and that  $H \times X^{ps} \rightarrow X^{ps} \times X^{ps}$ ,  $(h, x) \rightarrow (hx, x)$  is proper so the morphism  $X^{ps} \rightarrow Y = X^{ps}/H$  is affine and  $Y$  is separated hence quasi-factorial. The main purpose of this note is to give the appropriate generalization of this result in arbitrary characteristics. Since  $H$  contains a normal series with successive quotients isomorphic to the additive group  $G_a$ , one would expect the answer to lie in  $G_a$ -actions. This is indeed the case. A first guess might be to replace *locally trivial* by locally

trivial in the finite radical topology. However, the example 3 of [2] gives a counterexample to this conjecture.

It is important to note here that without the hypothesis that  $X$  be quasi-factorial, the action of  $H$  on  $X^{ps}$  need not be proper! (See Example 2, p. 727 in [2].)

A point  $x \in X$  will be called *finitely-stable* or *f-stable* if there exists an open affine neighborhood  $V$  of  $x$  invariant under the action of  $H$  and an  $H$ -equivariant finite morphism  $H \times S \rightarrow V$  for some affine variety  $S$ . Let  $X^{fs}$  denote the set of finitely stable points of  $X$ . In the definition we may assume without loss of generality that  $S$  is normal.

LEMMA 1.2.  $X^{fs}$  is contained in  $X^s$ .

*Proof.* It suffices to show that if  $H \times S \rightarrow V$  is a finite surjective  $H$ -morphism with  $V$  normal then  $V/H$  exists and is affine. By [3; p. 539] we can find a Seshadri cover  $Z \rightarrow V$  of  $V$  with respect to  $H$  such that  $k(Z)$  is the normal closure of  $k(H \times S)$  in an algebraic closure of  $k(V)$ . It follows that  $Z$  is the normalization of  $H \times S$  in  $k(Z)$  so in particular is affine. Moreover,  $Z \rightarrow H \times S$  is a Seshadri cover of  $H \times S$ . The action of  $H \times S$  is easily seen to be proper so the action of  $H$  on  $Z$  is proper. Then  $W = Z/H \rightarrow S$  is finite so  $W$  is affine. By Theorem 7.1 of [3]  $V/H$  exists and is affine.

LEMMA 1.3. Let  $Z \rightarrow X$  be a Seshadri cover of  $X$ . If the action of  $H$  on  $Z$  is proper, then the action of  $H$  on  $X$  is proper.

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} H \times Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ H \times X & \xrightarrow{\Phi} & X \times X \end{array}$$

The vertical and upper horizontal maps are finite hence  $\Phi$  is finite hence proper.

The following lemma is the key to our description of  $X^{ps}$ . It describes the situation locally when  $H = G_a$ .

LEMMA 1.4. Let  $V$  be a factorial affine variety on which  $G_a$  acts. Let  $R$  denote the coordinate ring of  $V$ . Then the following conditions are equivalent:

- (1) There exists a variety  $S$  and a finite surjective  $G_a$ -equivariant morphism  $p: G_a \times S \rightarrow V$

- (2) *There is an element  $g \in R$  such that  $\tilde{\sigma}(g)$  is monic in  $R(\lambda) = k[G_a \times V]$  where  $\tilde{\sigma}$  is the comorphism for the action of  $G_a$  on  $V$ .*

*Proof.* Suppose first that (1) holds. Let  $G_a$  act diagonally on  $G_a \times V$ . Then  $1 \times p: G_a \times S \rightarrow G_a \times V$  is finite and  $G_a$  equivariant. Let  $W$  be the image of  $1 \times p$ . Then  $W$  is a  $G_a$ -stable subvariety of  $G_a \times V$  of codimension one. Thus  $W$  is defined by a single irreducible invariant polynomial  $F(T)$  in  $R[T]$ . The composition of  $1 \times p$  with the second projection  $G_a \times V \rightarrow V$  is the original morphism  $p$ . Hence the restriction of the second projection to  $W$  is a finite morphism. It follows that  $F(T)$  can be taken monic in  $T$ . Write  $F(T) = a_0 + a_1T + \cdots + T^n$  with  $a_i \in R$ .

Let  $\hat{\sigma}: R[T] \rightarrow R[T][\lambda] = R[T, \lambda]$  denote the comorphism for the action of  $G_a$  on  $G_a \times V$ . If  $\Sigma b_i T^i \in R[T]$ . Then  $\sigma(\Sigma b_i T^i) = \Sigma \tilde{\sigma}(b_i)(T + \lambda)^i$ . Using the fact that  $\hat{\sigma}(F(T)) = F(T)$  we find

$$\begin{aligned} (T + \lambda)^n + \hat{\sigma}(a_{n-1})(T + \lambda)^{n-1} + \cdots + \hat{\sigma}(a_0) \\ = T^n + a_{n-1} + \cdots + a_1T + a_0 \end{aligned}$$

Taking for  $T$  the value  $-\lambda$  we see that  $\hat{\sigma}(a_0) = (-\lambda)^n + a_{n-1}(-\lambda)^{n-1} + \cdots + a_0$ . Thus  $g = (-1)^n a_0$  satisfies  $\hat{\sigma}(g)$  is monic in  $\lambda$  and (2) holds.

Conversely suppose  $g \in R$  with  $\hat{\sigma}(g) = g + g_1\lambda + \cdots + g_{n-1}\lambda^{n-1} + \lambda^n$ . Let  $W$  be the closed subset of  $G_a \times V$  defined by

$$G(T) = g + g_1(-T) + \cdots + g_{n-1}(-T)^{n-1} + (-T)^n = 0.$$

Note that if  $(\mu, p) \in W$  then

$$\begin{aligned} g(-\mu \cdot p) &= g(p) + g_1(p)(-\mu) + \cdots + g_{n-1}(p)(-\lambda)^{n-1} + (-\mu)^n \\ &= G(T)(\mu, p) = 0. \end{aligned}$$

Conversely if  $g(-\mu \cdot p) = 0$  then  $(\mu, p) \in W$ . Now let  $G_a$  act diagonally on  $G_a \times V$  then if  $(\mu, p) \in W$  and  $\lambda \in G_a$  we have

$$\begin{aligned} G(T)(\lambda \cdot (u, p)) &= G(T)(\lambda + u, \lambda \cdot p) \\ &= g((-\lambda - u) \cdot (\lambda p)) \\ &= g(-\mu \cdot p) = 0. \end{aligned}$$

Thus  $W$  is  $G_a$ -stable. The mapping  $W \rightarrow V$  obtained by restricting the second projection  $G \times V \rightarrow V$  to  $W$  is finite since  $G(T)$  is monic. Replacing  $W$  by a suitable irreducible component if necessary, we obtain a  $G_a$ -stable closed subvariety  $W$  of  $G_a \times V$  such that the mapping  $W \rightarrow V$  is finite and  $G_a$ -stable closed subvariety  $W$  of  $G_a \times V$  such that the mapping  $W \rightarrow V$  is finite and  $G_a$ -equivariant. Finally, since  $G_a \times V$  is trivial as a  $G_a$ -space so also is any  $G_a$ -stable subvariety so that  $W \simeq G_a \times S$  for some variety  $S$ . This gives the desired implication (2) implies (1) and completes the proof of the lemma.

**THEOREM 1.5.** *Let  $X$  be a quasi-factorial variety on which the connected unipotent group  $H$  acts. Then  $H$  acts properly on  $X^{fs}(H)$ . In particular,  $Y = X^{fs}(H)/H$  is quasi-factorial and  $q: X^{fs}(H) \rightarrow Y$  is an affine morphism.*

*Proof.* We argue by induction on  $\dim H$ . Assume the result holds for connected subgroups  $N \subseteq H$  with  $0 < \dim N < \dim H$  and let  $N$  be such a subgroup which is normal in  $H$ . Recall, [1; Sec. 3] that  $H \simeq N \times (H/N)$  as an  $N$ -space. It follows that  $X^{fs}(H) \simeq X^{fs}(N)$  and by the inductive assumption  $H/N$  acts properly on  $Y_N^{fs}(H/N)$  where  $Y_N = X^{fs}(N)/N$ .

Let  $Z$  be a Seshadri cover of  $X^{fs}(H)$ . We have a commutative diagram:

$$\begin{array}{ccc}
 Z & \longrightarrow & X^{fs}(H) \\
 \downarrow & & \downarrow \\
 W_1 = Z/N & \longrightarrow & Y_N^{fs}(H/N) \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & Y
 \end{array}$$

where  $W$  and  $Y$  are quotients under the action of  $H/N$ . Since  $H/N$  acts properly on  $Y_N^{fs}(H/N)$  it also acts properly on  $W_1$ . Thus  $W$  is quasi-affine. But  $W = Z/H$  and  $Z$  is locally trivial. By [1, 1.9]  $H$  acts properly on  $Z$ . By Lemma 1.3,  $H$  acts properly on  $X^{fs}(H)$ .

To complete the proof we need only establish the result in the case  $H = G_a$ . By Lemma 1.4 we can find an affine open cover  $\{X_x\}$  of  $X^{fs}(H)$  consisting of  $H$ -stable open affines and an element  $g_x = R_x = k[X_x]$  with  $\tilde{\sigma}(g_x)$  monic in  $R_x[\lambda]$ . The map  $\Phi$  will be proper if it's finite. We consider the cover  $\{X_x \times X_\beta\}$  of  $X^{fs}(H) \times X^{fs}(H)$ . Then  $\Phi^{-1}(X_x \times X_\beta) = H \times X_x \cap X_\beta$  so  $\Phi$  is affine. Let  $B = \Gamma(X^{fs}(H), O_X)$  so that  $R_x = B[f_x^{-1}]$  with  $f_x \in A = B^H$ . Then  $k[X_x \cap X_\beta] = B[f_x^{-1} \cdot f_\beta^{-1}]$ . I claim the map

$$B[f_x^{-1}] \otimes B[f_\beta^{-1}] \xrightarrow{1 \otimes \tilde{\sigma}} B[f_x^{-1} \cdot f_\beta^{-1}][\lambda]$$

is finite. If  $b \in B[f_\alpha^{-1} \cdot f_\beta^{-1}]$  and  $b = s/f_\alpha^n f_\beta^m$  then  $b = (1 \otimes \tilde{\sigma})(s/f_\alpha^n \otimes 1/f_\beta^m)$  so  $B[f_\alpha^{-1} f_\beta^{-1}]$  is in the image of  $1 \otimes \tilde{\sigma}$ . Since  $(1 \otimes \tilde{\sigma})(1 \otimes g_\beta) = \tilde{\sigma}(g_\beta)$  is monic in  $\lambda$  the ring  $B[f_\alpha^{-1} \cdot f_\beta^{-1}][\lambda]$  is integral over the image of  $1 \otimes \tilde{\sigma}$ . It follows that  $\Phi$  is finite and the theorem is proved.

**COROLLARY 1.6.**  *$X^{fs}(H)$  contains every  $H$ -stable open subset of  $X$  on which  $H$  acts properly stably. In particular  $X^{fs}(H) = X^{ps}(H)$ .*

*Proof.* Let  $U \subseteq X$  be  $H$ -stable open and assume  $H$  acts properly stably on  $U$ . It follows that we can replace  $U$  by an affine open subset and assume  $Y = U/H$  is affine. If  $Z$  is a Seshadri cover of  $U$  then  $Z$  and  $W$  are affine and hence  $Z \simeq H \times W$ . Since  $Z \rightarrow U$  is finite,  $U \subset X^{fs}(H)$ . The theorem asserts that the action of  $H$  on  $X^{fs}$  is properly stable hence  $X^{fs}(H) \subset X^{ps}(H)$  and equality follows.

The extension of the results of [1] to arbitrary characteristics depends on the invariance under  $G$  of the properly stable points of  $R_u G$  for actions of arbitrary connected algebraic groups  $G$  on quasi-factorial varieties. The following lemma is a key technical tool for this.

**LEMMA 1.7** *Let  $G$  be a linear algebraic group,  $N$  a closed normal subgroup of  $G$  and  $X$  a quasi-factorial variety on which  $G$  acts. If  $U \subseteq X$  is an  $N$ -stable open subset on which  $N$  acts properly then  $N$  acts properly on  $gU$  for all  $g$  in  $G$ .*

*Proof.*  $NgU = gNU = gU$  so  $gU$  is  $N$ -stable. Now  $\Phi: N \times U \rightarrow U \times U$  is proper so finite. Let  $ad(g)$  denote conjugation by  $g$  in  $G$  so  $ad(g)(n) = gng^{-1}$  and denote by  $\lambda_g$  left multiplication by  $g$ . The following diagram is commutative with vertical arrows representing isomorphisms.

$$\begin{array}{ccc}
 N \times U & \xrightarrow{\Phi} & U \times U \\
 \downarrow ad(g) \times \lambda_g & & \downarrow \lambda_g \times \lambda_g \\
 N \times gU & \xrightarrow{\Phi_g} & gU \times gU
 \end{array}$$

Thus  $\Phi_g$  is finite hence proper.

Note that if  $G$  is unipotent and  $U \subset X^s$  (for the action of  $N$ ) then  $gU \subset X^s$  for all  $g \in G$ . For a proof see [1; Proposition 2.4].

**THEOREM 1.7.** *Let  $N$  be a closed connected normal subgroup of the unipotent group  $G$  and  $X$  a quasi-factorial variety on which  $G$  acts. Let  $X_0 = X^{ps}(N)$ ,  $Y_0 = X_0/N$  and  $q: X_0 \rightarrow Y_0$  the quotient map. Then  $X_0$  is  $G$  stable and  $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N))$ .*

*Proof.* The lemma and the preceding note imply  $GX_0 = X_0$ . We saw in the proof of Theorem 1.5 that  $X^{ps}(G) \subseteq X_0$  and it is evidently  $N$ -stable. Its

image  $Y_1$  in  $Y_0$  is thus open,  $G/N$  stable and easily seen to be contained in  $Y_0^{ps}(G/N)$  (cf. [1; 2.4]). But if  $X_1 = q^{-1}(Y_0^{ps}(G/N))$ , then  $X_1$  is  $G$ -stable and clearly  $Y_0^{ps}(G/N)/(G/N) \simeq X_1/G$ . It remains only to show that  $G$  acts properly on  $X_1$ . This can be seen as follows:

Let  $T = X_1/G = Y^{ps}(G/N)/(G/N)$ . Then  $X_1 \rightarrow Y^{ps}(G/N)$  and  $Y^{ps}(G/N) \rightarrow T$  are affine maps because  $X_1 \subseteq X^{ps}(N)$  and  $N$  act properly on  $X_1$  and  $G/N$  acts properly on  $Y^{ps}(G/N)$ . Let  $T_2 \subset T$  be affine,  $Y_2 = \alpha^{-1}(T_2)$  where  $\alpha: Y^{ps}(G/N) \rightarrow T$  is the quotient map and finally let  $X_2$  be the inverse image of  $Y_2$  in  $X_1$ . If  $Z$  is a Seshadri cover of  $X_2$  and  $W = Z/G$  is its quotient then we have a commutative diagram

$$\begin{array}{ccc}
 Z & \longrightarrow & X_2 \\
 \downarrow & & \downarrow \\
 S & \dashrightarrow & Y_2 \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & T_2
 \end{array}$$

Since  $Z \rightarrow X_2$  is finite and  $N$  acts properly on  $X_2$  it acts properly on  $Z$ . But  $Z$  is locally trivial for the action of  $G$  hence also  $N$  so  $S = Z/N$  exists and is separated. But then the canonical map  $S \rightarrow Y_2$  is finite so  $S$  is actually affine. Since  $W = S/(G/N)$  is separated and  $W \rightarrow T_2$  is finite,  $W$  is affine. But  $Z \rightarrow W$  being locally trivial gives  $Z \simeq G \times W$ . Thus  $X_2 \subset X^{fs}(G) \subset X^{ps}(G)$ . Since  $X_1$  can be covered by such open affines it follows that  $X_1 \subseteq X^{ps}(G)$  and hence  $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N))$ .

**COROLLARY 1.8.** *Let  $X$  be a quasi-factorial variety on which the connected unipotent group  $G$  acts. If all stability groups for the action of  $G$  are finite then  $X^{ps}(G)$  is non-empty.*

*Proof.* It clearly suffices to establish the result when  $G = G_a$ . If  $f \in \Gamma(X, \mathcal{O}_X)$  is a nonconstant non-invariant function then  $\sigma(f) = f + f_1 T + \dots + f_k T^k$  with  $f_k$  invariant. Let  $X_0 = X_{f_k}$ . Lemma 2.4 implies  $X_0 \subseteq X^{fs}(G_a) = X^{ps}(G_a)$ .

**REMARK.** The results of [1] contained in Sections 3 and 4 now follow essentially from the arguments given there without any assumption on the characteristic of the ground field.

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