COMPOSITIO MATHEMATICA

D. BARLET

A. N. VARCHENKO

Around the intersection form of an isolated singularity of hypersurface

Compositio Mathematica, tome 67, nº 2 (1988), p. 149-163

<http://www.numdam.org/item?id=CM_1988__67_2_149_0>

Compositio Mathematica 67: 149–163 (1988)

Around the intersection form of an isolated singularity of hypersurface

D. BARLET* & A.N. VARCHENKO‡

Institut E. Cartan, Faculté des Sciences, Université Nancy I, BP 239, C.N.R.S. UA 750, F-54506 Vandoeuvre lès Nancy Cedex, France; ‡Math. Department, Moscow Gubkin Institute of Oil and Gas, Leninski Prospekt 65, 117917, GSP-1, Moscow, USSR

Received 1 June 1987; accepted 3 February 1988

Introduction

Let $P \in \mathbb{Q}\ [X_0, \ldots, X_n]$ be a quasi-homogeneous polynomial with an isolated singularity. For any pair of monomials m, m' the asymptotic expansion[§] when $s \to 0$ of the function

$$s \to \frac{1}{(2i\pi)^n} \int_{P=s} \varrho m.\ \bar{m}'\ \frac{dx}{dP} \wedge \overline{\frac{dx}{dP}}$$

where $\varrho \in C_c^\infty(\mathbb{C}^{n+1})$ satisfies $\varrho \equiv 1$ near 0, has at most one term which is not in $\mathbb{C}[\![s, \bar{s}]\!]$. Moreover this "non C^∞" term does not depend on the choice of ϱ. So let $c(m, m')$ be the coefficient of this term which will be equal now to $c(m, m')s^u\bar{s}^v$ if $u = v \neq 0$ modulo $\mathbb{Z}$ or to $c(m, m')s^u\bar{s}^v$ Log $s\bar{s}$ if $u, v \in \mathbb{N}$ (u, v depend on quasi-homogeneous weights of m and m' relative to the quasi-homogeneity type of P).

Now choose a monomial basis of the $\mathbb{Q}$-vector space

$$\mathbb{Q}[X_0, \ldots, X_n] \Big/ \left(\frac{\partial P}{\partial X_0}, \ldots, \frac{\partial P}{\partial X_n}\right)$$

and let λ be an eigenvalue of the monodromy operator acting on the cohomology of the Milnor fiber of P. The choice of λ corresponds to the choice of monomials in our basis having weights[¶] $\alpha, \alpha + 1, \ldots, \alpha + k$

---

§ The existence of an expansion (in a "general" context) is proved in [B.0]; here, use that $P(dx/dP)$ is C^∞ on $\mathbb{C}^{n+1}$ to fit the general result.

¶ Here the weight of the monomial m is by definition the quasi-homogeneous weight of $m(dx/dP)$, assuming the weight of P is 1.

where $e^{-2i\pi\alpha} = \lambda$: these monomials induce in cohomology a basis of the eigenspace of the monodromy, corresponding to the eigenvalue λ. Now let $(m_i)_{1 \leqslant i \leqslant N}$ be the list of monomials in our basis corresponding to λ and $\bar{\lambda}$ and define:

$$\Delta(\lambda) = \det_{1 \leqslant i,j \leqslant N} (c(m_i, m_j)).$$

Although the numbers $c(m_i, m_j)$ are in general highly transcendental, we have:

THEOREM. *For any λ, $\Delta(\lambda)$ is a rational number.*

The idea of the proof is to consider two $\mathbb{Q}$-lattices in the cohomology of the Milnor fiber of P (with complex coefficients). The first one, L_1 is just the lattice given by the integral cohomology; we shall call it the integral lattice. The second one, L_2, is defined by the cohomology classes induced by monomial forms on $\mathbb{C}^{n+1}$ or, more generally, by polynomial forms with rational coefficients; we shall call it the holomorphic lattice.

Then we shall prove that some nondegenerate bilinear form defined over $\mathbb{Q}$ (so having rational values on the integral lattice L_1) has a rational determinant on a basis of L_2. In fact we treat separately each pair of conjuguate eigenvalues of the monodromy operator.

For the eigenvalues $\neq 1$ the bilinear form is simply the intersection form, and its relation to the "residue form" given in [V3] achieves the computation of the determinant. Then the relation between the intersection form and the canonical hermitian form proved in [B1] allows us to relate $\Delta(\lambda)$ to the square of the "volume ratio" of the two lattices L_1 and L_2.

For the eigenvalue 1 we give the precise relationship between the residue form and the canonical hermitian form in order to use the same strategy. This is done by Loeser's Theorem which asserts that the natural hermitian extension of the residue form of [V3] is the canonical hermitian form of [B1] for any isolated singularity of a hypersurface ([L], Theorem 2).

As an application of the theorem in the eigenvalue 1 case, we obtain the following:

COROLLARY. *Let $\mathscr{H}$ be a smooth hypersurface in $\mathbb{P}_n(\mathbb{C})$ defined over $\mathbb{Q}$. Denote by $H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C})$ the primitive part of the cohomology of $\mathscr{H}$ in the middle dimension; denote by L_1 the topological $\mathbb{Q}$-lattice $H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C}) \cap H^{n-1}(\mathscr{H}, \mathbb{Q})$ and by L_2 the $\mathbb{Q}$-lattice generated in $H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C})$ by the residues of meromorphic differential n-forms on $\mathbb{P}_n(\mathbb{C})$ with rational coefficients and with poles in $\mathscr{H}$.*

Then for any $\mathbb{Q}$*-basis* B_1 *and* B_2 *of* L_1 *and* L_2*, respectively, we have*

$$\text{vol}\,(B_2/B_1)^2 \in (2i\pi)^{(n-1)\cdot l}\mathbb{Q}$$

where $l = \dim_{\mathbb{C}} H^{n-1}_{\text{pr}}(\mathscr{H}, \mathbb{C})$.

Section 1 is devoted to the relationship between the residue form and the intersection form. The theorem and its corollary are proved in §2.

This paper had been partially written when the second author was visiting Nancy University in June 1986. He wants to thank the E. Cartan Institute for hospitality.

§1

We consider the germ of a holomorphic function with an isolated singularity at the origin of $\mathbb{C}^{n+1}$ and let f: $X \to D$ be a Milnor representative of it. This means that D is a disc in $\mathbb{C}$ and that f induces a C^∞ locally trivial fibration of $X - f^{-1}(0)$ over $D^* = D - \{0\}$. We shall use a Scherk compactification of this situation [S] (see also [V3] for a detailed description).

So we have a projective map $\tilde{f}$: $Y \to D$ with an open relative imbedding $X \hookrightarrow Y$ over D, and Y gives a smooth polarized family of projective manifolds over D^*; the only critical point of $\tilde{f}$ is the origin in X.

Fix a base point $s_0 \in D^*$ and let $X(s_0)$ and $Y(s_0)$ denote the corresponding fibers of f and $\tilde{f}$. Then the Scherk compactification has the following properties:

1°) There is an exact sequence

$$0 \to \text{Inv}\; P^n(Y(s_0)) \to P^n(Y(s_0)) \xrightarrow{\;i^*\;} H^n(X(s_0)) \to 0 \tag{1}$$

where $P^n(Y(s_0))$ is the primitive part of the cohomology* $H^n(Y(s_0))$ with respect to the given polarization, where Inv $P^n(Y(s_0))$ is the monodromy invariant subspace of $P^n(Y(s_0))$ and where i* is the map induced by the inclusion i: $X(s_0) \hookrightarrow Y(s_0)$.

2°) For any class a in $H^n(X(s_0))_{=1}$ there exists a global $(n+1)$-holomorphic form ω on Y such that the asymptotic expansion of the section

$$s \to \left.\frac{\omega}{d\tilde{f}}\right|_{Y(s)}$$

* When the coefficients are not precised, they are in $\mathbb{C}$.

in an horizontal basis of the Gauss–Manin bundle $s \to P^n(Y(s))_{=1}$ is given by

$$\left.\frac{\omega}{\mathrm{d}f}\right|_{Y(s)} \sim s^k \left(\sum_{j=0}^{l} \tilde{a}_j (\mathrm{Log}\, s)^j \right) + \text{higher terms}$$

where $k \in \mathbb{N}$ and the $\tilde{a}_j$ are horizontal (multivalued) sections with $\mathrm{i}^*(\tilde{a}_0|_{Y(s)}) = a$.

Let us remark here for later use in §2 that if f is a polynomial defined over $\mathbb{Q}$, it is possible to choose a Scherk compactification defined over $\mathbb{Q}$ such that, if s_0 is fixed in $D^* \cap \mathbb{Q}$, for any class a in $H^n(X(s_0))_{=1}$ induced by monomial form $m(\mathrm{d}x/\mathrm{d}f)$, we can choose a global algebraic $(n+1)$-differential form ω on Y, defined over $\mathbb{Q}$, with property 2°) as before.

It will be important to notice that the $\mathbb{Q}$-lattice in $H^n(X(s_0))$ defined by classes induced by monomial forms, is invariant under algebraic change of coordinates in $\mathbb{C}^{n+1}$ defined over $\mathbb{Q}$.

We now introduce the monodromy operator T (resp. $\tilde{T}$) on $H^n(X(s_0))$ (resp. $P^n(Y(s_0))$). Denote by $H^n(X(s_0))_{=1}$ (resp. $P^n(Y(s_0))_{=1}$) the spectral subspace of $H^n(X(s_0))$ (resp. $P^n(Y(s_0))$) corresponding to the eigenvalue 1 and define

$$N = \frac{1}{2i\pi} \mathrm{Log}\, T \quad \text{on} \quad H^n(X(s_0))_{=1}$$

and

$$\tilde{N} = \frac{1}{2i\pi} \mathrm{Log}\, \tilde{T} \quad \text{on} \quad P^n(Y(s_0))_{=1}.$$

Observe that Ker $\tilde{N}$ is exactly the subspace Inv $P^n(Y(s_0))$ of monodromy invariant vectors in $P^n(Y(s_0))$. So we also have the exact sequence

$$0 \to \mathrm{Inv}\, P^n(Y(s_0)) \to P^n(Y(s_0))_{=1} \xrightarrow{\tilde{N}} \mathrm{Im}\, \tilde{N} \to 0. \tag{2}$$

But from the exact sequence (1) we also have, by monodromy invariance of i^*

$$0 \to \mathrm{Inv}\, P^n(Y(s_0)) \to P^n(Y(s_0))_{=1} \xrightarrow{\mathrm{i}^*} H^n(X(s_0))_{=1} \to 0. \tag{1 bis}$$

Comparison with (2) leads to a canonical isomorphism v: $H^n(X(s_0))_{=1} \to \mathrm{Im}\, \tilde{N}$ given by $v(a) = \tilde{N}(\tilde{a})$ for any $\tilde{a} \in P^n(Y(s_0))_{=1}$ such $\mathrm{i}^*(\tilde{a}) = a$. By monodromy invariance of the map i* we have

$$\mathrm{i}^*\nu(a) = N(a) \quad \forall a \in H^n(X(s_0))_{=1}. \tag{3}$$

Now let $\tilde{k}$ be the hermitian intersection form on $P^n(Y(s_0))_{=1}$ normalized by the following formula

$$\tilde{k}(u, v) = \frac{1}{(2i\pi)^n} \int_{Y(s_0)} u \wedge \bar{v} \quad \forall u, v \in P^n(Y(s_0))_{=1}. \tag{4}$$

LEMMA. *For any a, $b \in H^n(X(s_0))_{=1}$ and any $\tilde{a}$, $\tilde{b}$ in $P^n(Y(s_0))_{=1}$ such that* $\mathrm{i}^*\tilde{a} = a$ *and* $\mathrm{i}^*\tilde{b} = b$, *we have*

$$\tilde{k}(\nu a, \tilde{b}) = \tilde{k}(\tilde{a}, \nu b); \tag{5}$$

and this number depends only on a *and* b *and defines a nondegenerate hermitian form on $H^n(X(s_0))_{=1}$. We shall denote it by $\tilde{h}$.*

Proof. First note that monodromy invariance of $\tilde{k}$ (i.e., $\tilde{k}(\tilde{T}u, \tilde{T}v) = \tilde{k}(u, v)\, \forall u, v \in P^n(Y(s_0))_{=1}$) gives $\tilde{k}(\tilde{N}u, v) = \tilde{k}(u, \tilde{N}v)$ (infinitesimal invariance). To prove formula (5) it suffices to put $u = \tilde{a}$ and $v = \tilde{b}$.

Let $\tilde{h}(a, b)$ denote the number defined by formula (5); $\tilde{h}$ clearly defines an hermitian form on $H^n(X(s_0))_{=1}$ since $\tilde{k}$ is hermitian. To establish the nondegeneracy of $\tilde{h}$, it is enough to use nondegeneracy of $\tilde{k}$ on $P^n(Y(s_0))_{=1}$ which is well-known. So the lemma is proved.

THEOREM (Loeser [L]). *The restriction of the canonical hermitian form h introduced in* [B1] *to $H^n(X(s_0))_{=1}$ is equal to $\tilde{h}$.*

For a direct definition of the canonical hermitian form from asymptotics of integrals $\int_{f=s}\varphi$, $\varphi \in C_c^\infty(X)$ of type (n, n) we refer to [B1].

REMARK 1. Because $\tilde{k}$ induces the usual intersection form on the subspace $E_1^\infty = \mathrm{Image}(H_c^n(X(s_0))_{=1} \to H^n(X(s_0))_{=1})$, the theorem is compatible with the relation

$$k(Na, b) = h(a, b) \quad \forall a, b \in E_1^\infty \tag{6}$$

of [B1], Th. 3: indeed if $\mathrm{i}^*\tilde{a} = a$ and $\mathrm{i}^*\tilde{b} = b$ we have, for a, b in E_1^∞

$$\tilde{k}(\tilde{N}\tilde{a}, \tilde{b}) = k(Na, b) = \tilde{h}(a, b).$$

REMARK 2. The theorem shows that $\tilde{k}(\tilde{N}\tilde{a}, \tilde{b})$ does not depend on the choice of the Scherk' compactification: the canonical hermitian form is defined locally around the singular point.

REMARK 3. F. Loeser pointed out to us that his paper [L] (or formula (5)) gives a definition, in terms of the monodromy and variation, of the canonical hermitian form inside the Milnor fiber. This shows that it is a topological invariant of the singularity (see the final remark in [L]).

REMARK 4. On the rational cohomology $H^n(X(s_0), \mathbb{Q})$ of the Milnor fiber, there is a nondegenerate rational form given by intersection for eigenvalues of the monodromy which do not equal 1 and by $\tilde{k}(\tilde{N}\tilde{a}, \tilde{b})$ for the eigenvalue 1. There are two ways to extend it to $H^n(X(s_0), \mathbb{C})$: take a $\mathbb{C}$-bilinear extension or a hermitian extension. The first gives the residue form (see [V3], namely formulas (9) and (14)), the second the form introduced in [B1].

Proof. Let a and b be in $H^n(X(s_0))_{=1}$. Using property 2°) of the Scherk compactification, we can find ω and ω' which are global holomorphic $(n + 1)$-form on Y such that the first terms in the expansions of $\omega/\mathrm{d}f$ and $\omega'/\mathrm{d}f$ induce a and b, respectively, in $H^n(X(s_0))$*.

Let us consider the asymptotic expansions in an horizontal basis of the Gauss-Manin bundle associated to $s \to P^n(Y(s))$ of the sections defined by ω and ω' of this bundle (they are given by

$$s \to \frac{\omega}{\mathrm{d}f}\bigg|_{Y(s)} \quad \text{and} \quad s \to \frac{\omega'}{\mathrm{d}f}\bigg|_{Y(s)}$$

respectively). The first terms of these expansions are given by

$$\begin{cases} \dfrac{\omega}{\mathrm{d}f} \sim s^k(\tilde{a}_0 + \tilde{a}_1 \operatorname{Log} s + \cdots + \tilde{a}_{n-1} (\operatorname{Log} s)^{n-1}) + \ldots \\ \\ \dfrac{\omega'}{\mathrm{d}f} \sim s^{k'}(\tilde{b}_0 + \tilde{b}_1 \operatorname{Log} s + \cdots + \tilde{b}_{n-1} (\operatorname{Log} s)^{n-1}) + \ldots \end{cases} \tag{7}$$

where $k, k' \in \mathbb{N}$, the $\tilde{a}_i$ and $\tilde{b}_i$ are horizontal (multivalued) sections of the Gauss–Manin bundle $s \to P^n(Y(s))_{=1}$ and we have the relations (see [V1])

$$\tilde{N}\tilde{a}_i = \tilde{a}_{i+1} \quad \tilde{N}\tilde{b}_i = \tilde{b}_{i+1} \quad \forall i \geqslant 0 \tag{8}$$

* See property 2°) of Scherk compactification.

because the first terms must be invariant by $\tilde{T}$ (ω and ω' and uniform !). Then, by the definition of the canonical hermitian form in [B1], the value $h(a, b)$ is the coefficient of $s^k \bar{s}^{k'}$ Log $|s|^2$ in the expansion at $s = 0$ of the function

$$F(s) = \frac{1}{(2i\pi)^n} \int_{f=s} \frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}}.^*$$

But the difference between F and the function

$$G(s) = \frac{1}{(2i\pi)^n} \int_{\tilde{f}=s} \frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}} \qquad (\{\tilde{f} = s\} = Y(s))$$

is C^∞ because of the transversality of the fibers of $\tilde{f}$ to ∂X and because $\tilde{f}$ has no critical point in $Y - X$. So $h(a, b)$ is also the coefficient of $s^k \bar{s}^{k'}$ Log $|s|^2$ in the expansion at $s = 0$ of G. Now using (7) and (8) we obtain

$$(2i\pi)^n G(s) = s^k \bar{s}^{k'} \left(\int_{\tilde{f}=s} \tilde{a}_0 \wedge \overline{\tilde{b}_0} + \text{Log} s \int_{\tilde{f}=s} \tilde{N}\tilde{a}_0 \wedge \overline{\tilde{b}_0} \right.$$

$$\left. + \text{Log}\, \bar{s} \int_{\tilde{f}=s} \tilde{a}_0 \wedge \overline{\tilde{N}\tilde{b}_0} \right) + \cdots + \text{higher terms.}$$

But for any $s \neq 0$, we have

$$\tilde{k}(\tilde{N}\tilde{a}_0(s_0), b_0(s_0)) = \frac{1}{(2i\pi)^n} \int_{\tilde{f}=s} \tilde{N}\tilde{a}_0 \wedge \overline{\tilde{b}_0}$$

because of the horizontality of the intersection form.

Using formula (5), we see immediately that the coefficient of $s^k \bar{s}^{k'}$ Log $|s|^2$ in the expansion of G at $s = 0$ is

$$\tilde{k}(\tilde{N}\tilde{a}_0(s_0), \tilde{b}_0(s_0)) = \tilde{h}(i^*(\tilde{a}_0(s_0)), i^*(\tilde{b}_0(s_0))).$$

But the choice of ω and ω' is exactly such that we have

$$i^*(\tilde{a}_0(s_0)) = a \quad \text{and} \quad i^*(\tilde{b}_0(s_0)) = b.$$

So we have shown that $h = \tilde{h}$ on $H^n(X(s_0))_{=1}$, and the theorem is proved.

---

* Since ∂X is transversal to all fibers of f, up to a C^∞ function of s, this is the same as

$$\frac{1}{(2i\pi)^n} \int_{f=s} \varrho \frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}}$$

where $\varrho \in C_c^\infty(X)$, $\varrho = 1$ near 0.

COROLLARY. *For a, b in $H^n(X(s_0), \mathbb{Z})_{=1}$ we have*

$$(2i\pi)^{n-1}h(a, b) \in \mathbb{Q}.^*$$

The corollary is an obvious consequence of the proof of the theorem.

REMARK: It is not possible in general to replace $\mathbb{Q}$ by $\mathbb{Z}$ in the conclusion because the logarithm of a unipotent is not defined over $\mathbb{Z}$. But, of course, one can use the fact that $N^k = 0$ for some $k \leqslant n$ to be more precise about the denominators of these rational numbers!

The fact that $(2i\pi)^{n-1}h$ is real on $H^n(X(s_0), \mathbb{R})_{=1}$ was already proved by F. Loeser (see [L], corollary 1 of Theorem 2).

§2

Consider a quasi-homogeneous polynomial P with isolated singularity and assume that P is defined over $\mathbb{Q}$ (i.e., P has rational coefficients).

Let λ be an eigenvalue of the monodromy acting on the cohomology of the Milnor fiber of P. Set $k = \mathbb{Q}(\lambda)$ and $K = \mathbb{C}$. Denote by E_λ the eigenspace of the monodromy corresponding to the eigenvalue λ (E_λ is defined over k) and let

$$L_1 = E_\lambda + E_{\bar\lambda}.$$

Let L_2 be the k lattice in $V = L_1 \otimes_k \mathbb{C}$ generated by all monomial differential forms $m(\mathrm{d}x/\mathrm{d}P)$ whose weights correspond to the eigenvalues λ and $\bar\lambda$ of the monodromy (these weights satisfy $e^{-2i\pi w} = \lambda$ or $e^{-2i\pi w} = \bar\lambda$).

First assume that $\lambda \neq \bar\lambda$ and choose a monomial basis of

$$\mathbb{Q}[X_0, \ldots, X_n] \Big/ \left(\frac{\partial P}{\partial X_0}, \ldots, \frac{\partial P}{\partial X_n}\right).$$

Let $\alpha, \alpha + 1, \ldots, \alpha + k$ be the weights of the monomials in this basis satisfying $e^{-2i\pi w} = \lambda$ and let $\beta, \beta + 1, \ldots, \beta + l$ those satisfying $e^{-2i\pi w} = \bar\lambda$.

Let $H_\alpha, H_{\alpha+1}, \ldots, H_{\alpha+k}$ and $H_\beta, H_{\beta+1}, \ldots, H_{\beta+l}$ denote the corresponding k-subspaces of L_2 generated by the cohomology classes induced by these monomials (a monomial m induces $m(\mathrm{d}x/\mathrm{d}P)$ by definition).

---

* Of course, since h coincides on $H^n(X(s_0))_{\neq 1}$ with the intersection form, we have $(2i\pi)^n h(a, b) \in \mathbb{Z}$ for $a, b \in H^n(X(s_0), \mathbb{Z})_{\neq 1}$.

Then we have the following orthogonality relations for the k-bilinear intersection form on L_2 (see [V3])

$$\begin{cases} (H_{\alpha+i}, H_{\alpha+j}) = 0 & \forall i, j \\ (H_{\alpha+i}, H_{\beta+j}) = 0 & i + j < n - \alpha - \beta. \end{cases} \tag{9}$$

For monomials $\hat{a}$ and $\hat{b}$ which induce classes a and b in $H_{\alpha+i}$ and $H_{\beta+j}$ respectively with $(\alpha + i) + (\beta + j) = n$ we have from [V3] formula (9)

$$(a, b) = (-i)^i C_{n+1} \operatorname{Res}_{P,0}(\hat{a}\mathrm{d}x, \hat{b}\mathrm{d}x) \tag{10}$$

where C_{n+1} is an universal (explicitly given) constant depending only on n (see [V3], p. 35).

We shall only use the fact that $(2i\pi)^{-n} C_{n+1} \in \mathbb{Q}$.

So the restriction of the intersection form to L_2 is given by the following picture, where it is known that $k = l$ and $\alpha + \beta + k = n$ and where we have 0 in the upper left side and where the "skew diagonal" blocks are filled in with the help of formula (10).

Now, for the monomials $\hat{a}$ and $\hat{b}$, the numbers $\operatorname{Res}_{P,0}(\hat{a}\,\mathrm{d}x, \hat{b}\,\mathrm{d}x)$ are rational: for instance the residue can be defined as the value at 0 of the trace of the form $\hat{a}\hat{b}\,\mathrm{d}x$ via the finite map, defined over $\mathbb{Q}$, given by $\partial P/\partial X_0, \ldots, \partial P/\partial X_n$.

So the entries of the "skew diagonal" blocks are in $C_{n+1} \cdot \mathbb{Q}$; that is to say in $(2i\pi)^n \cdot \mathbb{Q}$. The determinant of the $\mathbb{C}$-bilinear extension of the intersection form in this basis B_2 of L_2 is in $(2i\pi)^{nd} \cdot \mathbb{Q}$ where $d = \dim(E_\lambda + E_{\bar{\lambda}})$.

Now let B_1 be a real $k = \mathbb{Q}(\lambda)$ basis of L_1 and denote by C the matrix of the intersection form in B_1. Then

1°) C has coefficients in $\mathbb{Q}(\lambda) \cap \mathbb{R}$ and the matrix D^*CD of the $\mathbb{C}$-bilinear intersection form in the basis B_2 (here $D = \mathrm{Gl}_d(\mathbb{C})$ is unknown) satisfies

2°) $\det(D^*CD) \in (2i\pi)^{nd} \cdot \mathbb{Q}$.

If we look now at the matrices of the hermitian intersection form (normalised by $1/(2i\pi)^n\, \alpha \cup \bar{\beta}$) in the bases B_1 and B_2 they are

$$\frac{1}{(2i\pi)^n} \mathrm{C} \quad \text{in} \quad B_1$$

and

$$D^* \frac{1}{(2i\pi)^n} C\bar{D} \quad \text{in} \quad B_2.$$

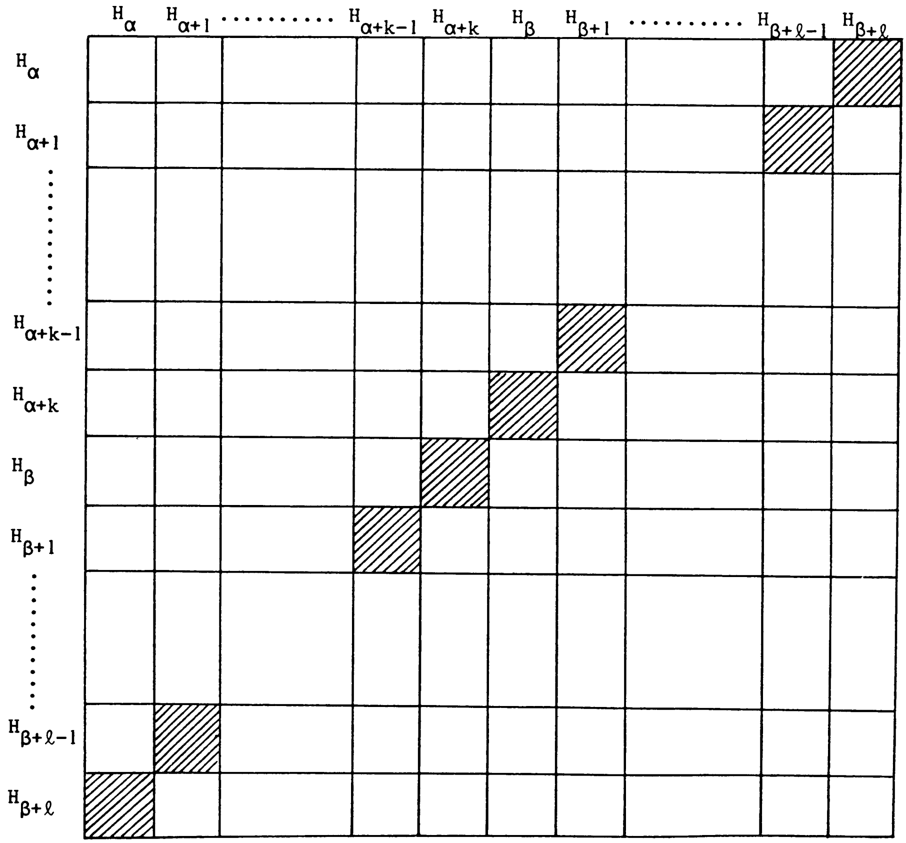

Fig.

Because the determinant of an hermitian form is real and C has real coefficients we get $(2i\pi)^{nd} \in \mathbb{R}$; that is to say nd is even (this is trivial in our case $\lambda \neq \bar{\lambda}$ because $d = 2 \dim E_\lambda$ is even).

So we deduce from 2°) that $(\det D)^2 \in \mathbb{R}$ and so $|\det D|^2 = \pm (\det D)^2$. So we get $\Delta(\lambda) = |\det D|^2 \det (1/(2i\pi)^n C) \in \mathbb{Q}$ by 2°).

This proves the result in this case.

For $\lambda = -1$ the fact that $\det (1/(2i\pi)^n C)$ is real shows that nd is even and we can argue along the same lines.

The case $\lambda = 1$ is slightly different.

First of all C is no longer real because of the corollary of Loeser's Theorem in §1. But of course $\det (1/(2i\pi)^n C)$ is still real, and so, if we know that $D^*[1/(2i\pi)^n] CD$ has a determinant in $\mathbb{Q}$ we can again conclude that $(\det D)^2 \in \mathbb{R}$ and so $|\det D|^2 = \pm (\det D)^2$ and so $\Delta(1) \in \mathbb{Q}$.

So for the case $\lambda = 1$ we can again choose a decomposition $L_2 = \oplus_0^k H_{\alpha+i}$ corresponding to integral weights in our basis, but we have *to change the polynomial* P^* to have a nice Scherk compactification (as in §1) in order to use results of [V3] in the case of a degenerate intersection form (§3). Of course this can be done algebraically over $\mathbb{Q}$** and so we can conclude again that the determinant of the $\mathbb{C}$-bilinear intersection form in our basis B_2 belongs to

$$(2i\pi)^{nd} \cdot \mathbb{Q} \quad \text{where} \quad d = \dim_{\mathbb{Q}} E_1.$$

Now we can finish the proof in this case as before if, instead of [B1] Theorem 3, we use the Theorem of §1 and [V3], §3.

This completes the proof of the Theorem.

REMARK 1. Let f be any polynomial having an isolated singularity at $0 \in \mathbb{C}^{n+1}$ and defined over $\mathbb{Q}$. For fixed λ define the integral lattice $L_1 = E_\lambda + E_{\bar{\lambda}}$ and let L_2 be the lattice defined by the cohomology classes induced by spectral projections on $L_1 \otimes_k \mathbb{C}$ of polynomial $(n+1)$-forms with rational coefficients. Then it is possible to prove again that vol $(B_2/B_1)^2 \in \mathbb{Q}(\lambda)$ for any $\mathbb{Q}(\lambda)$ basis of L_1 and L_2. Just replace the picture we have used to compute the determinant in our monomial basis of L_2, by the picture given in [V3], §4, no. 6. But of course, it is more complicated to define $\Delta(\lambda)$ (the determinant of the canonical hermitian form in a basis of L_2) in such a case, because its expression in terms of asymptotics of integrals on the fibers of f is much more involved.

REMARK 2. Let $P \in \mathbb{Q}\,[X_0, \ldots, X_n, \xi_1, \ldots, \xi_p]$ a family of quasi-homogeneous polynomial (parametrized by ξ) with an isolated singularity at the origin. Assume that the Milnor number $\mu(\xi)$ is constant for ξ near 0 (in $\mathbb{C}^p$). Then working over the field $\mathbb{Q}(\xi)$ leads to an analogous theorem. In fact, the only point where ξ appears (because monomials are defined over $\mathbb{Q}$) is in the residue. Computing trace via the map

$$\left(\frac{\partial P_\xi}{\partial X_0}, \ldots, \frac{\partial P_\xi}{\partial X_n}\right)$$

leads to coefficients in $\mathbb{C}(\xi)$. Then the use of conjugation leads to $\Delta(\lambda)^2 \in \mathbb{Q}(\xi, \bar{\xi})$.

* By an algebraic change of coordinates defined over $\mathbb{Q}$.

** As noticed before, this preserves the L_2 lattice.

EXAMPLE (see [B3]). Let $P_\xi(X, Y, Z) = X^3 + Y^3 + Z^3 + 3\xi XYZ$ and look at $\lambda = 1$. Then L_2 is generated by monomials 1 and XYZ. The remark above says concretely the following:

Let $\omega = X\mathrm{d}Y \wedge \mathrm{d}Z + Y\mathrm{d}Z \wedge \mathrm{d}X + Z\mathrm{d}X \wedge \mathrm{d}Y$ and for $\varrho \in C_c^\infty(\mathbb{C}^3)$ such that $\varrho \equiv 1$ near 0 we have the following asymptotics when $s \to 0$

$$\frac{1}{(2i\pi)^2}\int_{P_\xi = s} \varrho\omega \wedge \bar{\omega} \sim c_{1,1}(\xi)\,|s|^2 \operatorname{Log} s\bar{s} \mod \mathbb{C}\,[\![s, \bar{s}]\!]$$

$$\frac{1}{(2i\pi)^2}\int_{P_\xi = s} \varrho\omega \wedge \overline{XYZ\omega} \sim c_{1,2}(\xi)\, s\bar{s}^2 \operatorname{Log} s\bar{s} \mod \mathbb{C}\,[\![s, \bar{s}]\!]$$

$$\frac{1}{(2i\pi)^2}\int_{P_\xi = s} \varrho XYZ\omega \wedge \bar{\omega} \sim c_{2,1}(\xi)\, s^2\bar{s} \operatorname{Log} s\bar{s} \mod \mathbb{C}\,[\![s, \bar{s}]\!]$$

$$\frac{1}{(2i\pi)^2}\int_{P_\xi = s} \varrho XYZ\omega \wedge \overline{XYZ\omega} \sim c_{2,2}(\xi)\,|s|^4 \operatorname{Log} s\bar{s} \mod \mathbb{C}\,[\![s, \bar{s}]\!].$$

Then the square of

$$\Delta_\xi(1) = \det \begin{vmatrix} c_{1,1}(\xi) & c_{2,1}(\xi) \\ c_{1,2}(\xi) & c_{2,2}(\xi) \end{vmatrix}$$

is in $\mathbb{Q}(\xi, \bar{\xi})$ (the rationals function on ξ and $\bar{\xi}$ with coefficients in $\mathbb{Q}$).

The function $c_{1,1}(\xi)$ is given in [B3]:

$$c_{1,1}(\xi) = \frac{2}{\pi}\frac{|A(\xi)|}{|1 + \xi^3|}$$

where $A(\xi)$ is the area of the fundamental parallelogram of the elliptic curve $X^3 + Y^3 + Z^3 + 3\xi XYZ = 0$ in $\mathbb{P}_2(\mathbb{C})$. The coefficients $c_{1,2}(\xi)$, $c_{2,1}(\xi)$ and $c_{2,2}(\xi)$ can be deduced from $c_{1,1}(\xi)$ by the following remark:

We have $\mathrm{d}\omega/\mathrm{d}\xi = -3XYZ\omega$ on the family $P_\xi = s$ (s fixed). This gives

$$\frac{\partial}{\partial \xi} c_{1,1} = -3c_{2,1} \quad \text{and} \quad \frac{\partial^2 c_{1,1}}{\partial\xi\partial\bar{\xi}} = 9c_{2,2}$$

and so

$$\Delta_\xi(1) = 9\left(c_{1,1}(\xi)\frac{\partial^2 c_{1,1}(\xi)}{\partial\xi\partial\bar{\xi}} - \left|\frac{\partial c_{1,1}(\xi)}{\partial\xi}\right|^2\right).$$

As one can clearly see in the previous example the matrix of the canonical hermitian form in a basis of the holomorphic lattice L_2 is not "skew triangular" and has highly transcendantal entries. This comes from the conjuguation map which does not preserve the L_2 lattice.

Proof of the corollary. Denote by $(X_0, \ldots, X_n, t)$ homogeneous coordinates on $\mathbb{P}_{n+1}(\mathbb{C})$ and identify $\mathbb{P}_n(\mathbb{C})$ with the hyperplane $\{t = 0\}$ in $\mathbb{P}_{n+1}$. Let $P \in \mathbb{Q}[X_0, \ldots, X_n]$ be an irreducible homogeneous polynomial of degree δ such that $\{P = 0\} \cap \mathbb{P}_n = \mathscr{H}$. Then $C(\mathscr{H}) = \{P = 0\}$ is the cone over $\mathscr{H}$ in $\mathbb{P}_{n+1}$.

We shall consider the smooth hypersurface V of $\mathbb{P}_{n+1}$ defined by $P(X) = t^\delta$.

$$\text{Set } \omega = \sum_{j=0}^{n} (-1)^j \left(\frac{X_j}{t}\right) \mathrm{d}\left(\frac{X_0}{t}\right) \wedge \cdots \wedge \widehat{\mathrm{d}\left(\frac{X_j}{t}\right)} \wedge \cdots \wedge \mathrm{d}\left(\frac{X_n}{t}\right).$$

Then for $m \in \mathbb{C}[X_0, \ldots, X_n]$ a monomial of degree $k\delta - (n+1)$ $(k \in \mathbb{N}^*)$, the meromorphic differential form (defined over $\mathbb{Q}$) on $\mathbb{P}_{n+1}$

$$\frac{m\omega}{P^k}$$

with poles in $\{P = 0\}$, satisfies

$$\mathrm{d}\left(\frac{m\omega}{P^k}\right) = 0.$$

Then we have the commutative diagram of restrictions and residues (see Leray [$\mathscr{L}$])

$$\begin{array}{ccc} H^n(\mathbb{P}_{n+1} - C(\mathscr{H}), \mathbb{C}) & \xrightarrow{\ \mid_{\mathbb{P}_n}\ } & H^n(\mathbb{P}_n - \mathscr{H}, \mathbb{C}) \\ \big\downarrow \mid_V & & \big\downarrow \mathrm{Res}_{\mathscr{H}}^{\mathbb{P}_n} \\ H^n(V - \mathscr{H}, \mathbb{C}) & \xrightarrow{\ \mathrm{Res}_{\mathscr{H}}^{V}\ } & H^{n-1}(\mathscr{H}, \mathbb{C}) \end{array}$$

since

$$C(\mathscr{H}) \cap V = C(\mathscr{H}) \cap \mathbb{P}_n = \mathscr{H}.$$

As is known (see [G], Theorem 8.3), the image of the residue map $\mathrm{Res}_{\mathscr{H}}^{\mathbb{P}_n}$ is the primitive part $H_{\mathrm{pr}}^{n-1}(\mathscr{H}, \mathbb{C})$, so the lattice L_2 is generated (as a $\mathbb{Q}$ vector space) by elements of the following type

$$\mathrm{Res}^V_{\mathscr{H}}\left(\frac{m\omega}{P^k}\bigg|_V\right)$$

for $m \in \mathbb{Q}[X_0, \ldots, X_n]$ monomial of degree $k\delta - (n+1)$.

Now it is clear that the map $2i\pi\ \mathrm{Res}^V_{\mathscr{H}}$ (which is defined over $\mathbb{Q}$ as the adjoint of the tube map $H_{n-1}(\mathscr{H}, \mathbb{Q}) \to H_{n+1}(V - \mathscr{H}, \mathbb{Q})$) sends the two lattices of the monodromy invariant part of the cohomology of the Milnor fiber $V - \mathscr{H}$ of the affine hypersurface $\{P = 0\}$ in $\mathbb{C}^{n+1}$, to the lattices L_1 and $2i\pi L_2$ of the group $H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C})$. So, as a by product of the proof of the theorem, we obtain

$$\mathrm{vol}\,(B_2/B_1)^2 \in (2i\pi)^{(n-1)l}\mathbb{Q}$$

where $l = \dim_{\mathbb{C}} H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C})$, for any $\mathbb{Q}$-basis B_1 and B_2 of L_1 and L_2 respectively.

Remark. This implies of course, that the determinant in any basis B_2 of L_2 of the hermitian intersection form on $H^{n-1}_{\mathrm{pr}}(\mathscr{H}, \mathbb{C})$ normalized by $1/(2i\pi)^{n-1}\int_{\mathscr{H}} u \wedge \bar{v}$ is rational.

Example. For an elliptic curve in $\mathbb{P}_2(\mathbb{C})$ defined over $\mathbb{Q}$ ($Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$ with $g_2, g_3 \in \mathbb{Q}$) we have $n = 2, l = 2$ and L_2 is generated by $(Z/Y)\mathrm{d}(X/Z)$ and$(X/Y)\mathrm{d}(X/Z)$.

The matrix of this basis in the toplogical basis coming from the standard parametrisation

$$\frac{X}{Z} = \wp \quad \frac{Y}{Z} = \wp' \quad \text{is just} \quad \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$$

where ω_1, ω_2 are the periods and η_1, η_2 the coperiods; the classical Legendre equation is

$$\eta_1\omega_2 - \eta_2\omega_1 = 2i\pi \quad (\mathrm{Im}(\omega_2/\omega_1) > 0)$$

which is compatible with the statement of the corollary; namely $((\eta_2\omega_1 - \eta_1\omega_2))^2 \in (2i\pi)^2\mathbb{Q}$.

References

[B.0] D. Barlet: Développements asymptotiques des fonctions obtenues par intégration dans les fibres. *Inv. Math.* 68 (1982) 129–174.

[B.1] D. Barlet: La forme hermitienne canonique sur la cohomologie de la fibre de Milnor d'une hypersurface à singularité isolée. *Inv. Math.* 81 (1985) 115–153.

[B.2] D. Barlet: le Calcul de la forme hermitienne canonique pour $X^a + Y^b + Z^c$. In *Lecture Note* n° 1198, pp. 35–46, Séminaire Dolbeault/Lelong/Skoda (1983/1984).

[B.3] D. Barlet: Le calcul de la forme hermitienne canonique pour un polynôme homogène à singularité isolée. *Journal de Crelles* 362, (1985) 179–196.

[G] P. Griffiths: On the periods of certain rational integrals I. *Ann. Math.* 90 (1969) 460–495.

[$\mathscr{L}$] J. Leray: Le calcul différentiel et intégral sur une variété analytique complexe (problème de Cauchy III). *Bull. Soc. Math. France* 87 (1959).

[L] F. Loeser: A propos de la forme hermitienne canonique d'une singularité isolée d'hypersurface. *Bull. Soc. Math. France* 114 (1986) 385–392.

[S] J. Scherk: On the monodromy theorem for isolated hypersurface singularities. *Inv. Math.* 58 (1980).

[V.1] A.N. Varchenko: Gauss–Manin connection of isolated singular point and Bernstein polynomial. *Bull. Sci. math.* (2) 104, (1980) 205–223.

[V.2] A.N. Varchenko: The complex singularity index does not change along the stratum μ = const. *Funk. Anal. i Prilozken* 16(a) (1982) 1–12.

[V.3] A.N. Varchenko: On the local residue and the intersection form on the vanishing cohomology. *Math. USSR Izvestiya*, 26(1) (1986) 31–52, translated from иЭв Akaq Hayk CCCP Cep. Matem. Tom. 45, n° 1 (1985).