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Foliations admitting transverse systems of differential equations

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Introduction

Riemannian foliations have been for a long time the subject of particular attention and at present we know a lot about their properties. It turns out that many of these properties are the consequence of two facts, i.e., that the geodesics are global on compact manifolds and that if a geodesic is orthogonal to the foliation at one point then it is orthogonal to the foliation at any point of its domain. A quick look at the equation of the geodesic for the Levi–Civita connection of a bundle-like metric reveals that this equation is of a special form which we shall call a foliated system of differential equations.

Many properties of Riemannian foliations have the foliations of a much larger class, the class of foliations admitting "foliated" differential equations with some additional properties like completeness of the equation and smooth dependence on the initial condition. For example, the equation of the geodesic is complete iff the geodesics are global. It is obvious that on compact Riemannian manifolds the equation of the geodesic of the Levi–Civita connection is complete, and it is not coincidental that for non-compact manifolds customarily one assumes that the metric is complete.

In addition to Riemannian foliations foliated equations admit, among others, transversely affine, transversely homogeneous, conformal and ∇-G-foliations. Up to now they have been considered separately. In all these cases we take the equation of the geodesic of some transverse connection and the completeness of this equation means precisely that geodesics are global. To indicate that the completeness plays an important role we provide two examples of transversely affine foliations, but first we prove a general property of complete (in our sense) transversely affine foliations.

PROPOSITION 1. *Let F be a complete transversely affine foliation. Then there exists a covering $\tilde{M}$ of M such that the lifted foliation $\tilde{F}$ is defined by a locally trivial fibre bundle $f\colon \tilde{M} \to \mathfrak{R}^q$.*

Proof. It is well known that affine foliations are developable; therefore there exists a covering space $\tilde{M}$ of M such that the lifted foliation $\tilde{F}$ to $\tilde{M}$ is given by a submersion f into $\mathfrak{R}^q$. Transverse geodesics in $\tilde{M}$ project onto geodesics in $\mathfrak{R}^q$, thus onto straight lines. Since the transverse geodesics are global, their images by f are whole lines and, therefore, the submersion f is surjective. Moreover, Theorem 1 of our paper assures that f is a locally trivial fibre bundle. □

The foliation of Example 1 is a transversely affine, not Riemannian, foliation whose equation of the geodesic is complete.

EXAMPLE 1 (cf. [3]). Let us consider the space $\mathfrak{R}^{2q+1}$ with coordinates $x_1, \ldots, x_q, y_1, \ldots, y_q, z$ and take the following form w with values in $\mathfrak{R}^{q+1}$

$$w = \begin{pmatrix} dx_1 \\ \vdots \\ dx_q \\ dz + x_1 dy_1 \ldots + x_q dy_q \end{pmatrix}$$

The Pfaff system $w = 0$ is integrable and of rank $q + 1$; therefore it defines a foliation F of codimension $q + 1$ on $\mathfrak{R}^{2q+1}$. It is a transversely affine foliation. The foliation F is defined by a global submersion $f(x_1, \ldots, x_q, y_1, \ldots, y_q, z) = (x_1, \ldots, x_q, z + x_1 y_1 + \cdots + x_q y_q)$ which is a trivial fibre bundle.

The foliation F is left invariant by a group $\mathbf{A}$ with generators $g_1, \ldots, g_q, h_1 \ldots h_q, k$ acting as follows:

(1) g_i maps x_i onto $x_i + 1$, z onto $z - y_i$ and leaves other coordinates invariant,
(2) h_i maps y_i onto $y_i + 1$ and leaves other coordinates invariant,
(3) k maps z onto $z + 1$ and leaves other coordinates invariant.

Therefore the foliation F projects onto the foliation $F_{\mathbf{A}}$ of the compact quotient manifold $\mathfrak{R}^{2q+1}/\mathbf{A} = \mathfrak{T}^{2q+1}_{\mathbf{A}}$. The manifold $\mathfrak{T}^{2q+1}_{\mathbf{A}}$ is a locally trivial fibre bundle over $\mathfrak{T}^q$ with standard fibre $\mathfrak{T}^{q+1}$. In each fibre the foliation $F_{\mathbf{A}}$ induces a codimension 1 foliation whose leaves are diffeomorphic to $\mathfrak{R}^s \times \mathfrak{T}^{q-s}$. Thus, indeed, the leaves have the same universal covering space.

The following is an example of a transversely affine foliation which is not complete in our meaning (cf. [10], [7]).

EXAMPLE 2 (cf. [3], [4]). Let us consider the sphere S^n as embedded in $\mathfrak{R}^{n+1}$. On the manifold $S^n \times S^1$ we take 1-forms $w_i = dx_i - x_i d\theta$, then the differential form $w_1 \wedge \cdots \wedge w_{n+1}$ is without singularities. It is equal namely to $d\bar{x}_1 \ldots dx_{n+1} - d\theta \wedge w$, where w is the volume form of S^n:

$$w = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots \wedge dx_n$$

Therefore the Pfaff system $\{w_j = 0\}_{j=1}^{s}$ defines a transversely affine foliation of codimension s on the manifold $S^n \times S^1$.

For $s = 1$ the foliation has one compact leaf $S^{n-1} \times S^1$, the other leaves are diffeomorphic to $\mathfrak{R}^n$.

One can easily check that the foliation of $S^n \times \mathfrak{R}$ does not fibre over $\mathfrak{R}$, and the leaves do not have the common universal covering space.

From our point of view the transversely affine foliations of Examples 1 and 2 belong to two different classes of foliations, i.e. the system of differential equations, (the equation of the geodesic), in Example 1 is transversely complete, which is not the case of Example 2. These two examples indicate that it is worth-while to study foliated systems of differential equations.

To complete the introduction we provide a method of producing examples of foliations with transverse systems of equations.

Let us take a manifold T with a system of differential equations $\mathbf{E}$. Let $\mathbf{Diff}(T, \mathbf{E})$ be the group of global automorphisms of the system $\mathbf{E}$. A homomorphism $h: \pi_1(B) \to \mathbf{Diff}(T, \mathbf{E})$ of the fundamental group of a manifold B allows us to define a system of differential equations on the fibre bundle $\tilde{B} \times_h T$. Let $p_2: \tilde{B} \times T \to T$ be the projection on the second factor. Then the system $p_2^* \mathbf{E}$ is a transverse system for the foliation F defined by the submersion p_2. Moreover, the equivalence relation '$\sim h$': $(x\alpha, t) \sim_h (x, h(\alpha)t)$ preserve the system $p_2^* \mathbf{E}$, hence the projected system $\mathbf{E}_h$ on $\tilde{B} \times_h T$ is a transverse system for the foliation F_h. It is easy to see that if the system $\mathbf{E}$ is complete, then the system $\mathbf{E}_h$ is complete as well, and the solutions are tangent to the fibres. Other properties of the system $\mathbf{E}$ translate in a similar and good way.

It is not difficult to find systems of differential equations on $\mathfrak{R}^q$ or $\mathfrak{T}^q$ with non-trivial "symmetry groups" as groups of automorphisms of such systems are called in the theory of differential equations. The references are quite numerous, for example [9].

For convenience sake all the objects are smooth and the manifolds connected.

1. Preliminaries

For us a system **E** of differential equations of order k on a manifold M is a fibre subbundle of $\mathbf{J}^k(\mathfrak{R}, M)$ – the bundle of k-jets of smooth mappings from $\mathfrak{R}$ into M. Therefore, to learn more about the structure of foliations admitting "foliated" systems of differential equations, we have to look closer at bundles of jets on foliated manifolds.

First of all we shall recall some facts about transverse bundles to the foliation (cf. [12], [13]).

A leaf curve α: $[0, 1] \to M$, $\alpha(0) = x$, $\alpha(1) = y$ defines a holonomy isomorphism T_α: $N(M, F)_x \to N(M, F)_y$ of the fibres of the normal bundle, as well as the fibres of the following transverse bundles:

$$N^r(T_\alpha)\colon N^r(M, F)_x \to N^r(M, F)_y.$$

$N^r(M, F)$ – the normal bundle of order r.

The total spaces of these bundles $N^r(M, F)$ admit foliations F^r, respectively, of the same dimension as F. Their leaves project onto leaves of the foliation F, and they are covering spaces of these leaves. This allows us to interpret the mappings $N^r(T_\alpha)$ in the following way. Let $\xi \in N^r(M, F)_x$, then the vector $N^r(T_\alpha)(\xi) \in N^r(M, F)_y$ is the end of the lift $\tilde{\alpha}$ of the leaf curve α to the vector ξ obtained by lifting the curve α to the leaf of the foliation F^r passing through ξ.

Let Q be a subbundle of TM transverse to F. By Q^k we shall denote the subset (subbundle) of $T^k(M)$ of the k-tangent bundle of the manifold M, consisting of k-jets of curves tangent to Q.

Since the bundle Q^r is isomorphic to $N^r(M, F)$, all the above remarks are also true for this bundle, as well as for any foliated subbundle of Q^r.

In the same way as $N^r(M, F)$ we define the space $\mathbf{J}^k(\mathfrak{R}, M; F)$, i.e., the bundle of transverse k-jets of mappings of $\mathfrak{R}$ into M. $\mathbf{J}^k(\mathfrak{R}, M; F)$ is a fibre bundle over both $\mathfrak{R}$ and M. Its fibre over any point v of $\mathfrak{R}$ is diffeomorphic with $N^k(M, F)$. On the manifold $\mathbf{J}^k(\mathfrak{R}, M; F)$ there is a foliation F_k which induces on each fibre the foliation F^k.

The bundle $\mathbf{J}^k(\mathfrak{R}, Q)$ of k-jets of mappings f from $\mathfrak{R}$ into M tangent to Q is isomorphic to $\mathbf{J}^k(\mathfrak{R}, M; F)$ and thus admits a foliation F_k of the same dimension as F.

Now we shall recall the definition of a system of differential equations.

Definition 1. A subbundle **E** of $\mathbf{J}^k(\mathfrak{R}, M)$ is called a system of differential equations of order k on the manifold M.

A mapping $f\colon \mathfrak{R} \to M$ of a connected domain is a solution of the system **E** if the mapping $\tilde{f}\colon \operatorname{dom} f \ni t \mapsto j_t^k f \in \mathbf{J}^k(\mathfrak{R}, M)$ is a section of **E**.

Let r be an integer smaller than or equal to k, $0 \leqslant r \leqslant k$. For each such an r the system **E** defines a subset $\mathbf{E}_0^r$ of $T^r(M)$.

$\mathbf{E}_0^r = \{j_0^r f \circ \tau_t : f$ is a solution of **E** at $t \in \mathfrak{R}$, τ_t – the translation in $\mathfrak{R}$ by the vector $t\}$. The set $\mathbf{E}_0^r$ is called the set of initial conditions of the system **E** of order r.

A system **E** is called a USP (Unique Solution Property) system if there exists $0 \leqslant r \leqslant k$ such that the set $\mathbf{E}_0^r$ is a subbundle of $T^r(M)$ and for any pair $(t, \xi) \in \mathfrak{R} \times \mathbf{E}_0^r$ there exists exactly one solution f in a neighbourhood of t such that $j_0^r f \circ \tau_t = \xi$. Moreover, we assume that the solutions depend smoothly on the initial condition.

Let r be the smallest integer having the above property, then the bundle $\mathbf{E}_0^r$ is called the bundle of initial conditions of the system **E**.

We say that solutions of the system **E** depend smoothly on the initial condition if for any smooth mapping $f\colon W \to E_0^r$, W an open subset of $\mathfrak{R}^m$, φ_t – the solution of **E** with the initial condition $f(t)$, the mapping φ: $W \times \mathfrak{R} \to M$ defined as $\varphi(t, v) = \varphi_t(v)$ is a smooth mapping.

A system **E** is called transitive if for any tangent vector X there exists a solution f of the system **E** such that $X \in \operatorname{im} d_0 f$.

2. Foliated systems of differential equations

It is our aim to explain the influence of transverse systems of differential equations on the structure of foliations. To have any relation between properties of these two objects on the manifold M they must be in some way compatible, i.e., the system should be "adapted" or "foliated". We shall work with the following definition of a foliated system of differential equations.

DEFINITION 2. A system of differential equations **E** is called foliated if there exists a subbundle Q supplementary to TF such that the set $\mathbf{J}^k(\mathfrak{R}, Q) \cap \mathbf{E} = \mathbf{E}_Q$ is a foliated subbundle of $\mathbf{J}^k(\mathfrak{R}, Q)$.

The foliated subbundle $\mathbf{E}_Q$ defines a system $\mathbf{E}_T$ of differential equations on the transverse manifold T, i.e., $\mathbf{E}_T \subset \mathbf{J}^k(\mathfrak{R}, T)$, and $\mathbf{E}_Q$ is the subbundle of $\mathbf{J}^k(\mathfrak{R}, Q)$ corresponding to $\mathbf{E}_T$ (cf. [12]). The holonomy pseudogroup **H** is a pseudogroup of automorphism of the system $\mathbf{E}_T$.

Since we are interested in the transverse structure of the foliation, therefore we shall look only at solutions transverse to the foliation. The following definitions will be very useful.

DEFINITION 3. A solution $f\colon \mathfrak{R} \longrightarrow M$ of $\mathbf{E}$ is said to be tangent to Q if $\operatorname{im} d_t f \subset Q$ for any $t \in \operatorname{dom} f$.

Let the system $\mathbf{E}_T$ be USP and $\mathbf{E}_0^r(T)$ be its bundle of initial conditions. The corresponding subbundle of Q^r we denote by $\mathbf{E}_0^r(Q)$ and it is called the bundle of transverse initial conditions. A foliated system $\mathbf{E}$ is called a TUSP (Transverse Unique Solution Property) system if solutions with initial conditions from the bundle $\mathbf{E}_0^r(Q)$ are unique and depend smoothly on the initial condition.

A solution is said to be transverse if its initial condition belongs to the bundle of transverse initial conditions.

A systems $\mathbf{E}$ is (transversely) complete if any (transverse) solution can be extended to a global one.

A foliated system $\mathbf{E}$ is transversely transitive if for any vector X of the bundle Q there exists a transverse solution f of the system $\mathbf{E}$ such that $\operatorname{im} \mathrm{d}_0 f \ni X$.

A curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is called a solution curve of the system $\mathbf{E}$ if there exists a solution f of the system $\mathbf{E}$ at $x = \gamma(0)$ and a curve $\bar{\gamma}\colon (-\varepsilon, \varepsilon) \to \mathfrak{R}$, $\bar{\gamma}(0) = 0$ such that $\gamma = f \circ \bar{\gamma}$.

A curve $\gamma\colon [0, 1] \to M$ is called a piecewise solution curve of the system $\mathbf{E}$ if there exists a sequence of numbers $t_0 = 0 < t_1 < \cdots < t_{m+1} = 1$ such that for $i = 0, \ldots, m$ the curve $\gamma \mid [t_i, t_{i+1}]$ is a solution curve.

REMARK. If the system $\mathbf{E}$ is transversely transitive, then any two leaves of the connected manifold M can be joined by a piecewise solution curve.

LEMMA 1. *Let F be a simple foliation given by a submersion $p\colon M \to T$ and $\mathbf{E}$ be a foliated TUSP system of differential equations on (M, F). Denote by $\mathbf{E}_T$ the induced system on the manifold T. Let ξ_0 be an element of the set $\mathbf{E}_0^r(T)$ of initial conditions of the system $\mathbf{E}_T$ over a point x_0. If f is a solution of the system $\mathbf{E}_T$ such that $j_0^r f \circ \tau_t = \xi_0$, then for any $x \in p^{-1}(x_0)$ $\xi = (N^r(p)_x)^{-1}(\xi_0) \in \mathbf{E}_0^r(Q)$ there exists a solution f_x of the system $\mathbf{E}_Q$ such that $j_0^r f_x \circ \tau_t = \xi$.*

Proof. The mapping f is the solution of the system $\mathbf{E}_T$ with the initial condition ξ_0, so its lift f_x at x tangent to Q is a solution of the system $\mathbf{E}_Q$. As $p \circ f_x = f$ in a neighbourhood of t and $N^r(p)_x(\xi) = \xi_0$ the r-jet of $f_x \circ \tau_t$ at 0 must be ξ. □

COROLLARY. *Let F be a simple foliation defined by a submersion $p\colon M \to T$ and let $\mathbf{E}$ be a transversely complete, TUSP foliated system. Let f_1* and *f_2 be two transverse solutions of $\mathbf{E}$ such that $pf_1(0) = pf_2(0)$ and $N^r(p)(j_0^r f_1) = N^r(p)(j_0^r f_2)$. Then for any t of the intersection* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$ *of the domains of f_1 and f_2, the points $f_1(t)$ and $f_2(t)$ belong to the same fibre of p.*

Proof. The mappings pf_1 and pf_2 are solutions of the system $\mathbf{E}_T$ with the initial condition $N^r(p)(j_0^r f_1)$. From Lemma 1 it follows that pf_1 and pf_2 are equal on the intersection of domains. □

LEMMA 2. *Let* **E** *be a TUSP foliated system of differential equations. Let* α: $[0, s_0] \to E_0^r(Q)$ *be a leaf curve and* f_s *the solution of the system* **E** *with the initial condition* $\alpha(s)$ *at* 0. *If for any* $s \in [0, s_0]$ *the solution* f_s *is defined on compact connected neighbourhood* W *of* 0 *in* R, *then for any* t *of* W *the points* $f_s(t)$, $s \in [0, s_0]$ *belong to the same leaf of the foliation* F.

Proof. Since the system **E** is a TUSP system, the mapping F: $[0, s_0] \times W \to M$, $F(s, t) = f_s(t)$, is a smooth mapping. The set $F([0, s_0] \times W)$ is compact and we can cover it by a finite number of adapted charts. Let us choose an $s_1 \in [0, s_0]$ and adapted charts $(U_1, \varphi_1), \ldots, (U_m, \varphi_m)$ covering the set $f_{s_1}(W)$. Then there exist $\varepsilon > 0$ and compact sets $K_1, \ldots, K_v$ covering W such that each set $F([s_1 - \varepsilon, s_1 + \varepsilon] \times K_i)$ is contained in some U_j for $j = 1, \ldots, m$. Corollary of Lemma 1 ensures that for any $t \in K_i$ the points $f_s(t)$, $s \in [s_1 - \varepsilon, s_1 + \varepsilon]$ belong to the same leaf of the foliation F. Thus for any $t \in W$ the points $f_s(t)$, $s \in [s_1 - \varepsilon, s_1 + \varepsilon]$ belong to the same leaf of F. As we can cover the interval $[0, s_0]$ with intervals $[s_1 - \varepsilon, s_1 + \varepsilon]$ having the required property, the lemma has been proved. □

Let us consider the bundle $\mathbf{E}_0^r(Q) \times \mathfrak{R}$. This manifold is foliated by the product of the foliation F^r and the foliation by the points of $\mathfrak{R}$. To any pair $(\xi, t) \in \mathbf{E}_0^r(Q) \times \mathfrak{R}$ we can associate the point $f_\xi(t) \in M$, where f_ξ is the solution of the system **E** with the initial condition ξ at 0. This correspondence defines a smooth mapping Exp: $\mathbf{E}_0^r(Q) \times \mathfrak{R} \to M$. In the case of the equation of the geodesic this mapping becomes the real exponential mapping.

LEMMA 3. *Let* x *be a point of* M, $\xi \in \mathbf{E}_0^r(Q)_x$ *and* L_ξ *be the leaf of the foliation* F^r *passing through* ξ. *Then for any* $t \in \mathfrak{R}$ *the mapping* Exp $| L_\xi \times \{t\}$: $L_\xi \to L_t$ *is a covering,* (L_t *is the leaf of* F *passing through* Exp (ξ, t)).

Proof. Lemma 2 ensures that Exp $(L_\xi \times \{t\})$ is contained in a leaf of F. It is sufficient to show that the mapping in question is a local diffeomorphism and that it has the property of lifting curves. Let y be any point of L_t and α be a leaf curve linking x_t = Exp (ξ, t) to y. Let $\xi_t = j_t^r f_\xi \in \mathbf{E}_0^r(Q)_{x_t}$ and α_t be the lift of α to ξ_t. Then the correspondence $s \mapsto j_0^r f_{\alpha_t(s)}$, is a curve in the leaf L_ξ, as $j_0^r f_{\alpha_t(0)} = \xi (f_{\alpha_t(s)}$ is the solution of **E** with the initial condition $\alpha_t(s)$ at t). The fact that transverse solutions project onto solutions of the system $\mathbf{E}_T$ ensures that the mapping is a local diffeomorphism. □

Actually, we have proved the following.

COROLLARY (of the proof). *Let x be a point M, $\xi \in \mathbf{E}_0^r(Q)_x$, L_ξ and L_{ξ_t} be the leaves of the foliation F^r passing through ξ and ξ_t, respectively. Then for any $t \in R$, the leaves L_ξ and L_{ξ_t} are diffeomorphic.*

From the above corollary we immediately obtain the following proposition.

PROPOSITION 2. *Let (M, F) be a foliated manifold admitting a transversely complete, transversely transitive, foliated TUSP system of differential equations. Then the leaves of the foliation F have the common universal covering space.*

Proof. Corollary to Lemma 3 asserts that two leaves joined by a solution curve have the same universal covering space. This fact coupled with the remark that for a transversely transitive system any two leaves can be linked by a piecewise solution curve, completes the proof. □

Lemma 3 leads us to the formulation of the following proposition.

PROPOSITION 3. *Let $\mathbf{E}$ be a transversely complete, foliated TUSP system on a foliated manifold (M, F). Then the mapping* Exp: $\mathbf{E}_0^r(Q) \times \mathfrak{R} \to M$ *is smooth and foliated.*

It results from Proposition 3 that any global foliated section X of the bundle $\mathbf{E}_0^r(Q)$, for any $t \in \mathfrak{R}$, defines a global foliated mapping $\exp_{X,t}$: $M \to M$, $\exp_{X,t}(x) = \mathrm{Exp}\,(X(x), t)$. We denote the semigroup generated by mappings of this form by $\mathbf{Map}_{\mathbf{E}}(M, F)$. Then, as one can easily show, we obtain the following proposition.

PROPOSITION 4. *Let $\mathbf{E}$ be a transversely complete, foliated TUSP system of differentiated equations on a foliated manifold (M, F). Then the leaf of F passing through a point x is a covering space of any leaf passing through the $\mathbf{Map}_{\mathbf{E}}(M, F)$-orbit of the point x.*

Let $X_1, \ldots, X_w$ be foliated sections of $\mathbf{E}_0^r(Q)$ over an open subset U. Then we can define the following mapping:

$$\exp_X^w\colon \mathfrak{R}^w \times U \to M,$$

$$\exp_X^w (t_1, \ldots, t_w, x) = \mathrm{Exp}\,(X_w, t_w) \circ \ldots \mathrm{Exp}\,(X_1, t_1)(x)$$

where $\mathrm{Exp}\,(X_i, t_i)(x) = \mathrm{Exp}\,(X_i(x), t_i)$.

The smooth mapping $\exp^w_X$ is foliated for the product foliation of F/U and the foliation by points of $\mathfrak{R}^w$.

Before proceeding further we need the following notation. Let π^r: $T^r(M) \to T(M)$ be the natural projection. Then π^r maps $\mathbf{E}^r_0(Q)$ into Q.

We conclude our considerations with the following theorem, which is a generalization of Herman's theorem about Riemannian submersions (cf. [6]).

THEOREM 1. *Let h: $M \to N$ be a submersion of a manifold M of dimension n into a connected manifold N of dimension q. If for the foliation F defined by the submersion h there exists a transversely complete, transversely transitive, foliated TUSP system* $\mathbf{E}$ *of differential equations, then the submersion h: $M \to N$ is a locally trivial fibre bundle.*

Proof. Let us consider two mappings Exp_Q: $\mathbf{E}^r_0(Q) \times \mathfrak{R} \to M$ and Exp_{TN}: $\mathbf{E}^r_0(N) \times \mathfrak{R} \to N$ defined as above. The second mapping Exp_{TN} is defined by the induced system $\mathbf{E}_N$ which is, obviously, a foliated system for the foliation by points. Lemma 1 ensures that the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{E}^r_0(Q) \times \mathfrak{R} & \xrightarrow{\mathrm{Exp}_Q} & M \\ N^r(h) \times \mathrm{id} \downarrow & & \downarrow h \\ \mathbf{E}^r_0(N) \times \mathfrak{R} & \xrightarrow{\mathrm{Exp}_{TN}} & N \end{array}$$

Since the system $\mathbf{E}$ is transversely transitive, the system $\mathbf{E}_N$ is also transitive. Therefore for any point $x_0 \in N$, there exist a neighbourhood U and sections $\hat{X}_1, \ldots, \hat{X}_q$ of $\mathbf{E}^r_0(N)$ over U such that for any point $x \in U$ the vectors $\pi^r(\hat{X}_1(x)), \ldots, \pi^r(\hat{X}_q(X))$ span TN_x. Thus for some neighbourhood W of 0 in $\mathfrak{R}^q$ the mapping $\exp^q_{\hat{X}} | W \times \{x_0\}$ is a diffeomorphism on the image $\tilde{W}$. As the foliation defined by h is without holonomy, the sections $\hat{X}_1, \ldots, \hat{X}_q$ define the foliated sections $X_1, \ldots, X_q$ of $\mathbf{E}^r_0(Q)$ over $h^{-1}(U)$. The image of the mapping $\exp^q_X | W \times h^{-1}(x_0)$ is precisely $h^{-1}(\tilde{W})$. For each $t \in W$, the mapping $\exp^q_X|\{t\} \times h^{-1}(x_0)$ is a diffeomorphism of $h^{-1}(x_0)$ onto $h^{-1}(\exp^q_{\hat{X}}(t, x_0))$ as the leaves of the foliation F^r of $\mathbf{E}^r_0(Q)$ are diffeomorphic to the corresponding leaves of F. The fact that $\exp^q_{\hat{X}} | W \times \{x_0\}$ is a diffeomorphism on the image ensures that the mapping $\exp^q_{\hat{X}} | W \times h^{-1}(x_0)$ is itself a diffeomorphism on the image, which precisely means that the submersion h is a locally trivial fibre bundle. □

Having proved a theorem about submersions we go back to the study of foliated manifolds and, in particular, of the universal covering of the manifold itself. But first some preparatory explanations are necessary.

Let α: $[0, 1] \to M$ be any leaf curve, and ξ_0 be an element of the fibre $\mathbf{E}_0^r(Q)_{\alpha(0)}$ of the bundle of transverse initial conditions $\mathbf{E}_0^r(Q)$ over $\alpha(0)$. The bundle of transverse initial conditions $\mathbf{E}_0^r(Q)$ is foliated by F^r. Thus the curve α admits a lift $\tilde{\alpha}$ to ξ_0 such that the curve $\tilde{\alpha}$ is a leaf curve. In this way we have obtained a differentiable field of initial conditions along α.

Let us assume that the TUSP foliated system **E** is transversely complete. If at a point x_0 of the manifold M we have a pair of curves α: $[0, 1] \to M$, σ: $[0, \varepsilon] \to M$ where α is a leaf curve and σ is a solution curve, then there exists a mapping κ: $[0, 1] \times [0, \varepsilon] \to M$ such that $\kappa|[0, 1] \times \{0\} = \alpha$, $\kappa|\{0\} \times [0, \varepsilon] = \sigma$, for any $t \in [0, \varepsilon]$ $\kappa|[0, 1] \times \{t\}$ is a leaf curve, and for any $v \in [0, 1]$, $\kappa|\{v\} \times [0, \varepsilon]$ is a solution curve tangent to Q. Since σ is a solution curve there is a solution f: $\mathfrak{R} \to M$ of the system **E** at 0 and a curve γ: $[0, \varepsilon] \to \mathfrak{R}$ for which $\sigma = f \circ \gamma$. Denote by ξ_0 the initial condition of the solution f, i.e., $\xi_0 = j_0^r f$. Let $\tilde{\alpha}$ be the lift of the curve α to ξ_0. Then the mapping κ: $[0, 1] \times [0, \varepsilon] \to M$, $\kappa(v, t) = f_v \circ \gamma(t)$, where f_v is the solution of the system **E** with the initial condition $\tilde{\alpha}(v)$, has the required properties because of Lemma 2. Moreover, if we take at a point x_0 a pair of curves α: $[0, 1] \to M$, σ: $[0, \varepsilon] \to M$ such that the curve α is a leaf curve and σ is a piecewise solution curve, i.e., there is a sequence $t_0 = 0 < t_1 < \cdots < t_{m+1} = 1$ for which $\sigma|[t_i, t_{i+1}]$, $i = 0, \ldots, m$ is a solution curve of the system **E** tangent to Q, then there exists a mapping κ: $[0, 1] \times [0, \varepsilon] \to M$ with the same properties as above, but with the following change: for any $v \in [0, 1]$ the curve $\kappa|\{v\} \times [t_i, t_{i+1}]$ is a solution curve of the system **E** tangent to the bundle Q.

Now we shall deal with the universal covering space of a foliated manifold admitting a foliated system of differential equations. First of all we prove a preparatory lemma.

LEMMA 4. *Let σ: $[0, 1] \to M$ be a curve. Then σ is homotopic, relative to its ends, to a curve of the form $\beta * \alpha$ such that α is a leaf curve, and β is a piecewise solution curve of the system* E *tangent to the bundle* Q.

Proof. For any point x of the manifold M there exist foliated sections $X_1, \ldots, X_q$ defined on a neighbourhood of x, a neighbourhood V of x in the leaf passing through x and a neighbourhood W of 0 in $\mathfrak{R}^q$ such that the mapping $\exp_X^q|W \times V$: $W \times V \to M$ is a diffeomorphism on the image. By taking smaller W and V we can assume that both sets are contractible. Then it is obvious that the lemma is true for curves contained in $\exp_X^q(W \times V)$.

The lemma results easily from the following two facts:

(i) any curve σ can be covered by a finite number of sets of the form as above;

(ii) a curve of the form $\alpha * \beta$ is homotopic, relative to the ends, to a curve of the form $\beta' * \alpha'$ where α, α' are leaf curves and β, β' are piecewise solution curves. (It is a consequence of the considerations preceding the lemma). □

Using a standard method (cf. [2]), we can prove the following proposition.

PROPOSITION 5. *Let (M, F) be a foliated manifold. Let* **E** *be a transversely complete, transversely transitive, foliated TUSP system of differential equations. If the bundle Q is integrable, then the universal covering space $\tilde{M}$ of the manifold M is diffeomorphic to $\tilde{L} \times \tilde{G}$ where $\tilde{L}$ is the universal covering space of a leaf L of the foliation F, and $\tilde{G}$ is the universal covering space of a leaf G to the foliation* Q.

COROLLARY 1. *If the foliation F is of codimention 1, then the universal covering space $\tilde{M}$ is diffeomorphic to $\tilde{L} \times \mathfrak{R}$ where $\tilde{L}$ is the universal covering space of leaves of the foliation.*

COROLLARY 2. *If M is a compact manifold and the foliation F is of codimention 1, then the fundamental group $\pi_1(M)$ of the manifold M is infinite.*

Proof. If the group $\pi_1(M)$ were finite, then the universal covering space $\tilde{M}$ would be compact and homeomorphic to $\tilde{L} \times \mathfrak{R}$; contradiction. □

If the foliation F is Riemannian and Q a supplementary foliation, then the leaves of F and Q intersect one another; this can be proved by a well known method, (cf. [1]), also in our case.

PROPOSITION 4. *Let (M, F) be a foliated manifold with a transversely complete, transversely transitive foliated TUSP system* **E** *of differential equations. If the normal bundle Q is integrable, then any leaf L of the foliation F intersects any leaf K of the foliation Q.*

3. Examples

1. Let T be a transverse manifold of the foliation F and **H** the holonomy pseudogroup on T. Let us assume that there exists a G-connection on T of which the holonomy pseudogroup **H** is a pseudogroup of affine transformations. Thus in the induced foliated G-structure there is a transversely projectable connection. Such foliations are called ∇-G-foliations and have been studied, among others, by P. Molino and the author (cf. [7], [10], [11] and [13]). In [7] P. Molino proved Proposition 2 of this paper for foliations admitting complete transversely projectable connections. This class of foliations, in particular, includes Riemannian and transversely affine foliations.

To such foliations we can associate a foliated system of differential equations; namely, let us take a supplementary subbundle Q. The connection ∇ defines a covariant differentiation on the vector bundle Q. We can extend this operator to the whole tangent bundle by choosing any covariant differentiation on the bundle tangent to the leaves. The equation of the geodesic of the connection determined by this operator of covariant differentiation is a foliated one.

This class of foliations also includes transversely parallelisable foliations. If F is a transversely parallelisable foliation, we choose a subbundle Q and vector fields $X_1, \ldots, X_q$, sections of the subbundle Q, defining the transverse parallelism. As the connection ∇ we choose a connection making the vector fields parallel, thus segments of the flows of vector fields X_i are geodesics.

2. A Riemannian foliation F is a foliation equipped with a bundle-like metric g. The equation of the geodesic of the Levi–Civita connection of the Riemannian metric g is a foliated equation.

3. We can do the same for connections of higher order, as for linear connections in Example 1. For the equation of geodesic in this case see [5].

Final remarks

It is possible to develop a similar theory for partial differential equations. All proofs go as for ordinary differential equations with the exception of Lemma 1. If the subbundle Q is not integrable, lifts of solutions of the system $\mathbf{E}_T$ do not always exist. With the additional assumptions of integrability of the supplementary subbundle Q, all the properties proved in this paper for ordinary differential equations are true for partial ones. The proofs have been so formulated that the reader should not have any difficulties in adapting them to the required case. Instead of the normal bundle of order k, $N^k(M, F)$ (Q^k), it is necessary to consider the bundle of transverse (p, s)-velocities, $N_s^p(M, F)$ (Q_s^p) (cf. [12]).

Actually, as one can verify quite easily, the global existence of solutions is not really necessary. It is sufficient to know that for any leaf curve α: $[0, s_0] \to \mathbf{E}_0^r(Q)$, the solutions $f_{\alpha(s)}$, $s \in [0, s_0]$ have the same non-empty domain.

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References

1. R.A. Blumenthal and J.J. Hebda: De Rham decomposition theorem for foliated manifolds. *Ann. Inst. Fourier* 33 (1983) 183–198.
2. R.A. Blumenthal and J.J. Hebda: Ehresmann connection for foliations. *Indiana Un. Math. J.* 33(4) (1984) 597–611.
3. Bobo Seke: Sur les structures transversalement affines des feuilletages de codimension un. *Ann. Inst. Fourier* 30(1) (1980) 1–29.
4. Bobo Seke: *Structures transverses affines trivialisables.* Publ. IRMA, Strasbourg, Vol. 188, p. 109.
5. J. Gancarzewicz: Geodesics of order 2. *Prace Mat.* 19 (1977) 121–136.
6. R. Herman: A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. *Proc. A.M.S.* 11 (1960) 232–242.
7. P. Molino: Propriétés cohomologiques et propriétés topologiques de feuilletages à connexion transverse projectable. *Topology* 12 (1973) 317–325.
8. P. Molino: Etude des feuilletages transversalement complets et applications. *Ann. Scient. Ec. Norm. Sup.* 10 (1977) 289–307.
9. L.V. Ovsiannikov: *Group Analysis of Differential Equations.* Academic Press, New York (1982).
10. R. Wolak: On ∇-G-foliations. *Suppl. Rend. Circolo Mat. Palermo* 6 (1984), 329–341.
11. R. Wolak: Normal bundles of foliations of order k. *Demons. Math.* 18(4) (1985) 977–994.
12. R. Wolak: Transverse structures of foliations. *Suppl. Rend. Cir. Mat. Palermo* 9 (1985) 227–243.
13. R. Wolak: Some remarks on ∇-G-foliations. In: L.A. Cordero (ed.) *Proceedings, V-th Colloquium on Differential Geometry*, Santiago de Compostela, Spain (1984); Pitman (1985) pp. 276–289.