COMPOSITIO MATHEMATICA

E. M. OPDAM

Root systems and hypergeometric functions III

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E.M. OPDAM
*Department of Mathematics, University of Leiden, Niels Bohrweg 1, 2333 CA Leiden,
The Netherlands (present address: Department of Mathematics, University of Utrecht,
Budapestlaan 6, De Uithof, Postbus 80010, 3508 TA Utrecht, The Netherlands)*

1. Introduction

We keep notations as in [HO] and [H]. In those papers we studied the
following differential operator $L(\ell)$ associated with a (possibly non-reduced)
root system R:

$$L(\ell) = \sum_{j=1}^{n} \partial(X_j)^2 - \sum_{\alpha \in R_+} \ell_\alpha \frac{1 + h^\alpha}{1 - h^\alpha} \partial(X_\alpha), \tag{1.1}$$

where $\ell = (\ell_\alpha) \in K \cong \mathbb{C}^m$ and m the number of Weyl group orbits in R. A
crucial role in [HO] was played by the hypothesis that the "Harish-Chandra
homomorphism"

$$\gamma(\ell): \mathbb{D}(\ell, H) = (\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}))^{W,L} \to \mathbb{C}[\mathfrak{h}^*]^W \tag{1.2}$$

is an isomorphism onto. For R of type A_2 or BC_2 this hypothesis is known
from the work of Koornwinder. Under the assumption that the map (1.2)
is onto we investigated in [HO] the system of differential equations

$$P\phi = \gamma(P)(\lambda)\phi, \quad P \in \mathbb{D}(H), \quad \lambda \in \mathfrak{h}^* \text{ fixed.} \tag{1.3}$$

Note that if the parameter $\ell \in \mathbb{C}^m$ is fixed it is sometimes dropped in the
notation. It turns out that, viewed on a toroidal completion of the torus H,
the system (1.3) is holonomic on H^{reg} with simple singularities along the
hyperplanes at infinity. For generic parameters we have shown in [HO] that
the system (1.3) has a Weyl group invariant, analytic solution on some
tubular neighbourhood of $A \subset H$ intersected with H^{reg}. This solution is
unique up to a constant and with a suitable normalization we denote this
solution by $F(\lambda, \ell; h)$, and call it the (multivariable) hypergeometric function
associated with the root system R.

In order to avoid the hypothesis that (1.2) is an isomorphism a different construction of $F(\lambda, \ell; h)$ is given in $[H]$ using the Riemann-Hilbert correspondence. In our example the Riemann-Hilbert correspondence amounts to the fact that – at least for generic parameters λ and ℓ – the function $F(\lambda, \ell; h)$ is characterized by the single differential equation

$$L\phi = (\lambda - \varrho, \lambda + \varrho)\phi \qquad (1.4)$$

together with the monodromy representation. The fundamental group $\pi_1(W\backslash H^{\mathrm{reg}})$ of the complement of the discriminant has been described in terms of generators and relations by v.d. Lek and Looijenga ([L]). The crucial point, four our purpose, of their description is that the relations (braid relations and push relations) allow a reduction to rank two.

The main goal of this paper is to prove that the Harish-Chandra homomorphism (1.2) is onto for R a root system of type G_2, thereby filling in the missing link in the construction of $F(\lambda, \ell; h)$ for an arbitrary root system R. In order to do this we study the concept of shift operators, which is a multivariable analogue of relations of the form

$$\frac{d}{dz} F(\alpha, \beta, \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z) \qquad (1.5)$$

$$((z - 1)\frac{d}{dz} + (\alpha + \beta - \gamma)) F(\alpha, \beta, \gamma; z) = \left(\alpha + \beta - \gamma - \frac{\alpha\beta}{\gamma}\right)$$

$$\times F(\alpha, \beta, \gamma + 1; z). \qquad (1.6)$$

For the root systems of rank two we give a complete description of the shift operators. A first application is a proof of the Harish-Chandra isomorphism for type G_2. A second application is the verification of the conjecture that $F(\lambda, \ell; e) = 1$ in the case that either $\lambda \in P_-$ and $k_\alpha \geqslant 0$ (the multivariable Jacobi polynomials) or $F(\lambda, \ell; h)$ "contiguous" to a case where the conjecture is known (e.g., the group case).

2. Shift operators for rank one and rank two root systems

Let R be a possibly non reduced root system. Let m be the number of conjugacy classes of roots in R. Denote by $K \cong \mathbb{C}^m$ the ℓ_α – parameter space (see [HO], Section 2). As in [HO] we write P for the weight lattice of R^0 (the "inmultiplicable" roots of R) and H for the torus with characterlattice P.

The Weyl denominator Δ is the function on H given by

$$\Delta(h) = \prod_{\alpha \in R^0} (h^{\alpha/2} - h^{-\alpha/2}). \tag{2.1}$$

Put $H^{\text{reg}} = \{h \in H \mid \Delta(h) \neq 0\}$, the set of regular points of H.

DEFINITION 2.1. Let $D \in \mathbb{C}[K] \otimes \mathcal{O}(H^{\text{reg}}) \otimes \mathcal{U}(\mathfrak{h})$ (where $\mathcal{O}(H^{\text{reg}})$ means the ring of holomorphic functions on H^{reg}). We will call D a shift operator with shift $\ell \in K$ if it satisfies:
(a) $D(\ell)(L(\ell) + (\varrho(\ell), \varrho(\ell))) = (L(\ell + \ell) + (\varrho(\ell + \ell), \varrho(\ell + \ell)))D(\ell)$
  $(\forall \ell \in K)$
(b) D has a convergent asymptotic expansion of the form:

$$D = h^{\varrho(\ell)} \sum_{\mu \geqslant 0} h^\mu \cdot \partial(p_\mu) \text{ on } A_- \text{ (observe that we assume that } \varrho(\ell) \in P\text{)}.$$

We will use the notation $\mathbb{S}(\ell)$ for the $\mathbb{C}[K]$ – module of all shift operators with shift ℓ.

PROPOSITION 2.2. *Let $D \in \mathbb{S}(\ell)$. Then D is W-invariant and, viewed on $W \backslash H$, it is contained in $\mathbb{C}[K] \otimes_\mathbb{C} \mathbb{A}_n$ (where $\mathbb{A}_n$ is the Weyl algebra in n variables with coefficients in $\mathbb{C}$).*
  Proof. If λ satisfies $(\lambda, \kappa^\vee) \neq 1$, $\forall \kappa \in Q_+$, $\kappa \neq 0$ then $\phi(\lambda + \varrho(\ell), \ell)$ is the unique solution of the equation:

$$L(\ell)\phi = (\lambda - \varrho(\ell), \lambda + \varrho(\ell))\phi \text{ of the form:} \tag{2.2}$$

$$\phi(\lambda + \varrho(\ell), \ell; h) = \sum_{\mu \geqslant \lambda + \varrho(\ell)} \Gamma_\mu \cdot h^\mu, \quad \Gamma_{\lambda + \varrho(\ell)} = 1 \text{ (see [HO], Section 3)}.$$

$$\tag{2.3}$$

Therefore it is clear that:

$$D(\ell)\phi(\lambda + \varrho(\ell), \ell; h) = p_0(\ell; \lambda + \varrho(\ell)) \cdot \phi(\lambda + \varrho(\ell + \ell), \ell + \ell; h) \tag{2.4}$$

if D satisfies Definition 2.1. If we restrict ourselves to the case $\lambda + \varrho(\ell) \in P_-$ and $k \geqslant 0$, $\phi(\lambda + \varrho(\ell), \ell; h)$ is in fact a W-invariant Fourier polynomial $P(\lambda + \varrho(\ell), \ell)$ (the so called Jacobi polynomials) (see [HO], Definition 3.13). So if $\lambda + \varrho(\ell), \lambda + \varrho(\ell + \ell) \in P_-$ equation (2.4)

becomes

$$D(\ell)P(\lambda + \varrho(\ell), \ell) = p_0(\ell; \lambda + \varrho(\ell)) \cdot P(\lambda + \varrho(\ell + \ell), \ell + \ell).$$
$$(2.5)$$

If $\lambda + \varrho(\ell) \in P_-$ but $\lambda + \varrho(\ell + \ell) \notin P_-$ then $p_0(\ell; \lambda + \varrho(\ell)) = 0$, because otherwise $\phi(\lambda + \varrho(\ell + \ell), \ell + \ell; h)$ would be an element of $\mathcal{O}(H^{\mathrm{reg}})$ while rank one reduction shows that this cannot be the case. Thus (2.5) holds for all ℓ generic, $\lambda + \varrho(\ell) \in P_-$. Because the Jacobi polynomials form a basis for the space of W-invariant Fourier polynomials Proposition 2.2 follows.   □

We will use the following notation for elements of the Weyl algebra: if $\lambda \in P_-, \lambda = -\Sigma_{i=1}^{n} k_i \cdot \lambda_i$ then

$$z^\lambda = z_1^{k_1} \cdot \ldots \cdot z_n^{k_n} \tag{2.6}$$

$$(\partial/\partial z)^\lambda = (\partial/\partial z_1)^{k_1} \cdot \ldots \cdot (\partial/\partial z_n)^{k_n}. \tag{2.7}$$

We obtain a P-gradation on $\mathbb{A}_n$:

$$\mathbb{A}_n = \bigoplus_{\lambda \in P} \mathbb{A}_n^\lambda \text{ where } \mathbb{A}_n^\lambda \text{ is the span of the elements } z^{\lambda + \mu}(\partial/\partial z)^\mu$$

with $\lambda + \mu, \mu \in P_-$. $\tag{2.8}$

So

$$\mathbb{A}_{n,\lambda} = \bigotimes_{\mu \geqslant \lambda} \mathbb{A}_n^\lambda \text{ is a filtration on } \mathbb{A}_n. \tag{2.9}$$

PROPOSITION 2.3. *Let $D \in \mathbb{S}(\ell)$. Then $D \in \mathbb{A}_{n,\varrho(\ell)}$.*
    Proof. Write $D = \Sigma_{\lambda, \mu \in P_-} a_{\lambda, \mu} z^\lambda (\partial/\partial z)^\mu$. Use induction on (μ, μ) and formula (2.5) to prove:

$$a_{\lambda, \mu} \neq 0 \Rightarrow \lambda - \mu \geqslant \varrho(\ell). \tag{2.10} \quad □$$

Let $\mathscr{S}$ be the algebra of functions on H^{reg} generated by the functions:

$$\begin{cases} h^\lambda, & \lambda \in P \\ (1 - h^\alpha)^{-1}, & \alpha \in R_+. \end{cases} \tag{2.11}$$

PROPOSITION 2.4. *Suppose $D \in \mathbb{C}[K] \otimes \mathcal{S} \otimes \mathfrak{U}(\mathfrak{h})$ is a differential operator that satisfies (a) of Definition 2.1. Then $D \in \mathbb{S}(\ell)$.*

Proof. We have to show that D has an asymptotic expansion as required in (b) of Definition 2.1. In order to do so we consider the set Ω of subsets of P that are of the form $S + Q_+$ for some finite subset S of P. It is easy to see that Ω is in one to one correspondence with the set of non ordered finite subsets of P and that Ω is closed under the operation of taking finite unions. Therefore there exists a unique finite non ordered subset $S(D)$ of P such that:

$$D = \sum_{\kappa \in S(D) + Q_+} h^\lambda . \partial(p_\lambda), \quad p_\lambda \neq 0 \text{ if } \kappa \in S(D). \tag{2.12}$$

We must show that $S(D) = \{\varrho(\ell)\}$. If we let D operate on $\phi(\lambda + \varrho(\mathcal{k}), \mathcal{k}; h)$ we obtain:

$$D\phi(\lambda + \varrho(\mathcal{k}), \mathcal{k}; h) = \sum_{\kappa \in \lambda + \varrho(\mathcal{k}) + S(D) + Q_+} \Gamma'_\kappa . h^\kappa,$$

with $\Gamma'_\kappa \neq 0$ if $\kappa \in \lambda + \varrho(\mathcal{k}) + S(D)$ $\qquad(2.13)$

and this has to be an eigenfunction for $L(\mathcal{k} + \ell)$ with eigenvalue $(\lambda + \varrho(\mathcal{k} + \ell), \lambda - \varrho(\mathcal{k} + \ell))$. So

$$\kappa \in S(D) \Rightarrow (\lambda - 2\varrho(\ell) - \varrho(\mathcal{k}) + \kappa, \lambda + \varrho(\mathcal{k}) + \kappa) = (\lambda - \varrho(\mathcal{k} + \ell),$$

$$\lambda + \varrho(\mathcal{k} + \ell)), \forall \lambda$$

$$\tag{2.14}$$

$$\Leftrightarrow 2(\lambda, \kappa - \varrho(\ell)) + (\kappa - \varrho(\ell), \kappa - \varrho(\ell)) = 0, \forall \lambda$$

$$\Leftrightarrow \kappa = \varrho(\ell). \qquad \square$$

PROPOSITION 2.5. *Consider the covering $\pi: H^{\text{reg}} \to \mathbb{C}^n \backslash \{\Delta^2 = 0\}$ (see [HO], Section 2). Let $D \in \mathbb{A}_n$. Then $\pi^*(D) \in (\mathcal{S} \otimes \mathfrak{U}(\mathfrak{h}))^W$. In particular; if $D \in \mathbb{C}[K] \otimes \mathbb{A}_n$ and D satisfies (a) of Definition 2.1 then $D \in \mathbb{S}(\ell)$.*

Proof. Observe that z_i, $\partial(X_j)z_i$ and Δ^{-1} are in $\mathcal{S}$. This is sufficient to prove $\pi^*(D) \in (\mathcal{S} \otimes \mathfrak{U}(\mathfrak{h}))$. The W-invariance of $\pi^*(D)$ is trivial. $\qquad \square$

For the rank one and two root systems we will give a basis B of K and $\forall \mathcal{b} \in B$ a shift operator $G(\mathcal{b})$ with shift $\mathcal{b}$. In fact these operators form generators for all shift operators of these root systems, as will be shown in

Section 3. We have found these operators using ad hoc methods and computer calculations. The results can be checked (in principle at least) by elementary but lengthy calculations. We will present the operators as elements of $\mathbb{A}_n$.

Table 2.6
The case $\underline{BC_1}$: *We have roots* $\{\pm\alpha, \pm 2\alpha\}$. *Take* $(\alpha, \alpha) = 1$. *As coordinate on* $\overline{W\backslash H} \cong \mathbb{C}$ *we use* $z = \frac{1}{2} - \frac{1}{4}(h^\alpha + h^{-\alpha})$. *Put* $\ell_1 = \ell_\alpha$, $\ell_2 = \ell_{2\alpha}$. *Then we have* (*see also* [HO], *Section 4*):

$$L(\ell) = z(z - 1)\partial^2 + \{(1 + \ell_1 + 2\ell_2)z$$

$$- (\tfrac{1}{2} + \ell_1 + \ell_2)\}\,\partial \; (\text{where } \partial = d/dz). \tag{2.15}$$

and:

$$(\varrho(\ell), \varrho(\ell)) = (\tfrac{1}{2}\ell_1 + \ell_2)^2 \tag{2.16}$$

The weight function $\delta(\ell; a)$ *becomes* (*in the coordinate z*)

$$\delta(\ell; z) = 2^{2\ell_1 + 4\ell_2} \cdot |z|^{\ell_1 + \ell_2} \cdot |1 - z|^{\ell_2} \tag{2.17}$$

and:

$$\Delta^2(z) = -16 \cdot z \cdot (1 - z). \tag{2.18}$$

We have the following shift operators $G(\ell_i)$ $(i = 1, 2)$

$$\underline{\ell_1 = (0, 1):} \quad -\tfrac{1}{4}\partial$$

$$\underline{\ell_2 = (2, -1):} \; (z - 1)\partial + (\ell_2 - \tfrac{1}{2}).$$

If we take $\zeta = h^\alpha + h^{-\alpha}$ *as coordinate the operators become*:

$$\underline{\ell_1}: d/d\zeta$$

$$\underline{\ell_2}: (\zeta + 2)d/d\zeta + (\ell_2 - \tfrac{1}{2}).$$

The rank two cases: *As in ([HO], Section 2) we will use*

$$z_i = \sum_{w \in (W/\text{stab}(\lambda_i))} h^{-w\lambda_i} \quad (i = 1, 2)$$

as coordinates on $W \backslash H \cong \mathbb{C}^2$. *It is convenient to write* ∂_i *instead of* $\partial/\partial z_i$.
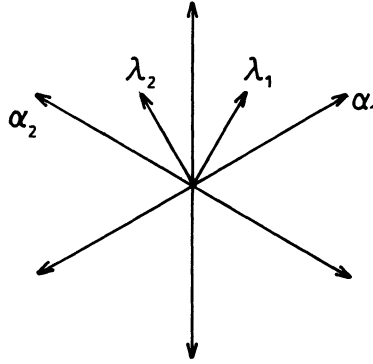
$\underline{A_2}$:

Fig. 1.

Let $k = k_\alpha$. *Take* $(\alpha, \alpha) = 3 (\forall \alpha \in A_2)$. *In the coordinates* (z_1, z_2) *we have*:

$$L(k) = (z_1^2 - 3z_2)\partial_1^2 + (z_1 z_2 - 9)\partial_1\partial_2 + (z_2^2 - 3z_1)\partial_2^2$$

$$+ (3k_1 + 1)(z_1\partial_1 + z_2\partial_2) \tag{2.19}$$

and

$$(\varrho(k), \varrho(k)) = 3k^2. \tag{2.20}$$

The weight function $\delta(k; a)$ *becomes*:

$$\delta(k; z) = |z_1^2 z_2^2 - 4z_1^3 - 4z_2^3 + 18z_1 z_2 - 27|^k \tag{2.21}$$

and the square of the Weyl denominator Δ:

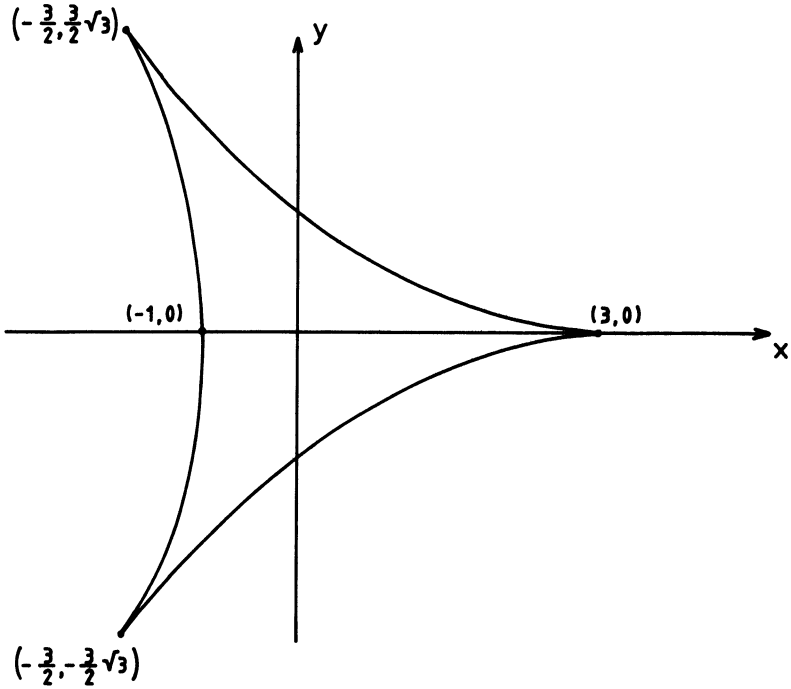$$\Delta^2(z) = z_1^2 z_2^2 - 4z_1^3 - 4z_2^3 + 18z_1 z_2 - 27. \tag{2.22}$$

Fig. 2. The locus $\Delta^2(z) = 0$, drawn in the plane $z_1 = \bar{z}_2 = x + iy$, $x, y \in \mathbb{R}$.

There exists a shift operator $G(\ell)$ for:

$$\underline{\ell = 1}: \partial_1^3 + \partial_2^3 + z_1\partial_1^2\partial_2 + z_2\partial_1\partial_2^2 + (\ell + 2)\partial_1\partial_2.$$

This operator was found by Vretare (see [V], Section 8).

BC_2

Define $\ell_1 = \ell_{\alpha_1}$, $\ell_2 = \ell_{\alpha_2}$, $\ell_3 = \ell_{(1/2)\alpha_2}$. Take $(\alpha_1, \alpha_1) = 2$.
We have:

$$
\begin{aligned}
L(\ell) = {} & (z_1^2 - 2z_2 - 8)\partial_1^2 + (2z_1z_2 - 8z_1)\partial_1\partial_2 + (2z_2^2 - 4z_1^2 + 8z_2)\partial_2^2 \\
& + ((2\ell_1 + 2\ell_2 + \ell_3 + 1)z_1 + 4\ell_3)\partial_1 \\
& + (2\ell_3z_1 + 2(\ell_1 + 2\ell_2 + \ell_3 + 1)z_2 + 8\ell_1)\partial_2. \qquad (2.23)
\end{aligned}
$$
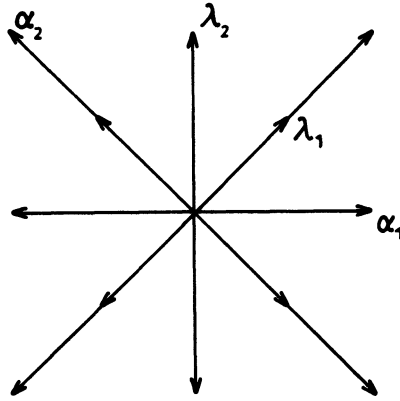
Fig. 3.

and

$$(\varrho(\ell), \varrho(\ell)) \;=\; \ell_1^2 + 2\ell_2^2 + \tfrac{1}{2}\ell_3^2 + 2\ell_1\ell_2 + \ell_1\ell_3 + 2\ell_2\ell_3. \qquad (2.24)$$

The weight function $\delta(\ell; a)$ in the z coordinates:

$$\delta(\ell, z) \;=\; |z_1^2 - 4z_2|^{\ell_1} . |2z_1 + z_2 + 4|^{\ell_2} . |-2z_1 + z_2 + 4|^{\ell_2 + \ell_3} \tag{2.25}$$

and the square of the Weyl denominator:

$$\Delta^2(z) \;=\; (z_1^2 - 4z_2)(2z_1 + z_2 + 4)(-2z_1 + z_2 + 4) \quad \text{(see figure 4)}. \tag{2.26}$$

We have the following shift operators $G(\ell_i)$:

$$\underline{\ell_1 = (1, 0, 0)}: \quad z_1\partial_1^2 + 2(z_2 + 4)\partial_1\partial_2 + 4z_1\partial_2^2$$

$$+ (2\ell_2 + \ell_3 + 1)\partial_1 - 2\ell_3\partial_2$$

$$\underline{\ell_2 = (0, 1, 0)}: \quad \partial_1^2 + z_1\partial_1\partial_2 + z_2\partial_2^2 + (\ell_1 + 1)\partial_2$$

$$\underline{\ell_3 = (0, -1, 2)}: \quad (2z_1 + z_2 + 4)(\partial_1^2 + z_1\partial_1\partial_2 + z_2\partial_2^2)$$

$$+ (\ell_2 - \tfrac{1}{2})(z_1 + 4)\partial_1 + ((2\ell_1 + 2\ell_2 + 1)z_1$$

$$+ (\ell_1 + 2\ell_2)z_2 + 4\ell_1 + 4)\partial_2 + (\ell_2 - \tfrac{1}{2})^2$$

$$+ \ell_1(\ell_2 - \tfrac{1}{2}).$$

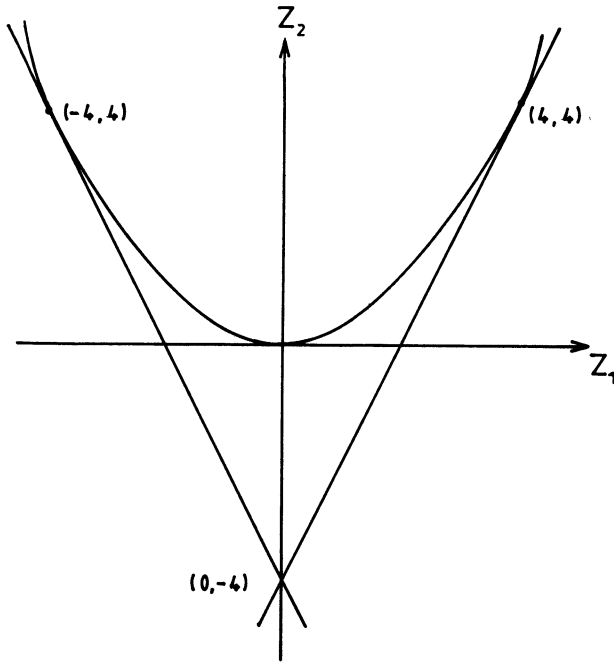Fig. 4. The locus $\Delta^2(z) = 0$ for BC_2 drawn in the plane $(z_1, z_2) \in \mathbb{R}^2$.

For ℓ_1 the shift operator was found by Koornwinder (see [K], page 61, formula (5.4)). For ℓ_2 the operator was found by Sprinkhuizen (see [S], page 505, formula (4.4)).
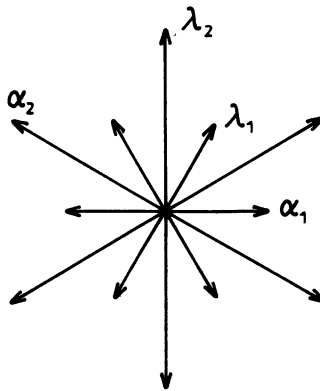
$\underline{G_2}$

Fig. 5.

Take $\ell_1 = \ell_{\alpha_1}$, $\ell_2 = \ell_{\alpha_2}$. Let $(\alpha_1, \alpha_1) = 1$. Then:

$$
\begin{aligned}
L(\ell) \;=\;& (z_1^2 - 3z_1 - z_2 - 12)\partial_1^2 + (-6z_1^2 + 3z_1 z_2 + 12z_2 + 36)\partial_1 \partial_2 \\
&+ (-3z_1^3 + 9z_1 z_2 + 3z_2^2 + 27z_1 + 9z_2)\partial_2^2 \\
&+ ((2\ell_1 + 3\ell_2 + 1)z_1 + 6\ell_1)\partial_1 + ((3\ell_1 + 6\ell_2 + 3)z_2 \\
&+ 6\ell_1 z_1 + 18\ell_2)\partial_2 \tag{2.27}
\end{aligned}
$$

and

$$
(\varrho(\ell), \varrho(\ell)) \;=\; \ell_1^2 + 3\ell_2^2 + 3\ell_1 \ell_2. \tag{2.28}
$$

The weight function $\delta(\ell; a)$ takes the form:

$$
\delta(\ell; z) \;=\; |z_1^2 - 4z_2 - 12|^{\ell_1} \cdot | - 4z_1^3 + z_2^2 + 12z_1 z_2 + 24z_2 + 36z_1 + 36|^{\ell_2} \tag{2.29}
$$

and

$$
\Delta^2(z) \;=\; (z_1^2 - 4z_2 - 12) \cdot (-4z_1^3 + z_2^2 + 12z_1 z_2 + 24z_2 + 36z_1 + 36) \tag{2.30}
$$

(see Fig. 6). We have shift operators $G(\ell_i)$ for:

$$
\begin{aligned}
\underline{\ell_1 = (1, 0)}:\;\; & (2z_1^2 + 18z_1 + z_2 + 30)\partial_1^3 \\
&+ 9(z_1 z_2 + 4z_1^2 + z_2 + 6z_1 - 6)\partial_1^2 \partial_2 \\
&+ 9(2z_1^3 + z_2^2 + 3z_1 z_2 + 9z_2 + 18)\partial_1 \partial_2^2 \\
&+ 27(z_1^2 z_2 - z_2^2 + 2z_1^2 - 5z_2 - 6)\partial_2^3 \\
&+ (\ell_1 + 9\ell_2 + 6)(z_1 + 3)\partial_1^2 + 3((\ell_1 + 6\ell_2 + 6)z_2 \\
&+ 2(\ell_1 + 9\ell_2 + 12)z_1 + 6(3\ell_2 + 6))\partial_1 \partial_2 \\
&+ 9((\ell_1 + 3\ell_2 + 6)z_1^2 - (2\ell_1 + 3\ell_2 + 6)z_2 \\
&- 2\ell_1 z_1 - 3(\ell_1 + 3\ell_2 + 6))\partial_2^2 \\
&+ (3\ell_1 \ell_2 + 9\ell_2^2 + \ell_1 + 9\ell_2 + 2)\partial_1 - 6\ell_1(3\ell_2 + 1)\partial_2
\end{aligned}
$$

$\underline{\ell_2 = (0, 1)}$: $(z_1 + 2)\partial_1^3 + (z_1^2 + 3z_1 + 2z_2 + 6)\partial_1^2\partial_2$

$$+ (3z_1^2 + 3z_1z_2 + 9z_1 - 6z_2 - 18)\partial_1\partial_2^2$$

$$+ (z_1^3 - 3z_1z_2 - 9z_1 + 2z_2^2 + 12z_2 + 18)\partial_2^3$$

$$+ (\ell_1 + \ell_2 + 2)\partial_1^2 + ((2\ell_1 + \ell_2 + 4)z_1 - 6\ell_1)\partial_1\partial_2$$

$$+ ((9\ell_1 + 3\ell_2 + 18) + (3\ell_1 + \ell_2 + 6)z_2 - 3\ell_1z_1)\partial_2^2$$

$$+ (\ell_1^2 + 3\ell_1 + \ell_1\ell_2 + \ell_2 + 2)\partial_2$$
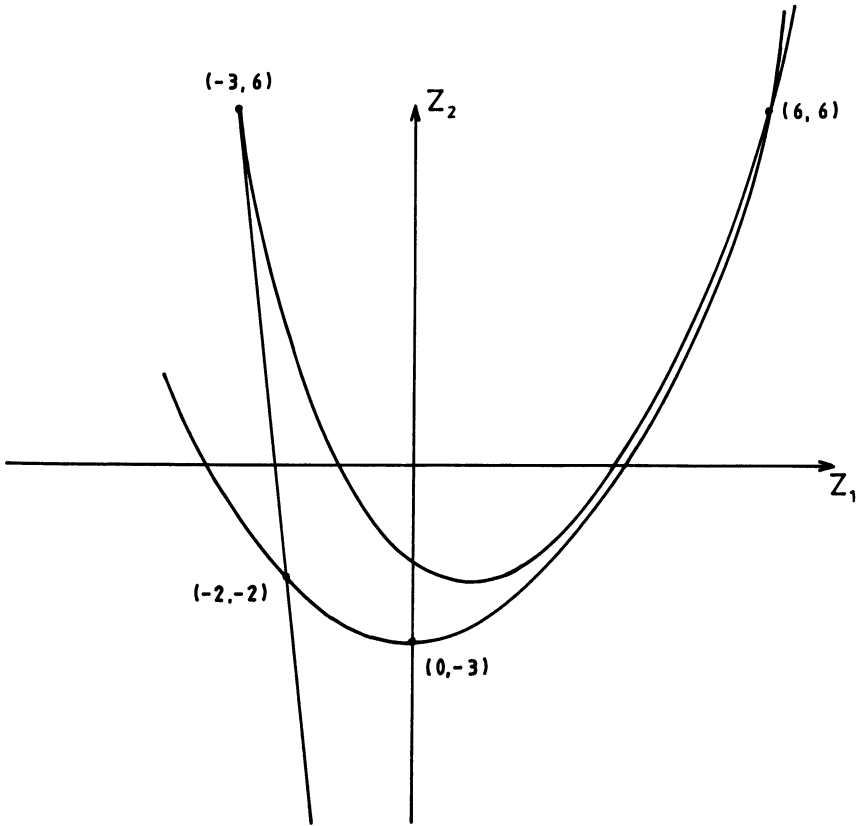
Fig. 6. The locus $\Delta^2(z) = 0$ drawn in the plane $(z_1, z_2) \in \mathbb{R}^2$ for the root system G_2. ((6,6) is intersection point of order 3).

REMARKS. A: The Jacobi polynomials corresponding to these root systems are orthogonal on a proper region bounded by $\{\Delta^2(z) = 0\}$ (see Figs 2, 4, and 6) with respect to the measure $\delta(\ell; z) \cdot |\Delta^2(z)|^{-1/2} \cdot dz$.

B: If one tries to verify the operators $G(\ell_i)$ for G_2 it is helpful to observe that these operators reduce to the operator $G(1)$ for A_2 (contained as subsystem of long or short roots in G_2) when they are restricted to certain lines in the parameter space K.

3. Some algebraic properties of shift operators

In this section we will study the structure of the spaces $\mathbb{S}(\ell) \subset \mathbb{C}[K] \otimes \mathbb{A}_n$ and the relations between them. Using the operators given in the previous section this leads to a complete description of $\mathbb{S}(\ell)$, in particular of $\mathbb{S}(0)$ (the commutant of L in $\mathbb{C}[K] \otimes \mathbb{A}_n$), for the rank two root systems. For $\ell \in K$ fixed we use the following notation:

$$\begin{aligned}
\mathbb{S}(\ell, \ell) &= \{D \in (\mathscr{S} \otimes \mathfrak{U}(\mathfrak{h}))^W \mid D(L(\ell) + (\varrho(\ell), \varrho(\ell))) \\
&= (L(\ell + \ell) + (\varrho(\ell + \ell), \varrho(\ell + \ell)))D\}
\end{aligned}$$

DEFINITION 3.1. (Generalization of Definition 2.5 of [HO]). Let $D \in \mathbb{S}(\ell, \ell)$. Then $D = h^{\varrho(\ell)} \Sigma_{\mu \geqslant 0} h^\mu \cdot \partial(p_\mu)$. We call the mapping:

$$\eta = \eta(\ell) = \eta(\ell, \ell): \mathbb{S}(\ell, \ell) \to \mathbb{C}[\mathfrak{h}^*]$$

$$D \to \{\lambda \to p_0(\lambda + \varrho(\ell))\} \tag{3.1}$$

the Harish-Chandra mapping. (Remark: if $\ell = 0$ this is just the mapping $\gamma(\ell)$ as defined in [HO] (Definition 2.5)).

PROPOSITION 3.2. $\eta(\ell, \ell)$ is injective and $\deg(D) = \deg(p_0)$ (with $\deg(D)$ we mean the usual degree of D as differential operator).

Proof. Analogous to Lemma 2.7 from [HO] we obtain that $D = h^{\varrho(\ell)} \Sigma_{\mu \geqslant 0} h^\mu \partial(p_\mu) \in \mathbb{S}(\ell, \ell)$ if and only if the polynomials $p_\mu \in \mathbb{C}[\mathfrak{h}^*]$ satisfy the recurrence relations:

$$\begin{aligned}
(\mu, \mu + 2\lambda - 2\varrho(\ell)) \cdot p_\mu(\lambda) &= 2 \sum_{\alpha \in R_+} \sum_{j \geqslant 1} \{(\ell_\alpha + \ell_\alpha) \\
&\times (\alpha, \lambda + \varrho(\ell) + \mu - j\alpha)p_{\mu - j\alpha}(\lambda) \\
&- \ell_\alpha(\alpha, \lambda)p_{\mu - j\alpha}(\lambda + j\alpha)\}.
\end{aligned} \tag{3.2}$$

The proposition follows from (3.2). $\square$

Consider the following weight function on A

$$\delta(\text{\textit{k}}; a) \;=\; \prod_{\alpha \in R_+} (|a^{-\alpha/2} - a^{\alpha/2}|)^{2k_\alpha} \tag{3.3}$$

(see [HO], Proposition 2.2). On A_- $\delta(\text{\textit{k}}; a)$ has a series expansion of the form:

$$\delta(\text{\textit{k}}; a) \;=\; a^{-2\varrho(\text{\textit{k}})} \cdot \left(1 + \sum_{\mu > 0} c_\mu h^\mu \right) \quad (c_\mu \in \mathbb{C}). \tag{3.4}$$

DEFINITION 3.3. For $D \in \mathbb{S}(\ell)$ we define:

$$\tilde{D}(\text{\textit{k}}) \;=\; \delta(\ell - \text{\textit{k}}) \circ D^*(\text{\textit{k}} - \ell) \circ \delta(\text{\textit{k}}), \tag{3.5}$$

viewed as differential operator on A^{reg}. (D^* means: formal transpose as differential operator on A with respect to the Haar measure da).

PROPOSITION 3.4. $\tilde{D} \in \mathbb{S}(-\ell)$.
    Proof. From (3.4) we conclude that $\tilde{D}$ has the right asymptotic expansion on A_-. Recall ([HO], Corollary 2.3) that $L(\text{\textit{k}})$ is symmetric with respect to the measure $\delta(\text{\textit{k}}; a)da$ on A, i.e., $L(\text{\textit{k}}) = \delta(-\text{\textit{k}}) \circ L^*(\text{\textit{k}}) \circ \delta(\text{\textit{k}})$. Hence:

$$\tilde{D}(\text{\textit{k}}) \circ (L(\text{\textit{k}}) + (\varrho(\text{\textit{k}}), \varrho(\text{\textit{k}}))) \;=\; \delta(\ell - \text{\textit{k}}) \circ D^*(\text{\textit{k}} - \ell) \circ \delta(\text{\textit{k}}) \circ (L(\text{\textit{k}})$$

$$+ (\varrho(\text{\textit{k}}), \varrho(\text{\textit{k}}))) \;=\; \delta(\ell - \text{\textit{k}}) \circ ((L(\text{\textit{k}}) + (\varrho(\text{\textit{k}}), \varrho(\text{\textit{k}}))) \circ D(\text{\textit{k}} - \ell))^* \circ \delta(\text{\textit{k}})$$

$$=\; \delta(\ell - \text{\textit{k}}) \circ (D(\text{\textit{k}} - \ell) \circ (L(\text{\textit{k}} - \ell) + (\varrho(\text{\textit{k}} - \ell), \varrho(\text{\textit{k}} - \ell))))^* \circ \delta(\text{\textit{k}})$$

$$=\; \delta(\ell - \text{\textit{k}}) \circ (L(\text{\textit{k}} - \ell) + (\varrho(\text{\textit{k}} - \ell), \varrho(\text{\textit{k}} - \ell)))^* \circ D^*(\text{\textit{k}} - \ell) \circ \varrho(\text{\textit{k}})$$

$$=\; (L(\text{\textit{k}} - \ell) + (\varrho(\text{\textit{k}} - \ell), \varrho(\text{\textit{k}} - \ell))) \circ \tilde{D}(\text{\textit{k}}). \qquad \square$$

PROPOSITION 3.5. $\eta(\tilde{D}(\text{\textit{k}}))(\lambda) = \eta(D(\text{\textit{k}} - \ell))(-\lambda)$ if $D \in \mathbb{S}(\ell)$.
    Proof. On A_- we have:

$$\tilde{D}(\text{\textit{k}}) \;=\; h^{2\varrho(\text{\textit{k}} - \ell)} \prod_{\alpha \in R} (1 - h^\alpha)^{2(\ell_\alpha - k_\alpha)} \circ \left(\sum_{\mu \geqslant 0} \partial(p_\mu(-\lambda)) h^\mu \right) h^{\varrho(\ell)} \circ h^{-2\varrho(\text{\textit{k}})}$$

$$\times \prod_{\alpha \in R} (1 - h^\alpha)^{2k_\alpha} \;=\; h^{-\varrho(\ell)} \sum_{\mu \geqslant 0} h^\mu \partial(q_\mu(\lambda))$$

with $q_0(\lambda) = p_0(-\lambda + 2\varrho(\mathcal{k}) - \varrho(\ell))$. In other words:

$$\eta(\tilde{D}(\mathcal{k}))(\lambda) = q_0(\lambda + \varrho(\mathcal{k})) = p_0(-\lambda + \varrho(\mathcal{k} - \ell))$$

$$= \eta(D(\mathcal{k} - \ell))(-\lambda). \qquad \square$$

THEOREM 3.6. $\eta(0, \mathcal{k}): \mathbb{S}(0, \mathcal{k}) \to \mathbb{C}[\mathfrak{h}^*]^W$ *is an isomorphism* $(\forall \mathcal{k} \in K)$ *for all rank two root systems.*

Proof. For A_2 and BC_2 this was proved by Koornwinder in his thesis (see [K]). For G_2 we take the operator $G = \tilde{G}((0, 1), (\mathcal{k}_1, \mathcal{k}_2 + 1)) \circ G((0, 1), (\mathcal{k}_1, \mathcal{k}_2))$ (see Table 2.6). Obviously $G \in \mathbb{S}(0)$ (according to Proposition 3.4). $\eta(G(0, 1))$ can be calculated with the formula:

$$G(0, 1)(z^\lambda) = \eta(G(0, 1))(\lambda - \varrho(\mathcal{k})) \cdot z^{\lambda + \varrho(\ell)} + \sum_{\mu > \lambda + \varrho(\ell)} d_\mu \cdot z^\mu. \qquad (3.6)$$

We obtain $(\lambda = n_1 \lambda_1 + n_2 \lambda_2)$:

$$\eta(G(0, 1))(\lambda) = -(n_2 + \mathcal{k}_2)(n_1 + n_2 + \mathcal{k}_2)(n_1 + 2n_2 + \mathcal{k}_2). \qquad (3.7)$$

Using Proposition 3.5 we see:

$$\eta(G)(\lambda) = (n_2 + \mathcal{k}_2)(n_1 + n_2 + \mathcal{k}_2)(n_1 + 2n_2 + \mathcal{k}_2) \cdot$$

$$(-n_2 + \mathcal{k}_2)(-n_1 - n_2 + \mathcal{k}_2)(-n_1 - 2n_2 + \mathcal{k}_2)$$

$$= -n_2^2(n_1 + n_2)^2 (n_1 + 2n_2)^2 + \text{terms of lower degree}$$
$$\text{in } n_1, n_2. \qquad (3.8)$$

Now it is easy to see that $\eta(G)$ and $\eta(L)$ generate $\mathbb{C}[\mathfrak{h}^*]^W$. $\qquad \square$

We have bilinear mappings:

$$\Pi_{\ell_1, \ell_2}: \mathbb{S}(\ell_1) \times \mathbb{S}(\ell_2) \to \mathbb{S}(\ell_1 + \ell_2)$$

$$(D_1(\mathcal{k}), D_2(\mathcal{k})) \to D_1(\mathcal{k} + \ell_2) \circ D_2(\mathcal{k}). \qquad (3.9)$$

So we can view $\mathbb{S}(\ell)$ as a module over the algebra $\mathbb{S}(0)$. We will study this module structure in the sequel of this section. Observe that

$$\eta(\Pi_{\ell_1, \ell_2}(D_1, D_2)) = \eta(D_1(\mathcal{k} + \ell_2)) \circ \eta(D_2(\mathcal{k})). \qquad (3.10)$$

LEMMA 3.7 (rank one case). *Suppose D is a rank one shift operator with shift* $\ell = (\ell_\alpha, \ell_{2\alpha}) = (\ell_1, \ell_2)$. *D is an element of* $\mathbb{A}_1$, *so we can write* $D = \sum_{j=0}^m a_j \cdot \partial^j$ *(with a_j polynomials in z and k). Then $m \geqslant$* max $(|\ell_1 + \ell_2|, |\ell_2|)$ *and* $a_m = c \cdot z^{1/2(m-(\ell_1+\ell_2))} \cdot (1-z)^{1/2(m-\ell_2)}$ *with* ℓ_1, $m - \ell_2 \in 2\mathbb{Z}$, $c \in \mathbb{C}[K]\backslash\{0\}$.

Proof. Looking at the highest order part of the equation $D(k) \circ (L(k) + (\varrho(k), \varrho(k))) = (L(k + \ell) + (\varrho(k + \ell), \varrho(k + \ell))) \circ D(k)$ we get:

$$-ma_m \cdot (1-2z) + 2z(1-z) \cdot a_m' = (-(\ell_1 + \ell_2)(1-z) + \ell_2 z) \cdot a_m$$

$$\Leftrightarrow 2a_m^{-1} \cdot a_m' = (m - (\ell_1 + \ell_2)) \cdot z^{-1} - (m - \ell_2) \cdot (1-z)^{-1}$$

$$\Leftrightarrow a_m = c \cdot z^{1/2(m-(\ell_1+\ell_2))} \cdot (1-z)^{1/2(m-\ell_2)} \quad \text{(with } c \in \mathbb{C}[K]\backslash\{0\}).$$

Because $\tilde{D}$ is also in the Weyl algebra we have:

$$m \geqslant \max(|\ell_1 + \ell_2|, |\ell_2|) \tag{3.11}$$

$$\ell_1, m - \ell_2 \in 2\mathbb{Z}. \tag{3.12} \quad \square$$

As we have seen in Table 2.6 we have in rank one:

$$G(0, 1) = -\tfrac{1}{4}\partial$$

$$G(2, -1) = (z-1)\cdot\partial + (k_2 - \tfrac{1}{2})$$

$$G(0, -1) = \tilde{G}(0, 1) = 4(z(z-1)\partial + (k_1 + 2k_2 - 1)z \tag{3.13}$$

$$- (k_1 + k_2 - \tfrac{1}{2}))$$

$$G(-2, 1) = \tilde{G}(2, -1) = -z\partial - (k_1 + k_2 - \tfrac{1}{2}).$$

Since $\ell_1 \in 2\mathbb{Z}$ we can decompose ℓ with respect to $(0, 1)$ and $(2, -1)$ with integral coefficients:

$$(\ell_1, \ell_2) = (\ell_1/2) \cdot (2, -1) + (\ell_1/2 + \ell_2) \cdot (0, 1)$$

$$= a_1 v_1 + a_2 v_2$$

with $a_i \in \mathbb{Z}_+$ and $v_1 = \pm(2, -1)$, $v_2 = \pm(0, 1)$. \tag{3.14}

Observe that $\max\left(|\ell_1 + \ell_2|, |\ell_2|\right) = |\ell_1/2| + |\ell_1/2 + \ell_2|$. Define

$$G(\ell) \;=\; G(\ell, \mathit{k}) \;=\; G(v_1, \mathit{k} + (a_1 - 1)v_1 + a_2 v_2) \circ \cdots \circ$$

$$G(v_1, \mathit{k} + a_2 v_2) \circ G(v_2, \mathit{k} + (a_2 - 1)v_2) \circ \cdots \circ G(v_2, \mathit{k}). \tag{3.15}$$

The operator

$$G(\ell) \circ L^{1/2(m - |\ell_1/2| - |\ell_1/2 + \ell_2|)} \tag{3.16}$$

is a shift operator with shift ℓ and highest order part:

$$c \cdot z^{1/2(m - (\ell_1 + \ell_2))} \cdot (1 - z)^{1/2(m - \ell_2)} \cdot \partial^m \tag{3.17}$$

With induction on m we conclude:

PROPOSITION 3.8. *Every rank one shift operator $D \in \mathbb{S}(\ell)$ is of the form*: $D = G(\ell, \mathit{k}) \circ P(L(\mathit{k}))$ *with P a polynomial in one variable.*   □

COROLLARY 3.9. *In the rank one situation we have*:
(a) *If $\ell \in \mathbb{Z} \cdot (0, 1) + \mathbb{Z} \cdot (2, -1)$ then $\mathbb{S}(\ell)$ is a free rank one $\mathbb{S}(0)$ module with generator $G(\ell)$.*
(b) *If $\ell \notin \mathbb{Z} \cdot (0, 1) + \mathbb{Z} \cdot (2, -1)$ then $\mathbb{S}(\ell) = \{0\}$.*
(c) *Between the generators $G(\ell, \mathit{k})$ we have the relations*:

$$G(\ell + m, \mathit{k}) \;=\; G(\ell, \mathit{k} + m) \circ G(m, \mathit{k}) \;=\; G(m, \mathit{k} + \ell) \circ G(\ell, \mathit{k})$$
$$\text{if } \mathrm{sign}\,(\ell_1) \;=\; \mathrm{sign}\,(m_1)$$

and $\mathrm{sign}\,(\tfrac{1}{2}\ell_1 + \ell_2) = \mathrm{sign}\,(\tfrac{1}{2}m_1 + m_2)$.   □

Define $r(\ell, \mathit{k}; x)$ as the polynomial such that

$$r(\ell, \mathit{k}; (((2\alpha)^{\vee}, \lambda)) \;=\; \eta(G(\ell))(\lambda). \tag{3.18}$$

So we have

$$\begin{cases} r((0, 1); x) &=\; -(x + \tfrac{1}{2}\mathit{k}_1 + \mathit{k}_2) \\[2mm] r((2, -1); x) &=\; -(x + \tfrac{1}{2}\mathit{k}_1 + \tfrac{1}{2}). \end{cases}$$

and for other values of ℓ, $r(\ell; x)$ can be calculated using Proposition 3.5, (3.10) and (3.15). For convenience we define $r(\ell, \ell; x) \equiv 0$ if $\ell \notin \mathbb{Z}.(0, 1) + \mathbb{Z}.(2, -1)$.

The rank one calculations enable us to describe what one may expect for the higher rank situation. Let R be a possibly non reduced root system. R^0 is reduced and we choose a basis of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of R^0. Denote by C_i ($i = 1, \ldots, m_0$) the conjugacy classes of R^0. For $\ell \in K$ we write ℓ_i for the multiplicity function restricted to $(C_i \cup \frac{1}{2}C_i) \cap R$. The following theorem generalizes Proposition 2.9 in [HO].

THEOREM 3.10. *Let* $D \in \mathbb{S}(\ell)$. *Then* $\eta(D)$ *is of the following form*:

$$\eta(D)(\lambda) = \left[\prod_{i=1}^{m_0} \prod_{\alpha \in C_{i+}} r(\ell_i, \ell_i; (\alpha^\vee, \lambda)) \right] \cdot p(\lambda). \tag{3.19}$$

with $p \in \mathbb{C}[\mathfrak{h}^*]^W$, *where* $r(\ell_i, \ell_i; x)$ *is the polynomial defined by (3.18). In particular:* $\mathbb{S}(\ell) = \{0\}$ *if not all* ℓ_i *are in* $\mathbb{Z}.(0, 1) + \mathbb{Z}.(2, -1)$. *(with* $\ell_i = (\ell_{i,\alpha/2}, \ell_{i,\alpha})$).

Proof. We may assume that $\ell_i \in \mathbb{Z}.(0, 1) + \mathbb{Z}.(2, -1)$ ($\forall i$) because, if this is not the case, rank one reduction with respect to a suitable simple root shows that D has to be 0. Let us consider the rational function

$$p(\lambda) = \left\{ \frac{\eta(D)(\lambda)}{\prod_{i=1}^{m_0} \prod_{\alpha \in C_{i+}} r(\ell_i, \ell_i; (\alpha^\vee, \lambda))} \right\}. \tag{3.20}$$

First we will show that p is W-invariant. Take a simple reflection r_j in $\{\alpha_j = 0\}$, α_j simple root of R^0, $\alpha_j \in C_i$ for certain i. By rank one reduction (as in [HO], proof of Proposition 2.9) we see that $\eta(D)(\lambda)$ is divisible by $r(\ell_i, \ell_i; (\alpha_j^\vee, \lambda))$ and that the remainder is r_j-invariant (see Proposition 3.8). The expression

$$\prod_{\substack{i=1 \\ }}^{m_0} \prod_{\substack{\alpha \in C_{i+} \\ \alpha \neq \alpha_j}} r(\ell_i, \ell_i; (\alpha^\vee, \lambda)) \tag{3.21}$$

is also r_j-invariant because $R^0_+ \setminus \{\alpha_j\}$ is r_j-invariant. The conclusion is that $p(\lambda)$ is W-invariant. The set of poles P of $p(\lambda)$ is a W-invariant set of hyperplanes contained in:

$$\bigcup_{i=1}^{m_0} \bigcup_{\substack{\alpha \in C_{i+} \\ \alpha \text{ not simple}}} \{r(\ell_i, \ell_i; (\alpha^\vee, \lambda)) = 0\} \tag{3.22}$$

and therefore P is empty. Hence $p(\lambda)$ is a polynomial.

COROLLARY 3.11. *Let* $R = \bigcup_{i=1}^{m} C_i$ *be the decomposition of* R *in conjugacy classes and define* $e_i \in K$ *by* $e_i(C_j) = \delta_{ij}$. *Let* $B = \{\theta_i\}$ *be the following basis of* K

$$\begin{cases} \theta_i = e_i \text{ if } 2C_i \cap R = \phi \text{ for a conjugacy class } C_i \text{ in } R. \\ \\ \theta_i = (2e_i - e_j) \text{ if } 2C_i = C_j \\ \quad \text{ for conjugacy classes } C_i, C_j \text{ in } R. \end{cases} \tag{3.23}$$

Then $\mathbb{S}(\ell) = 0$ *unless* $\ell \in \mathbb{Z} . B$, *and for* $\ell \in \mathbb{Z} . B$ *we have*:

$$\deg(D) \geqslant \sum_{\alpha \in R_+^0} \max(|\ell_\alpha|, |\ell_{\alpha/2} + \ell_\alpha|). \tag{3.24}$$

□

We obtain from 3.10, Proposition 3.2, 3.4 and 3.5:

COROLLARY 3.12. *Suppose that*:
(a) $\gamma(\ell)(\mathbb{S}(0, \ell)) = \mathbb{C}[\mathfrak{h}^*]^W (\forall \ell)$ *(see [HO] Conjecture 2.10)*
(b) *For* $i = 1, \ldots, m$ *we have an operator* $G(\theta_i) \in \mathbb{S}(\theta_i)$ *with* $\deg(G(\theta_i)) = \sum_{\alpha \in R_+} (\theta_i)_\alpha$.
Then $\forall \ell \in \mathbb{Z} . B$, $\mathbb{S}(\ell)$ *is a free rank one* $\mathbb{S}(0)$ *module generated by an operator* $G(\ell)$ *with*

$$\eta(\ell, \ell)(G(\ell, \ell))(\lambda) = \prod_{i=1}^{m_0} \prod_{\alpha \in C_{i+}} r(\ell_i, \ell_i; (\alpha^\vee, \lambda)) \left(\text{with } R^0 = \bigcup_{i=1}^{m_0} C_i \right) \tag{3.25}$$

(so (3.24) is an equality for $D = G(\ell)$). *The generators* $G(\ell)$ *are products of the operators* $G(\theta_i)$ *and* $\tilde{G}(\theta_i) = G(-\theta_i)$ *as in the rank one case (see (3.15)). They satisfy the relations*:

$$G(\ell + m) = G(\ell, \ell + m) \circ G(m, \ell) = G(m, \ell + \ell) \circ G(\ell, \ell) \tag{3.26}$$

if the coefficients of ℓ *and* m *in their decomposition with respect to* B *all have the same sign. If* $p \in \mathbb{C}[\mathfrak{h}^*]^W$ *and* $D(\ell) = \gamma(\ell)^{-1}(p)$ *then*

$$G(\ell, \ell) \circ D(\ell) = D(\ell + \ell) \circ G(\ell, \ell) \tag{3.27}$$

□

THEOREM 3.13 (the rank two case). *For all rank two root systems the conditions (a) and (b) of Corollary 3.12 are satisfied.*   □

REMARK 3.14. For BC_3 Vretare has found one shift operator (see [V], section 7, Theorem 7.1) namely $G(0, 1, 0)$) (with conventions as for BC_2, see Table 2.6). Beerends has found the generator $G(1)$ for A_3 ([B]). Together with the result of Sekiguchi that $\gamma \colon \mathbb{S}(0) \to \mathbb{C}[\mathfrak{h}^*]^W$ is an isomorphism for A_n (see [Se]) this implies that (a) and (b) of Corollary 3.12 are also satisfied for $R = A_3$.

It is easy to calculate $\eta(G(\ell_i))$ for the shift operators given in Table 2.6. We obtain in accordance with formula (3.25):

Table 3.15. *Let* $\lambda = n_1\lambda_1 + n_2\lambda_2$. *For the operators* $G(\ell_i)$ *of Table 2.6* $\eta(G(\ell_i))$ *equals*:

$\underline{A_2}$	$\ell = 1$:	$-(n_1 + \ell)(n_2 + \ell)(n_1 + n_2 + \ell)$
$\underline{BC_2}$	$\ell_1 = (1, 0, 0)$:	$(n_1 + \ell_1)(n_1 + 2n_2 + \ell_1)$
	$\ell_2 = (0, 1, 0)$:	$(n_2 + \ell_2 + \tfrac{1}{2}\ell_3)(n_1 + n_2 + \ell_2 + \tfrac{1}{2}\ell_3)$
	$\ell_3 = (0, 2, -1)$:	$(n_2 + \tfrac{1}{2}\ell_3 + \tfrac{1}{2})(n_1 + n_2 + \tfrac{1}{2}\ell_3 + \tfrac{1}{2})$
$\underline{G_2}$	$\ell_1 = (1, 0)$:	$-(n_1 + \ell_1)(n_1 + 3n_2 + \ell_1)(2n_1 + 3n_2 + \ell_1)$
	$\ell_2 = (0, 1)$:	$-(n_2 + \ell_2)(n_1 + n_2 + \ell_2)(n_1 + 2n_2 + \ell_2)$.

REMARK 3.16. A calculation shows that the expression

$$\frac{\eta(G(-\ell, \ell))(\lambda) \cdot c(\lambda, \ell)}{c(\lambda, \ell - \ell)} \quad (\ell \in \mathbb{Z}_+ \cdot B), \tag{3.28}$$

where $c(\lambda, \ell)$ is Harish Chandra's c-function (see [HO], section 6), does not depend on λ. (In the next section we will give its precise value for rank $(R) = 2$).

4. Behaviour of shift operators at the identity element

Let $\mathscr{T}$ be the ring of functions on $\mathfrak{h}^{reg}$ generated by the ring $\mathbb{C}[\mathfrak{h}]$ and the functions $1/\alpha$ ($\alpha \in R$). We denote by E the Euler vectorfield on $\mathfrak{h}$:

$$E(X) = \sum_{i=1}^{n} (X_i, X)\partial(X_i) \quad (X \in \mathfrak{h}) \tag{4.1}$$

where $\{X_1, \ldots, X_n\}$ is an orthogonal basis for $\mathfrak{a}$. The operator $\mathrm{ad}(E)$ acts semisimply on the algebra of differential operators $\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h})$ and its eigenvalues are in $\mathbb{Z}$. So we have a $\mathbb{Z}$-gradation on $\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h})$:

$$\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h}) = \sum_{k \in \mathbb{Z}}^{\oplus} (\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h}))_k$$

with

$$D \in (\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h}))_k \Leftrightarrow \mathrm{ad}(E)(D) = k . D. \tag{4.2}$$

(Notation: h.d. $(D) = k$, the homogeneous degree of D). For example:

$$\left.\begin{array}{lll} \text{h.d.}(\lambda) & = 1 & \forall \lambda \in \mathfrak{h}^* \\[2mm] \text{h.d.}(1/\alpha) & = -1 & \forall \alpha \in R \\[2mm] \text{h.d.}(\partial(X)) & = -1 & \forall X \in \mathfrak{h} \end{array}\right\} \tag{4.3}$$

If $f \in \mathscr{S}$ (see (2.11)) then f has a unique series expansion:

$$f = \sum_{k \geqslant k_0} f_k \tag{4.4}$$

that is convergent in some neighbourhood of $0 \in \mathfrak{h}$ with (possible) exception of $\{\Pi_{\alpha \in R_+} \alpha = 0\}$ and which satisfies:

(a) $f_k \in \mathscr{T}_k$.

$$\tag{4.5}$$

(b) $\exists N \in \mathbb{N}: \forall k: \left(\prod_{\alpha \in R_+} \alpha\right)^N . f_k$ is a polynomial.

Essentially this is just the powerseries expansion at $0 \in \mathfrak{h}$ of f. If $f_{k_0} \neq 0$ we define the lowest homogeneous degree of f (l.h.d.) as:

$$\text{l.h.d.} \; (f) \;\; = \;\; k_0 \tag{4.6}$$

and the lowest homogeneous part (l.h.p.) as:

$$\varepsilon(f) \;\; = \;\; \text{l.h.p.} \; (f) \;\; = \;\; f_{k_0}. \tag{4.7}$$

If $D \in \mathscr{S} \otimes \mathfrak{U}(\mathfrak{h})$ we can expand the coefficients functions in $\mathscr{S}$ as described in (4.4), (4.5) to obtain the unique series expansion

$$D \;\; = \;\; \sum_{k \geqslant k_0} D_k \;\; (\text{with } D_{k_0} \neq 0) \tag{4.8}$$

such that

(a) $D_k \in (\mathscr{S} \otimes \mathfrak{U}(\mathfrak{h}))_k \; \forall k$

$$\tag{4.9}$$

(b) $\forall f$ analytic in a neighbourhood of $0 \in \mathfrak{h}$, $Df = \sum_{k \geqslant k_0} D_k f$ is a convergent series in a neighbourhood of $0 \in \mathfrak{h}$ with exception of $\{\Pi_{\alpha \in R_+} \; \alpha = 0\}$.

So we obtain a mapping

$$\varepsilon \colon \mathscr{S} \otimes \mathfrak{U}(\mathfrak{h}) \to \mathscr{T} \otimes \mathfrak{U}(\mathfrak{h})$$

$$\tag{4.10}$$

$$D \to D_{k_0}$$

k_0 is called the lowest homogeneous degree of D (l.h.d.$(D) = k_0$). ε has the following properties: let D_1, $D_2 \in \mathscr{S} \otimes \mathfrak{U}(\mathfrak{h})$, with l.h.d.$(D_i) = k_i$ $(i = 1, 2)$. Then:

(i) if $k_1 < k_2$ then $\varepsilon(D_1 + D_2) = \varepsilon(D_2)$,
  if $k_1 = k_2$ and $\varepsilon(D_1) + \varepsilon(D_2) \neq 0$ then $\varepsilon(D_1 + D_2) = \varepsilon(D_1) + \varepsilon(D_2)$.
(ii) $\varepsilon(D_1 D_2) = \varepsilon(D_1)\varepsilon(D_2)$.

PROPOSITION 4.1. $\varepsilon(L(\ell)) = \sum_{j=1}^{n} \partial(X_j)^2 + 2 \cdot \sum_{\alpha \in R_+} \ell_\alpha \cdot \alpha^{-1} \cdot \partial(X_\alpha) \in (\mathscr{T} \otimes \mathfrak{U}(\mathfrak{h}))_{-2}$.

Proof. The function $(\alpha/2)\mathrm{cth}\,\alpha/2$ is analytic in a neighbourhood of $0 \in \mathfrak{h}$ and its value at 0 is 1. Hence $\varepsilon(\mathrm{cth}\,\alpha/2) = 2/\alpha$. □

LEMMA 4.2. *Consider the following elements of* $\mathcal{T} \otimes \mathfrak{U}(\mathfrak{h})$:

$$
\left.
\begin{aligned}
f(\xi) &= \varepsilon(L(\xi)) \in (\mathcal{T} \otimes \mathfrak{U}(\mathfrak{h}))_{-2} \\[2mm]
h(\xi) &= E + n/2 + \sum_{\alpha \in R_+} \xi_\alpha \in (\mathcal{T} \otimes \mathfrak{U}(\mathfrak{h}))_0 \\[2mm]
e(\xi) &= -\tfrac{1}{4}(X, X) \in (\mathcal{T} \otimes \mathfrak{U}(\mathfrak{h}))_2
\end{aligned}
\right\}
\tag{4.12}
$$

(*where* (. , .) *is the inner product on* $\mathfrak{a}$).
  They satisfy the $\mathfrak{sl}(2)$ *commutation relations, i.e.*

$$
\left.
\begin{aligned}
[h, e] &= 2e \\[2mm]
[h, f] &= -2f \\[2mm]
[e, f] &= h.
\end{aligned}
\right\}
\tag{4.13}
$$

Proof. The first two relations follow from the definition of homogeneous degree. The relation $[e, f] = h$ is verified by an easy direct calculation. □

PROPOSITION 4.3. *Suppose* $D \in \mathbb{S}(0)$ *is a differential operator of degree* m. *Then the following holds*:

(a) $D \in \partial(p) + \mathcal{S} \otimes \mathfrak{U}_{m-1}$ (*where* $\mathfrak{U}_k$ *means: the elements of* $\mathfrak{U}(\mathfrak{h})$ *of degree* $\leqslant k$). *Here* p *is a homogeneous* W *invariant polynomial of degree* m *on* $\mathfrak{h}^*$
(b) l.h.d. $(D) = -m$.
(c) $\varepsilon(D) = (2^m m!)^{-1} (\mathrm{ad}(f))^m(p^*)$ (*with* $p^* \in \mathbb{C}[\mathfrak{h}]$ *defined by* $p^*(X) = p(X^*)$ *and* $X^*(Y) = (X, Y) \forall X, Y \in \mathfrak{h}$).

Proof. (a) was already proved in [HO], Corollary 2.8. For the proof of (b) we consider the operator $\varepsilon(D)$. It follows from (4.11) (ii) that $[f, \varepsilon(D)] = 0$. Furthermore we see from (a) that l.h.d. $(D) \leqslant -m$, or equivalently: $[h, \varepsilon(D)] = -m' . \varepsilon(D)$ with $m' \geqslant m$. Now we apply $\mathfrak{sl}(2)$ representation theory to conclude that $(\mathrm{ad}(e))^m(\varepsilon(D)) \neq 0$. Hence: $0 \leqslant \deg((\mathrm{ad}(e))^m (\varepsilon(D))) \leqslant \deg(\varepsilon(D)) - m = \deg(\varepsilon(D)) - \deg(D) \leqslant 0$ and we obtain that $\deg(\varepsilon(D)) = m$. By virtue of (4.11) (i) this means that l.h.d. $(D) = -m$. With induction on m one proves: $(\mathrm{ad}(e))^m(\partial(p)) = (m!/2^m) . p^*$ (see also

[HC] page 99): it is clear for $m = 1$ and for general m one has:

$$(\mathrm{ad}(e))^m(\partial(p)) = -\tfrac{1}{4} \cdot \sum_{i=1}^{n} (\mathrm{ad}(e))^{m-1} [(X_i^*)^2, \partial(p)].$$

We have:

$$[(X_i^*)^2, \partial(p)] = -(2X_i^* \cdot \partial(\partial(X_i^*)p) + \partial(\partial(X_i^*)^2 p)).$$

The last term $\partial(X_i^*)^2 p$ has degree $(m - 2)$ so from the induction hypothesis we get:

$$(\mathrm{ad}(e))^{m-1} (\partial(\partial(X_i^*)^2 p)) = 0.$$

Thus:

$$(\mathrm{ad}(e))^m(\partial(p)) = \tfrac{1}{4} \cdot \sum_{i=1}^{n} (\mathrm{ad}(e))^{m-1}(2X_i^* \cdot \partial(\partial(X_i^*)p))$$

$$= \tfrac{1}{2} \cdot \sum_{i=1}^{n} X_i^*(\mathrm{ad}(e))^{m-1}(\partial(\partial(X_i^*)p))$$

$$= \tfrac{1}{2} \cdot \sum_{i=1}^{n} \frac{(m-1)!}{2^{m-1}} \cdot X_i^*(\partial(X_i)p^*)$$

$$= \frac{(m-1)!}{2^m} \, \mathrm{ad}(E)(p^*) = \frac{m!}{2^m} p^*.$$

Using again $\mathfrak{sl}(2)$ representation theory we see: $(\mathrm{ad}(f))^m(p^*) = 2^m m! \varepsilon(D)$

$\square$

Let us again assume (a) and (b) of Corollary 3.12, so we have, for all $\ell \in \mathbb{Z} . B$, the generator $G(\ell)$ for $\mathbb{S}(\ell)$.

THEOREM 4.4. *If $\ell \in \mathbb{Z}_{\leqslant 0} . B$. Then we have:*

$$\mathrm{l.h.d.}(G(\ell)) = 0. \tag{4.14}$$

Proof. Consider the operator

$$G(-\ell, \mathit{k} + \ell) \circ G(\ell, \mathit{k}) \in \mathbb{S}(0). \tag{4.15}$$

From Proposition 3.2 and Corollary 3.12 it follows that:

$$\deg(G(\ell)) \;=\; \deg(G(-\ell)) \;=\; -\sum_{\alpha \in R_+} \ell_\alpha. \tag{4.16}$$

So we conclude from (4.11) (ii) and Proposition 4.3 that:

$$\text{l.h.d.}(G(\ell, \textit{k})) + \text{l.h.d.}(G(-\ell, \textit{k} + \ell)) - 2 \cdot \sum_{\alpha \in R_+} \ell_\alpha \;=\; 0. \tag{4.17}$$

Furthermore from Definition 3.3:

$$\text{l.h.d.}(G(\ell, \textit{k})) - \text{l.h.d.}(G(-\ell, \textit{k} + \ell)) + 2 \cdot \sum_{\alpha \in R_+} \ell_\alpha \;=\; 0 \tag{4.18}$$

because

$$\text{l.h.d.}(\delta(\ell)) \;=\; 2 \cdot \sum_{\alpha \in R_+} \ell_\alpha. \tag{4.19}$$

The Theorem follows from (4.17) and (4.18).   □

COROLLARY 4.5. *Let $f \in C^\infty(\Omega)$ where Ω is a neighbourhood of $e \in A$. Suppose that $G(\ell) \cdot f \in C(\Omega)$ with $\ell \in \mathbb{Z}_{\leqslant 0} \cdot B$. Then:*

$$G(\ell)(f)(e) \;=\; G(\ell)(1)(e) \cdot f(e). \tag{4.20}$$

Here $G(\ell)(1)(e)$ denotes the constant term of $G(\ell)$ evaluated at the identity element e of A.

Proof. Write $G(\ell) = \Sigma_k f_k \cdot \partial(X^k)$ with $X^k = X_1^{k_1} \ldots \ldots X_n^{k_n}$. Theorem 4.4 insures that

$$\text{l.h.d.}(f_k) \geqslant |k| \quad \text{if} \quad f_k \neq 0. \tag{4.21}$$

Choose a vector $X \in \mathfrak{a}$ outside the set $\{\Pi_{\alpha \in R_+}\, \alpha = 0\}$. Because $G(\ell)f \in C(\Omega)$ we have:

$$
\begin{aligned}
G(\ell)(f)(e) &= \lim_{t \to 0} G(\ell)(f)(\exp tX) \\[2mm]
&= \sum_k \lim_{t \to 0} (f_k(\exp tX) \cdot \lim_{t \to 0}((\partial(X^k)(f)(\exp tX))) \\[2mm]
&= \sum_k \lim_{t \to 0} (f_k(\exp tX))\partial(X^k)f(e)
\end{aligned}
$$

In view of (4.21) we see that $\lim_{t \to 0} f_k (\exp tX) = 0$ if $k \neq 0$.   □

Theorem 1.1 of [H] and Theorem 3.6 of the previous section guarantee us the existence and uniqueness of a W-invariant analytic solution on a tubular neighbourhood of $A \subset H$ of the equation

$$L(\ell)\phi = (\lambda - \varrho(\ell), \lambda + \varrho(\ell))\phi \tag{4.22}$$

(if $(\lambda, \alpha) \neq 0, \forall \alpha \in R$, $(\lambda, \kappa^\vee) \neq 1, \forall \kappa \in Q\backslash\{0\}$ and $\frac{1}{2} + \ell_\alpha + \ell_{2\alpha} \notin \mathbb{Z}_-$, $\forall \alpha \in R$) of the form:

$$F(\lambda, \ell; h) = \sum_{w \in W} c(w\lambda) \phi(w\lambda + \varrho; h). \tag{4.23}$$

Here $\phi(w\lambda + \varrho)$ is a solution of (4.22) as described in (2.3), and

$$c(\lambda, \ell) = c_0 \cdot \sum_{\alpha \in R_{0+}} c_\alpha(\lambda, \ell) \tag{4.24}$$

with

$$c_\alpha(\lambda, \ell) = \frac{2^{(\lambda, \alpha^\vee)}\Gamma(-(\lambda, \alpha^\vee))}{\Gamma(-\frac{1}{2}(\lambda, \alpha^\vee) + \frac{1}{2}\ell_\alpha + \ell_{2\alpha})\Gamma(-\frac{1}{2}(\lambda, \alpha^\vee) + \frac{1}{2}\ell_\alpha + \frac{1}{2})} \tag{4.25}$$

and $c_0 \in \mathbb{C}$ such that:

$$c(-\varrho(\ell), \ell) = 1. \tag{4.26}$$

We will use the shift operators given in Section 2 and Corollary 4.5 to calculate, in the rank two cases, the value of $F(\lambda, \ell; e)$ for those values of λ, ℓ that can be shifted to parameter values for which this value is known. It is easy to see that for $D \in \mathbb{S}(\ell)$:

$$D(\ell)F(\lambda, \ell) = \frac{\eta(\ell, \ell)(D(\ell))(\lambda) \cdot c(\lambda, \ell)}{c(\lambda, \ell + \ell)} \cdot F(\lambda, \ell + \ell) \tag{4.27}$$

On the other hand we know that for $\ell \in \mathbb{Z}_- \cdot B$:

$$D(\ell)F(\lambda, \ell; h)|_{h=e} = D(\ell)(1)(e) \cdot F(\lambda, \ell; e) \tag{4.28}$$

(see Corollary 4.5). Consequently:

$$F(\lambda, \mathit{k}; e) \; = \; \left(\frac{\eta(\ell, \mathit{k})(D(\mathit{k}))(\lambda) \cdot c(\lambda, \mathit{k})}{D(\mathit{k})1(e) \cdot c(\lambda, \mathit{k} + \ell)} \right) \cdot F(\lambda, \mathit{k} + \ell; e) \, (\ell \in \mathbb{Z}_- \cdot B).$$

(4.29)

A calculation gives the following values for $D(\mathit{k})(1)(e)$, with $D(\mathit{k}) = \tilde{G}(\ell, \mathit{k})$, $G(\ell)$ the operators given in Table 2.6:

Table 4.6

$\underline{A_2}$	$\underline{\ell = 1:}$	$-6(2\mathit{k} - 1)(3\mathit{k} - 1)(3\mathit{k} - 2)$
$\underline{BC_2}$	$\underline{\ell = (1, 0, 0):}$	$4(2\mathit{k}_1 - 1)(2\mathit{k}_1 + 2\mathit{k}_2 + 2\mathit{k}_3 - 1)$
	$\underline{\ell = (0, 1, 0):}$	$4(2\mathit{k}_2 + 2\mathit{k}_3 - 1)(2\mathit{k}_1 + 2\mathit{k}_2 + 2\mathit{k}_3 - 1)$
	$\underline{\ell = (0, 2, -1):}$	$\frac{1}{4}(2\mathit{k}_2 + 2\mathit{k}_3 - 1)(2\mathit{k}_1 + 2\mathit{k}_2 + 2\mathit{k}_3 - 1)$
$\underline{G_2}$	$\underline{\ell = (1, 0):}$	$-6(2\mathit{k}_1 - 1)(3\mathit{k}_1 + 3\mathit{k}_2 - 1)$
		$\times \; (3\mathit{k}_1 + 3\mathit{k}_2 - 2)$
	$\underline{\ell = (0, 1):}$	$-6(2\mathit{k}_2 - 1)(3\mathit{k}_1 + 3\mathit{k}_2 - 1)$
		$\times \; (3\mathit{k}_1 + 3\mathit{k}_2 - 2).$

From Table 4.6 and Table 3.15 and formulas (4.24), (4.25) and (4.26) we check that:

$$\frac{\eta(\ell, \mathit{k})(G(-\ell, \mathit{k}))(\lambda) \cdot c(\lambda, \mathit{k})}{G(-\ell, \mathit{k})1(e) \cdot c(\lambda, \mathit{k} - \ell)} \; = \; 1 \quad \text{for } \ell \in B.$$

(4.30)

THEOREM 4.7. *Let R be of type A_2, BC_2 or G_2. Then $F(\lambda, \mathit{k}; e) = 1$ for:*

(a) *λ generic, $\mathit{k} \in \mathit{k}' + \mathbb{Z} \cdot B$ with k' the multiplicity function of a symmetric space G/K with restricted root system of type R (see [HO], Remark 2.11).*

(b) *$\lambda + \varrho(\mathit{k}) \in P_-$, $\mathit{k}_\alpha \geqslant 0$ (Jacobi polynomial case).*

Proof. We can reduce (a) to $\mathit{k} = \mathit{k}'$ by repeated application of (4.29) and (4.30). In this case (a) is known to be true ($F(\lambda, \mathit{k}'; a)$ is the spherical

function for the symmetric space G/K). For G_2 and BC_2 we can prove (b) in the same manner but now we use (4.29) and (4.30) to shift to the situation $\lambda + \varrho(\ell) = 0$, in which case (b) is obvious. To prove (b) for A_2 we have to relate A_2 to G_2. If we think of A_2 as the long roots in G_2 we see that $P_A = P_G$ (P_X is the weight lattice of X) and that

$$F_G(\lambda, (0, \ell)) \;=\; d \cdot (F_A(\lambda, \ell) + F_A(\bar{\lambda}, \ell)) \tag{4.31}$$

for $\lambda \in P_{G,+}$ and certain $d \in \mathbb{C}$.

d can be calculated as follows:

$$c_G((0, \ell), \lambda) \;=\; d \cdot c_A(\ell, \lambda) \Leftrightarrow$$

$$d = \lim_{\ell' \to 0} \prod_{\substack{\alpha \in G_{2,+} \\ \alpha \text{ short}}} c_\alpha((\ell', \ell), \lambda) \quad \text{with} \quad c_\alpha((\ell', \ell), -\varrho(\ell', \ell)) = 1$$

$$= \lim_{\ell' \to 0} \prod_{\substack{\alpha \in G_{2,+} \\ \alpha \text{ short}}} \frac{2^{(\lambda + \varrho, \alpha^\vee)} \Gamma(-\lambda, \alpha^\vee)) \Gamma(\tfrac{1}{2}(\varrho, \alpha^\vee) + \tfrac{1}{2}\ell') \Gamma(\tfrac{1}{2}(\varrho, \alpha^\vee) + \tfrac{1}{2}\ell' + \tfrac{1}{2})}{\Gamma((\varrho, \alpha^\vee)) \Gamma(-\tfrac{1}{2}(\lambda, \alpha^\vee) + \tfrac{1}{2}\ell') \Gamma(-\tfrac{1}{2}(\lambda, \alpha^\vee) + \tfrac{1}{2}\ell' + \tfrac{1}{2})}.$$

Using the duplication formula for the Γ-function this becomes:

$$d \;=\; \lim_{\ell' \to 0} \prod_{\substack{\alpha \in G_{2,+} \\ \alpha \text{ short}}} \frac{\Gamma(-\lambda, \alpha^\vee)) \Gamma((\varrho, \alpha^\vee) + \ell')}{\Gamma((\varrho, \alpha^\vee)) \Gamma(-(\lambda, \alpha^\vee) + \ell')} \;=\; \lim_{\ell' \to 0} \frac{\Gamma(2\ell')}{\Gamma(\ell')} \;=\; \frac{1}{2}.$$

Now for A_2 (b) follows from (4.31) with $d = 1/2$.   □

Acknowledgements

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