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On the numbers of solutions of weighted unit equations

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§0. Introduction

Let K be a finitely generated extension field of $\mathbb{Q}$, let Γ be a finitely generated subgroup of the multiplicative group K^* of non-zero elements of K, and let $\alpha_1, \ldots, \alpha_n \in K^*$. Many problems of number theory lead to equations of the types

$$x_1 + \cdots + x_n = 1 \quad \text{in} \quad x_1, \ldots, x_n \in \Gamma \tag{0}$$

or more generally

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \quad \text{in} \quad x_1, \ldots, x_n \in \Gamma, \tag{1}$$

where, in particular, Γ is the unit group of a subring of K which is finitely generated over $\mathbb{Z}$. Hence the above equations are called *unit equations* and *weighted unit equations*, respectively. For a general survey on these equations and their applications we refer to Evertse, Győry, Stewart and Tijdeman [7].

For $n = 2$, several results are known about the number of solutions of (1). In 1985, Evertse and Győry [6] derived in case $n = 2$ an explicit upper bound for the number of solutions of (1) which is independent of α_1 and α_2. Two tuples $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n)$ with non-zero entries in K^* are called *Γ-equivalent* if there are $\varepsilon_1, \ldots, \varepsilon_n \in \Gamma$ such that $\beta_i = \alpha_i \varepsilon_i$ for $i = 1, \ldots, n$. Obviously, the number of solutions of (1) does not change when $(\alpha_1, \ldots, \alpha_n)$ is replaced by a Γ-equivalent tuple. In 1987, Evertse, Győry, Stewart and Tijdeman [8] proved that in case $n = 2$ (1) has at most two solutions for all but finitely many Γ-equivalence classes of pairs $(\alpha_1, \alpha_2) \in (K^*)^2$.

In the present paper we shall partly generalize the results mentioned above to the case $n > 2$. We shall prove (cf. Theorem 1 in §1) that the number of solutions of (1) with

$$\sum_{j \in J} \alpha_j x_j \neq 0 \text{ for each non-empty subset } J \text{ of } \{1, \ldots, n\} \tag{2}$$

can be bounded above by a number (not explicitly computable by our method) which does not depend on $\alpha_1, \ldots, \alpha_n$. This provides a refinement of a theorem of Evertse [2] and van der Poorten and Schlickewei [11] which says that (1) has only finitely many solutions with property (2). By using Theorem 1 we shall also extend our result on equation (1) to systems of weighted unit equations (cf. Theorem 2). Our Theorem 1 will be deduced in §4 from the following generalization (cf. Theorem 4 in §1) of the result of Evertse, Győry, Stewart and Tijdeman [8] mentioned above: apart from the tuples $(\alpha_1, \ldots, \alpha_n)$ belonging to at most finitely many Γ-equivalence classes depending only on n and Γ, the set of solutions of (1) is contained in the union of fewer than $2^{(n+1)!}$ proper linear subspaces of K^n. We shall derive this bound in §3, by combining a generalization of the method applied in [8] for the case $n = 2$ with the result of [2] and [11] quoted above. Finally, we shall give an application of Theorem 4 to the case $n = 3$ (cf. Theorem 5 in §1).

We shall consider equation (1) also over function fields. Let K be a function field of transcendence degree 1 over an algebraically closed field $\Bbbk$ of characteristic 0, let Γ be the group of S-units in K where S is a finite set of valuations on K, and let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of K. A solution $(x_1, \ldots, x_n)$ of (1) is called degenerate if $\alpha_1 x_1, \ldots, \alpha_n x_n$ belong to $\Bbbk$. As an analogue of Theorem 4, we shall derive an explicit upper bound for the minimal number of proper linear subspaces of K^n in the union of which the set of non-degenerate solutions of (1) is contained (cf. Theorem 6 in §2). This bound is also independent of $\alpha_1, \ldots, \alpha_n$. In deriving our bound we shall use an effective upper bound of Brownawell and Masser [1] for the heights of solutions of (1) and a "higher dimensional gap principle".

Applications of our results are given in the recent papers [13] and [5].

§1. Weighted unit equations over finitely generated fields

Let K be a finitely generated extension field of $\mathbb{Q}$, $n \geqq 1$ an integer, $\alpha_1, \ldots, \alpha_n$ elements of K^*, and Γ a finitely generated multiplicative subgroup of K^*. A solution $(x_1, \ldots, x_n)$ of the weighted unit equation

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \quad \text{in} \quad x_1, \ldots, x_n \in \Gamma \tag{1}$$

is called non-degenerate if

$$\sum_{j \in J} \alpha_j x_j \neq 0 \text{ for each non-empty subset } J \text{ of } \{1, \ldots, n\} \tag{2}$$

and *degenerate* otherwise. It is clear that if Γ is infinite and if (1) has a degenerate solution then (1) has infinitely many degenerate solutions. For non-degenerate solutions we have the following result.

THEOREM 1. *The number of non-degenerate solutions of* (1) *is at most* $C_1 = C_1(n, \Gamma)$, *where* C_1 *is a number depending only on n and* Γ.

This is a refinement of a result of Evertse [2] and van der Poorten and Schlickewei [11] (cf. Lemma 1 in §3 and the Remark made after the lemma) on the finiteness of the number of non-degenerate solutions of (1).

For $n = 1$, Theorem 1 is trivial. For $n = 2$, the solutions of (1) are all non-degenerate. In the case $n = 2$ an explicit expression for C_1 has been evaluated by Evertse and Győry [6]. As we mentioned above, for $n > 2$ we are not able to make explicit the bound C_1 occurring in Theorem 1.

Let $k \geqq 1$ be an integer. As a generalization of equation (1), consider the *system of weighted unit equations*

$$\left.\begin{array}{l} \alpha_{11}x_1 + \cdots + \alpha_{1n}x_n = \alpha_{10} \\ \quad\vdots \\ \alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n = \alpha_{k0} \end{array}\right\} \text{ in } \quad x_1, \ldots, x_n \in \Gamma \tag{3}$$

where $\alpha_{ij} \in K$ for $i = 1, \ldots, k$ and $j = 0, \ldots, n$, and at least one of the α_{i0} is different from zero. We shall say that a solution $(x_1, \ldots, x_n)$ of (3) is *degenerate* if $\Sigma_{j \in J}\, \alpha_{ij}x_j = 0$ for $i = 1, \ldots, k$ and for a proper non-empty subset J of $\{1, \ldots, n\}$, and *non-degenerate* otherwise. In §4 we shall deduce from Theorem 1 the following

THEOREM 2. *The number of non-degenerate solutions of* (3) *is at most* $C_2 = C_2(n, \Gamma)$, *where* C_2 *is a number depending only on n and* Γ.

For $k = 1$ Theorem 2 reduces to Theorem 1, hence Theorem 2 and Theorem 1 are equivalent.

In the remainder of this section it will be assumed that $n \geqq 2$. In §4 we shall prove in an elementary way that Theorem 1 is equivalent also to the following theorem.

THEOREM 3. *All solutions of* (1) *are contained in the union of at most* $C_3 = C_3(n, \Gamma)$ $(n - 1)$*-dimensional linear subspaces of* K^n, *where* C_3 *is a number depending only on and* Γ.

The tuples $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n)$ in $(K^*)^n$ are called *Γ-equivalent* if $\beta_i = \alpha_i \varepsilon_i$ for some $\varepsilon_i \in \Gamma$ for $i = 1, \dots, n$. It is obvious that the number of (non-degenerate) solutions of (1) as well as the minimal number of $(n-1)$-dimensional subspaces of K^n containing all the solutions of (1) remain unchanged if $(\alpha_1, \dots, \alpha_n)$ is replaced by a Γ-equivalent tuple. The main result of this section is as follows.

THEOREM 4. *For all but finitely many Γ-equivalence classes of tuples $(\alpha_1, \dots, \alpha_n) \in (K^*)^n$, the set of solutions of* (1) *is contained in the union of fewer than $2^{(n+1)!}$ $(n-1)$-dimensional linear subspaces of K^n.*

As we shall see in §4, Theorem 3 easily follows from Theorem 4 and Lemma 1 of §3.

It is possible to generalize Theorems 1 to 4 to the case that K is any subfield of $\mathbb{C}$ and Γ is any subgroup of finite rank of $\mathbb{C}^*$. For the proofs it suffices to replace Lemma 1 of §3 on unit equations as we use it by a more general version due to Laurent [9] (cf. Remark 2 in §3).

For $n = 2$, Theorem 4 implies that apart from finitely many Γ-equivalence classes of pairs $(\alpha_1, \alpha_2) \in (K^*)^2$, the equation

$$\alpha_1 x_1 + \alpha_2 x_2 = 1 \quad \text{in} \quad x_1, x_2 \in \Gamma \tag{4}$$

has fewer than 2^6 solutions. We note that Evertse, Győry, Stewart and Tijdeman [8] proved a similar result with the upper bound 2 which is already best possible.

For $n = 3$, a result of similar type can be deduced from Theorem 4. To state this result we need some further notation. Let $\{z_1, \dots, z_q\}$ be a transcendence basis of K over $\mathbb{Q}$. Then K is a finite extension of degree d, say, of the field $K_0 = \mathbb{Q}(z_1, \dots, z_q)$. The polynomial ring $\mathcal{O} = \mathbb{Z}[z_1, \dots, z_q]$ is a unique factorization domain in which the prime elements are the rational primes and the primitive irreducible non-constant polynomials in $\mathcal{O}$. To every prime element π of $\mathcal{O}$ corresponds an (additive) valuation v_π on K_0 with the property that $v_\pi(\pi) = 1$ and $v_\pi(a/b) = 0$ if a, b are elements of $\mathcal{O}$ not divisible by π. Thus we have a set of pairwise inequivalent valuations on K_0 with value group $\mathbb{Z}$. Each of these valuations can be extended in at most d different ways to K. Let m_K denote the set of these extensions, and let S be a finite subset of m_K of cardinality s. One can show that

$$\Gamma_S := \{\alpha \in K : v(\alpha) = 0 \quad \text{for all} \quad v \in m_K \backslash S\} \tag{5}$$

is a finitely generated multiplicative subgroup of K^*. Moreover, every finitely generated multiplicative subgroup of K^* can be embedded in some subgroup Γ_S of K^*.

THEOREM 5. *For all but finitely many Γ_S-equivalence classes of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in (K^*)^3$, the equation*

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 1 \quad in \quad x_1, x_2, x_3 \in \Gamma_S \tag{6}$$

has fewer than $2^{26} \times 7^{3d+2s}$ non-degenerate solutions.

Theorem 4 implies that apart from finitely many Γ_S-equivalence classes of $\boldsymbol{\alpha} \in (K^*)^3$, the solutions of (6) are contained in the union of at most $2^{4!}$ proper linear subspaces of K^3. However, the non-degenerate solutions (x_1, x_2, x_3) contained in such a subspace satisfy a weighted unit equation of the form

$$\beta_{i_1} x_{i_1} + \beta_{i_2} x_{i_2} = 1 \quad \text{in} \quad x_{i_1}, x_{i_2} \in \Gamma_S \tag{7}$$

for some distinct $i_1, i_2 \in \{1, 2, 3\}$ and some $\beta_{i_1}, \beta_{i_2} \in K^*$. Now Theorem 1 of Evertse and Győry [6] implies that the number of solutions of (7) is at most $4 \times 7^{3d+2s}$ and hence Theorem 5 follows.

We note that in view of an example given by Evertse, Győry, Stewart and Tijdeman (cf. [8], §§0,5) the bound occurring in Theorem 5 cannot be replaced by a bound which is polynomial in terms of s.

The above results suggest the following

CONJECTURE. *It is possible to give an explicit expression $C(n)$, in terms of n only, such that for every $\alpha_1, \ldots, \alpha_n \in K^*$, the number of non-degenerate solutions of the equation*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \quad in \quad x_1, \ldots, x_n \in \Gamma_S$$

is at most $C(n)^{d+s}$.

§2. Weighted unit equations over function fields

Let K be a function field of transcendence degree 1 with algebraically closed constant field $\mathbb{k}$ of characteristic 0. Thus K is a finite extension of $\mathbb{k}(t)$, where t is a transcendental element of K over $\mathbb{k}$. The field K can be endowed with a set M_K of additive valuations with value group $\mathbb{Z}$ for which

$$\mathbb{k} = \{0\} \cup \{\alpha \in K : v(\alpha) = 0 \quad \text{for each } v \text{ in } M_K\} \tag{8}$$

holds. Let S be a finite subset of M_K. An element α of K is called an *S-unit* if $v(\alpha) = 0$ for all v in $M_K \backslash S$. The S-units form a multiplicative group which is denoted by U_S. The group U_S contains $\Bbbk^*$ as a subgroup and $U_S/\Bbbk^*$ is finitely generated. We shall prove the following analogue of Theorem 4 for the function field case. As usual, e denotes the number 2.71828 . . . .

THEOREM 6. *Let K, $\Bbbk$, S be as above. Let g be the genus of $K/\Bbbk$, s the cardinality of S, and $n \geqq 2$ an integer. Then for every $\alpha_1, \ldots, \alpha_n \in K^*$, the set of solutions of*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \text{ in } x_1, \ldots, x_n \in U_S$$
$$\text{with } \alpha_1 x_1, \ldots, \alpha_n x_n \text{ not all in } \Bbbk \tag{9}$$

is contained in the union of at most

$$\{\log (g + 2)\} \times \{e(n + 1)\}^{(n+1)s+2}$$

$(n - 1)$-dimensional linear subspaces of K^n.

Note that when $\alpha_1, \ldots, \alpha_n$ are not all S-units, the condition that $\alpha_1 x_1, \ldots, \alpha_n x_n$ do not all belong to $\Bbbk$ holds for all S-units $x_1, \ldots, x_n$. If $\alpha_1, \ldots, \alpha_n$ are all S-units, then the solutions of (9) with $\alpha_1 x_1, \ldots, \alpha_n x_n \in \Bbbk$ are not contained in the union of finitely many $(n - 1)$-dimensional linear subspaces of K^n.

For $n = 2$, Theorem 6 gives the upper bound

$$\{\log (g + 2)\} (3e)^{3s+2}$$

for the number of solutions of (9). We note that in case $n = 2$ Evertse [4] established the upper bound 2×7^{2s} for the number of solutions, a bound which is better and independent of g.

Theorem 6 will be proved in §5. The proof will involve the following results: a "gap principle" by which one can estimate the minimal number of $(n - 1)$-dimensional subspaces of K^n containing all solutions of (9) with bounded heights; and an explicit upper bound, due to Brownawell and Masser [1], for the heights of the non-degenerate solutions of (9). (We mention that, independently, J.F. Voloch [15], Thm. 4) derived the same upper bound for the heights of the solutions $x_1, \ldots, x_n$ of (9), but subject to the slightly stronger condition that $\alpha_1 x_1, \ldots, \alpha_n x_n$ are linearly independent over $\Bbbk$. The bound obtained in [1] and [15] sharpens an earlier inequality of Mason [10].) We remark that by using the method of proof of Theorem 4 instead of the gap principle just mentioned it is possible to get a bound similar to that of Theorem 6, but with a much worse dependence on the number of unknowns n.

REMARK. The higher dimensional gap principle for weighted S-unit equations over function fields has an analogue for number fields (cf. §5, Lemma 6) which gives an upper bound for the number of subspaces containing the solutions of (1) with small height if Γ is the group U_S of S-units in some algebraic number field K. By combining this gap principle with a mean to deal with the large solutions, such as a quantitative version of Schlickewei's p-adic subspace theorem over number fields, it would be possible to obtain an explicit upper bound, independent of the coefficients, for the number of subspaces containing all solutions (or for the number of non-degenerate solutions) of (1) with $\Gamma = U_S$. Just before this paper went to press, at the Conference on Diophantine Approximations in Oberwolfach (14–18 March, 1988), Schlickewei announced that he succeeded in making his subspace theorem quantitative. He used this together with a gap principle that he obtained independently of us, to show that (1) has at most $(C(n, d)s)^{s^6}$ non-degenerate solutions, where $\Gamma = U_S$ as above, s is the cardinality of S, $d = [K:\mathbb{Q}]$ and $C(n, d)$ is some explicit function of n and d.

§3. Proof of Theorem 4

We shall use the same notation as in Section 1. Theorem 4 will be proved by an extension to the case $n > 2$ of the method used in [8] for the case $n = 2$. The basic idea of our proof is as follows. Any $n + 1$ solutions $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})$, $i = 0, \ldots, n$, of (1) satisfy the equation

$$\Delta(\mathbf{X}_0, \ldots, \mathbf{X}_{n-1}, \mathbf{X}_n) = \begin{vmatrix} X_{01} & \ldots & X_{0n} & 1 \\ \vdots & & \vdots & \vdots \\ X_{n-1,1} & \ldots & X_{n-1,n} & 1 \\ X_{n-1} & \ldots & X_{nn} & 1 \end{vmatrix} = 0.$$

We shall show that if the set of solutions of (1) is not contained in the union of fewer than $2^{(n+1)!}$ proper subspaces of K^n, then there are a subsum $\Sigma = \Sigma_0 + \Sigma_1 X_{n-1} + \cdots + \Sigma_n X_{nn}$ of the polynomial Δ, solutions $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$ of (1) with $\tilde{\Sigma}_k = \Sigma_k(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) \neq 0$ $(k = 0, \ldots, n)$, and n linearly independent solutions $\mathbf{y}_0, \ldots, \mathbf{y}_{n-1}$ of (1) such that $l_\Sigma(\mathbf{y}_i) = 0$, $l_{\Sigma'}(\mathbf{y}_i) \neq 0$ for each proper non-empty subsum Σ' of Σ for $i = 0, \ldots, n - 1$, where $l_\Sigma = \tilde{\Sigma}_0 + \tilde{\Sigma}_1 X_{n1} + \cdots + \tilde{\Sigma}_n X_{nn}$ and $l_{\Sigma'}$ is defined similarly. But $l_\Sigma(\mathbf{y}_i)$ is the sum of at most $(n+1)!$ elements from Γ, hence, applying Lemma 1 below with $i = 0, \ldots, n-1$ one can prove that the y_{ij}/y_{0j} $(i = 0, \ldots, n-1; j = 1, \ldots, n)$ belong to a finite subset of K^* depending only on n and Γ. Since $(y_{i1}/y_{01}, \ldots, y_{in}/y_{0n})$ for $i = 0, \ldots, n-1$ are linearly independent and, by (1),

$$\sum_{j=1}^{n} (\alpha_j y_{0j})(y_{ij}/y_{0j}) = 1 \quad \text{for} \quad i = 0, \ldots, n-1,$$

it will follow that the $\alpha_j y_{0j}$ $(j = 1, \ldots, n)$ belong to a finite subset of K^* which depends only on n and Γ. This will prove the assertion of Theorem 4.

We now turn to the detailed proof of Theorem 4. We shall need two lemmas.

LEMMA 1. *Let $n \geqq 1$ be an integer. The equation*

$$x_1 + \cdots + x_n = 1 \quad \textit{in} \quad x_1, \ldots, x_n \in \Gamma$$

has only finitely many non-degenerate solutions.

REMARK 1. In case that Γ is contained in an algebraic number field (algebraic number field case), this lemma has been proved independently by Evertse [2] and van der Poorten and Schlickewei [11]. In their paper, van der Poorten and Schlickewei gave a rough sketch of a specialization argument by which this lemma in the general case can be deduced from Lemma 1 in the algebraic number field case. Up to now, no complete exposition of this specialization argument has been published. For the sake of completeness only, we shall show in the Appendix how to derive Lemma 1 in the general case from Lemma 1 in the algebraic number field case. However, our arguments will be different from those of van der Poorten and Schlickewei. Instead of a specialization argument we shall use our Theorem 6 on S-unit equations over function fields.

REMARK 2. Using the above version of Lemma 1, Laurent [9] generalized Lemma 1 to the case that K is any subfield of $\mathbb{C}$ and Γ is any subgroup of finite rank of $\mathbb{C}^*$.

To state Lemma 2 we need some further notation. Next let $n \geqq 2$ be an integer. For every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (K^*)^n$, denote by $l_{\boldsymbol{\alpha}}$ the linear polynomial $\alpha_1 X_1 + \cdots + \alpha_n X_n - 1$ and by H_α the $(n-1)$-dimensional linear subvariety (i.e., a hyperplane not necessarily containing $\mathbf{0}$) of K^n which consists of $\mathbf{y} \in K^n$ with $l_\alpha(\mathbf{y}) = 0$. For every subset $\mathscr{S}$ of H_α we define $M(\mathscr{S}, H_\alpha)$ as the smallest number of hyperspaces (i.e. $(n-1)$-dimensional linear subspaces) of K^n, the union of which contains $\mathscr{S}$. If $k \geqq 1$ is an integer and

$$P(X_1, \ldots, X_k) = \sum_{(i_1, \ldots, i_k) \in I} a_{i_1, \ldots, i_k} X_1^{i_1} \ldots X_k^{i_k} \in K[X_1, \ldots, X_k]$$

with a finite, non-empty subset I of k-tuples of non-negative integers, then by a subsum of P we mean a polynomial

$$Q(X_1, \ldots, X_k) = \sum_{(i_1, \ldots, i_k) \in J} a_{i_1, \ldots, i_k} X_1^{i_1} \ldots X_k^{i_k}$$

where J is a non-empty subset of I.

LEMMA 2. *Let $\boldsymbol{\alpha} \in (K^*)^n$ and let $\mathscr{S}$ be a subset of H_α with $M(\mathscr{S}, H_\alpha) \geqq 2^{(n+1)!}$. Then for $k = 1, \ldots, n$ the following holds: there are $\mathbf{x}_0 = (x_{01}, \ldots, x_{0n}), \ldots, \mathbf{x}_{k-1} = (x_{k-1,1}, \ldots, x_{k-1,n}) \in \mathscr{S}$ such that all subsums of all $(k \times k)$ subdeterminants of the matrix*

$$A_{k,n} := \begin{pmatrix} 1 & X_{01} & \ldots & X_{0n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & X_{k-1,1} & \ldots & X_{k-1,n} \end{pmatrix}$$

are non-zero for $X_{01} = x_{01}, \ldots, X_{k-1,n} = x_{k-1,n}$.

Proof of Lemma 2. We shall proceed by induction on k. For $k = 1$ the lemma is obvious. Suppose that the lemma holds for $k = p - 1$ where $p \geqq 1$ (induction hypothesis). We shall prove it for $k = p$.

For convenience, we shall assume throughout this section that $X_{00} = X_{10} = \cdots = X_{n0} = 1$ and $x_{00} = \cdots = x_{n0} = 1$. By the induction hypothesis there are $\mathbf{x}_0 = (x_{01}, \ldots, x_{0n}), \ldots, \mathbf{x}_{p-2} = (x_{p-2,1}, \ldots, x_{p-2,n}) \in \mathscr{S}$ such that all subsums of all $(p-1) \times (p-1)$ subdeterminants of $A_{p-1,n}$ are non-zero at $x_{01}, \ldots, x_{p-2,n}$. Let B be a subsum of some $p \times p$ subdeterminant of A_{pn},

$$\begin{vmatrix} X_{0,i_1} & \ldots & X_{0,i_p} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ X_{p-1,i_1} & \ldots & X_{p-1,i_p} \end{vmatrix},$$

say. Then B can be written as $B_{t_1}X_{p-1,t_1} + \cdots + B_{t_p}X_{p-1,t_p}$, where each B_{t_i} is either identically 0 or, up to sign, a subsum of some $(p-1) \times (p-1)$ subdeterminant of $A_{p-1,n}$. To the subsum B we associate the linear polynomial

$$l_B(\mathbf{X}) = \tilde{B}_{t_1}X_{t_1} + \cdots + \tilde{B}_{t_p}X_{t_p},$$

where $\tilde{B}_{t_i}$ is obtained from B_{t_i} by replacing $X_{00}, \ldots, X_{p-2,n}$ by $x_{00}, \ldots, x_{p-2,n}$, respectively. At least one of the polynomials B_{t_i} is not identically 0, whence is, up to sign, a subsum of some $(p-1) \times (p-1)$ subdeterminant of $A_{p-1,n}$. Therefore, none of the polynomials l_B is identically 0. Moreover, each polynomial l_B is linearly independent of l_α, since each l_B has at most $p \leqq n$ non-zero terms, whereas l_α has $n+1$ non-zero terms. There are at most $\binom{n+1}{p}$ $p \times p$ subdeterminants of $A_{p,n}$, and each subdeteminant has less than $2^{p!}$ subsums. This implies that there are at most $\binom{n+1}{p}2^{p!}$ polynomials l_B. But $\binom{n+1}{p}2^{p!} < 2^{(n+1)!}$, hence, by the assumption $M(\mathscr{S}, H_\alpha) \geqq 2^{(n+1)!}$, there must be an $\mathbf{x}_{p-1}$ in $\mathscr{S}$ with $l_B(\mathbf{x}_{p-1}) \neq 0$ for each subsum B of each $p \times p$ subdeterminant of $A_{p,n}$. This implies at once that $B(\mathbf{x}_0, \ldots, \mathbf{x}_{p-1}) \neq 0$ for each subsum B of each $p \times p$ subdeterminant of $A_{p,n}$. □

Proof of Theorem 4. We have to prove that $M(\Gamma^n \cap H_\alpha, H_\alpha) < 2^{(n+1)!}$ for all but finitely many Γ-equivalence classes of $\alpha \in (K^*)^n$. Suppose that $M(\Gamma^n \cap H_\alpha, H_\alpha) \geqq 2^{(n+1)!}$ for some $\alpha \in (K^*)^n$. We shall show that the set of α having this property is contained in the union of at most finitely many Γ-equivalence classes.

Each subsum Σ of the determinant

$$\Delta = \begin{vmatrix} X_{00} & \cdots & X_{0n} \\ \vdots & & \\ X_{n0} & \cdots & X_{nn} \end{vmatrix}$$

can be written as $\Sigma_0 X_{n0} + \Sigma_1 X_{n1} + \cdots + \Sigma_n X_{nn}$ where Σ_i is either identically 0 or, up to sign, a subsum of some $n \times n$ subdeterminant of $A_{n,n}$ for $i = 0, \ldots, n$. Σ is called *full* if none of the polynomials $\Sigma_0, \ldots, \Sigma_n$ is identically 0. By Lemma 2 there are $\mathbf{x}_0 = (x_{01}, \ldots, x_{0n}), \ldots, \mathbf{x}_{n-1} = (x_{n-1,1}, \ldots, x_{n-1,n})$ in $\Gamma^n \cap H_\alpha$ such that each subsum of each $n \times n$ subdeterminant of $A_{n,n}$ is different from zero in $x_{01}, \ldots, x_{n-1,n}$. To each subsum Σ of Δ we associate the linear polynomial $l_\Sigma = \tilde{\Sigma}_0 X_0 + \tilde{\Sigma}_1 X_1 + \cdots + \tilde{\Sigma}_n X_n$ where $\tilde{\Sigma}_i$ is obtained from Σ_i by substituting $x_{00}, \ldots, x_{n-1,n}$ for $X_{00}, \ldots, X_{n-1,n}$, respectively. By our choice of $\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}$, none of the polynomials l_Σ is identically 0. Further, for every $\mathbf{y}$ in $\Gamma^n \cap H_\alpha$ there is a subsum Σ

such that

$$l_\Sigma(\mathbf{y}) = 0, \; l_{\Sigma'}(\mathbf{y}) \neq 0 \text{ for each subsum } \Sigma' \text{ of } \Sigma \quad \text{with} \quad \Sigma' \neq \Sigma. \tag{10}$$

This follows from the fact that $\Delta(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}, \mathbf{y}) = 0$ for each $\mathbf{y}$ in $\Gamma^n \cap H_\alpha$. Elements $\mathbf{y}$ of $\Gamma^n \cap H_\alpha$ with property (10) are said to be Σ-*associated.*

The set of subsums of Δ is divided into three classes:

(I) the subsums which are not full;

(II) the full subsums Σ for which the set of Σ-associated $\mathbf{y}$ in $\Gamma^n \cap H_\alpha$ is contained in a single hyperspace of K^n;

(III) the full subsums Σ for which $\Gamma^n \cap H_\alpha$ contains n linearly independent Σ-associated elements $\mathbf{y}_0, \ldots, \mathbf{y}_{n-1}$.

If Σ is not a full subsum of Δ, then obviously l_Σ is linearly independent of l_α. Further, Δ has fewer than $2^{(n+1)!}$ subsums. Hence the set of elements $\mathbf{y}$ in $\Gamma^n \cap H_\alpha$ which are Σ-associated for some Σ in class I or II is contained in the union of fewer than $2^{(n+1)!}$ hyperspaces of K^n. Therefore, class III is non-empty. We shall show that this implies that $\boldsymbol{\alpha}$ must belong to one of at most finitely many Γ-equivalence classes depending on n and Γ only.

Let Σ be a subsum of Δ in class III. Then Σ can be written as

$$\Sigma\,(\mathbf{X}_0, \ldots, \mathbf{X}_n) = \sum_{i=0}^{n} \left\{ \sum_{k \in J_i} P_k(\mathbf{X}_0, \ldots, \mathbf{X}_{n-1}) \right\} X_{ni},$$

where each P_k is a monomial, multiplied by 1 or -1, and each index set J_i is non-empty. Consider the equation

$$\left.\begin{array}{l} \displaystyle\sum_{i=0}^{n} \sum_{k \in J_i} z_{ik} = 0, \\ \displaystyle\sum_{i=0}^{n} \sum_{k \in L_i} z_{ik} \neq 0 \text{ for each tuple of subsets} \\ L_0, \ldots, L_n \text{ of } J_0, \ldots, J_n \text{ respectively, for which} \\ \text{at least one } L_i \text{ is non-empty and at least one} \\ L_i \text{ is different from } J_i \end{array}\right\} \begin{array}{l} \text{in } z_{ik} \in \Gamma \\ \text{for } i = 0, \ldots, n \\ \text{and } k \in J_i \end{array} \tag{11}$$

The number of terms in equation (11) is at most $(n+1)!$. Hence, by Lemma 1, each quotient z_{ik}/z_{jl} must belong to a finite set $\mathscr{J}_1 \subset K^*$ which depends only on n and Γ. Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a Σ-associated element of $\Gamma^n \cap H_{\boldsymbol{\alpha}}$, and let $y_0 = 1$. Then $z_{ik} = P_k(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})y_i$ $(i = 0, \ldots, n, k \in J_i)$ is a solution of (11). Hence for each i, k, j, l,

$$\frac{P_k(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})y_i}{P_l(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1})y_j} \in \mathscr{I}_1.$$

But this implies that for any two Σ-associated $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{y}' = (y_1', \ldots, y_n')$ in $\Gamma^n \cap H_{\boldsymbol{\alpha}}$ and all i, j in $\{0, \ldots, n\}$ we have

$$\frac{y_2}{y_j} \Big/ \frac{y_2'}{y_j'} \in \mathscr{I}_2, \tag{12}$$

where $y_0' = 1$ and $\mathscr{I}_2$ is a finite subset of K^*, depending only on n and Γ, which is obtained by taking all quotients of the elements of $\mathscr{I}_1$. Let now $\mathbf{y}_0, \ldots, \mathbf{y}_{n-1}$ be n linearly independent, Σ-associated elements of $\Gamma^n \cap H_{\boldsymbol{\alpha}}$. These exist, since by assumption, Σ belongs to class III. Let $\mathbf{y}_i = (y_{i1}, \ldots, y_{in})$ for $i = 0, \ldots, n-1$, and put

$$\beta_j = \alpha_j y_{0j} \qquad \text{for } j = 1, \ldots, n;$$

$$w_{ij} = y_{ij}/y_{0j} \qquad \text{for } i = 0, \ldots, n-1, \quad j = 1, \ldots, n;$$

and

$$\mathbf{w}_i = (w_{i1}, \ldots, w_{in}) \quad \text{for } i = 0, \ldots, n-1.$$

Then $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ is Γ-equivalent to $\boldsymbol{\alpha}$. Further, $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{n-1}$ are linearly independent in K^n, and, in view of $\mathbf{y}_0, \ldots, \mathbf{y}_{n-1} \in H_{\boldsymbol{\alpha}}$,

$$\sum_{j=1}^{n} \beta_j w_{ij} = 1 \quad \text{for } i = 0, \ldots, n-1.$$

By Cramer's rule, $\beta_1, \ldots, \beta_n$ are uniquely determined by the w_{ij}. But, by (12), each w_{ij} belongs to $\mathscr{I}_2$. Hence the tuple $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ belongs to a finite subset of $(K^*)^n$ which depends only on n and Γ. Since $\boldsymbol{\alpha}$ is Γ-equivalent to $\boldsymbol{\beta}$, this implies that $\boldsymbol{\alpha}$ belongs to one of at most finitely many Γ-equivalence classes depending only on n and Γ. This completes the proof of Theorem 4. □

§4. Equivalence of Theorem 1 and Theorem 3. Proofs of Theorems 3 and 2

In this section we shall first show that Theorems 1 and 3 are equivalent. Then we shall deduce Theorem 3 from Theorem 4, and Theorem 2 from Theorem 1.

We shall use the notation introduced in §1. In particular, K is a finitely generated extension field of $\mathbb{Q}$, and Γ is a finitely generated multiplicative subgroup of K^*. Our aim is to prove that the following two statements are equivalent:

(i) *For* $n \geqq 1$, *and for every* $(\alpha_1, \ldots, \alpha_n) \in (K^*)^n$, *the equation*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \quad \textit{in} \quad x_1, \ldots, x_n \in \Gamma \tag{1}$$

has at most $C_1(n, \Gamma)$ *non-degenerate solutions, where* $C_1(n, \Gamma)$ *depends only on* n *and* Γ;

(ii) *For* $n \geqq 2$, *and for every* $(\alpha_1, \ldots, \alpha_n) \in (K^*)^n$, *the set of solutions of equation* (1) *is contained in the union of at most* $C_3(n, \Gamma)$ $(n - 1)$*-dimensional linear subspaces of* K^n, *where* $C_3(n, \Gamma)$ *depends only on* n *and* Γ.

Proof. First we shall prove the implication (i) ⇒ (ii). By (i), the set of non-degenerate solutions of (1) is contained in at most $C_1(n, \Gamma)$ $(n - 1)$-dimensional linear subspaces of K^n. Further, the set of degenerate solutions of (1) is contained in fewer than 2^n $(n - 1)$-dimensional linear subspaces of K^n. This proves (ii).

We shall now show that (ii) ⇒ (i). We shall prove (i) by induction on the number of unknowns n of (1). For $n = 1$, (i) is trivially true with $C_1(1, \Gamma) = 1$. Let $p \geqq 2$, and suppose that (i) has been proved for all positive integers $n < p$. We shall show that this implies (i) for $n = p$.

By (ii), the set of non-degenerate solutions of (1) is contained in the union of at most $C_3(p, \Gamma)$ $(p - 1)$-dimensional linear subspaces of K^p. Let V denote one of these subspaces, and consider the non-degenerate solutions $(x_1, \ldots, x_p)$ of (1) contained in V. There exist $\gamma_1, \ldots, \gamma_p \in K$, not all zero, such that

$$\gamma_1 x_1 + \cdots + \gamma_p x_p = 0 \quad \text{identically on } V.$$

This implies that each solution $(x_1, \ldots, x_p) \in V$ of (1) satisfies

$$\delta_{i_1} x_{i_1} + \cdots + \delta_{i_s} x_{i_s} = 1 \tag{13}$$

for some positive integer s with $s < p$, where $\delta_{i_1}, \ldots, \delta_{i_s}$ are non-zero elements of K. For every non-degenerate solution $(x_1, \ldots, x_p)$ of (1) which satisfies (13) there is a non-empty subset J of $\{1, \ldots, s\}$ such that

$$\sum_{j \in J} \delta_{i_j} x_{i_j} = 1, \quad \sum_{j \in L} \delta_{i_j} x_{i_j} \neq 0 \text{ for each non-empty subset } L \text{ of } J. \tag{14}$$

Now the induction hypothesis implies that the number of tuples $(x_{i_j} : j \in J)$ where $(x_1, \ldots, x_p)$ is a solution of (1) with property (14) is at most $C_1(|J|, \Gamma)$ (here $|J|$ denotes the cardinality of J). But it follows again from the induction hypothesis that the number of non-degenerate solutions of (1) with given values for $x_{i_j} (j \in J)$ is at most $C_1(p - |J|, \Gamma)$. The total number of non-degenerate solutions of (1) with property (14) is therefore bounded above by $C_1(|J|, \Gamma) \cdot C_1(p - |J|, \Gamma)$. Further, the number of tuples $(\gamma_1, \ldots, \gamma_p)$ and $(\delta_{i_1}, \ldots, \delta_{i_s})$ considered above is at most $C_3(p, \Gamma)$, and the number of subsets of $\{1, \ldots, s\}$ of cardinality j is at most $\binom{p}{j}$. Hence it follows that the number of non-degenerate solutions of (1) is at most

$$C_1(p, \Gamma) := C_3(p, \Gamma) \sum_{j=1}^{p-1} \binom{p}{j} C_1(j, \Gamma) C_1(p - j, \Gamma).$$

This completes the proof of (i). □

Proof of Theorem 3. We may suppose without loss of generality that K is generated over $\mathbb{Q}$ by the elements of Γ. Indeed, if (1) has at most $n - 1$ linearly independent solutions then all solutions of (1) are contained in a single $(n - 1)$-dimensional linear subspace of K^n. Otherwise, if there are n linearly independent solutions of (1) then, by Cramer's rule, the coefficients $\alpha_1, \ldots, \alpha_n$ of (1) are uniquely determined by the coordinates of these solutions and hence belong to the field $\mathbb{Q}(\Gamma)$.

It follows from Theorem 4 that there exists a finite set $\mathscr{J}$ of elements of K^n, depending only on n and Γ, with the following property: If the set of solutions of (1) is not contained in the union of fewer than $2^{(n+1)!}$ $(n - 1)$-dimensional linear subspaces of K^n, then $(\alpha_1, \ldots, \alpha_n)$ is Γ-equivalent to one of the elements of $\mathscr{J}$. Further, denoting by $\Gamma_{\mathscr{J}}$ the multiplicative subgroup of K^* generated by Γ and the coordinates of the elements of $\mathscr{J}$, we have $\alpha_1, \ldots, \alpha_n \in \Gamma_{\mathscr{J}}$. It is clear that the group $\Gamma_{\mathscr{J}}$ is finitely generated and depends only on n and Γ. For any solution $(x_1, \ldots, x_n)$ of (1), $(\alpha_1 x_1, \ldots, \alpha_n x_n)$ is a solution of the equation

$$y_1 + \cdots + y_n = 1 \quad \text{in} \quad y_1, \ldots, y_n \in \Gamma_{\mathscr{J}}. \tag{15}$$

It follows now from Lemma 1 and the equivalence of Theorem 1 and Theorem 3 that the set of solutions of (15) is contained in the union of at most $C_4(n, \Gamma)$ $(n - 1)$-dimensional linear subspaces of K^n. This implies, however, that the set of solutions of (1) is also contained in the union of at most $C_4(n, \Gamma)$ $(n - 1)$-dimensional subspaces of K^n. This completes the proof of Theorem 3. □

Proof of Theorem 2. We shall proceed by induction on n. For $n = 1$, Theorem 2 is true with $C_2(1, \Gamma) = 1$. Let $p \geqq 2$ be an integer, and suppose that the assertion has been already proved for all positive integers $n < p$. As we shall see, this implies Theorem 2 for $n = p$.

We may suppose without loss of generality that $\alpha_{10} \neq 0$ and hence that $\alpha_{10} = 1$. For every non-degenerate solution $(x_1, \ldots, x_p)$ of (3) there exists a non-empty subset J of $\{1, \ldots, p\}$ such that

$$\sum_{j \in J} \alpha_{1j} x_j = 1, \quad \sum_{j \in L} \alpha_{1j} x_j \neq 0 \text{ for each non-empty subset } L \text{ of } J. \tag{16}$$

It follows from Theorem 1 that the number of tuples $(x_j : j \in J)$ where $(x_1, \ldots, x_p)$ is a non-degenerate solution of (3) with property (16) is at most $C_1(|J|, \Gamma)$. Further, the induction hypothesis implies that (3) has at most $C_2(p - |J|, \Gamma)$ non-degenerate solutions with given values for x_j $(j \in J)$, where $C_2(0, \Gamma) = 1$. Hence the number of non-degenerate solutions of (3) having property (16) is at most $C_1(|J|, \Gamma) \times C_2(p - |J|, \Gamma)$. Finally, taking into consideration all possible non-empty subsets J of $\{1, \ldots, p\}$, we obtain that the number of non-degenerate solutions of (3) is bounded above by

$$C_2(p, \Gamma) := \sum_{j=1}^{p} \binom{p}{j} C_1(j, \Gamma) C_2(p - j, \Gamma).$$

This proves Theorem 2. □

§5. Proof of Theorem 6

We use the notation introduced in §2. In particular, K is a finite extension of $\mathbb{k}(t)$, where $\mathbb{k}$ is an algebraically closed field of characteristic 0 and t is a transcendental element of K over $\mathbb{k}$. Further, g denotes the genus of $K/\mathbb{k}$, and M_K the maximal set of additive valuations on K with value group $\mathbb{Z}$. Then M_K satisfies (8), as well as the Sum Formula:

$$\sum_{v \in M_K} v(\alpha) = 0 \quad \text{for every } \alpha \text{ in } K^*.$$

Let S be a finite subset of M_K of cardinality s. If $S = \emptyset$ (i.e. $s = 0$) then (8) implies that the S-units in K are just the elements of $\Bbbk^*$. In this case it follows from (9) that

$$\alpha_1' x_1 + \cdots + \alpha_n' x_n = 0 \text{ for all solutions of (9) in } S\text{-units } x_1, \ldots, x_n,$$

where $\alpha_1', \ldots, \alpha_n'$ denote the derivatives of $\alpha_1, \ldots, \alpha_n$, respectively, with respect to t. This shows that for $S = \emptyset$ all solutions of (9) are contained in a single proper subspace of K^n. Therefore, it suffices to prove Theorem 6 for non-empty S.

By $\mathbb{P}^n(K)$ we denote the n-dimensional projective space over K, that is the collection of lines of the vector space K^{n+1} through the origin; these lines in K^{n+1} will be called *projective points*. The projective point through $\mathbf{x} = (x_1, \ldots, x_n) \neq \mathbf{0}$ is denoted by $\langle \mathbf{x} \rangle$ or $(x_0 : \ldots : x_n)$, where $x_0, \ldots, x_n$ are called the homogeneous coordinates of the projective point; they are determined up to a common factor. $\mathbb{P}^n(\Bbbk)$ ($n \geqq 2$) denotes the set of lines in K^{n+1} through the points in $\Bbbk^{n+1}\backslash\{\mathbf{0}\}$. Thus

$$\mathbb{P}^n(\Bbbk) = \{(\lambda\xi_0 : \lambda\xi_1 : \ldots : \lambda\xi_n) : (\xi_0, \ldots, \xi_n) \in \Bbbk^{n+1}\backslash\{\mathbf{0}\}, \lambda \in K^*\}.$$

A projective subspace of $\mathbb{P}^n(K)$ is the collection of lines of a linear subspace of K^{n+1}.

For each $\mathbf{x} = (x_0, \ldots, x_n) \in K^{n+1}\backslash\{\mathbf{0}\}$ and $v \in M_K$ we put

$$v(\mathbf{x}) = \min (v(x_0), \ldots, v(x_n)).$$

Further, for each $\langle \mathbf{x} \rangle \in \mathbb{P}^n(K)$ we define the height $H(\langle \mathbf{x} \rangle)$ by

$$H(\langle \mathbf{x} \rangle) = -\sum_{v \in M_K} v(\mathbf{x}).$$

By the Sum Formula, $H(\langle \mathbf{x} \rangle)$ is well-defined, that is independent of the choice of the projective coordinates of $\langle \mathbf{x} \rangle$. It is easy to check that

$$\mathbb{P}^n(\Bbbk) = \{\langle \mathbf{x} \rangle \in \mathbb{P}^n(K) : H(\langle \mathbf{x} \rangle) = 0\},$$

and

$$H(\langle \mathbf{x} \rangle) \geqq 1 \quad \text{for every} \quad \langle \mathbf{x} \rangle \in \mathbb{P}^n(K)\backslash\mathbb{P}^n(\Bbbk).$$

Let $\mathscr{H}$ be the $(n-1)$-dimensional projective subspace of $\mathbb{P}^n(K)$ defined by

$$\mathscr{H} = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n(K) : x_0 + \cdots + x_n = 0\}.$$

For any $\mathbf{v} = (v_0, \ldots, v_n) \in (K^*)^{n+1}$ and any finite, non-empty subset S of M_K, let $\mathscr{S}(\mathbf{v})$ be the subset of $\mathscr{H}$ defined by

$$\mathscr{S}(\mathbf{v}) = \left\{ \langle \mathbf{x} \rangle = (x_0 : \ldots : x_n) \in \mathscr{H} : \langle \mathbf{x} \rangle \notin \mathbb{P}^n(\Bbbk), \right.$$
$$\left. v\left(\frac{x_0}{v_0}\right) = \cdots = v\left(\frac{x_n}{v_n}\right) \text{ for every } v \in M_K \backslash S \right\}.$$

If $\alpha_1 = -v_1/v_0, \ldots, \alpha_n = -v_n/v_0$ then there is a one-to-one relationship between the elements of $\mathscr{S}(\mathbf{v})$ and the solutions of the equation

$$\alpha_1 y_1 + \cdots + \alpha_n y_n = 1 \text{ in } S\text{-units } y_1, \ldots, y_n \text{ with } \alpha_1, y_1, \ldots, \alpha_n y_n \text{ not all in } \Bbbk. \tag{17}$$

Namely, a solution $(y_1, \ldots, y_n)$ of (17) corresponds to the projective point $(v_0 : v_1 y_1 : \ldots : v_n y_n)$ in $\mathscr{S}(\mathbf{v})$. In order to prove Theorem 6, it suffices to prove the following

THEOREM 6′. *Let K, $\Bbbk$ be as above, and let S be a finite, non-empty subset of M_K of cardinality s. Then for every $\mathbf{v} \in (K^*)^{n+1}$, the set of projective points in $\mathscr{S}(\mathbf{v})$ is contained in the union of at most*

$$\{\log (g + 2)\} \times \{e(n + 1)\}^{(n+1)s+2}$$

$(n - 2)$-dimensional projective subspaces of $\mathscr{H}$.

In the proof of Theorem 6′ several lemmas will be needed. The first is a combinatorial lemma.

LEMMA 3. *Let B be a real number with $0 < B < 1$, let $q \geqq 1$ be an integer, and put $R(B) = (1 - B)^{-1} B^{B/(B-1)}$. Then there exists a set $\mathscr{W}$ of cardinality at most* $\max(1, (2B)^{-1}) R(B)^{q-1}$, *consisting of tuples $(\Gamma_1, \ldots, \Gamma_q)$ with $\Gamma_j \geqq 0$ for $j = 1, \ldots, q$ and $\Sigma_{j=1}^q \Gamma_j = B$, with the following property: for every set* *of reals* $G_1, \ldots, G_q, M$ *with* $G_j \geqq 0$ *for* $j = 1, \ldots, q$ *and* $\Sigma_{j=1}^q G_j \geqq M$ *there exists a tuple* $(\Gamma_1, \ldots, \Gamma_q) \in \mathscr{W}$ *such that*

$$G_j \geqq \Gamma_j M \quad \textit{for} \quad j = 1, \ldots, q.$$

Proof. Put $F_j = e^{-G_j}$, $\Lambda = e^{-M}$. Then $0 < F_j \leqq 1$ for $j = 1, \ldots, q$ and $\Pi_{j=1}^q F_j \leqq \Lambda$. Further, $G_j \geqq \Gamma_j M$ for $j = 1, \ldots, q$ if and only if $F_j \leqq \Lambda^{\Gamma_j}$ for $j = 1, \ldots, q$. Now Lemma 3 follows at once from Lemma 4 of [3]. □

For each $\mathbf{v} = (v_0, \ldots, v_n) \in (K^*)^{n+1}$ and each finite, non-empty subset S of M_K, put

$$A_S(\mathbf{v}) = \sum_{v \in M_K \setminus S} \sum_{i=0}^{n} \{v(v_i) - v(\mathbf{v})\}.$$

The quotient $A_S(\mathbf{v})/H(\mathbf{v})$ can be considered as some kind of S-defect for $\mathbf{v}$, in the sense of Vojta's conjectures (cf. [14], Chapter 3). $A_S(\mathbf{v})$ measures, in some respect, how far $\mathbf{v}$ is away from a tuple of S-units. The quantity $A_S(\mathbf{v})$ will play an important rôle in the proof of Theorem 6′. In the sequel, we fix a tuple $\mathbf{v} = (v_0, \ldots, v_n) \in (K^*)^{n+1}$ and a finite, non-empty subset S of M_K of cardinality s. The next lemma states that projective points in $\mathscr{S}(\mathbf{v})$ cannot have large heights.

LEMMA 4. *For every* $\langle \mathbf{x} \rangle = (x_0 : \ldots : x_n) \in \mathscr{S}(\mathbf{v})$ *with non-vanishing subsums:*

$$\sum_{i \in J} x_i \neq 0 \textit{ for each proper, non-empty subset } J \textit{ of } \{0, \ldots, n\}, \tag{18}$$

we have $H(\langle x \rangle) \leqslant \frac{1}{2} n(n-1)\{2g + s + A_S(\mathbf{v})\}$.

Proof. If $v(v_0) = v(v_1) = \cdots = v(v_n)$ for all v in $M_K \setminus S$ then $A_S(\mathbf{v}) = 0$ and $v(x_0) = \cdots = v(x_n)$ for all $(x_0 : \ldots : x_n) \in \mathscr{S}(\mathbf{v})$ and v in $M_K \setminus S$. Further, we have $\langle \mathbf{x} \rangle = (1 : x_1/x_0 : \ldots : x_n/x_0)$ where the coordinates are all S-units. Hence Lemma 4 follows at once from Theorem B of Brownawell and Masser [1]. Now suppose that there are v in $M_K \setminus S$ for which $v(v_0), \ldots, v(v_n)$ are not all equal, and let S' be the set of these v. Let s' denote the cardinality of S'. For each v in S' we have, noting that its value group is $\mathbb{Z}$,

$$\sum_{i=0}^{n} \{v(v_i) - v(\mathbf{v})\} \geqq \max(v(v_0), \ldots, v(v_n)) - \min(v(v_0), \ldots, v(v_n))$$

$$\geqq 1.$$

Hence

$$A_s(\mathbf{v}) \geqq \sum_{v \in S'} \sum_{i=0}^{n} \{v(v_i) - v(\mathbf{v})\} \geqq s'.$$

Now Lemma 4 follows by applying Theorem B of Brownawell and Masser [1] in the same way as above, with $S \cup S'$ instead of S. □

The next lemma (a higher dimensional "gap principle") is our main tool to estimate the number of $(n-2)$-dimensional projective subspaces of $\mathscr{H}$ containing the projective points of $\mathscr{S}(\mathbf{v})$ with heights not exceeding some given bound.

LEMMA 5. *Let B be a real number with $n/(n+1) < B < 1$, and let $P > 0$. Then the set of projective points $\langle \mathbf{x} \rangle$ of $\mathscr{S}(\mathbf{v})$ with*

$$P \leqq H(\langle \mathbf{x} \rangle) < \frac{1-B}{n-1} A_S(\mathbf{v}) + \left\{1 + \frac{(n+1)B - n}{n-1}\right\} P \tag{19}$$

is contained in the union of at most $(n+1)^s\{e/(1-B)\}^{ns-1}$ $(n-2)$-dimensional projective subspaces of $\mathscr{H}$.

Proof. Let $\langle \mathbf{x} \rangle = (x_0 : x_1 : \ldots : x_n) \in \mathscr{S}(\mathbf{v})$. First we shall show that

$$\sum_{v \in S} \sum_{i=0}^{n} \{v(x_i) - v(\mathbf{x})\} = (n+1)H(\langle \mathbf{x} \rangle) - A_S(\mathbf{v}). \tag{20}$$

Indeed, by the Sum Formula and the definition of the height we have

$$\sum_{v \in S} \sum_{i=0}^{n} \{v(x_i) - v(\mathbf{x})\} = (n+1)H(\langle \mathbf{x} \rangle) - \sum_{v \in M_K \setminus S} \sum_{i=0}^{n} \{v(x_i) - v(\mathbf{x})\}. \tag{21}$$

Further, since $\langle \mathbf{x} \rangle \in \mathscr{S}(\mathbf{v})$ we have $v(x_0/v_0) = \cdots = v(x_n/v_n)$. Thus

$$\sum_{i=0}^{n} \{v(x_i) - v(\mathbf{x})\} = \sum_{i=0}^{n} \{v(v_i) - v(\mathbf{v})\} \text{ for all } v \text{ in } M_K \setminus S.$$

Together with (21) and the definition of $A_S(\mathbf{v})$ this implies (20). For $v \in S$, let $i_v \in \{0, \ldots, n\}$ be a fixed element such that $v(x_{i_v}) = v(\mathbf{x})$. Then (20) can be rewritten as

$$\sum_{v \in S} \sum_{i \neq i_v} \{v(x_i) - v(\mathbf{x})\} = (n+1)H(\langle\mathbf{x}\rangle) - A_S(\mathbf{v}) \tag{22}$$

where $i \neq i_v$ is used as an abbreviation for $i \in \{0, \ldots, n\} \setminus \{i_v\}$.

We now apply Lemma 3 to (22). We infer that there is a set $\mathscr{W}_0$ of tuples $(\Gamma_{i_v})_{v \in S, i \neq i_v}$ of cardinality at most $R(B)^{ns-1}$ with $\Gamma_{iv} \geqq 0$ for all $v \in S$, $i \neq i_v$ and $\Sigma_{v \in S} \Sigma_{i \neq i_v} \Gamma_{iv} = B$, such that there is a tuple $(\Gamma_{iv})_{v \in S, i \neq i_v} \in \mathscr{W}_0$ with

$$v(x_i) - v(\mathbf{x}) \geqq \Gamma_{iv}((n+1)H(\langle\mathbf{x}\rangle) - A_S(\mathbf{v})) \text{ for all } v \in S, i \neq i_v. \tag{23}$$

It is easy to check that $R(B) \leqq e/(1-B)$ for all B with $0 < B < 1$. Taking into consideration that there are at most $(n+1)^s$ possibilities for the tuple $(i_v)_{v \in S}$, we infer that each $\langle\mathbf{x}\rangle = (x_0 : \ldots : x_n) \in \mathscr{S}(\mathbf{v})$ satisfies at least one of at most $(n+1)^s(e/(1-B))^{ns-1}$ systems of inequalities of type (23). Hence it suffices to prove that the set of projective points $\langle\mathbf{x}\rangle = (x_0 : \ldots : x_n)$ in $\mathscr{S}(\mathbf{v})$ satisfying both (19) and (23) for fixed tuples $(i_v)_{v \in S}$, $(\Gamma_{iv})_{v \in S, i \neq i_v}$ is contained in a single $(n-2)$-dimensional projective subspace of $\mathscr{H}$.

Suppose that the projective points in $\mathscr{S}(\mathbf{v})$ satisfying both (19) and (23) for some fixed tuples $(i_v)_{v \in S}$, $(\Gamma_{iv})_{v \in S, i \neq i_v}$ are not contained in a single $(n-2)$-dimensional projective subspace of $\mathscr{H}$. Then there are $\langle\mathbf{x}_1\rangle, \ldots, \langle\mathbf{x}_n\rangle$ in $\mathscr{S}(\mathbf{v})$ satisfying (19) and (23) such that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent over K. Suppose that $\mathbf{x}_i = (x_{i0}, \ldots, x_{in})$ for $i = 1, \ldots, n$, and that $H(\langle\mathbf{x}_1\rangle) \leqq H(\langle\mathbf{x}_2\rangle) \leqq \cdots \leqslant H(\langle\mathbf{x}_n\rangle)$. For each v in M_K, let

$$\Delta_v = v\left(\begin{vmatrix} x_{1,\sigma(1)} & x_{1,\sigma(2)} & \cdots & x_{1,\sigma(n)} \\ \vdots & & & \\ x_{n,\sigma(1)} & x_{n,\sigma(2)} & \cdots & x_{n,\sigma(n)} \end{vmatrix}\right) - \sum_{i=1}^{n} v(\mathbf{x}_i)$$

where $|A|$ denotes the determinant of matrix A, and σ is some permutation of $(0, \ldots, n)$. (σ also acts on 0). Since $\langle\mathbf{x}_1\rangle, \ldots, \langle\mathbf{x}_n\rangle \in \mathscr{H}$, Δ_v is independent of σ. Further, Δ_v is independent of the choice of the homogeneous coordinates of the projective points $\langle\mathbf{x}_1\rangle, \ldots, \langle\mathbf{x}_n\rangle$, respectively. Since $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent over K, the determinant in the definition of Δ_v is non-zero. Hence the Sum Formula implies that

$$\sum_{v \in M_K} \Delta_v = \sum_{i=1}^{n} H(\langle\mathbf{x}_i\rangle). \tag{24}$$

We now estimate Δ_v from below. First take $v \in S$, and let σ be a permutation of $(0, \ldots, n)$ with $\sigma(0) = i_v$. The determinant in the definition of Δ_v has $n!$ terms, each being of the type

$$\pm x_{\tau(1),\sigma(1)} x_{\tau(2),\sigma(2)} \cdots x_{\tau(n),\sigma(n)},$$

where τ is some permutation of $(1, \ldots, n)$. For one of the permutations τ we have

$$\Delta_v \geqq v(x_{\tau(1),\sigma(1)} \cdots x_{\tau(n),\sigma(n)}) - \sum_{i=1}^{n} v(\mathbf{x}_i) = \sum_{i=1}^{n} \{v(x_{\tau(i),\sigma(i)}) - v(\mathbf{x}_{\tau(i)})\}. \tag{25}$$

But by (23) and $H(\mathbf{x}_{\tau(i)}\rangle) \geqq H(\langle \mathbf{x}_1 \rangle)$, we have

$$v(x_{\tau(i),\sigma(i)}) - v(x_{\tau(i)}) \geqq \Gamma_{\sigma(i),v}((n+1)H(\langle \mathbf{x}_{\tau(i)} \rangle) - A_S(\mathbf{v}))$$

$$\geqq \Gamma_{\sigma(i),v}((n+1)H(\langle \mathbf{x}_1 \rangle) - A_S(\mathbf{v})) \quad \text{for} \quad i = 1, \ldots, n.$$

Together with (25) this implies that for each v in S,

$$\Delta_v \geqq \left(\sum_{i=1}^{n} \Gamma_{\sigma(i),v}\right)((n+1)H(\langle \mathbf{x}_1 \rangle) - A_S(\mathbf{v}))$$

$$= \left(\sum_{i \neq i_v} \Gamma_{iv}\right)((n+1)H(\langle \mathbf{x}_1 \rangle) - A_S(\mathbf{v})).$$

Hence

$$\sum_{v \in S} \Delta_v \geqq B((n+1)H(\langle \mathbf{x}_1 \rangle) - A_S(\mathbf{v})).$$

Now take $v \in M_K \setminus S$. Let σ be a permutation of $(0, \ldots, n)$ such that $v(v_{\sigma(0)}) = v(\mathbf{v}) = \min(v(v_0), \ldots, v(v_n))$. Again, there is a permutation τ of $(1, \ldots, n)$ such that (25) holds:

$$\Delta_v \geqq \sum_{i=1}^{n} \{v(x_{\tau(i),\sigma(i)}) - v(\mathbf{x}_{\tau(i)})\}.$$

But since $v(x_{i0}/v_0) = \cdots = v(x_{in}/v_n)$ for $i = 1, \ldots, n$, we have

$$v(x_{\tau(i),\sigma(i)}) - v(\mathbf{x}_{\tau(i)}) = v(v_{\sigma(i)}) - v(\mathbf{v}) \quad \text{for} \quad i = 0, \ldots, n,$$

and $v(v_{\sigma(0)}) - v(v) = 0$, so that

$$\Delta_v \geqq \sum_{i=0}^{n} \{v(v_{\sigma(i)}) - v(\mathbf{v})\} = \sum_{i=0}^{n} \{v(v_i) - v(\mathbf{v})\} \quad \text{for} \quad v \in M_K \setminus S.$$

This implies that

$$\sum_{v \in M_K \setminus S} \Delta_v \geqq A_S(\mathbf{v}).$$

By combining this inequality with (24) and (25) we infer that

$$\sum_{i=1}^{n} H(\langle \mathbf{x}_i \rangle) \geqq B((n+1)H(\langle \mathbf{x}_1 \rangle) - A_S(\mathbf{v})) + A_S(\mathbf{v}),$$

whence

$$(n-1)H(\langle \mathbf{x}_n \rangle) \geqq \sum_{i=2}^{n} H(\langle \mathbf{x}_i \rangle)$$
$$\geqq \{B(n+1) - 1\}H(\langle \mathbf{x}_1 \rangle) + (1 - B)A_S(\mathbf{v}).$$

Since by assumption $H(\langle \mathbf{x}_1 \rangle) \geqq P$, we obtain

$$H(\langle \mathbf{x}_n \rangle) \geqq \frac{1-B}{n-1} A_S(\mathbf{v}) + \left\{1 + \frac{B(n+1) - n}{n-1}\right\} P$$

which contradicts the fact that $\langle \mathbf{x}_n \rangle$ satisfies (19). Thus our assumption that the projective points in $\mathscr{S}(\mathbf{v})$ satisfying both (19) and (23) do not belong to a single $(n-2)$-dimensional projective subspace of $\mathscr{H}$ leads to a contradiction. This completes the proof of Lemma 5. □

Remark. Let K be an algebraic number field. Then it is possible to choose a set of multiplicative valuations $\{|\ |\}_{v \in M_K}$ on K which are normalized in the usual way and which satisfy the Product Formula $\Pi_{v \in M_K} |\alpha|_v = 1$. For each v in M_K, put $v(\alpha) = \log |\alpha|_v$ for $\alpha \in K^*$, $v(0) = \infty$. Let S be a finite subset of M_K of cardinality s containing all archimedean valuations. Define the height H and the quantity $A_S(\mathbf{v})$, the set $\mathscr{S}(\mathbf{v})$, and the projective space $\mathscr{H}$ in the same way as for function fields. Then one can derive the following analogue of Lemma 5 for algebraic number fields, in essentially the same way as for function fields.

LEMMA 6. *Let B be a real number with $n/(n+1) < B < 1$, and let $P > 0$. Then the set of projective points $\langle \mathbf{x} \rangle$ of $\mathscr{S}(\mathbf{v})$ with*

$$P \leqq H(\langle \mathbf{x} \rangle) < \frac{1-B}{n-1} A_S(\mathbf{v}) - \frac{1}{n-1} \log n! + \left\{ 1 + \frac{(n+1)B - n}{n-1} \right\} P$$

is contained in the union of at most $(n+1)^s (e/(1-B))^{ns-1}$ $(n-2)$-dimensional projective subspaces of $\mathscr{H}$.

In the algebraic number field case, two other "higher dimensional gap principles" have been established recently by J.H. Silverman ([12], Thm. 8) and W.M. Schmidt (announced at the Number Theory Conference in Budapest, 20–25 July, 1987).

Proof of Theorem 6′. The projective points $(x_0 : x_1 : \ldots : x_n)$ in $\mathscr{S}(\mathbf{v})$ for which $\Sigma_{j \in J} x_j = 0$ for some proper, non-empty subset J of $\{0, \ldots, n\}$ obviously belong to at most 2^{n+1} $(n-2)$-dimensional projective subspaces of $\mathscr{H}$. In the sequel we consider the remaining solutions, i.e. those with

$$\sum_{j \in J} x_j \neq 0 \text{ for each proper, non-empty subset } J \text{ of } \{0, \ldots, n\}. \tag{18}$$

Let B be a real with $n/(n+1) < B < 1$ which will be specified later. It follows by repeated application of Lemma 5 that for each integer $k > 0$ and each Q with

$$Q > \frac{1-B}{(n+1)B-n} A_S(\mathbf{v}),$$

the set of projective points $\langle \mathbf{x} \rangle$ of $\mathscr{S}(\mathbf{v})$ with

$$Q - \frac{1-B}{(n+1)B-n} A_S(\mathbf{v}) \leqq H(\langle \mathbf{x} \rangle)$$

$$< \left\{ 1 + \frac{(n+1)B-n}{n-1} \right\}^k Q - \frac{1-B}{(n+1)B-n} A_S(\mathbf{v}) \tag{26}$$

is contained in the union of at most

$$k \times (n+1)^s \left(\frac{e}{1-B} \right)^{ns-1}$$

$(n-2)$-dimensional subspaces of $\mathscr{H}$. Together with Lemma 4 and the fact that each $\langle \mathbf{x} \rangle$ in $\mathbb{P}^n(K) \setminus \mathbb{P}^n(\Bbbk)$ has height $\geqq 1$ we obtain the following:

Let

$$Q = 1 + \frac{1-B}{(n+1)B-n} A_S(\mathbf{v})$$

and let k be the smallest integer such that

$$\left\{1 + \frac{(n+1)B-n}{n-1}\right\}^k \left(1 + \frac{1-B}{(n+1)B-n} A_S(\mathbf{v})\right)$$

$$> \tfrac{1}{2}n(n-1)(2g+s+A_S(\mathbf{v})) + \frac{1-B}{(n+1)B-n} A_S(\mathbf{v}). \tag{27}$$

Then the set of projective points of $\mathscr{S}(\mathbf{v})$ is contained in the union of at most

$$2^{n+1} + k \times (n+1)^s \left(\frac{e}{1-B}\right)^{ns-1} \tag{28}$$

$(n-2)$-dimensional projective subspaces of $\mathscr{H}$. (Here we included also the solutions with certain vanishing subsums). We choose

$$B := 1 - \frac{1}{(n+1)\left(1 + \frac{1}{ns}\right)}.$$

Then $[(n+1)B-n]/(n-1) = 1/[(ns+1)(n-1)]$, whence $n/(n+1) < B < 1$ and

$$\log\left(1 + \frac{(n+1)B-n}{n-1}\right) > \frac{1}{2(ns+1)(n-1)}. \tag{29}$$

Further, since $s \geqq 1$,

$$\frac{\tfrac{1}{2}n(n-1)(2g+s+A_S(\mathbf{v})) + \{[1-B]/[(n+1)B-n]\}A_S(\mathbf{v})}{1 + \{[1-B]/[(n+1)B-n]\}A_S(\mathbf{v})}$$

$$= \frac{\tfrac{1}{2}n(n-1)(2g+s+A_S(\mathbf{v})) + (ns/(n+1))A_S(\mathbf{v})}{1 + (ns/(n+1))A_S(\mathbf{v})}$$

$$\leqq \tfrac{1}{2}(n^2+1)(2g+s).$$

In view of (27) and (29) this implies that

$$k \leqq 2(n - 1)(ns + 1) \log (\tfrac{1}{2}(n^2 + 1)(2g + s)) + 1.$$

Now a straightforward computation shows that the number in (28) is bounded above by

$$2^{n+1} + [2(n - 1)(ns + 1) \log \{\tfrac{1}{2}(n^2 + 1)(2g + s)\} + 1](n + 1)^s \times$$
$$\times \left\{e(n + 1)\left(1 + \frac{1}{ns}\right)\right\}^{ns}$$
$$\leqq \frac{3e(n - 1)ns \log \{\frac{1}{2}(n^2 + 1)(2g + s)\}}{e^{s+2}(n + 1)^3} \times \{e(n + 1)\}^{(n+1)s+2}$$
$$\leqq \{\log (g + 2)\} \times \{e(n + 1)\}^{(n+1)s+2}.$$

This completes the proof of Theorem 6′. □

Appendix

In this Appendix we shall prove Lemma 1 of §3 in the general case, supposing that this Lemma is true in the algebraic number field case. Let Γ be a finitely generated multiplicative subgroup of the group of non-zero elements of a field of characteristic 0, and let K be the smallest subfield containing Γ. Then K is finitely generated over $\mathbb{Q}$ and has finite transcendence degree over $\mathbb{Q}$, q say. By the equivalence of Theorem 1 and Theorem 3 shown in §4, it suffices to prove that the set of solutions of the equation

$$x_1 + \cdots + x_n = 1 \quad \text{in} \quad x_1, \ldots, x_n \in \Gamma \tag{A}$$

is contained in the union of at most finetely many $(n - 1)$-dimensional linear subspaces of K^n. We shall prove this assertion by induction on the transcendence degree q.

If $q = 0$, then our assertion follows from Lemma 1 in the algebraic number field case (cf. [2], [11]) and from the equivalence of Theorem 1 and Theorem 3. Suppose that our assertion holds for all $q < p$, where $p \geqq 1$. Assume that K has transcendence degree p. There is a subfield K' in K for which K/K' has transcendence degree 1. Denote by $\Bbbk$ the algebraic closure of K', and by L the composite of the fields K and $\Bbbk$ (in a fixed algebraic closure of K). Then L is a function field with constant field $\Bbbk$. By the induction hypothesis, the solutions $(x_1, \ldots, x_n)$ of (A) with $x_1, \ldots, x_n \in \Gamma \cap \Bbbk^*$ are contained in only finitely many $(n-1)$-dimensional linear subspaces of K^n. Further, by Theorem 6, noting that Γ can be contained in the group of S-units for some finite set of valuations S on L, the solutions $(x_1, \ldots, x_n)$ of (A) for which at least one of $x_1, \ldots, x_n$ does not belong to $\Gamma \cap \Bbbk^*$ are contained in finitely many $(n-1)$-dimensional linear subspaces of L^n. But this implies that the solutions in question belong to finitely many $(n-1)$-dimensional linear subspaces of K^n. Thus our assertion is proved for $q = p$. □

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