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K-theory, λ-rings, and formal groups

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Introduction

In [C] the author showed how one can compute the algebraic K-group $K_2(R, I)$ when R has a structure of λ-ring; this relied on a presentation of that group found by H. Maazen and J. Stienstra [Ms] and F. Keune [K]. The purpose of this paper is to show that the same technique works for a more general group $K_{2,F}(R, I)$ in which also a formal group F is involved. An instance of this more general group occurs in §17 of [S], which inspired the present paper.

DEFINITION 0.1. Let F be a one-dimensional formal group over $\mathbf{Z}$. Let R be a commutative ring and let I be a nilpotent ideal. Then $K_{2,F}(R, I)$ is the abelian group with as generators the symbols $\langle a, b\rangle_F$ for $(a, b) \in I \times R \cup R \times I$ and relations

$$\langle a, 1\rangle_F \qquad \text{for } a \in I$$

$$\langle a, b\rangle_F + \langle a, c\rangle_F - \langle a, a^{-1}F(ab, ac)\rangle_F \qquad \text{for } (a, b, c) \in I \times R \times R \cup R \times I \times I$$

$$\langle a, bc\rangle_F + \langle b, ac\rangle_F - \langle ab, c\rangle_F \qquad \text{for } (a, b, c) \in I \times R \times R \cup R \times I \times R \cup R \times R \times I$$

where $a^{-1}F(ab, ac)$ has to be interpreted in the obvious way.

If F is the multiplicative formal group defined by $F(X, Y) = X + Y - XY$ then $K_{2,F}(R, I)$ coincides with the group $K_2(R, I)$ from algebraic K-theory. If F is the additive formal group defined by $F(X, Y) = X + Y$ then $K_{2,F}(R, I)$ coincides with the group $K_{2,L}(R, I)$ considered in [C] and [LQ]. As explained there it is a cyclic homology group of (R, I); it is isomorphic to $\Omega_{R,I}/\delta I$ if the natural projection $R \to R/I$ splits.

To formulate the main theorem we introduce the notation $G(R, I)^{\text{top}}$ for the inverse limit $\lim G(R/J^N, (I + J^N)/J^N)$ if G is a functor from pairs (R, I) as above to abelian groups and J is an ideal describing a topology on R.

THEOREM 0.2. *Let F be a one dimensional formal group over $\mathbf{Z}$ which is strongly isomorphic to a special one. Let R be a λ-ring and let I and J be ideals such that (R, J, I) is admissible. Then there is a homomorphism L_F: $K_{2,F}(R, I)^{\text{top}} \to K_{2,L}(R, I)^{\text{top}}$ such that*

$$L_F\langle a, b\rangle_F = \langle a, b\rangle_L + \textit{higher order terms}.$$

The notion of special formal group and of strong isomorphism are defined in the next section. For the definition of admissible and other notions connected with λ-rings we refer to [C].

§1. Generalities about formal groups

Let A be a commutative ring. A one-dimensional commutative formal group F over A is a formal power series $F(X, Y) = \Sigma f_{ij} X^i Y^j$ with $f_{ij} \in A$ such that

$$\begin{aligned} F(X, Y) &= X + Y + \textit{terms of degree} > 1 \\ F(X, Y) &= F(Y, X), \\ F(X, F(Y, Z)) &= F(F(X, Y), Z). \end{aligned}$$

By substitution into F one can define a structure of abelian group on any topologically nilpotent ideal in a complete commutative A-algebra. In particular this applies to the formal power series over A with vanishing constant term. In this paper we use the words "formal group" in the understanding that we always mean a one-dimensional commutative one.

If F and $\tilde{F}$ are both formal groups then an isomorphism G from $\tilde{F}$ to F means a formal power series $G(X) = \Sigma\, g_n X^n$ with $g_n \in A$ such that

$$\begin{aligned} G(X) &= X + \textit{terms of degree} > 1, \\ G\tilde{F}(X, Y) &= F(G(X), G(Y)). \end{aligned}$$

In the case that $A = \mathbf{Z}$ one can find integers γ_n such that G is an infinite sum of terms X^n, each occurring γ_n times; the sum being taken in the F sense.

DEFINITION 1.1. The isomorphism G is called a strong isomorphism if n divides γ_n for every n.

PROPOSITION 1.2. *Let $\tilde{F}$ and F be formal groups over $\mathbf{Z}$, and let G: $\tilde{F} \to F$ be a strong isomorphism. Then G induces an equivalence G_*: $K_{2,\tilde{F}} \to K_{2,F}$ such that*

$$G_* \langle a, b\rangle_{\tilde{F}} = \langle a, a^{-1}G(ab)\rangle_F$$

Proof. We show that the three relations are satisfied.

1) $G_* \langle a, 1\rangle_{\tilde{F}} = \langle a, a^{-1}G(a)\rangle_F = \Sigma\, \gamma_n\langle a, a^{-1}a^n\rangle_F = \Sigma\, n^{-1}\gamma_n\langle a^n, 1\rangle_F = 0$ since it follows from the third relation that $\langle a^n, 1\rangle_F = n\langle a, a^{n-1}\rangle_F$.

2) $G_* \langle a, b\rangle_{\tilde{F}} + G_* \langle a, c\rangle_{\tilde{F}} = \langle a, a^{-1}G(ab)\rangle_F + \langle a, a^{-1}G(ac)\rangle_F = \langle a, a^{-1}F(G(ab), G(ac))\rangle_F = G_* \langle a, a^{-1}G^{-1}F(G(ab), G(ac))\rangle_{\tilde{F}} = G_* \langle a, a^{-1}\tilde{F}(ab, ac)\rangle_{\tilde{F}}$

3) $G_* \langle a, bc\rangle_{\tilde{F}} + G_* \langle b, ac\rangle_{\tilde{F}} = \langle a, a^{-1}G(abc)\rangle_F + \langle b, b^{-1}G(abc)\rangle_F = \langle ab, a^{-1}b^{-1}G(abc)\rangle_F = G_* \langle ab, c\rangle_{\tilde{F}}$ ●

For any formal group over $\mathbf{Q}$ there is an isomorphism f from F to the additive formal group; this is called the logarithm of the formal group. Let p be a prime number and let F be a formal group over $\mathbf{Q}$ with logarithm $F(X) = \Sigma\, f_n X^n$; then the formal group with logarithm $\Sigma\, f_{p^e} X^{p^e}$ is called the p-typical formal group associated to F. If F is defined over $\mathbf{Z}$ then its p-typification is (for these facts see [H]).

If $f = \Sigma\, f_n X^n$ is a logarithm for F and $\tilde{f} = \Sigma\, \tilde{f}_n X^n$ is one for $\tilde{F}$ and if G is an isomorphism as above then we have $\tilde{f} = fG = \Sigma\, \gamma_n f(X^n)$. Writing this out yields $\tilde{f}_n = \Sigma_{mk=n}\, \gamma_m f_k$ hence $(\Sigma\, \tilde{f}_n n^s) = (\Sigma\, \gamma_m m^s)(\Sigma\, f_k k^s)$ in the language of formal Dirichlet series introduced in [C]. So if F and $\tilde{F}$ are strongly isomorphic then their p-typifications are.

DEFINITION 1.3. A formal group F is called special if its logarithm $f(X) = \Sigma\, f_n X^n$ satisfies $f_{mk} = f_m f_k$ for every m, k.

So the p-typification of a special formal group is again special. Therefore the p-typification of a formal group satisfying the conditions of Theorem 0.2 again satisfies those conditions.

§2. The λ-operations associated to F

In [C] we introduced certain λ-operations λ^n, θ^n and η^n. In this § we introduce F-twisted versions λ_F^n, θ_F^n and η_F^n of these operations.

Recall that the element a of a λ-ring is called one-dimensional if $\lambda^n(a) = 0$ for $n > 1$.

Proposition 2.1. *Let F be a formal group over $\mathbf{Z}$. There exists a unique sequence $\{\lambda_F^n\}$ of λ-operations such that $\lambda_t^F = \Sigma_{n=1}^{\infty} t^n \lambda_F^n$ satisfies*
1) $\lambda_t^F(a + b) = F(\lambda_t^F(a), \lambda_t^F(b))$
2) $\lambda_t^F(a) = ta$ if a is one-dimensional.

Proof. Recall that the universal λ-ring U can be embedded in the inverse limit of polynomial rings $\mathbf{Z}[s_1, s_2, \ldots, s_n]$ so that the canonical element $u \in U$ corresponds to $\Sigma_{n=1}^{\infty} s_n$. Now $F(ts_1, ts_2, \ldots)$ is the required element λ_t^F. ●

Proposition 2.2. *Let F be a formal group over $\mathbf{Q}$ with logarithm f. Let R be a λ-ring containing $\mathbf{Q}$, and let $a \in R$. Then*

$$f(\lambda_t^F(a)) = \sum_{n=1}^{\infty} f_n t^n \psi^n(a).$$

Proof. From the definition of logarithm and the first property of λ_t^F it follows that $f\lambda_t^F$ is additive. Moreover the statement is true on one-dimensional elements. Therefore it is true on $\mathbf{Q}[s_1, s_2, \ldots]$ and thus on $U \otimes \mathbf{Q}$. ●

Now we define the F-twisted version of the operations η^n in §3 of [C].

Lemma 2.3. *If R is a λ-ring and $a \in R$ is not a zero-divisor then*

$$\lambda_{ta}^F = \sum_{n=1}^{\infty} t^n a^n \lambda_F^n\colon R[[t]] \to taR[[t]] \text{ is a bijection.}$$

Proof. This follows easily from the first property of λ_t^F together with the fact that

$$\lambda_F^n(t^m R[[t]]) \subseteq t^{nm} R[[t]].$$ ●

Definition 2.4. Let F be a formal group over $\mathbf{Z}$. The λ-operations η_F^n in two variables are defined by the condition that

$$\lambda_{ta}^F\left(\sum_{n=1}^{\infty} t^{n-1}\eta_F^n(a, b)\right) = tab.$$

We write $\theta_F^n(b)$ for $\eta_F^n(1, b)$.

By applying f on both sides in this definition and using Proposition 2.3 we get the following generalisation of Proposition 3.3 of [C]:

OBSERVATION 2.5. For elements a, b in a λ-ring R containing $\boldsymbol{Q}$ one has

$$f_n a^{n-1} b^n = \sum_{m|n} f_m a^{m-1} \psi^m (\eta_F^{n/m}(a, b)).$$

From this it is clear that $\eta_F^n(a, b)$ is of degree $n - 1$ in a and of degree n in b. Therefore we can rewrite the formula 2.4 as

$$\lambda_{ta}^F(\eta_F(ta, b)) = tab \quad \text{where} \quad \eta_F(ta, b) = \sum_{n=1}^{\infty} \eta_F^n(ta, b).$$

PROPOSITION 2.6. *If F is a formal group over Z and a, b, c are elements of a λ-ring then*

$$\eta_F(ta, b) + \eta_F(ta, c) = \eta_F(ta, (ta)^{-1} F(tab, tac)).$$

Proof. This follows from Lemma 2.3 since both sides have the same image under λ_{ta}^F by Definition 2.4 and the first property of λ^F. ●

Using the formalism of formal Dirichlet series introduced in §8 of [C] we can rewrite some of the foregoing identities as relations between such series.

DEFINITION 2.7. Let F be a formal group over $\boldsymbol{Z}$. Then $Y_F \in DS(U)$ and $H_F \in DS(U_2)$ are defined by the formulas

$$Y_F(a) = \sum n f_n a^{n-1} n^s, \; H_F(a, b) = \sum \eta_F^n(a, b) n^s$$

Now Observation 2.5 can be reformulated as $bY_F(ab) = Y_F(a) \cdot TH_F(a, b)$. The formula at the end of §1 can be rewritten as $Y_{\bar{F}}(1) = T\Gamma \cdot Y_F(1)$. Henceforth we write Y_F for $Y_F(1)$. ●

Since we have taken a generalisation of Proposition 3.8 of [C] as definition of the η_F^n, the generalisation of Definition 3.1 of [C] becomes a proposition:

PROPOSITION 2.8. *If F is a formal group over $\boldsymbol{Z}$ and a, b are elements of a λ-ring then*

$$\eta_F^n(a, b) = \sum_m \eta_F^m(a, a^{(n-m)/m}) \psi^m \theta_F^{n/m}(b) \quad \text{for } n > 1.$$

Here the sum extends over all $m \neq n$ dividing n.

Proof. Property 2.5 determines the operations η_F^n uniquely: we may assume that a, b are the canonical elements in U_2, and the right hand side is of the form $\eta_F^n(a, b)$ + *terms involving* $\eta_F^k(a, b)$ *with* $k < n$.

Therefore it is sufficient to show that the right hand side of the above statement satisfies the same identity. This follows easily by rearranging sums and using the induction hypothesis. ●

COROLLARY 2.9. *The operations η_F^n are elements of the subring V_2 of the ring U_2 of λ-operations in two variables as defined in §2 in [C].*

Therefore the theory of §5 of [C] tells us the following. If (R, J, I) is admissible then $\eta_F(a, b) = \Sigma_{n=1}^{\infty} \eta_F^n(a, b)$ converges in the J-topology for $a \in I$ or $b \in I$. Moreover Proposition 2.6 implies that in that situation one has

$$\eta_F(a, b) + \eta_F(a, c) = \eta_F(a, a^{-1}F(ab, ac)).$$

§3. The relation between η_F and η

In this § the operations η_F^n are expressed in terms of the η^m.

DEFINITION 3.1. Let F be a formal group over $\mathbf{Q}$ with logarithm f. Then the λ-operations $C_F^{n,m} \in U \otimes \mathbf{Q}$ are defined by

$$\sum_{k|m} f_k a^{k-1} \psi^k (C_F^{n,m/k}(a)) = n f_{nm} a^{m-1}$$

In particular $C_F^{n,1}(a) = nf_n$; and if F is special then $C_F^{n,m} = 0$ for $m > 1$.

PROPOSITION 3.2. *Let F be a formal group over $\mathbf{Q}$ and let a, b be elements of a λ-ring containing $\mathbf{Q}$. Then*

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a) \psi^m (\eta^{n/m}(a, b)).$$

Proof. Again it is sufficient to show that the right hand side satisfies identity 2.5. This follows easily by rearranging the sum and applying Definition 3.1 and Proposition 2.5 for the η^m. ●

We now show that in fact $C_F^{n,m} \in U$ if F is a formal group over $\mathbf{Z}$.

LEMMA 3.3

a) *Suppose that* $\xi^m \in U \otimes \mathbf{Q}$ *and that the operation* $(a, b) \to \Sigma_{m|n}\, \xi^m(a)\psi^m(\theta^{n/m}(b))$ *is element of* U_2*; then in fact* $\xi^m \in U$.
b) *Suppose that* $\xi \in U \otimes \mathbf{Q}$ *and that the operation* $a \to \xi(a)\psi^k(a')$ *is in* U. *Then in fact* $\xi \in U$.

Proof

a) If the operations ξ^m are integral for $m < M$ then we may suppose that they vanish in that range. Now apply the above operation to the ring $U[t_1, t_2]/(t_1^{M+1})$ and take for a the canonical element $u \in U$ and $b = t_1 + t_2$ *or* t_2 respectively, and take the difference of the results. Then the mth term vanishes also for $m > M$, and the Mth term is

$$\xi^M(u)\{\theta^{n/M}(t_1^M + t_2^M) - \theta^{n/M}(t_2^M)\} = \xi^M(u)t_1^M t_2^{n-M}.$$

If this is integral then $\xi^M(u)$ must be.
b) The universal λ-ring U is the polynomial ring over $\mathbf{Z}$ freely generated by the $\lambda^n(u)$. Therefore a product of two elements can only be a multiple of an integer > 1 if one of the factors is. But the element of U corresponding to the operation $a \to \psi^k(a')$ is not divisible by any integer > 1 as is easily seen by applying this operation on the ring $\mathbf{Z}[t]$ and the element t. ●

PROPOSITION 3.4. *Let F be a formal group over* $\mathbf{Z}$. *Then the operations* $C_F^{n,m}$ *are in* U_2.

Proof. By substituting Definition 3.1 of [C] into Proposition 3.2 and rearranging terms we get

$$\eta_F^n(a, b) = \sum_{m|n} \xi^m(a)\psi^m(\theta^{n/m}(b))$$

where

$$\xi^m(a) = \begin{cases} \sum_{k|m} C_F^{n/k,k}(a)\psi^k(\eta^{m/k}(a, a^{n-m/m}) & \text{for } m < n \\ C_F^{1,m} & \text{for } m = n \end{cases}$$

So by Lemma 3.3a these operations are integral. From this it follows by induction on n and Lemma 3.3b that the operations $C_F^{n,m}$ are integral ●

COROLLARY 3.5. *Let F be a formal group over* $\mathbf{Z}$ *and let a, b be elements of a λ-ring. Then again*

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a)\psi^m(\eta^{n/m}(a, b)).$$

REMARK 3.6. Proposition 3.4 implies that the numbers $C_F^{n,m}(1) \in \mathbf{Q}$ are integers. We will abbreviate these to $C_F^{n,m}$. They are determined by the formula $\Sigma_{k|m} f_k C_F^{n,m/k} = nf_{nm}$ which is exactly the formula in [D] expressing that nf_n is a "lexoid function". The meaning of these numbers for us becomes clear by putting $a = 1$ in Proposition 3.2; that yields the formula

$$\theta_F^n(b) = \sum_{m|n} C_F^{n/m,m} \psi^m(\theta^{n/m}(b)).$$

§4. Some identities for the numbers C_F

In this § the prime p is fixed. First we prove some general relations between the numbers $C_F^{p^i,p^j}$; then we use these to draw some consequences for these numbers from the hypothesis of Theorem 0.2.

LEMMA 4.1 (see [D]). *Let F be a formal group over $\mathbf{Q}$. Then*

$$C_F^{p^i,p^j} = pC_F^{p^{i-1},p^{j+1}} + C_F^{p^{i-1},1} C_F^{p,p^j}$$

for every (i, j) with $i > 0$.

Proof. Let $f(X) = \Sigma_n f_n X^n$ be the logarithm of F. If we substitute the identities

$$C_F^{p^i,p^k} = pC_F^{p^{i-1},p^{k+1}} + C_F^{p^{i-1},1} C_F^{p,p^k} \text{ for } 0 \leqslant k \leqslant j - 1$$

into the identity

$$\sum_{k=0}^{j} C_F^{p^i,p^k} f_{p^{j-k}} = p^i f_{p^{i-j}} = pC_F^{p^{i-1},1} f_{p^{j+1}} + p \sum_{k=0}^{j} C_F^{p^{i-1},p^{k+1}} f_{p^{j-k}}$$

then we get

$$C_F^{p^i,p^j} + C_F^{p^{i-1},1} \sum_{k=0}^{j-1} C_F^{p,p^k} f_{p^{j-k}} = pC_F^{p^{i-1},p^{j+1}} + pC_F^{p^{i-1},1} f_{p^{j+1}}.$$

Now the statement follows by comparing this with the identity $pf_{p^{j+1}} = \Sigma_{k=0}^{j} C_F^{p,p^k} f_{p^{j-k}}$ ●

LEMMA 4.2. *Let F be a formal group over $\mathbf{Q}$. Then*

$$C_F^{p^i,p^j} = \sum_{k=0}^{i-1} p^k C_F^{p,p^{j+k}} C_F^{p^{i-k-1},1} \text{ for every } (i,j).$$

Proof. Immediate from Lemma 4.1 by induction on i. ●

LEMMA 4.3. *Let F be a formal group over $\mathbf{Q}$ and let p be prime. Then for every (i, j, k) with $k \leqslant i$ one has*

$$C_F^{p^i,p^j} - \sum_{m=0}^{k} p^m C_F^{p^{i-k},p^m} C_F^{p^{k-m},p^j} = \begin{cases} p^k C_F^{p^{i-k},p^{k+j}} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

Proof. We use induction on k; for $k = 0$ the statement is empty. For $k > 0$ we apply Lemma 4.1 for (i, j), the induction hypothesis for $(i - 1, 0, k - 1)$, Lemma 4.1 for $(k - m, j)$ and the induction hypothesis for $(i - 1, j + 1, k - 1)$. ●

Now we deduce from the condition on F in theorem 0.2. a relation for the integers $C_F^{n,m}$. First we recall a few notations from §9 of [C].

DEFINITION 4.4. If $\Xi = \Sigma\, \xi_n n^s$ is a Dirichlet series then we write

$$\Xi_P = \sum \xi_{p^e}(p^e)^s, \quad T_P\Xi = \sum p^{v_p(n)} \xi_n n^s.$$

PROPOSITION 4.5. *Let F be a p-typical formal group over $\mathbf{Z}$ strongly isomorphic to a special one. Then p^h divides $C_F^{p^i,p^h}$ for every h.*

Proof. By Remark 3.6 one has $p^i f_{p^{e+i}} = \Sigma_{d=0}^{e} C_F^{p^i,p^{e-d}} f_{p^d}$. Therefore

$$Y_F - \sum_{j=0}^{i-1} p^j f_{p^j}(p^j)^s = Y_F \sum_h p^h C_F^{p^i,p^h}(p^{h+i})^s,$$

so

$$\sum_h p^h C_F^{p^i,p^h}(p^{h+i})^s = 1 - Y_F^{-1} \sum_{j=0}^{i-1} p^j f_{p^j}(p^j)^s.$$

If F is strongly isomorphic to a special formal group with logarithm $\tilde{f}$ then one can rewrite Y_F^{-1} as

$$T\Gamma_P \cdot Y_{\tilde{F}}^{-1} = T\Gamma_P \cdot (1 - p\tilde{f}_p p^s)$$

and the right hand side is of the form

$$1 - \sum_{k=0}^{\infty} p^k \gamma_{p^k}(p^k)^s \cdot \sum_{m=0}^{i} \xi_m(p^m)^s$$

where

$$\sum_{m=0}^{i} \xi_m(p^m)^s = (1 - p\tilde{f}_p p^s) \cdot \sum_{j=0}^{i-1} p^j f_{p^j}(p^j)^s$$

has integral coëfficients. Now the statement follows from the fact that $p^k \gamma_{p^k}$ is divisible by p^{2k} and thus by p^{2h} if $k + m = h + i$. ●

LEMMA 4.6. *Let F be any formal group, and a element of a λ-ring. Then*

$$Y_F(a) = \sum a^{p^h q - 1} Y_{F,P}(a^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s.$$

In particular $Y_F = Y_{F,P} \cdot Y_{F,C} = Y_{F,C} \cdot Y_{F,P}$ where $Y_{F,C} = \Sigma\, p^h C_F^{q,p^h}(p^h q)^s$. Here the sums are over all q indivisible by p and over all h.

Proof. According to Remark 3.6 one has $qf_{p^e q} = \Sigma_{d=0}^{e} C_F^{q,p^{e-d}} f_{p^d}$. Substituting this into the formula $Y_F(a) = \Sigma\, p^e q f_{p^e q} a^{p^e q - 1}(p^e q)^s$ and writing h for $e - d$ gives the result. ●

PROPOSITION 4.7. *Let F be a formal group over* **Z** *strongly isomorphic to a special one. Then* p^h *divides* C_F^{q,p^h} *if q is indivisible by p.*

Proof. Suppose that F is strongly isomorphic to $\tilde{F}$ and that $\tilde{F}$ is special. Then according to 2.7 one has $Y_{\tilde{F}} = T\Gamma \cdot Y_F$ where $T\Gamma$ is in the image of T_P^2. This implies that $Y_{\tilde{F},P} = T\Gamma_P \cdot Y_{F,P}$ where $T\Gamma_P$ is in the image of T_P^2. By lemma 4.6 one has $Y_{F,C} = Y_F \cdot Y_{F,P}^{-1} = T\Gamma^{-1} \cdot Y_{\tilde{F}} \cdot T\Gamma_P \cdot Y_{\tilde{F},P}^{-1} = T\Gamma^{-1} \cdot \Gamma_P \cdot Y_{\tilde{F},C}$. However $Y_{\tilde{F},C}$ is of the form $\Sigma_q\, C_{\tilde{F}}^{q,1} q^s$ and thus in the image of T_P^2. Therefore $Y_{F,C}$ is in the image of T_P^2. Writing this out one gets the statement. ●

§5. Some p-primary congruences

Now we use Proposition 4.5 to generalise the contents of §10 of [C] to the F-twisted case.

DEFINITION 5.1. Let F be a formal group over $\boldsymbol{Z}$. If $i \leqslant e$ then we write $\chi_F^{p^e,p^i}$ for the λ-operation defined by

$$\chi_F^{p^e,p^i}(a) \;=\; \theta_F^{p^e}(a) \;-\; \sum_{m=0}^{i} p^m C_F^{p^{e-i},p^m} a^{p^e-p^{i-m}} \theta_F^{p^{i-m}}(a)$$

PROPOSITION 5.2. *Let F be a p-typical formal group over $\boldsymbol{Z}$ which is strongly isomorphic to a special one. Let a be the canonical element of U. Then*

$$\chi_F^{p^e,p^i}(a) \equiv \sum_{j=0}^{i}\sum_{m=0}^{i-j} p^m C_F^{p^{e-i},p^m} C_F^{p^{i-j-m},p^j} \psi^{p^j}(\chi^{p^{e-j},p^{i-j-m}}(a)) \text{ modulo } p^i V.$$

Proof. If one expresses the θ_F in terms of the θ as in Remark 3.6 then the right hand side of Definition 5.1 becomes

$$\sum_{j=i+1}^{e} C_F^{p^{e-j},p^j} \psi^{p^j}(\theta^{p^{e-j}}(a))$$

$$+ \sum_{j=0}^{i}\left\{C_F^{p^{e-j},p^j} - \sum_{m=0}^{i-j} p^m C_F^{p^{e-i},p^m} C_F^{p^{i-j-m},p^j}\right\} \psi^{p^j}(\theta^{p^{e-j}}(a))$$

$$- \sum_{j=0}^{i}\sum_{m=0}^{i-j} p^m C_F^{p^{e-i},p^m} C_F^{p^{i-j-m},p^j}\{a^{p^e-p^{i-m}} - \psi^{p^j}(a^{p^{e-j}-p^{i-j-m}})\}\psi^{p^j}(\theta^{p^{i-j-m}}(a))$$

$$+ \sum_{j=0}^{i}\sum_{m=0}^{i-j} p^m C_F^{p^{e-i},p^m} C_F^{p^{i-j-m},p^j} \psi^{p^j}\{\theta^{p^{e-j}}(a) - a^{p^{e-j}-p^{i-j-m}}\theta^{p^{i-j-m}}(a)\}.$$

The last sum is the right hand side of the statement. From Propositions 4.3 and 4.5 and the fact that the λ-operation $x \to x^{p^{j+k}} - \psi^{p^j}(x^{p^k})$ is in $p^{k+1}V$ it follows that the first three terms are in $p^i V$. ●

COROLLARY 5.3. *In the situation of Proposition 5.2 one has*

$$\chi_F^{p^e,p^i}(a) \equiv \begin{cases} 0 & \text{if } p \text{ is odd} \\ C_F^{2^{e-i},1} \sum_{j=0}^{i} C_F^{2^{i-j},2^j} \psi^{2^j}(\chi^{2^{e-j},2^{i-j}}(a)) & \text{if } p = 2 \end{cases}$$

Proof. It follows from Lemma 10.5 of [C] that χ^{p^e,p^i} is in $p^i V$ if p is odd and in $p^{i-1} V$ if $p = 2$. ●

DEFINITION 5.4. Let F be a formal group over $\boldsymbol{Z}$. Then the λ-operations τ_F^n are defined by $\Sigma\ \tau_F^n(a, b)n^s = H_F(1, a) \cdot TH_F(a, b)$. In other words

$$\tau_F^n(a, b) = \sum_{m|n} \frac{n}{m}\, \theta_F^m(a)\psi^m \eta_F^{n/m}(a, b).$$

From this one deduces easily that

$$\tau_F^n(a, b) = \theta_F^n(a)\psi^n(b) + \sum \frac{n}{m}\, \tau_F^m(a, a^{(n-m)/m})\psi^m \theta_F^{n/m}(b)$$

where the sum extends over all $m < n$ dividing n.

DEFINITION 5.5. Let p be a prime, and let F be a formal group over $\boldsymbol{Z}$. Then the operation $\varepsilon_F^{p^e}$ is defined by

$$\varepsilon_F^{p^e}(a, b) = \sum_{i=0}^{e} C_F^{p^{e-i}p^i}\psi^{p^i}(\varepsilon^{p^{e-i}}(a, b))$$

where ε is defined as in Definition 10.7 of [C] i.e.

$$\varepsilon^{p^e}(a, b) = \begin{cases} 0 & \text{if } p > 2 \text{ or } e \leqslant 1 \\ 2^{e-1}(ab)^{2^e-4}\theta^2(a)^2\theta^2(b)^2 & \text{if } p = 2 \text{ and } e \geqslant 2 \end{cases}$$

PROPOSITION 5.6. *Let p be a prime and let F be a p-typical formal group strongly isomorphic to a special one. Let a, b be the canonical elements of U_2; then one has*

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a)b^{p^e} + \varepsilon_F^{p^e}(a, b) \textit{ modulo } p^e W_2.$$

Proof. We use induction in e; the case $e = 0$ is trivial so assume $e > 0$. We claim that

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a)b^{p^e} + \sum_{k=0}^{e} [C_F^{p^{e-k},1}\varepsilon_F^{p^k}(a, a^{p^{e-k}}) - \chi_F^{p^e,p^k}(a)]\, p^{e-k}\psi^{p^k}\theta_F^{p^{e-k}}(b).$$

To prove the claim we apply the induction hypothesis and get

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i}\tau_F^{p^i}(a, a^{p^{e-i}-1})\psi^{p^i}\theta_F^{p^{e-i}}(b)$$

$$= \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i}[\theta_F^{p^i}(a)a^{p^e-p^i} + \varepsilon_F^{p^i}(a, a^{p^{e-i}-1}) \text{ modulo } p^i\mathrm{W}]\psi^{p^i}\theta_F^{p^{e-i}}(b)$$

$$\equiv \sum_{i=0}^{e} p^{e-i}\theta_F^{p^i}(a)a^{p^e-p^i}\psi^{p^i}\theta_F^{p^{e-i}}(b) + \sum_{i=0}^{e-1} p^{e-i}\varepsilon_F^{p^i}(a, a^{p^{e-i}-1})\psi^{p^i}\theta_F^{p^{e-i}}(b).$$

The first sum can be rewritten as

$$\sum_{i=0}^{e}\sum_{j=0}^{e-i} p^{e-i}\theta_F^{p^i}(a)a^{p^e-p^i}C_F^{p^{e-i-j},p^j}\psi^{p^{i+j}}\theta^{p^{e-i-j}}(b)$$

$$= \sum_{k=0}^{e}\left[\sum_{j=0}^{k} p^j\theta_F^{p^{k-j}}(a)a^{p^e-p^{k-j}}C_F^{p^{e-k},p^j}\right]p^{e-k}\psi^{p^k}\theta^{p^{e-k}}(b)$$

$$= \sum_{k=0}^{e}[\theta_F^{p^e}(a) - \chi_F^{p^e,p^k}(a)]p^{e-k}\psi^{p^k}\theta^{p^{e-k}}(b)$$

and by Lemma 2.2 of [C] the terms involving $\theta_F^{p^e}(a)$ add up to $\theta_F^{p^e}(a)b^{p^e}$.

In the second sum the value of $\psi^{p^i}\theta_F^{p^{e-i}}(b)$ only matters modulo pW since its cofactor is in $p^{e-1}W$. So we may replace it by $C_F^{p^{e-i},1}\psi^{p^i}\theta^{p^{e-i}}(b)$. This establishes the claim.

Now consider the bracket expression in the claim, modulo p^kW. For $p > 2$ it vanishes according to Corollary 5.3 and the definition of ε_F, and the proof is finished. So let $p = 2$. For $k > 1$ the χ_F term yields

$$-C_F^{2^{e-k},1}\Bigg[C_F^{2,2^{k-1}}\psi^{2^{k-1}}(a^{2^{e-k+1}-4}\theta^2(a)^2)$$

$$+ \sum_{j=0}^{k-2} 2^{k-j-1}C_F^{2^{k-j},2^j}\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2 + a^{2^{e-j}-8}\theta^2(a)^4)\Bigg]$$

and the ε_F term yields

$$C_F^{2^{e-k},1}\sum_{j=0}^{k-2} C_F^{2^{k-j},2^j}\psi^{2^j}\varepsilon_F^{2^{k-j}}(a, a^{2^{e-k}-1})$$

$$= C_F^{2^{e-k},1}\sum_{j=0}^{k-2} C_F^{2^{k-j},2^j}\psi^{2^j}(2^{k-j-1}(a^{2^{e-k}})^{2^k-4}\theta^2(a)^2\theta^2(a^{2^{e-k}-1})^2).$$

Again the value of $\theta^2(a^{2^{e-k}-1})^2$ only matters modulo $2W$ so we may replace it by $a^{2^{e-k+2}-8}\theta^2(a)^2$. The expression thus becomes

$$-\,C_F^{2^{e-k},1}\sum_{j=0}^{k-1} C_F^{2^{k-j},2^j}\,2^{k-j-1}\,\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$$

For $k = 1$ one gets the same result; for $k = 0$ one gets zero. Finally we may replace $C_F^{2^{e-k},1}\,C_F^{2^{k-j},2^j}$ by $C_F^{2^{e-j},2^j}$ since it is the same modulo 2^{j+1}. We thus find the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\tau_F^{p^e}(a, b)$ to be

$$-\,2^{e-j-1}\,C_F^{2^{e-j},2^j}\sum_{k=j+1}^{e-1}\psi^{2^k}\theta^{2^{e-k}}(b)$$

and according to Lemma 10.6 of [C] this may be rewritten as

$$-\,2^{e-j-1}\,C_F^{2^{e-j},2^j}(b^{2^{e-j}-4}\theta^2(b)^2)$$

which is exactly the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\varepsilon_F^{2^e}(a, b)$. ●

§6. Proof of the theorem

LEMMA 6.1. *Let p be a prime. Let F be a p-typical formal group over $\mathbf{Z}$ strongly isomorphic to a special one. If a, b are the canonical elements in U_2 then*

$$\theta_F^{p^e}(ab) \equiv \theta_F^{p^e}(a)b^{p^e} + a^{p^e}\theta_F^{p^e}(b) \text{ modulo } p^e W_2$$

Proof. In view of Proposition 4.5 and the fact that $\theta_F^{p^e} = \Sigma_{i=0}^{e}\, C_F^{p^{e-i},p^i}\psi^{p^i}\theta^{p^{e-i}}$ we only have to show that

$$\psi^{p^i}(\theta^{p^{e-i}}(ab)) \equiv \psi^{p^i}(\theta^{p^{e-i}}(a))b^{p^e} + a^{p^e}\psi^{p^i}(\theta^{p^{e-i}}(b)) \text{ modulo } p^{e-i}W_2$$

But this follows from the fact that

$$\theta^{p^j}(ab) = \theta^{p^j}(a)b^{p^j} + a^{p^j}\theta^{p^j}(b) - p^j\theta^{p^j}(a)\theta^{p^j}(b)$$

for all j. ●

PROPOSITION 6.2. *Let p be a prime. Let F be a p-typical formal group over* $\mathbf{Z}$ *strongly isomorphic to a special one. Then for each natural number d there exist* $B_F \in DS(W_d)$ *such that*

$$TB_F(a_1, \ldots, a_d) = \sum_{i=1}^{d} H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{i=1}^{d} a_j\right)$$

if $a_1, \ldots, a_d$ *are elements of a λ-ring R.*

Proof. For $d = 2$ we have to prove that there exists a sequence of operations $\beta_F^{p^e}$ of two variables such that $p^e \beta_F^{p^e}(a, b) = \tau_F^{p^e}(a, b) + \tau_F^{p^e}(b, a) - \theta_F^{p^e}(ab)$. But Proposition 5.6 and 6.1 accomplish just that.
The statement for $d > 2$ follows from the one for $d = 2$ as was proven in Proposition 9.1 of [C]. ●

To prove a similar result in the general case we need a generalisation of Lemma 9.2 of [C].

LEMMA 6.3. *Let F be a formal group over* $\mathbf{Z}$*. If R is a λ-ring and* $a, b \in R$ *then*

$$TH_F(1, a) = Y_{F,C}^{-1} \cdot \sum TH_{F,P}(1, a^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s,$$

$$TH_F(a, b) = \left[\sum TH_{F,P}(a, a^{p^h q - 1}) p^h C_F^{q,p^h}(p^h q)^s\right]^{-1}$$

$$\times \left[\sum TH_{F,P}(a, a^{p^h q - 1} b^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s\right]$$

Here the sum is over all q prime to p and over all h.

Proof. One has $TH_F(1, a) = Y_F^{-1} \cdot a Y_F(a)$ and $TH_{F,P}(1, a^{p^h q}) = Y_{F,P}^{-1} \cdot a^{p^h q} Y_{F,P}(a^{p^h q})$ according to Remark 2.7. The substitution of these identities into Lemma 4.6 yields the first statement. According to 2.7 one has also

$$TH_F(1, ab) = Y_F^{-1} \cdot ab Y_F(ab) = [Y_F^{-1} \cdot a Y_F(a)] \cdot [Y_F(a)^{-1} \cdot b Y_F(ab)]$$

$$= TH_F(1, a) \cdot TH_F(a, b).$$

Similarly one has

$$TH_{F,P}(1, a^{p^h q} b^{p^h q}) = TH_{F,P}(1, a) \cdot TH_{F,P}(1, a^{p^h q - 1} b^{p^h q}).$$

The combination of these identities with the first statement yields the second one. ●

PROPOSITION 6.4. *Let F be a formal group over $\mathbf{Z}$ strongly isomorphic to a special one. Then there exist $B_F \in DS(W_d)$ such that*

$$TB_F(a_1, \ldots, a_d) = \sum_{i=1}^{d} H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{j=1}^{d} a_j\right)$$

if $a_1, \ldots, a_d$ are elements of a λ-ring R.

Proof. (Essentially Proposition 9.3 of [C]). Let a, b be the canonical elements in U_2. Let p be a prime. By Lemma 6.3 one has

$$TH_F(1, a) \cdot T^2 H_F(a, b) - Y_{F,C}^{-1} \cdot TH_{F,P}(1, a)$$

$$\times \sum p^{2h} q C_F^{q,p^h} T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q})(p^h q)^s$$

$$= Y_{F,C}^{-1} \cdot \left[\sum [TH_{F,P}(1, a^{p^h q}) - p^h q TH_{F,P}(1, a)\right.$$

$$\left. \times T^2 H_{F,P}(a, a^{p^h q - 1})] p^h C_F^{q,p^h} (p^h q)^s\right]$$

$$\times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1}) p^{2h} q C_F^{q,p^h} (p^h q)^s\right]^{-1}$$

$$\times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) p^{2h} q C_F^{q,p^h} (p^h q)^s\right].$$

Here all four factors are in the image of T_P^2; for this one needs Proposition 4.7 for the first two factors and the $d = p^h q$ case of Proposition 6.2 for the second factor. So the whole difference is in the image of T_P^2. Therefore modulo $T_P^2 DS(W_2)$ one has

$$TH_F(1, a) \cdot T^2 H_F(a, b) + TH_F(1, b) \cdot T^2 H_F(b, a) - TH_F(1, ab)$$

$$\equiv Y_{F,C}^{-1} \cdot \sum [p^h q TH_{F,P}(1, a) \cdot T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) + p^h q TH_{F,P}(1, b)$$

$$\times T^2 H_{F,P}(b, a^{p^h q} b^{p^h q - 1}) - TH_{F,P}(1, a^{p^h q} b^{p^h q})] p^h C_F^{q,p^h} (p^h q)^s$$

and by using Proposition 6.2 for $d = 2p^h q$ one sees that this is in $T_P^2 DS(W_2)$. Thus the coëfficient of n^s in this Dirichlet series is in $p^{2v_p(n)} W_2$. Since that is the case for every prime p the Chinese Remainder Theorem says that the coëfficient is in $n^2 W_2$. In other words the Dirichlet series is in $T^2 DS(W_2)$. This finishes the proof for $d = 2$; the cases $d > 2$ follow from the case $d = 2$ as in Proposition 9.1 of [C]. ●

DEFINITION 6.5. Let F be a formal group over $\mathbf{Z}$. Then the operations v_F^n and the Dirichlet series N are defined by

$$\sum v_F^n(a, b)n^s = N_F(a, b) = H_F(1, a) \cdot \delta H_F(a, b).$$

In other words

$$v^n(a, b) = \sum_{mk=n} \theta_F^m(a)\phi^m(\delta\eta_F^k(a, b)).$$

Here the ϕ^m are the operations on differential forms introduced in §4 of [C].

PROPOSITION 6.6. *Let F be a formal group over $\mathbf{Z}$ which is strongly isomorphic to a special one, and let $a_1, \ldots, a_d$ be elements of a λ-ring R. Then*

$$\sum_{i=1}^{d} N_F\left(a_i, \prod_{j \neq i} a_j\right) = \delta B_F(a_1, \ldots, a_d)$$

Proof. This is an easy consequence of Proposition 6.4. For the details of the proof see Proposition 8.6 of [C]. ●

Now we prove the main theorem. Consider the expression $v_F(a, b) = \Sigma_{n=1}^{\infty} v_F^n(a, b)$. Using exactly the same arguments as in §6 of [C] one finds that it converges if $(a, b) \in I \times R \cup R \times I$ and that Corollary 2.9 and Proposition 6.6 imply that it maps the relations in Definition 0.1 to the zero element of $(\Omega_{R,I}/\delta I)^{\text{top}}$. By the same reasoning as in §7 of [C] one can lift the resulting map to $K_{2,L}(R, I)^{\text{top}}$.

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