COMPOSITIO MATHEMATICA

YOSHIHIRO OHNITA

On stability of minimal submanifolds in compact symmetric spaces

Compositio Mathematica, tome 64, nº 2 (1987), p. 157-189
<http://www.numdam.org/item?id=CM_1987__64_2_157_0>

Compositio Mathematica 64: 157–189 (1987)

On stability of minimal submanifolds in compact symmetric spaces

YOSHIHIRO OHNITA
Department of Mathematics, Tokyo Metropolitan University, Fukasawa, Setagaya, Tokyo, 158, Japan (current address until 1 September 1988: Max-Plank-Institüt für Mathematik, Gottfried-Claren-Strasse 26, D-5300 Bonn 3, FRG)

Received 22 January 1987; accepted 8 April 1987

Dedicated to Professor Ichiro Satake on his 60th birthday

Introduction

A minimal subvariety M of a Riemannian manifold N is called *stable* if the second variation for the volume of M is nonnegative for every variation of M in N. Stable minimal subvarieties in compact rank one symmetric spaces have been classified (cf. [S], [L-S], [O1], [H-W]). It is an interesting and important problem to find all stable minimal subvarieties in each symmetric space. In this paper we discuss the stability of certain minimal subvarieties in compact symmetric spaces. First we reformulate the algorithm of determining the indices and nullities of compact totally geodesic submanifolds in compact symmetric spaces (cf. [C-L-N]). Using this method we determine the indices and nullities of all compact totally geodesic submanifolds in compact rank one symmetric spaces and Helgason spheres (cf. [H1]) in all compact irreducible symmetric spaces. Moreover we prove a nonexistence theorem for stable rectifiable currents of certain degree on some simply connected compact symmetric spaces by the method of Lawson and Simons [L-S].

1. Jacobi operator of minimal submanifolds

Let M be an m-dimensional compact minimal submanifold (without boundary) immersed in a Riemannian manifold (N, h) and denote the immersion by $\varphi: M \to N$. If the second derivative of the volume Vol $(M, \varphi_t^* h)$ at $t = 0$ is nonnegative for every smooth variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$, then we say that φ is *stable* and M is a stable minimal submanifold of N. We denote by g, A, B, $\nabla^\perp$ and R^N the Riemannian metric on M induced from h through φ, the shape operator, the second fundamental form, the normal connection of φ and the curvature tensor of (N, h), respectively. For any vector field $V \in \Gamma(\varphi^{-1}T(N))$ we choose a smooth variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$ and the variational vector field $(\partial/\partial t)\varphi_t(x)|_{t=0} = V_x$ $(x \in M)$. Then the classical second variational formula is given as follows (cf. [S]);

$$(d^2/dt^2) \operatorname{Vol}(M, \varphi_t^* h)|_{t=0} = \int_M \langle \mathfrak{J}(V^N), V^N \rangle \, dv,$$

where dv denotes the Riemannian measure of (M, g) and V^N the component of V normal to M. Here $\mathfrak{J}$ is defined as

$$\mathfrak{J} = -\Delta^\perp - A_\varphi + R_\varphi,$$

where $\Delta^\perp = \Sigma_{i=1}^m \nabla^{\perp 2}_{e_i, e_i}$ and A_φ, R_φ are smooth sections of End $(N(M))$ defined by $\langle A_\varphi(u), v\rangle = \operatorname{Tr}_g(A_u A_v)$, $\langle R_\varphi(u), v\rangle = \Sigma_{i=1}^m \langle R^N(e_i, u)e_i, v\rangle$ for $u, v \in N_x(M)$. $\{e_i\}$ denotes an orthonormal basis of $T_x(M)$. $\mathfrak{J}$ is a self-adjoint strongly elliptic linear differential operator of order 2 acting on the space $\Gamma(N(M))$ of all smooth sections of the normal bundle $N(M)$, called the *Jacobi operator* of φ. $\mathfrak{J}$ has discrete eigenvalues $\mu_1 < \mu_2 < \cdots \to \infty$. We put $E_\mu = \{V \in \Gamma(N(M)); \mathfrak{J}(V) = \mu V\}$. The number $\Sigma_{\mu<0} \dim(E_\mu)$ is called the *index* of φ or the index of M in N, denoted by $i(\varphi)$ or $i(M)$. Clearly, φ is stable if and only if $i(\varphi) = 0$. The number $\dim(E_0)$ is called the *nullity* of φ or the nullity of M in N, denoted by $n(\varphi)$ or $n(M)$. A vector field V in E_0 is called a *Jacobi field* of φ. We define a subspace P of $\Gamma(N(M))$ by

$$P = \{X^N; X \text{ is a Killing vector field on } N\}.$$

It is known that $P \subset E_0$ (cf. [S]). We call the dimension of P the *Killing nullity* of φ, denoted by $n_k(\varphi)$ or $n_k(M)$.

Here we give a lower bound for the nullity of compact minimal submanifolds in homogeneous Riemannian manifolds. Let N be an n-dimensional homogeneous Riemannian manifold and M an m-dimensional compact minimal submanifold immersed in N. Let $I_0(N)$ be the largest connected isometry group of N and denote by $\mathfrak{g}$ its Lie algebra, that is, the Lie algebra of all Killing vector fields on N. For $x \in N$, we denote by K_x and $\mathfrak{k}_x$ the isotropy subgroup of $I_0(N)$ at x and its Lie algebra, respectively. We define a function α on M by

$$\alpha(x) = \dim \{V \in \mathfrak{k}_x; A^V(T_x(M)) \subset T_x(M)\}$$

for $x \in M$, where $A^V(X) = \nabla^N_X V$ ($X \in T_x(N)$). Put $\alpha(M) = \mathrm{Min}\,\{\alpha(x); x \in M\}$. For $V \in \mathfrak{g}$ and $x \in M$, we denote by V^T_x (resp. V^N_x) the compornent of V_x tangent (resp. normal) to M. Then we get the following.

PROPOSITION 1.1.

$$n(M) \geqq n_k(M) \geqq (n - m) + \dim(\mathfrak{k}_x) - \alpha(M).$$

Proof. We fix any point x of M and define a linear mapping Ψ: $\mathfrak{g} \to N_x(M) \oplus \mathrm{Hom}\,(T_x(M), N_x(M))$ by $\Psi(V) = (V^N_x, \nabla^\perp V^N_x)$. Put $\mathfrak{g}^0 = \{V \in \mathfrak{g};\ V^N = 0\}$ and we have $\mathfrak{g}^0 \subset \mathrm{Ker}\,(\Psi)$. Hence we have $n_k(M) = \dim(\mathfrak{g}/\mathfrak{g}^0) \geqq \dim(\mathfrak{g}/\mathrm{Ker}\,(\Psi)) = \dim(\Psi(\mathfrak{g}))$. Let $\{\xi_1, \ldots, \xi_{n-m}\}$ be an orthonormal basis for $N_x(M)$. We choose $V_\alpha \in \mathfrak{g}$ ($\alpha = 1, \ldots, n - m$) so that $(V^N_\alpha)_x = \xi_\alpha$. For $V \in \mathfrak{k}_x$, we have $\Psi(V) = (0, \nabla^\perp V^N_x)$ and $\nabla^\perp_X V^N = (\nabla^N_X V^N)^N = (\nabla^N_X V - \nabla^N_X V^T)^N = (\nabla^N_X V)^N - B(X, V^T) = (A^V(X))^N$ for $X \in T_x(M)$. Hence $\mathfrak{k}_x \cap \mathrm{Ker}\,(\Psi) = \{V \in \mathfrak{k}_x;\ A^V(T_x(M)) \subset T_x(M)\}$. Thus $\dim(\Psi(\mathfrak{k}_x)) = \dim(\mathfrak{k}_x) - \dim(\mathfrak{k}_x \cap \mathrm{Ker}\,(\Psi)) = \dim(\mathfrak{k}_x) - \alpha(x)$. We take a basis $\{\Psi(W_1), \ldots, \Psi(W_l)\}$ of $\Psi(\mathfrak{k}_x)$. Then $\{\Psi(V_1), \ldots, \Psi(V_{n-m}), \Psi(W_1), \ldots, \Psi(W_l)\}$ are linearly independent. Hence $\dim(\Psi(\mathfrak{g})) \geqq (n - m) + \dim(\Psi(\mathfrak{k}_x))$. Thus we get $n_k(M) \geqq (n - m) + \dim(\mathfrak{k}_x) - \alpha(x)$. Q.E.D.

REMARK: The idea of this proof is inspired by that of [S, p. 87].

2. Homogeneous vector bundles and Casimir operators

In this section we recall some results from the theory of homogeneous vector bundles.

Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ and K a closed subgroup of G with the Lie algebra $\mathfrak{k}$. We consider a homogeneous space $M = G/K$ and a natural principal K-bundle $\pi: G \to M$. The coset K is called the origin of M and will be denoted by o. For a finite dimensional representation (σ, W) of K, we get a G-homogeneous vector bundle $(E = G \times_\sigma W, \pi_E, M)$ associated with the principal bundle (G, π, M). All G-homogeneous vector bundles over M are obtained in this fashion. We denote by τ the action of G on E and M. Let $\Gamma(E)$ be the vector space of all smooth sections of E on M. We denote by $C^\infty(G; W)_K$ the vector space of all smooth W-valued functions on G satisfying $f(uk) = \sigma(k^{-1})(f(u))$ for $u \in G$ and

$k \in K$. G acts on $\Gamma(E)$ and $C^\infty(G; W)_K$, respectively, by

$$(L_g\xi)_x = \tau_g(\xi(g^{-1}x)), \quad (L_g f)_u = f(g^{-1}u)$$

for $\xi \in \Gamma(E)$, $f \in C^\infty(G; W)_K$, $g, u \in G$ and $x \in M$. For any $\xi \in \Gamma(E)$ we define $\tilde{\xi} \in C^\infty(G; W)_K$ by $(\xi)_{\pi(u)} = u(\tilde{\xi}(u))$ for $u \in G$. The map A: $\xi \to \tilde{\xi}$ of $\Gamma(E)$ to $C^\infty(G; W)_K$ is a linear isomorphism preserving the actions of G. For any mapping D: $\Gamma(E) \to \Gamma(E)$, we denote by $\tilde{D}$ the map $A \cdot D \cdot A^{-1}$: $C^\infty(G; W)_K \to C^\infty(G; W)_K$.

Assume that G is compact and (σ, W) is a finite dimensional unitary representation of K. Then E is a complex vector bundle with the hermitian fibre metric induced by the Hermitian inner product of W. Let $L^2(E)$ and $L^2(G; W)_K$ be the completion of $\Gamma(E)$ and $C^\infty(G; W)_K$ relative to the L^2-inner products induced by the normalized Haar measure of G and the Hermitian metric of E. The map $\xi \to \tilde{\xi}$ extends to a unitary isomorphism of $L^2(E)$ to $L^2(G; W)_K$ preserving the actions of G. Let $D(G)$ denote the set of all equivalence classes of finite dimensional irreducible complex representations of G. Let for each $\lambda \in D(G)$, $(\varrho_\lambda, V_\lambda)$ be a fixed representation of λ. For each $\lambda \in D(G)$ we assign a map A_λ from $V_\lambda \otimes \mathrm{Hom}_K(V_\lambda, W)$ to $C^\infty(G; W)_K$ by the rule $A_\lambda(v \otimes L)(g) = L(\varrho_\lambda(g^{-1})v)$. Here $\mathrm{Hom}_K(V_\lambda, W)$ denotes the space of all linear maps L of V_λ into W so that $\sigma(k) \cdot L = L \cdot \varrho_\lambda(k)$ for all $k \in K$. For $\lambda \in D(G)$, set $\Gamma_\lambda(E) = A^{-1}(A_\lambda(V_\lambda \otimes \mathrm{Hom}_K(V_\lambda, W)))$. By virtue of the Peter-Weyl theorem and the Frobenius reciprocity the following proposition holds.

PROPOSITION 2.1. (cf. [Wa]): *$L^2(E)$ is the unitary direct sum $\Sigma_{\lambda \in D(G)} \oplus \Gamma_\lambda(E)$. Moreover the algebraic direct sum $\Sigma_{\lambda \in D(G)} \Gamma_\lambda(E)$ is uniformly dense in $\Gamma(E)$ relative to the uniform topology.*

Let (,) be an $\mathrm{ad}(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. We choose an orthonormal basis $\{X_1, \ldots, X_m\}$ of $\mathfrak{g}$ relative to (,). The Casimir differential operator of G relative to (,) is defined by

$$\mathscr{C}f = \sum_{i=1}^{m} X_i(X_i f)$$

for $f \in C^\infty(G)$. Then we have $\mathscr{C}(C^\infty(G; W)_K) \subset C^\infty(G; W)_K$.

PROPOSITION 2.2 (cf. [Wa]): *On each $\Gamma_\lambda(E)$, $\mathscr{C}$ is a constant operator $a_\lambda I$. Here a_λ is the eigenvalue of the Casimir operator of the representation λ relative to (,).*

We do not suppose that G is compact. Let Γ be a connection in the principal bundle (G, π, M, K) which is invariant by the left translations of G. Γ determines a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ such that $\mathrm{ad}(K)\mathfrak{m} = \mathfrak{m}$ (cf. [K-N,I, p. 103]). Γ induces the covariant differentiation ∇^E on E. For a vector X on M, we denote by X^* the horizontal lift of X to G relative to Γ. For $\xi \in \Gamma(E)$ and $X \in \Gamma(T(M))$, we have

$$(\nabla^E_X \xi)^\sim \;=\; X^* \tilde{\xi} \qquad \text{(cf. [K-N,I, p. 115])}. \tag{1.1}$$

Let g be a G-invariant Riemannian metric on M. Then g defines an $\mathrm{ad}(K)$-invariant inner product $B_\mathfrak{m}$ on $\mathfrak{m}$. We denote by ∇ the Riemannian connection of g. For $X, Y, Z \in \mathfrak{m}$, we have

$$g_o(\nabla_Y X, Z) \;=\; (1/2)\{B_\mathfrak{m}([X, Y]_\mathfrak{m}, Z) + B_\mathfrak{m}(X, [Z, Y]_\mathfrak{m}) + B_\mathfrak{m}([Z, X]_\mathfrak{m}, Y)\} \qquad \text{(cf. [K-N,II, p. 201])}. \tag{2.2}$$

Here $[X, Y]_\mathfrak{m}$ denotes the $\mathfrak{m}$-compornent of $[X, Y]$ in $\mathfrak{g}$.

The (rough) Laplacian Δ^E relative to ∇^E is defined by

$$\Delta^E \xi \;=\; \sum_{i=1}^{m} \nabla^{E2}_{e_i, e_i} \xi,$$

where $\{e_i\}$ is a local orthonormal frame field on M. Let $\{X_1, \ldots, X_m\}$ be an orthonormal basis of $\mathfrak{m}$ relative to $B_\mathfrak{m}$,

PROPOSITION 2.3.

$$\tilde{\Delta}^E \;=\; \sum_{i=1}^{m} \{X_i X_i - (\mathrm{trace}(\mathrm{ad}\, X_i))X_i\}.$$

If G is compact, more generally unimodular, then

$$\tilde{\Delta}^E \;=\; \sum_{i=1}^{m} X_i X_i.$$

Proof. We fix points $x \in M$ and $u \in G$ with $x = \pi(u)$. We define an orthonormal frame field $\{e_1, \ldots, e_m\}$ in a neighborhood U of x so that $(e_i)_{u\exp(tX)\cdot o} = d\tau_u \cdot d\tau_{\exp(tX)}(X_i) = d\pi(X_i)_{u\exp(tX)}$ for $i = 1, \ldots, m$, $X \in \mathfrak{m}$

and $|x| < \varepsilon$. By (2.1) we have for any $\xi \in \Gamma(E)$,

$$\begin{aligned}(\tilde{\Delta}^E\tilde{\xi})(u) &= (\Delta^E\xi)^\sim(u) \\ &= \sum_{i=1}^{m} \{\nabla^E_{e_i}(\nabla^E_{e_i}\xi) - \nabla^E_{\nabla_{e_i}e_i}\xi\}^\sim(u) \\ &= \sum_{i=1}^{m} e_i^*(e_i^*\tilde{\xi})(u) - \left(\sum_{i=1}^{m} \nabla_{e_i}e_i\right)^*\tilde{\xi}(u).\end{aligned}$$

By (2.2) and $\mathrm{ad}(K)\mathfrak{m} = \mathfrak{m}$ we have for any $Z \in \mathfrak{m}$,

$$\begin{aligned}g_x\left(\sum_{i=1}^{m} \nabla_{e_i}e_i, d\tau_u(Z)\right) &= g_o\left(\sum_{i=1}^{m} \nabla_{X_i}X_i, Z\right) \\ &= \sum_{i=1}^{m} B_{\mathfrak{m}}([Z, X_i]_{\mathfrak{m}}, X_i) \\ &= \text{trace ad}(Z).\end{aligned}$$

Hence we get $\Sigma_{i=1}^m (\nabla_{e_i}e_i)_x = d\pi\,(\Sigma_{i=1}^m \text{trace}\,(\mathrm{ad}(X_i))(X_i)_u)$. Since $X_1, \ldots, X_n$ are horizontal with respect to Γ, we get $(\Sigma_{i=1}^m \nabla_{e_i}e_i)_u^* = \Sigma_{i=1}^m (\text{trace}(\mathrm{ad}\,(X_i))(X_i)_u$. Put $e_i^* = \Sigma_{j=1}^m c_i^j X_j$, where each c_i^j is a smooth function on $\pi^{-1}(U)$ and for each $u \in \pi^{-1}(U)$, $(c_i^j(u))$ is an orthogonal matrix. We compute

$$\begin{aligned}\sum_{i=1}^{m} e_i^*(e_i^*\tilde{\xi})(u) &= \sum_{i=1}^{m}\left(\sum_{j=1}^{m} c_i^j(u)\, X_j\right)\left(\sum_{k=1}^{m} c_i^k(u)\, X_k\right)\tilde{\xi}(u) \\ &= \sum_{i,j,k=1}^{m} \{c_i^j(u)c_i^k(u)(X_jX_k\tilde{\xi})(u) \\ &\quad + c_i^j(u)(X_jc_i^k)(u)(X_j\tilde{\xi})(u)\} \\ &= \sum_{j=1}^{m} X_j(X_j\tilde{\xi})(u) + \sum_{k=1}^{m}\left(\sum_{i,j=1}^{m} c_i^j(u)(X_jc_i^k)(u)\right)(X_k\tilde{\xi})(u).\end{aligned}$$

Since $(e_i^*)_{u\exp(tX)} = (X_i)_{u\exp(tX)}$ for any $X \in \mathfrak{m}$, we have $c_i^j(u\exp(tX)) = \delta_i^j$. Hence $(Xc_i^j)(u) = 0$ for any $X \in \mathfrak{m}$. Thus we get $\Sigma_{i=1}^m e_i^*(e_i^*\tilde{\xi})(u) = \Sigma_{j=1}^m X_j(X_j\tilde{\xi})(u)$. If G is unimodular, then trace $\mathrm{ad}(X) = 0$ for any $X \in \mathfrak{g}$.

Q.E.D.

Suppose that K is compact. We can extend B to an ad(K)-invariant inner product B on $\mathfrak{g}$ such that $B(\mathfrak{k}, \mathfrak{m}) = 0$. Let $\{X_{m+1}, \ldots, X_l\}$ be an orthonormal basis of $\mathfrak{k}$ relative to B. Then we get the following.

COROLLARY 2.4:

$$\tilde{\Delta}^E = \sum_{A=1}^{l} X_A X_A - \left(\sum_{\alpha=m+1}^{l} \sigma(X_\alpha)^2 \right) - \sum_{i=1}^{m} (\text{trace ad}\,(X_i))X_i.$$

Proof. For any integer α with $n + 1 \leqq \alpha \leqq m$, $\xi \in \Gamma(E)$ and $u \in G$, we compute

$$\begin{aligned} (X_\alpha \tilde{\xi})(u) &= (\mathrm{d}/\mathrm{d}t)\tilde{\xi}(u \exp(tX_\alpha))|_{t=0} \\ &= (\mathrm{d}/\mathrm{d}t)\sigma(\exp(tX_\alpha))^{-1}\tilde{\xi}(u)|_{t=0} \\ &= -(\sigma(X_\alpha) \cdot \tilde{\xi})(u), \end{aligned}$$

and

$$\begin{aligned} (X_\alpha^2 \tilde{\xi})(u) &= (\mathrm{d}/\mathrm{d}t)(X_\alpha \tilde{\xi})(u \exp(tX_\alpha))|_{t=0} \\ &= -(\mathrm{d}/\mathrm{d}t)(\sigma(X_\alpha) \cdot \tilde{\xi})(u \exp(tX_\alpha)|_{t=0} \\ &= (\sigma(X_\alpha)^2 \cdot \tilde{\xi})(u). \end{aligned}$$

Hence

$$\sum_{\alpha=m+1}^{l} X_\alpha^2 \tilde{\xi} = \sum_{\alpha=m+1}^{l} \sigma(X_\alpha)^2 \cdot \tilde{\xi}.$$ Q.E.D.

COROLLARY 2.5: *Assume that G is compact and g is a Riemannian metric on M induced by an* ad(G)*-invariant inner product* (,) *on* $\mathfrak{g}$. *Let $\mathscr{C}$ denote the Casimir differential operator of G relative to* (,). *Then we have*

$$\tilde{\Delta}^E = \mathscr{C} - \left(\sum_{\alpha=m+1}^{l} \sigma(X_\alpha)^2 \right).$$

REMARK: The endomorphism $\Sigma_{\alpha=m+1}^{l}\, \sigma(X_\alpha)^2$ coincides with the Casimir operator of the representation (σ, W) of K relative to (,).

3. Jacobi operator of totally geodesic submanifolds in compact symmetric spaces

In this section we explain a method for determining the stability of compact totally geodesic submanifolds in compact symmetric spaces. There seem to be inaccuracies in [C-L-N]. Here we reformulate this method by using results of Section 2.

Let N be a compact Riemannian symmetric space with the metric g_N and M an m-dimensional compact totally geodesic submanifold immersed in N. Then it is standard to show that the immersion $\varphi\colon M \to N$ is expressed as follows: There are compact symmetric pairs (U, L) and (G, K) with $N = U/L$ and $M = G/K$ so that

$$\begin{aligned} \varphi\colon M = G/K &\to N = U/L \\ uK &\to \varrho(u)L, \end{aligned}$$

where $\varrho\colon G \to U$ is an analytic homomorphism with $\varrho(K) \subset L$ and the injective differential $\varrho\colon \mathfrak{g} \to \mathfrak{u}$ which satisfies $\varrho(\mathfrak{m}) \subset \mathfrak{p}$. Here $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ are the canonical decompositions of $\mathfrak{u}$ and $\mathfrak{g}$, respectively. Denote by θ the involutive automorphism of the symmetric pair (U, L). We choose an $\mathrm{ad}(U)$-invariant inner product $(\ ,\ )$ on $\mathfrak{u}$ such that $(\ ,\ )$ induces g_N. We also denote by $(\ ,\ )$ the $\mathrm{ad}(G)$-invariant inner product on $\mathfrak{g}$ induced from $(\ ,\ )$ through ϱ. Let g be the G-invariant Riemannian metric on M induced by $(\ ,\ )$. Then $\varphi\colon (M, g) \to (N, g_N)$ is an isometric immersion. Let $N(M)$ be the normal bundle of φ and denote by $\Gamma(N(M))$ the vector space of all smooth sections of $N(M)$ on M. We can identify $\mathfrak{p}$, $\mathfrak{m}$ and $\varrho(\mathfrak{m})$ with $T_o(N)$, $T_o(M)$ and $\varphi_*(T_o(M))$, respectively. Let $\mathfrak{m}^\perp$ be the orthogonal compliment of $\varrho(\mathfrak{m})$ with $\mathfrak{p}$ relative to $(\ ,\ )$. We identify $\mathfrak{m}^\perp$ with $N_o(M)$. Define an orthogonal representation σ of K on $\mathfrak{m}^\perp$ by $\sigma(k)(v) = \mathrm{ad}(\varrho(k))v$ for $k \in K$ and $v \in \mathfrak{m}^\perp$. The normal bundle $N(M)$ is the homogeneous vector bundle $G \times_\sigma \mathfrak{m}^\perp$ over M associated with the principal K-bundle $\pi\colon G \to M$. As in Section 1 there is a linear isomorphism $\Gamma(N(M)) \cong C^\infty(G; \mathfrak{m}^\perp)_K$; $V \to \tilde{V}$. Let $\{X_1, \ldots, X_l\}$ be an orthonormal basis relative to $(\ ,\ )$ such that $\{X_1, \ldots, X_m\}$ is in $\mathfrak{m}$ and $\{X_{m+1}, \ldots, X_l\}$ is in $\mathfrak{k}$. Let $\mathscr{C}$ be the Casimir differential operator of G relative to $(\ ,\ )$.

Let Γ_0 be the canonical connection of $\pi\colon G \to G/K$ and ∇^0 the covariant differentiation of $G \times_\sigma \mathfrak{m}^\perp$ induced from Γ_0. We denote by Δ^0 the (rough) Laplacian defined from ∇^0 and the Riemannian connection ∇ of (M, g). Let $\nabla^\perp$ be the normal connection of φ on $N(M)$ and $\Delta^\perp$ the (rough) Laplacian defined from $\nabla^\perp$ and ∇. Since φ is totally geodesic, we have the following.

PROPOSITION 3.1: $\nabla^0 = \nabla^\perp$.

By Corollary 2.5 and Proposition 3.1, we get

$$(\Delta^\perp)^\sim = \mathscr{C} - \left(\sum_{\alpha=m+1}^{l} \sigma(X_\alpha)^2 \right). \tag{3.1}$$

Let $\mathfrak{k}^\perp$ be the orthogonal complement of $\varrho(\mathfrak{k})$ in $\mathfrak{l}$ relative to (,) and put $\mathfrak{g}^\perp = \mathfrak{k}^\perp + \mathfrak{m}^\perp$. Then the vector space $\mathfrak{g}^\perp$ is $\mathrm{ad}(\varrho(\mathfrak{g}))$-invariant and we have an orthogonal decomposition $\mathfrak{u} = \varrho(\mathfrak{g}) + \mathfrak{g}^\perp$ as $\mathrm{ad}(\varrho(G))$-modules. We denote by $(\mu, \mathfrak{g}^\perp)$ this representation of G on $\mathfrak{g}^\perp$. The Casimir operator of the representation μ relative to (,) is given by $C_\mu = \Sigma_{A=1}^{l} \mathrm{ad}(\varrho(X_A))^2 \in \mathrm{End}(\mathfrak{g}^\perp)$. Then the Jacobi operator $\mathfrak{J}$ of φ is expressed in terms of $\mathscr{C}$ and C_μ as follows:

LEMMA 3.2:

$$\tilde{\mathfrak{J}} = -\mathscr{C} + C_\mu.$$

Proof. Since φ is totally geodesic, we have $\mathfrak{J} = -\Delta^\perp + R_\varphi$. As the curvature tensor R^N of a symmetric space N is given by $R^N(X, Y)Z = -[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, we have

$$\begin{aligned}(\tilde{R}_\varphi(\tilde{V}))(u) &= -\sum_{i=1}^{m} [[\varrho(X_i), \tilde{V}(u)], \varrho(X_i)] \\ &= \sum_{i=1}^{m} (\mathrm{ad}(\varrho(X_i))^2 \cdot \tilde{V})(u)\end{aligned}$$

for $V \in \Gamma(N(M))$ and $u \in G$. Hence $\tilde{R}_\varphi = \Sigma_{i=1}^{m} \mathrm{ad}(\varrho(X_i))^2$. By (3.1) we get

$$\begin{aligned}\tilde{\mathfrak{J}} &= -\mathscr{C} + \left(\sum_{\alpha=m+1}^{l} \alpha(X_\alpha)^2 \right) + \left(\sum_{i=1}^{m} \mathrm{ad}(\varrho(X_i))^2 \right) \\ &= -\mathscr{C} + C_\mu.\end{aligned}$$

Q.E.D.

There is an orthogonal decomposition $\mathfrak{g}^\perp = \mathfrak{g}_1^\perp \oplus \cdots \oplus \mathfrak{g}_k^\perp$ such that each $\mathfrak{g}_i^\perp$ is an irreducible $\mathrm{ad}(\varrho(G))$-invariant subspace with $\theta(\mathfrak{g}_i^\perp) = \mathfrak{g}_i^\perp$. Then by Schur's lemma we have $C_\mu = a_i I$ on each $\mathfrak{g}_i^\perp$. Here a_i is the eigenvalue for the Casimir operator of the irreducible G-module $\mathfrak{g}_i^\perp$ relative to (,). Put $\mathfrak{g}_i^\perp = \mathfrak{k}_i^\perp + \mathfrak{m}_i^\perp$, where $\mathfrak{k}_i^\perp = \mathfrak{k}^\perp \cap \mathfrak{g}_i^\perp$ and $\mathfrak{m}_i^\perp = \mathfrak{m}^\perp \cap \mathfrak{g}_i^\perp$.

Then we have $\mathrm{ad}(\varrho(K))\mathfrak{k}_i^\perp = \mathfrak{k}_i^\perp$ and $\mathrm{ad}(\varrho(K))\mathfrak{m}_i^\perp = \mathfrak{m}_i^\perp$. Denote by $(\sigma_i, \mathfrak{m}_i^\perp)$ this representation of K on $\mathfrak{m}_i^\perp$. The decomposition $\mathfrak{m}^\perp = \mathfrak{m}_1^\perp \oplus \cdots \oplus \mathfrak{m}_k^\perp$ induces the decompositions

$$G \times {}_\sigma\mathfrak{m}^\perp = (G \times {}_{\sigma_1}\mathfrak{m}_1^\perp) \oplus \cdots \oplus (G \times {}_{\sigma_k}\mathfrak{m}_k^\perp)$$

and

$$\Gamma(G \times {}_\sigma\mathfrak{m}^\perp) = \Gamma(G \times {}_{\sigma_1}\mathfrak{m}_1^\perp) \oplus \cdots \oplus \Gamma(G \times {}_{\sigma_k}\mathfrak{m}_k^\perp).$$

By Lemma 3.2 we have for $V \in \Gamma(G \times {}_{\sigma_i}\mathfrak{m}_i^\perp)$,

$$\mathfrak{J}(\tilde{V}) = -\mathscr{C}(\tilde{V}) + a_i\tilde{V}. \tag{3.2}$$

THEOREM 3.3: *The index, nullity and Killing nullity of φ are given as follows:*
(i) $i(\varphi) = \Sigma_{i=1}^k \Sigma_{\lambda \in D(G), a_\lambda < a_i} m(\lambda) d_\lambda$,
(ii) $n(\varphi) = \Sigma_{i=1}^k \Sigma_{\lambda \in D(G), a_\lambda = a_i} m(\lambda) d_\lambda$,
(iii) $n_k(\varphi) = \Sigma_{i=1, \mathfrak{m}_i^\perp \neq \{0\}}^k \dim \mathfrak{g}_i^\perp$,
where $m(\lambda) = \dim \mathrm{Hom}_K(V_\lambda, (\mathfrak{m}_i^\perp)^{\mathbf{C}})$ *and d_λ denotes the dimension of the representation λ.*

4. Stability of Helgason spheres

Let (N, g_N) be an n-dimensional compact irreducible Riemannian symmetric space and $I_0(N)$ the largest connected Lie group of isometries of N. Let κ be the maximum of the sectional curvatures of N. By a theorem of E. Cartan, the same dimensional totally geodesic submanifolds of N of constant curvature 0 are all conjugate under $I_0(N)$. Helgason proved an analogous statement for the maximum curvature κ as follows.

THEOREM 4.1 ([H1]): *The space N contains totally geodesic submanifolds of constant curvature κ. Any two such submanifolds of the same dimension are conjugate under $I_0(N)$. The maximal dimension of such submanifolds is $1 + m(\delta)$ where $m(\delta)$ is the multiplicity of the highest restricted root δ. Also, $\kappa = 4\pi^2|\delta|^2$, where $|\ |$ denotes length. Except for the case when N is a real projective space, the submanifolds above of dimension $1 + m(\delta)$ are actually spheres.*

Assume that N is not a real projective space. Let S_κ be a maximal dimensional totally geodesic sphere of N with constant curvature κ. We call S_κ the

Helgason sphere of N. In this section we show that every Helgason sphere is a stable minimal submanifold of N. Put $m = 1 + m(\delta)$.

We begin with preliminaries on Lie algebras. Let (U, L) be a compact symmetric pair with $N = U/L$. Let $\mathfrak{u}$ and $\mathfrak{l}$ be the Lie algebras of U and L, respectively, and $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ the canonical decomposition. We choose the $\mathrm{ad}(U)$-invariant inner product (,) on $\mathfrak{u}$ so that (,) induces g_N. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{t}$ the maximal abelian subalgebra of $\mathfrak{u}$ containing $\mathfrak{a}$. We denote by $\mathfrak{u}^{\mathbf{C}}$ the complexification of $\mathfrak{u}$. For $\alpha \in \mathfrak{t}$, we define a subspace of $\mathfrak{u}^{\mathbf{C}}$ by

$$\tilde{\mathfrak{u}}_\alpha = \{X \in \mathfrak{u}^{\mathbf{C}};\ (\mathrm{ad}\, H)X = 2\pi\sqrt{-1}(\alpha, H)X \quad \text{for any} \quad H \in \mathfrak{t}\}.$$

$\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{u}$ with respect to $\mathfrak{t}$ if $\alpha \neq 0$ and $\tilde{\mathfrak{u}}_\alpha \neq \{0\}$. We denote by $\Sigma(U)$ the complete set of roots of with to $\mathfrak{t}$. The following proposition is well known (cf. [H2]).

PROPOSITION 4.2:
(1) *For* $\alpha \in \Sigma(U)$, *we have* $\dim \tilde{\mathfrak{u}}_\alpha = 1$.
(2) $\tilde{\mathfrak{u}}_0 = \mathfrak{t}^{\mathbf{C}}$.
(3) *There is a direct sum* $\mathfrak{u}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \Sigma_{\alpha \in \Sigma(U)} \tilde{\mathfrak{u}}_\alpha$, *called the root decomposition of* $\mathfrak{u}^{\mathbf{C}}$ *with respect to* $\mathfrak{t}$.
(4) *For* $\alpha, \beta \in \mathfrak{t}$, *we have* $[\tilde{\mathfrak{u}}_\alpha, \tilde{\mathfrak{u}}_\beta] \subset \tilde{\mathfrak{u}}_{\alpha+\beta}$. *In particular, if* $\alpha, \beta, \alpha + \beta \in \Sigma(U)$, *we have* $[\tilde{\mathfrak{u}}_\alpha, \tilde{\mathfrak{u}}_\beta] = \tilde{\mathfrak{u}}_{\alpha+\beta}$.

For $\gamma \in \mathfrak{a}$, we define a subspace of $\mathfrak{u}^{\mathbf{C}}$ by

$$\mathfrak{u}^{\mathbf{C}}_\gamma = \{X \in \tilde{\mathfrak{u}}^{\mathbf{C}};\ (\mathrm{ad}\, H)X = 2\pi\sqrt{-1}(\gamma, H)X \quad \text{for any} \quad H \in \mathfrak{a}\}.$$

$\gamma \in \mathfrak{a}$ is called a *root* of $(\mathfrak{u}, \mathfrak{l})$ (or a *restricted root* of $\mathfrak{u}$) with respect to $\mathfrak{a}$ if $\gamma \neq 0$ and $\mathfrak{u}^{\mathbf{C}}_\gamma \neq 0$. We denote be $\Sigma(U, L)$ the complete set of roots of $(\mathfrak{u}, \mathfrak{l})$. We have a direct sum $\mathfrak{u}^{\mathbf{C}} = \mathfrak{u}^{\mathbf{C}}_0 + \Sigma_{\gamma \in \Sigma(U,L)} \mathfrak{u}^{\mathbf{C}}_\gamma$, and $[\mathfrak{u}^{\mathbf{C}}_\gamma, \mathfrak{u}^{\mathbf{C}}_\delta] \subset \mathfrak{u}^{\mathbf{C}}_{\gamma+\delta}$. We have a direct sum $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}$, where $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{l}$. We put $\Sigma_0(U) = \Sigma(U) \cap \mathfrak{b}$. We define an involotive automorphism σ of $\mathfrak{t}$ as $\sigma(H_1 + H_2) = -H_1 + H_2$ for $H_1 \in \mathfrak{b}$ and $H_2 \in \mathfrak{a}$. A linear order $<$ of $\mathfrak{t}$ is called a *σ-order* if for any $\alpha \in \Sigma(U) - \Sigma_0(U)$ with $\alpha > 0$ we have $\sigma(\alpha) > 0$. There always is a σ-order of $\mathfrak{t}$. We fix a σ-order $<$. We denote by $\Sigma^+(U)$ and $\Sigma^+(U, L)$ the complete sets of positive roots of $\mathfrak{u}$ and $(\mathfrak{u}, \mathfrak{l})$, respectively. For $H \in \mathfrak{t}$, we denote by $\bar{H}$ the $\mathfrak{a}$-compornent of H with respect to (,). Then we have $\Sigma^+(U, L) = \{\bar{\alpha};\ \alpha \in \Sigma^+(U) - \Sigma_0(U)\}$. For $\gamma \in \mathfrak{a}$, we put

$$\mathfrak{l}_\gamma = \{X \in \mathfrak{l};\ (\mathrm{ad}\, H)^2 X = -4\pi^2(\gamma, H)^2 X \quad \text{for any} \quad H \in \mathfrak{a}\}$$

and

$$\mathfrak{p}_\gamma = \{X \in \mathfrak{p};\ (\mathrm{ad}\ H)^2 X = -4\pi^2(\gamma, H)^2 X \quad \text{for any} \quad H \in \mathfrak{a}\}.$$

The following propositions are well known (cf. [H2], [Te2]).

PROPOSITION 4.3:
(1) *We have orthogonal direct sums of* $\mathfrak{l}$ *and* $\mathfrak{p}$ *relative to* $(\ , \ )$;

$$\mathfrak{l} = \mathfrak{l}_0 + \sum_{\gamma \in \Sigma^+(U,L)} \mathfrak{l}_\gamma, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(U,L)} \mathfrak{p}_\gamma.$$

For each $\gamma \in \Sigma^+(U, L)$, *we have* $\dim \mathfrak{l}_\gamma = \dim \mathfrak{p}_\gamma$, *denoted by* $m(\gamma)$.
(2) *For* $\gamma \in \Sigma(U, L)$, *we have*

$$\mathfrak{l}_\gamma = \mathfrak{l} \cap (\mathfrak{u}_\gamma^{\mathbf{C}} + \mathfrak{u}_{-\gamma}^{\mathbf{C}}), \quad \mathfrak{p}_\gamma = \mathfrak{p} \cap (\mathfrak{u}_\gamma^{\mathbf{C}} + \mathfrak{u}_{-\gamma}^{\mathbf{C}})$$

and

$$\mathfrak{l}_0 = \{X \in \mathfrak{l};\ [X, \mathfrak{a}] = 0\}.$$

(3) *For* $\gamma, \delta \in \Sigma(U, L)$,

$$[\mathfrak{l}_\gamma, \mathfrak{l}_\delta] \subset \mathfrak{l}_{\gamma+\delta} + \mathfrak{l}_{\gamma-\delta}, \ [\mathfrak{l}_\gamma, \mathfrak{p}_\delta] \subset \mathfrak{p}_{\gamma+\delta} + \mathfrak{p}_{\gamma+\delta}$$

and

$$[\mathfrak{p}_\gamma, \mathfrak{p}_\delta] \subset \mathfrak{l}_{\gamma+\delta} + \mathfrak{l}_{\gamma-\delta}.$$

PROPOSITION 4.4: *For each* $\alpha \in \Sigma^+(U) - \Sigma_0(U)$, *we can choose* $S_\alpha \in \mathfrak{l}$, $T_\alpha \in \mathfrak{p}$ *satisfying the following properties:*
(1) *For each* $\gamma \in \Sigma^+(U, L)$, $\{S_\alpha;\ \alpha \in \Sigma^+(U) - \Sigma_0(U),\ \bar{\alpha} = \gamma\}$ *and* $\{T_\alpha;\ \alpha \in \Sigma^+(U) - \Sigma_0(U),\ \bar{\alpha} = \gamma\}$ *are orthonormal bases of* $\mathfrak{l}_\gamma$ and $\mathfrak{p}_\gamma$, *respectively.*
(2) $[H, S_\alpha] = 2\pi(\alpha, H)T_\alpha$, $[H, T_\alpha] = -2\pi(\alpha, H)S_\alpha$ *for any* $H \in \mathfrak{a}$.

$$\mathrm{Ad}(\exp H)S_\alpha = \cos 2\pi(\alpha, H)S_\alpha + \sin 2\pi(\alpha, H)T_\alpha$$

and

$$\mathrm{Ad}(\exp H)T_\alpha = -\sin 2\pi(\alpha, H)S_\alpha + \cos 2\pi(\alpha, H)T_\alpha$$

for any $H \in \mathfrak{a}$.

Assume that (U, L) is irreducible and N is not of constant curvature. Let δ_U be the highest root of $\Sigma^+(U)$. Then $\delta = \bar{\delta}_U$ is the highest root of $\Sigma^+(U, L)$ and the maximum of the sectional curvatures of (N, g_N) is $4\pi^2|\delta|^2$.

PROPOSITION 4.5 (cf. [H1]): *For each* $\alpha \in \Sigma^+(U) - \Sigma_0(U)$, *we have*
(1) $(\bar{\alpha}, \delta) = \varepsilon(\delta, \delta)$, *where* $\varepsilon = 0, 1/2$ or 1.
(2) *If* $(\bar{\alpha}, \delta) = (\delta, \delta)$, *then* $\bar{\alpha} = \delta$.

PROPOSITION 4.6 (cf. [H1]): $\mathrm{span}_{\mathbb{R}}\{\delta\} + \mathfrak{m}$ *is a Lie triple system of* (U, L) *which generates a maximal totally geodesic sphere* S_κ *of constant curvature* $\kappa = 4\pi^2|\delta|^2$.

Set

$$\mathfrak{k} = [\mathfrak{p}_\delta, \pi_\delta] + \mathfrak{l}_\delta. \quad \mathfrak{m} = \mathrm{span}_{\mathbb{R}}\{\delta\} + \mathfrak{p}_\delta$$

and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

Note that $[\mathfrak{p}_\delta, \mathfrak{p}_\delta] \subset \mathfrak{l}_0$. For $\varepsilon = 0, 1/2$ or 1, set

$$\mathfrak{l}(\varepsilon) = \sum_{(\gamma,\delta) = \varepsilon(\delta,\delta)} \mathfrak{l}_\gamma$$

and

$$\mathfrak{p}(\varepsilon) = \sum_{(\gamma,\delta) = \varepsilon(\delta,\delta)} \mathfrak{p}_\gamma.$$

Then we have

$$\mathfrak{l} = \mathfrak{l}_0 + \mathfrak{l}(0) + \mathfrak{l}(1/2) + \mathfrak{l}_\delta$$

and

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}(0) + \mathfrak{p}(1/2) + \mathfrak{p}_\delta.$$

Here $\mathfrak{l}(1) = \mathfrak{l}_\delta$ and $\mathfrak{p}(1) = \mathfrak{p}_\delta$.

Let G be an analytic subgroup of U generated by the Lie algebra $\mathfrak{g}$ and K an analytic subgroup of L generated by the Lie algebra $\mathfrak{k}$. Then S_κ is expressed as a homogeneous space G/K associated with the symmetric pair (G, K).

We denote by $\mathfrak{k}_0$ the orthogonal complement of $[\mathfrak{p}_\delta, \mathfrak{p}_\delta]$ in $\mathfrak{l}_0$ relative to the inner product $(\ , \ )$ and put $\delta^\perp = \{H \in \mathfrak{a}; (\delta, H) = 0\}$. Then we have

$$\mathfrak{l} = \mathfrak{k} + \mathfrak{k}_0 + \mathfrak{l}(0) + \mathfrak{l}(1/2)$$

and

$$\mathfrak{p} = \mathfrak{m} + \delta^\perp + \mathfrak{p}(0) + \mathfrak{p}(1/2).$$

Under the identification $T_o(N) = \mathfrak{p}$, we have

$$T_o(S_\delta) = \mathrm{span}_{\mathbb{R}}\{\delta\} + \mathfrak{p}_\delta, \ N_0(S_\delta) = \delta^\perp + \mathfrak{p}(0) + \mathfrak{p}(1/2).$$

It is clear that $(\mathrm{ad}\, X)(N_o(M)) \subset N_o(S_\kappa)$ and $\mathrm{ad}\,(k)(N_o(S_\kappa)) \subset N_o(S_\kappa)$ for any $X \in \mathfrak{k}$ and any $k \in K$.

We prepare some propositions.

PROPOSITION 4.7:
(1) $\mathrm{ad}\,(k)(X) = X$ *for* $k \in K$ *and* $X \in \delta^\perp + \mathfrak{p}(0)$,
(2) $\mathrm{ad}\,(K)(\mathfrak{p}(1/2)) = \mathfrak{p}(1/2)$.

Proof. By Proposition 4.4 (3) we have for any $\alpha \in \Sigma^+(U) - \Sigma_0(U)$ with $\bar{\alpha} = \delta$,

$$(\mathrm{ad}\, S_\alpha)\, \delta^\perp = \{0\},$$

$$(\mathrm{ad}\, S_\alpha)\, \mathfrak{p}(0) \subset \mathfrak{p} \cap N_o(S_\kappa) = \{0\}$$

and

$$(\mathrm{ad}\, S_\alpha)\, \mathfrak{p}(1/2) \subset \mathfrak{p}(1/2).$$

And we have

$$(\mathrm{ad}\, [\mathfrak{p}_\delta, \mathfrak{p}_\delta])\delta^\perp \subset [[\mathfrak{p}_\delta, \delta^\perp], \mathfrak{p}_\delta] = \{0\},$$

$$(\mathrm{ad}\, [\mathfrak{p}_\delta, \mathfrak{p}_\delta])\mathfrak{p}(0) \subset [[\mathfrak{p}_\delta, \mathfrak{p}(0)], \mathfrak{p}_\delta] \cap N_o(S_\kappa) \subset [\mathfrak{l}_\delta, \mathfrak{p}_\delta] \cap N_o(S_\kappa)$$
$$\subset \mathrm{span}_{\mathbb{R}}\{\delta\} \cap N_o(S_\kappa) = \{0\}$$

and

$$(\mathrm{ad}\,[\mathfrak{p}_\delta, \mathfrak{p}_\delta])\mathfrak{p}(1/2) \subset (\mathrm{ad}\,\mathfrak{l}_0)\mathfrak{p}(1/2) \subset \mathfrak{p}(1/2). \qquad \text{Q.E.D.}$$

PROPOSITION 4.8:
(1) $\mathrm{ad}\,(k)X = X$ *for* $k \in K$ *and* $X \in \mathfrak{k}_0 + \mathfrak{l}(0)$,
(2) $\mathrm{ad}\,(K)\mathfrak{k}(1/2) = \mathfrak{k}(1/2)$.

Proof. Using Proposition 4.3 (3) we compute

$$(\mathrm{ad}\,\mathfrak{k})\mathfrak{k}_0 \subset (\mathrm{ad}\,[\mathfrak{p}_\delta, \mathfrak{p}_\delta])\mathfrak{k}_0 + (\mathrm{ad}\,\mathfrak{l}_0)\mathfrak{k}_0$$
$$\subset ([\mathfrak{p}_\delta, \mathfrak{p}_\delta] + \mathfrak{l}_\delta) \cap (\mathfrak{k}_0 + \mathfrak{l}(0) + \mathfrak{l}(1/2)) = \{0\},$$

$$(\mathrm{ad}\,\mathfrak{k})\mathfrak{l}(0) \subset (\mathrm{ad}\,[\mathfrak{p}_\delta, \mathfrak{p}_\delta])\mathfrak{l}(0) + (\mathrm{ad}\,\mathfrak{l}_\delta)\mathfrak{l}(0)$$
$$\subset ([[\mathfrak{p}_\delta, \mathfrak{l}(0)], \mathfrak{p}_\delta] + \mathfrak{l}_\delta) \cap (\mathfrak{k}_0 + \mathfrak{l}(0) + \mathfrak{l}(1/2))$$
$$\subset ([\mathfrak{p}_\delta, \mathfrak{p}_\delta] + \mathfrak{l}_\delta) \subset (\mathfrak{k}_0 + \mathfrak{p}(0) + \mathfrak{p}(1/2)) = \{0\}$$

and

$$(\mathrm{ad}\,\mathfrak{k})\mathfrak{k}(1/2) \subset (\mathrm{ad}\,[\mathfrak{p}_\delta, \mathfrak{p}_\delta])\mathfrak{l}(1/2) + (\mathrm{ad}\,\mathfrak{l}_\delta)\mathfrak{l}(1/2)$$
$$\subset [[\mathfrak{p}_\delta, \mathfrak{l}(1/2)], \mathfrak{p}_\delta] + \mathfrak{l}(1/2)$$
$$\subset [\mathfrak{p}(1/2), \mathfrak{p}_\delta] + \mathfrak{l}(1/2) \subset \mathfrak{l}(1/2). \qquad \text{(Q.E.D.)}$$

PROPOSITION 4.9:
(1) $(\mathrm{ad}\,\mathfrak{m})(\mathfrak{k}_0 + \mathfrak{l}(0)) = \{0\}$,
(2) $(\mathrm{ad}\,\mathfrak{m})\,\mathfrak{l}(1/2) \subset \mathfrak{p}(1/2)$.

Proof. Using Proposition 4.3 (3), we compute

$$(\mathrm{ad}\,\mathfrak{m})\mathfrak{k}_0 \subset [\mathfrak{p}_\delta, \mathfrak{l}_0] \cap N_o(S_\kappa) \subset \mathfrak{p}_\delta \cap N_o(S_\kappa) = \{0\},$$

$$(\mathrm{ad}\,\mathfrak{m})\,\mathfrak{l}(0) \subset [\mathfrak{p}_\delta, \mathfrak{l}(0)] \cap N_o(S_\kappa) \subset \mathfrak{p}_\delta \cap N_o(S_\kappa) = \{0\}$$

and

$$(\mathrm{ad}\,\mathfrak{m})\,\mathfrak{l}(1/2) \subset \mathfrak{p}(1/2) + [\mathfrak{p}_\delta, \mathfrak{l}(1/2)] \subset \mathfrak{p}(1/2). \qquad \text{Q.E.D.}$$

Combining Propositions 4.7, 4.8 and 4.9, we obtain the following.

LEMMA 4.10:
(1) $(\mathrm{ad}\, \mathfrak{g})(\mathfrak{k}_0 + \mathfrak{l}(0) + \delta^\perp + \mathfrak{p}(0)) = \{0\}$, *and*
$(\mathrm{ad}\, \mathfrak{g})(\mathfrak{l}(1/2) + \mathfrak{p}(1/2)) \subset \mathfrak{l}(1/2) + \mathfrak{p}(1/2)$.
(2) $\mathrm{ad}\,(g)\, X = X$ *for any* $g \in G$ *and any* $X \in \mathfrak{k}_0 + \mathfrak{l}(0) + \delta^\perp + \mathfrak{p}(0)$, *and*
$\mathrm{ad}\,(G)(\mathfrak{l}(1/2) + \mathfrak{p}(1/2)) = \mathfrak{l}(1/2) + \mathfrak{p}(1/2)$.

Note that $\{(1/2\pi|\delta|)[T_\alpha, T_\beta]; \alpha, \beta \in \Sigma^+(U) - \Sigma_0(U), \alpha < \beta, \bar{\alpha} = \bar{\beta} = \delta\}$ is an orthonormal basis of $[\mathfrak{m}_\delta, \mathfrak{m}_\delta]$. Indeed, for $\alpha, \beta, \alpha', \beta' \in \Sigma^+(U) - \Sigma_0(U)$ with $\bar{\alpha} = \bar{\beta} = \bar{\alpha}' = \bar{\beta}' = \delta$, from Proposition 4.6 we have

$$
\begin{aligned}
([T_\alpha, T_\beta], [T_\alpha, T_{\beta'}]) &= (R^N(T_\alpha, T_\beta)T_{\beta'}, T_{\alpha'}) \\
&= 4\pi^2|\delta|^2(T_\alpha \wedge T_\beta, T_{\alpha'} \wedge T_{\beta'}).
\end{aligned}
$$

PROPOSITION 4.11: *For any* $\alpha, \beta \in \Sigma^+(U) - \Sigma_0(U)$ *which* $\bar{\alpha} = \bar{\beta} = \delta$ *and any* $X \in \mathfrak{l}(1/2) + \mathfrak{p}(1/2)$ *we have*
(1) $(\mathrm{ad}\, T_\alpha)^2 X = -\pi^2|\delta|^2 X$,
(2) $(\mathrm{ad}\, S_\alpha)^2 X = -\pi^2|\delta|^2 X$ *and*
(3) $(\mathrm{ad}\,(1/2\pi|\delta|)[T_\alpha, T_\beta])^2 X = -\pi^2|\delta|^2 X$.

Proof. Put $\delta_1 = \delta/|\delta|$. For any $\alpha \in \Sigma^+(U) - \Sigma_0(U)$ with $\bar{\alpha} = \delta$, since $[S_\alpha, T_\alpha] \in \mathfrak{m}_{2\delta} + \mathrm{span}_{\mathbb{R}}\{\delta\} = \mathrm{span}_{\mathbb{R}}\{\delta\}$ and $([S_\alpha, T_\alpha], \delta_1) = 2\pi|\delta|$, we have $[S_\alpha, T_\alpha] = 2\pi|\delta|\delta_1$. Proposition 4.4 implies

$$\mathrm{ad}\,(\exp tS_\alpha)\delta_1 = \cos(2\pi|\delta|t)\delta_1 - \sin(2\pi|\delta|t)T_\alpha$$

and

$$\mathrm{ad}\,(\exp tT_\alpha)\delta_1 = \cos(2\pi|\delta|t)\delta_1 + \sin(2\pi|\delta|t)S_\alpha$$

for any real number t. In particular we get

$$\mathrm{ad}\,(\exp(1/4\,|\delta|)S_\alpha)\delta_1 = -T_\alpha$$

and

$$\mathrm{ad}\,(\exp(1/4|\delta|)T_\alpha)\,\delta_1 = S_\alpha.$$

We put $k_\alpha = \exp(1/4|\delta|)S_\alpha \in K$ and $g_\alpha = \exp(1/4\pi|\delta|)T_\alpha \in G$. Then by Lemma 4.10 for $\alpha \in \Sigma^+(U) - \Sigma_0(U)$ with $\bar{\alpha} = \delta$ and $\xi \in \mathfrak{l}(1/2) + \mathfrak{p}(1/2)$ we compute

$$\begin{aligned}(\mathrm{ad}\ T_\alpha)^2\xi &= (\mathrm{ad}\ (\mathrm{ad}\ (k_\alpha)\delta_1))^2\xi \\ &= \mathrm{ad}\ (k_\alpha)\cdot(\mathrm{ad}\ \delta_1)^2\ (\mathrm{ad}(k_\alpha)^{-1}\xi) \\ &= -\mathrm{ad}\ (k_\alpha)(4\pi^2(|\delta|^2/4)\ \mathrm{ad}\ (k_\alpha)^{-1}\xi) \\ &= -\pi^2|\delta|^2\xi,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{ad}\ S_\alpha)^2\xi &= (\mathrm{ad}\ (\mathrm{ad}\ (g_\alpha)\delta_1))^2\xi \\ &= \mathrm{ad}\ (m_\alpha)\cdot(\mathrm{ad}\ \delta_1)^2(\mathrm{ad}\ (g_\alpha)^{-1}\xi) \\ &= -\pi^2|\delta|^2\xi.\end{aligned}$$

Let α, $\beta \in \Sigma^+(U) - \Sigma_0(U)$ with $\bar{\alpha} = \bar{\beta} = \delta$ and $\alpha \neq \beta$. Then there is an element $k \in K$ such that $\mathrm{ad}\ (k)\delta_1 = T_\alpha$ and $\mathrm{ad}\ (k)T_\alpha = T_\beta$. For any $\xi \in \mathfrak{l}(1/2) + \mathfrak{p}(1/2)$, we have

$$\begin{aligned}(\mathrm{ad}\ [T_\alpha, T_\beta])\xi &= (\mathrm{ad}\ [\mathrm{ad}\ (k)\delta_1, \mathrm{ad}\ (k)T_\beta])\xi \\ &= (\mathrm{ad}\ (k)\cdot\mathrm{ad}\ [\delta_1, T_\beta]\cdot\mathrm{ad}(k)^{-1})\xi \\ &= -2\pi|\delta|(\mathrm{ad}\ (k)\cdot\mathrm{ad}\ (S_\alpha)\cdot\mathrm{ad}(k)^{-1})\xi.\end{aligned}$$

Therefore by Lemma 4.10 we have

$$\begin{aligned}(\mathrm{ad}\ [T_\alpha, T_\beta])^2\xi &= 4\pi^2|\delta|^2(\mathrm{ad}\ (k)\cdot\mathrm{ad}\ (S_\alpha)^2\cdot\mathrm{ad}\ (k)^{-1})\xi \\ &= 4\pi^2|\delta|^2(-\pi^2|\delta|^2)\xi.\end{aligned}$$

Q.E.D.

Set

$$\mathfrak{g}^\perp = \mathfrak{k}_0 + \mathfrak{l}(0) + \mathfrak{l}(1/2) + \delta^\perp + \mathfrak{p}(0) + \mathfrak{p}(1/2),$$

$$\mathfrak{g}_0^\perp = \mathfrak{k}_0 + \mathfrak{l}(0) + \delta^\perp + \mathfrak{p}(0)$$

and

$$\mathfrak{g}_1^\perp = \mathfrak{l}(1/2) + \mathfrak{p}(1/2).$$

PROPOSITION 4.12: *The index, nullity and killing nullity of S_κ are given as follows;*

(1) $i(S_\kappa) = \Sigma_{\lambda \in D(G), a_\lambda > -(m(m+1)/2)\pi^2|\delta|^2} m(\lambda)\, d_\lambda$,

(2) $n(S_\kappa) = \Sigma_{\lambda \in D(G), a_\lambda = -(m(m+1)/2)\pi^2|\delta|^2} m(\lambda)\, d_\lambda$,

(3) $n_k(S_\kappa) = \dim(\delta^\perp + \mathfrak{p}(0)) + \dim \mathfrak{g}_1^\perp$
$= (n - m) + \dim \mathfrak{l}(1/2)$,

where $m(\lambda) = \dim \mathrm{Hom}_K(V_\lambda, \mathfrak{p}(1/2)^{\mathbf{C}})$.

Proof. By Lemma 4.11 the Casimir operator of the representation (ad, $\mathfrak{g}_0^\perp$) of G relative to (,) is a zero operator. It follows from Proposition 4.11 that

$$\sum_{\alpha \in \Sigma^+(U) - \Sigma_0(U), \bar{\alpha} = \delta} \{(\mathrm{ad}\, S_\alpha)^2 + (\mathrm{ad}\, T_\alpha)^2\}$$

$$+ \sum_{\alpha, \beta \in \Sigma^+(U) - \Sigma_0(U), \bar{\alpha} = \bar{\beta} = \delta} (\mathrm{ad}\,(1/2\pi|\delta|)[T_\alpha, T_\beta])^2$$

$$= -(m(m+1)/2)\pi^2|\delta|^2 I \qquad \text{on } \mathfrak{g}_1^\perp.$$

Hence the Casimir operator of the representation (ad, $\mathfrak{g}_1^\perp$) of G relative to (,) is the constant operator $-(m(m + 1)/2)\pi^2|\delta|^2 I$. Thus from Theorem 3.3 we get (1), (2) and (3).

THEOREM 4.13

(1) $i(S_\kappa) = 0$,

(2) $n(S_\kappa) = n_k(S_\kappa)$
$= (n - m) + \mathrm{card}\{\alpha \in \Sigma(U);\ \pm 2(\alpha, \delta) = (\delta, \delta)\}/2$.

Proof. Set $p = \{\alpha \in \Sigma(U);\ \pm 2(\alpha, \delta) = (\delta, \delta)\}$. Since $\dim \mathfrak{g}_1^\perp = p$, by Proposition 4.12 we get $n_k(S_\kappa) = (n - m) + p/2$. Let $\Lambda_1, \ldots, \Lambda_r$ (resp. $\Lambda_1', \ldots, \Lambda_s'$) be the fundamental weight system of $\mathfrak{g}$ (resp. $\mathfrak{k}$). We denote by $\lambda(m_1, \ldots, m_r)$ (resp. $\lambda'(l_1, \ldots, l_s)$) the representation of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with the highest weight $m_1\Lambda_1 + \cdots + m_r\Lambda_r$ (resp. $l_1\Lambda_1' + \cdots + l_s\Lambda_s'$). Here $m_1, \ldots, m_r, l_1, \ldots, l_s$ are nonnegative integers. Let $d(m_1, \ldots, m_r)$ (resp. $d'(l_1, \ldots, l_s)$) be the dimension of the representation $\lambda(m_1, \ldots, m_r)$ (resp. $\lambda'(l_1, \ldots, l_s)$).

In each case $\mathfrak{g}$ and $\mathfrak{k}$ are given as follows (cf. Table 1):

(1) If N is a group manifold, then $\mathfrak{g} = so(3) \otimes so(3)$ $(A_1 \oplus A_1)$ and $\mathfrak{k} = so(3)$ (A_1).
(2) If N is of type AII, then $\mathfrak{g} = so(6)$ (D_3) and $\mathfrak{k} = so(5)$ (B_2).

Table 1.

	$N = U/L$	dim N	m	κ	p
AI	$SU(l)/SO(l)$ $(l \geqq 3)$	$(l-1)(l+2)/2$	2	$1/l$	$4(l-2)$
AII	$SU(2l)/Sp(l)$ $(l \geqq 3)$	$(l-1)(2l+1)$	5	$1/4l$	$16(l-2)$
AIII	$SU(l)/S(U(k) \times U(l-k))$ $(l-k \geqq k \geqq 1)$	$2k(l-k)$	2	$1/l$	$4(l-2)$
BDI	$SO(l)/SO(k) \times SO(l-k)$ $(l \geqq 5, l-k \geqq k \geqq 2)$	$k(l-k)$	2	$1/(l-2)$	$4(l-4)$
BDII	$SO(l)/SO(l-1)$ $(l \geqq 3)$	$l-1$	$l-1$	$1/2(l-2)$	0
CI	$Sp(l)/U(l)$ $(l \geqq 2)$	$l(n+1)$	2	$1/(l+1)$	$4(l-1)$
CII	$Sp(l)/Sp(k) \times Sp(l-k)$ $(l-k \geqq k \geqq 1)$	$4l(l-k)$	4	$1/2(l+1)$	$8(l-2)$
DIII	$SO(2l)/U(l)$ $(l \geqq 5)$	$l(l-1)$	2	$1/2(l-1)$	$8(l-2)$
EI	$(e_6, sp(4))$	42	2	1/12	40
EII	$(e_6, su(6) + su(2))$	40	2	1/12	40
EIII	$(e_6, so(10) + R)$	32	2	1/12	40
EIV	(e_6, f_4)	26	9	1/24	32
EV	$(e_7, su(8))$	70	2	1/18	64
EVI	$(e_7, so(12) + su(2))$	64	2	1/18	64
EVII	$(e_7, e_6 + R)$	54	2	1/18	64
EVIII	$(e_8, so(16))$	128	2	1/30	112
EIX	$(e_8, e_7 + su(2))$	112	2	1/30	112
FI	$(f_4, sp(3) + su(2))$	28	2	1/9	28
FII	$(f_4, so(9))$	16	8	1/18	16
G	$(g_2, su(2) + su(2))$	8	2	1/4	8

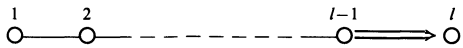
group manifold $N = L \times L/L$		dim N	m	κ	p
A_l	$SU(l+1)$ $(l \geqq 1)$	$l(l+2)$	3	$1/2(l+1)$	$8(l-1)$
B_l	Spin$(2l+1)$ $(l \geqq 2)$	$l(2l+1)$	3	$1/2(2l-1)$	$8(2l-3)$
C_l	$Sp(l)$ $(l \geqq 3)$	$l(2l+1)$	3	$1/2(l+1)$	$8(l-1)$
D_l	Spin $(2l)$ $(l \geqq 4)$	$l(2l-1)$	3	$1/2(2l-2)$	$8(2l-4)$
E_6		78	3	1/24	80
E_7		133	3	1/36	128
E_8		248	3	1/60	224
F_4		52	3	1/18	56
G_2		14	3	1/8	16

Here $m = 1 + m(\delta)$; $\kappa = 4\pi^2|\delta|^2$ and $p = \text{card}\,\{\alpha \in \Sigma(U);\ \pm 2(\alpha, \delta) = (\delta, \delta)\}$. Each compact irreducible symmetric space $N = U/L$ is equipped with the invariant Riemannian metric induced by the Killing–Cartan form of the Lie algebra of U.

(3) If N is of type CII, then $\mathfrak{g} = so(5)$ (B_2) and $\mathfrak{k} = so(4)$ $(A_1 \oplus A_1)$.
(4) If N is of type EIV, then $\mathfrak{g} = so(10)$ (D_5) and $\mathfrak{k} = so(9)$ (B_4).
(5) If N is of type FII, then $\mathfrak{g} = so(9)$ (B_4) and $\mathfrak{k} = so(8)$ (D_4).
(6) If N is otherwise, then $\mathfrak{g} = so(3)$ (A_1) and $\mathfrak{k} = so(2)$ (A_0).

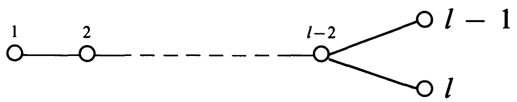
By the formula of Freudenthal (cf. [Te2]) we can determine the eigenvalues of Casimir operators. For example, the eigenvalues of Casimir operators relative to the Killing-Cartan metric are given as follows;

B_l: $so(2l + 1)$ $(l \geqq 1)$



$$\begin{cases} a_{\Lambda_i} = -i(2l + 1 - i)/(4l - 2) & \text{for} \quad 1 \leqq i \leqq l - 1, \\ a_{\Lambda_l} = -\, l(2l + 1)/4(4l - 2). \end{cases}$$

D_l: $so(2l)$ $(l \geqq 3)$



$$\begin{cases} a_{\Lambda_i} = -i(2l - 1)/4(l - 1) & \text{for } 1 \leqq i \leqq l - 2, \\ a_{\Lambda_{l-1}} = -l(2l - 1)/16(l - 1), \\ a_{\Lambda_l} = -l(2l - 1)/16(l - 1). \end{cases}$$

By examining the eigenvalues of Casimir operators case by case, we obtain the following:

$$\{\lambda \in D(\mathfrak{g}) \colon a_\lambda > -(m(m + 1)/2)\pi^2|\delta|^2\}$$

$$= \begin{cases} \{\lambda(0) \otimes \lambda(0)\} & \text{if } N \text{ is a group manifold,} \\ \{\lambda(0, 0, 0)\} & \text{if } N \text{ is of type } A\mathrm{II}, \\ \{\lambda(0, 0)\} & \text{if } N \text{ is of type } C\mathrm{II}, \\ \{\lambda(0, 0, 0, 0, 0,), \lambda(1, 0, 0, 0, 0)\} & \text{if } N \text{ is of type } E\mathrm{IV}, \\ \{\lambda(0, 0, 0, 0), \lambda(1, 0, 0, 0, 0)\} & \text{if } N \text{ is of type } F\mathrm{II}, \\ \{\lambda(0)\} & \text{if } N \text{ is otherwise.} \end{cases}$$

$$\{\lambda \in D(\mathfrak{g}); a_\lambda = -(m(m + 1)/2)\pi^2|\delta|^2\}$$

$$
= \begin{cases} \{\lambda(1) \otimes \lambda(0), \lambda(0) \otimes \lambda(1)\} & \text{if } N \text{ is a group manifold,} \\ \{\lambda(0, 1, 0), \lambda(0, 0, 1)\} & \text{if } N \text{ is of type } A\text{II,} \\ \{\lambda(0, 1)\} & \text{if } N \text{ if ot type } C\text{II,} \\ \{\lambda(0, 0, 0, 1, 0), \lambda(0, 0, 0, 0, 1)\} & \text{if } N \text{ is of type } E\text{IV,} \\ \{\lambda(0, 0, 0, 1)\} & \text{if } N \text{ is of type } F\text{II,} \\ \{\lambda(1)\} & \text{if } N \text{ is otherwise.} \end{cases}
$$

By the direct computations we obtain

$$
\mathfrak{p}(1/2)^{\mathbb{C}}
$$

$$
= \begin{cases} \oplus^{p} \lambda'(1) & \text{if } N \text{ is a group manifold,} \\ \oplus^{2(l-2)} \lambda'(0, 1) & \text{if } N \text{ is of type } A\text{II,} \\ \oplus^{l-2}((\lambda'(1) \oplus \lambda'(0)) \oplus (\lambda'(0) \oplus \lambda'(1))) & \text{if } N \text{ is of type } C\text{II,} \\ \lambda'(0, 0, 0, 1) & \text{if } N \text{ if of type } E\text{IV,} \\ \lambda'(0, 0, 0, 1) & \text{if } N \text{ is of type } F\text{II,} \\ \oplus^{p/2} \lambda_0'(1) & \text{if } N \text{ is otherwise.} \end{cases}
$$

According to the tables for the branching rules in [M–P] we have the following.

(1) $A_1 \oplus A_1 \supset A_1$; $\lambda(a) \otimes \lambda(0) = \lambda'(1)$, $\lambda(0) \otimes \lambda(1) = \lambda'(1)$.
(2) $D_3 \supset B_2$; $\lambda(0, 1, 0) = \lambda'(0, 1)$, $\lambda(0, 0, 1) = \lambda'(0, 1)$.
(3) $B_2 \supset A_1 \oplus A_1$; $\lambda(0, 1) = (\lambda'(1) \otimes \lambda'(0)) \oplus (\lambda'(0) \otimes \lambda'(1))$.
(4) $D_5 \supset B_4$; $\lambda(1, 0, 0, 0, 0) = \lambda'(0, 0, 0, 0) \oplus \lambda'(1, 0, 0, 0)$,
$\lambda(0, 0, 0, 1, 0) = \lambda'(0, 0, 0, 1)$,
$\lambda(0, 0, 0, 0, 1) = \lambda'(0, 0, 0, 1)$.
(5) $B_4 \supset D_4$; $\lambda(1, 0, 0, 0) = \lambda'(0, 0, 0, 0) \oplus \lambda'(1, 0, 0, 0)$,
$\lambda(0, 0, 0, 1) = \lambda'(0, 0, 0, 1) \oplus \lambda'(0, 0, 1, 0)$.
(6) $A_1 \supset A_0$; $\lambda(1) = \lambda_0(1) \oplus \lambda_0(-1)$.

Hence we get

$$\{\lambda \in D(\mathfrak{g});\ a_\lambda > -(m(m+1)/2)\pi^2|\delta|^2,\ \mathrm{Hom}_K(V_\lambda, \mathfrak{p}(1/2)^{\mathbf{C}}) \neq \{0\}\} = \varnothing.$$

and

$$\{\lambda \in D(\mathfrak{g});\ a_\lambda = -(m(m+1)/2)\pi^2|\delta|^2,\ \mathrm{Hom}_K(V_\lambda, \mathfrak{p}(1/2)^{\mathbf{C}}) \neq \{0\}\}$$

$$= \{\lambda \in D(\mathfrak{g});\ a_\lambda = -(m(m+1)/2)\pi^2|\delta|^2\}.$$

Thus we obtain $i(S_\kappa) = 0$. The decomposition of each $(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}}$ as a G-module in each case is given as follows:

(1) If N is a group manifold,

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \oplus^{p/4}((\lambda(1) \otimes \lambda(0)) \oplus (\lambda(0) \otimes \lambda(1))).$$

(2) If N is of type AII,

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \oplus^{2(l-2)}(\lambda(0, 1, 0) \oplus \lambda(0, 0, 1)).$$

(3) If N is of type CII,

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \oplus^{l-2}\lambda(0, 1).$$

(4) If N is of type EIV

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \lambda(0, 0, 0, 1, 0) \oplus \lambda(0, 0, 0, 0, 1).$$

(5) If N is of type FII,

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \lambda(0, 0, 0, 1).$$

(6) If N is otherwise,

$$(\mathfrak{g}_{\mathrm{I}}^{\perp})^{\mathbf{C}} = \oplus^{p/2}\lambda(1).$$

Therefore by Proposition 4.12 $n(S_\kappa) = \dim(\delta^\perp + \mathfrak{p}(0)) + \dim \mathfrak{g}_{\mathrm{I}}^{\perp}$.

Q.E.D.

REMARK:

(1) We know (cf. [Tel]) that

$$\pi_i(N) = \{0\} \quad \text{for } 2 \leqq i \leqq m-1.$$

and

$$\pi_m(N) = \begin{cases} \mathbb{Z} & \text{if } N = \text{group manifold, } BD\text{II, } A\text{II, } C\text{II, } E\text{IV, } F\text{II} \\ & \text{or Hermitian symmetric space,} \\ \mathbb{Z}_2 & \text{if } N = \text{otherwise.} \end{cases}$$

From our result and the second variational formula for a harmonic map we immediately see that if $m = 2$, the inclusion map $\iota: S_\kappa \to N$ is also stable as a harmonic map.

(2) By introducing a calibration by the fundamental 3-form on simple Lie groups, Tasaki [Ts] proved that if N is a group manifold, then S_κ is homologically volume minimizing in its real homology class.

5. A remark on certain submanifolds associated with Helgason spheres

Here we show a generalization of Proposition 3.8 in [O1, p. 214] and Lemma 5 in [O-T, p. 13]. Let $G(m, N) = \bigcup_{x \in N} G(m, T_x(N))$ be the Grassmann bundle of all nonoriented m-planes on an irreducible symmetric space N. Set $\xi_0 = \text{span}_{\mathbb{R}}\{\delta\} + \mathfrak{m}_\delta \subset \mathfrak{m} = T_o(N)$ and $m = \dim \xi_0$. Now we define a subbundle $\mathfrak{H}$ of $G(m, N)$ by

$$\mathfrak{H} = \{\xi \in G(m, N); \xi = (\mathrm{d}\tau)_o(\xi_0) \quad \text{for some} \quad \tau \in I_0(N)\}.$$

An m-dimensional submanifold M of N is called an *$\mathfrak{H}$-submanifold* if every tangent space of M belongs to $\mathfrak{H}$. For example, in case that M is a Hermitian symmetric space, an $\mathfrak{H}$-submanifold is a 2-dimensional complex submanifold. In case $N = P_2(\mathbb{C}\mathrm{ay})$, an $\mathfrak{H}$-submanifold is a Cayley submanifold (cf. [O1], and in case that N is a simple Lie group, an $\mathfrak{H}$-submanifold is a φ-submanifold, that is, a calibrated submanifold associated with the calibration defined by the fundamental 3-form φ of the Lie group N (cf. [Ts], [O-T]). In [O1] and [O-T] we showed that a Cayley submanifold and φ-submanifold are totally geodesic. The following generalizes theirs.

PROPOSITION 5.1: *Let M be an $\mathfrak{H}$-submanifold of N. Then M is a minimal submanifold of N. Moreover if $m = \dim M \geqq 3$, M is a totally geodesic submanifold of N.*

We prepare a lemma in order to show the above proposition.

LEMMA 5.2: *Let N be a locally symmetric space and M a curvature-invariant submanifold of N, that is, $R^N(X, Y)Z \in T_x(M)$ for any $x \in M$ and any X, Y, $Z \in T_x(M)$. Here we denote by R^N and B the curvature tensor of N and the second fundamental form of M, respectively. Then for any vectors $X, Y \in T_x(M)$ we have*

$$\begin{aligned} B(X, R^N(X, Y)Y) &+ B(Y, R^N(Y, X)X) \\ &= R^N(B(X, X), Y)Y + R^N(B(Y, Y)X)X \\ &\quad + R^N(X, B(X, Y))Y + R^N(Y, B(Y, X))X. \end{aligned}$$

Proof. R^N and B satisfy an equation

$$\begin{aligned} B(X, R^N(Y, Z)W) &= R^N(B(X, Y), Z)W + R^N(Y, B(X, Z))W \\ &\quad + R^N(Y, Z)B(X, W) \end{aligned}$$

for any $X, Y, Z, W \in T_x(M)$ (cf. [O1, p. 214]). It follows from this equation that

$$\begin{aligned} B(X, R^N(X, Y)Y) &= R^N(B(X, X), Y)Y + R^N(X, B(X, Y))Y \\ &\quad + R^N(X, Y)B(X, Y) \end{aligned}$$

and

$$\begin{aligned} B(Y, R^N(Y, X)X) &= R^N(B(Y, Y), X)X + R^N(Y, B(X, Y))X \\ &\quad + R^N(Y, X)B(X, Y). \end{aligned}$$

Adding these two equations, we get the desired equation. Q.E.D.

Proof of Proposition 5.1. We fix any point $x \in M$. We can identify $T_x(M)$ and $T_x(N)$ with ξ_0 and $\mathfrak{p}$, respectively. We regard the second fundamental form B as a symmetric bilinear form $B: \xi_0 \times \xi_0 \to \xi_0^{\perp}$, where $\xi_0^{\perp}$ denotes the orthogonal complement of ξ_0 in $\mathfrak{p}$. For any orthonormal vectors $X, Y \in \xi_0$, by $R^N(X, Y)Y = \kappa X$ and Lemma 5.2, we have

$$\begin{aligned} \kappa(B(X, X) + B(Y, Y)) &= -(\mathrm{ad}\ Y)^2 B(X, X) - (\mathrm{ad}\ X)^2 B(Y, Y) \\ &\quad + ((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ X)(\mathrm{ad}\ Y))B(X, Y). \end{aligned} \tag{5.1}$$

Since $\{\delta_1 = \delta/|\delta|, T_\alpha; \alpha \in \Sigma^+(U) - \Sigma_0(U), \bar{\alpha} = \delta\}$ is an orthonormal basis of ξ_0, transforming it by an isometry of N, if necessary, we may assume that $X = \delta_1$ and $Y = T_\alpha$. As $\xi_0^\perp = \delta^\perp + \mathfrak{p}(0) + \mathfrak{p}(1/2)$, we denote by B' and B'' the $(\delta^\perp + \mathfrak{p}(0))$- and $\mathfrak{p}(1/2)$-components of B, respectively. By Lemma 4.10 we have $(\mathrm{ad}\ X)^2 B' = (\mathrm{ad}\ Y)^2 B' = 0$. By Proposition 4.11 we have $(\mathrm{ad}\ X)^2 B'' = -\pi^2|\delta|^2 B'' = -(1/4)\kappa B''$ and $(\mathrm{ad}\ Y)^2 B'' = -(1/4)\kappa B''$. By Lemma 4.10 we have

$$\begin{aligned}&((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ X)(\mathrm{ad}\ Y))\, B(X, Y)\\ &\quad = ((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ X)(\mathrm{ad}\ Y))B''(X, Y).\end{aligned}$$

For any $\beta \in \Sigma^+(U) - \Sigma_0(U)$ with $2(\bar{\beta}, \delta) = (\delta, \delta)$, we compute

$$\begin{aligned}(\mathrm{ad}\ Y)(\mathrm{ad}\ X)T_\beta &= (\mathrm{ad}\ T_\alpha)(\mathrm{ad}\ \delta_1)T_\beta\\ &= (\mathrm{ad}\ T_\alpha)(-2\pi(\bar{\beta}, \delta_1)S_\beta)\\ &= -\pi|\delta|[T_\alpha, S_\beta],\end{aligned}$$

and

$$\begin{aligned}(\mathrm{ad}\ X)(\mathrm{ad}\ Y)T_\beta &= (\mathrm{ad}\ \delta_1)(\mathrm{ad}\ T_\alpha)T_\beta\\ &= [(\mathrm{ad}\ \delta_1)T_\alpha, T_\beta] + [T_\alpha, (\mathrm{ad}\ \delta_1)T_\beta]\\ &= -2\pi(\bar{\alpha}, \delta_1)[S_\alpha, T_\beta] - 2\pi(\bar{\beta}, \delta_1)[T_\alpha, S_\beta].\end{aligned}$$

Hence we get

$$((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ X)(\mathrm{ad}\ Y))T_\beta = -2\pi|\delta|([S_\alpha, T_\beta] - [S_\beta, T_\alpha]).$$

Since $S_\alpha = (X_\alpha + X_{-\alpha})/2$, $T_\alpha = \sqrt{-1}(X_\alpha - X_{-\alpha})/2$ and $\delta + \bar{\beta} \notin \Sigma(U, L)$, we have

$$\begin{aligned}[S_\alpha, T_\beta] - [S_\beta, T_\alpha] &= \sqrt{-1}([X_\alpha, X_\beta] - [X_{-\alpha}, X_{-\beta}])\\ &\in \mathfrak{u}^{\mathbf{C}}_{\delta+\bar{\beta}} + \mathfrak{u}^{\mathbf{C}}_{-(\delta+\bar{\beta})} = \{0\}.\end{aligned}$$

Hence we get $((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ X)(\mathrm{ad}\ Y))\ \mathfrak{p}(1/2) = \{0\}$. Thus $((\mathrm{ad}\ Y)(\mathrm{ad}\ X) + (\mathrm{ad}\ Y)(\mathrm{ad}\ X))\ B(X, Y) = 0$. Therefore (5.1) becomes

$$B'(X, X) + B'(Y, Y) + (3/4)(B''(X, X) + B''(Y, Y)) = 0.$$

It follows from this that $B(X, X) + B(Y, Y) = 0$ for any orthonormal vectors X, Y tangent to M.

Q.E.D.

6. Indices and nullities for totally geodesic submanifolds in compact rank one symmetric spaces

Using our method we can determine the indices, the nullities and the Killing nullities of all compact totally geodesic submanifolds in compact rank one symmetric spaces. In [O1] we stated these results partially. In this section we give their complete table.

PROPOSITION 6.1: *The index and the nullity of each compact totally geodesic submanifold M in a compact rank one symmetric space N are given as in* Table 2. *In each case the nullity is equal to the Killing nullity.*

REMARK: (1) The results of $N = S^n$ and $N = P_n(\mathbb{C})$ were shown by Simons [S] and Kimura [K], respectively. See also [Te3].

(2) Simons [S] showed that a great sphere S^m of S^n is a unique m-dimensional compact minimal submanifold of S^n with the lowest index and the lowest nullity. Kimura [K] showed that a complex projective subspace $P_m(\mathbb{C})$ of $P_n(\mathbb{C})$ is a unique m-dimensional compact minimal submanifold of $P_n(\mathbb{C})$ with the lowest nullity. These results for the nullity are

Table 2.

N	M	index	nullity
S^n	S^m $(1 \leqq m \leqq n-1)$	$n-m$	$(m+1)(n-m)$
$P_n(\mathbb{R})$	$P_m(\mathbb{R})$ $(1 \leqq m \leqq n-1)$	0	$(m+1)(n-m)$
$P_n(\mathbb{C})$	$P_m(\mathbb{R})$ $(1 \leqq m \leqq n)$	$m(m+1)/2$	$m(m+3)/2 + 2(m+1)(n-m)$
	$P_m(\mathbb{C})$ $(1 \leqq m \leqq n-1)$	0	$2(m+1)(n-m)$
$P_n(\mathbb{H})$	S^m $(1 \leqq m \leqq 3)$	$4-m$	$(m+1)(4-m) + 8(n-1)$
	$P_m(\mathbb{R})$ $(2 \leqq m \leqq n)$	$3m(m+1)/2$	$3m(m+3)/2$ $+ 4m(m+1)(n-m)$
	$P_m(\mathbb{C})$ $(2 \leqq m \leqq n)$	$2m(m+2)$	$(m+1)(m+2)$ $+ 4(m+1)(n-m)$
	$P_m(\mathbb{H})$ $(1 \leqq m \leqq n-1)$	0	$4(m+1)(n-m)$
$P_2(\mathbb{C}\mathrm{ay})$	S^m $(1 \leqq m \leqq 7)$	$8-m$	$(m+1)(8-m) + 16$
	$P_2(\mathbb{R})$	21	35
	$P_2(\mathbb{C})$	48	40
	$P_2(\mathbb{H})$	12	28
	$P_1(\mathbb{C}\mathrm{ay})$	0	16

generalized as follows. Let $\mathbb{F}$, $\mathbb{F}' = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{C}$ay. We denote by $P_n(\mathbb{F})$ an n-dimensional projective space over $\mathbb{F}$. Here $n = 1$ or 2 when $\mathbb{F} = \mathbb{C}$ay. Suppose that $\mathbb{F}' \subset \mathbb{F}$ and $P_m(\mathbb{F}')$ is a projective subspace of $P_n(\mathbb{F})$. We fix a point $o \in P_m(\mathbb{F}')$. We call that a submanifold M of $P_n(\mathbb{F})$ is of *type* $P_m(\mathbb{F}')$ if for any $x \in M$ there is an isometry τ of $P_n(\mathbb{F})$ such that $T_x(M) = (d\tau)_o(T_o(P_m(\mathbb{F}')))$. Clearly, $P_m(\mathbb{F}')$ is a submanifold of type $P_m(\mathbb{F}')$. For example, if $\mathbb{F} = \mathbb{C}$ and $\mathbb{F}' = \mathbb{C}$, then a submanifold of type $P_m(\mathbb{F}')$ is a complex submanifold of $P_n(\mathbb{C})$. If $\mathbb{F} = \mathbb{C}$ and $\mathbb{F}' = \mathbb{R}$, then a submanifold of type $P_m(\mathbb{F}')$ is a totally real submanifold of $P_n(\mathbb{C})$. Using the method of Proposition 1.1 we can show the following. Let M be a compact minimal submanifold of type $P_m(\mathbb{F}')$ immersed in $P_n(\mathbb{F})$ and denote by φ its immersion. Then $n(\varphi) \geqq n_k(\varphi) \geqq n(P_m(\mathbb{F}'))$. Moreover, $n_k(\varphi) = n(P_m(\mathbb{F}'))$ if and only if $\varphi(M)$ is congruent to $P_m(\mathbb{F}')$.

(3) An m-dimensional real projective subspace $P_m(\mathbb{R})$ of $P_n(\mathbb{C})$ is not always a totally real compact minimal submanifold with the lowest index. In fact, the totally real compact submanifold $M = SU(3)/SO(3) \cdot \mathbb{Z}_2$ embedded in $P_5(\mathbb{C})$ (cf. [N]) has $i(M) = 8(< 15 = i(P_5(\mathbb{R}))$ and $n(M) = n_k(M) = 27$. It seems that every n-dimensional totally real compact minimal submanifold M embedded in $P_n(\mathbb{C})$ with the parallel second fundamental form has $i(M) = \dim I_0(M)$ and $n(M) = n_k(M) = \dim I_0(P_n(\mathbb{C})) - \dim I_0(M)$.

7. Instability of rectifiable currents on compact symmetric spaces

In this section we show an instability theorem for rectifiable currents on compact minimal submanifolds in a sphere. Moreover using our result and the first standard minimal immersions of compact irreducible symmetric spaces, we show a nonexistence theorem for rectifiable stable currents of certain degrees on some compact symmetric spaces.

Let N be an n-dimensional compact Riemannian manifold and $\mathscr{R}_p(N, A)$ the group of rectifiable p-currents on N over A, where A is a finitely generated abelian group. We denote by $\mathscr{I}_p(N, A)$ the group of integral p-currents on N over A.

THEOREM 7.1 (Federer and Fleming [F-F], Fleming [F1]): *There is a natural isomorphism of the homology groups $H_*(\mathscr{I}_*(N, A))$ with the singular homology groups $H_*(N, A)$.*

THEOREM 7.2 (Federer and Fleming [F-F], Fleming [F1]): *For every nonzero homology class $\alpha \in H_p(\mathscr{I}_*(N, A)) \cong H_p(N, A)$ there is a $\mathscr{S} \in \alpha$ of at least mass in the sense that $0 < \mathbb{M}(\mathscr{S}) \leqq \mathbb{M}(\mathscr{S}')$ for all $\mathscr{S}' \in \alpha$.*

An element $\mathscr{S} \in \mathscr{R}_p(N, A)$ is called *stable* if for every smooth vector field V on N with the flow φ_t there is an $\varepsilon > 0$ such that $\mathbb{M}(\varphi_{t*}\mathscr{S}) \geqq \mathbb{M}(\mathscr{S})$ for all $|t| < \varepsilon$.

THE GENERALIZED PRINCIPLE OF SYNGE

If there are no nonzero stable currents in $\mathscr{R}_p(N, A)$, *then* $H_p(N, A) = \{0\}$. *If* $p = 1$ *and* $A = \mathbb{Z}$, *then not only does* $H_1(N, \mathbb{Z}) = \{0\}$ *but* N *is also simply connected. If* $p = 2$ *and* $A = \mathbb{Z}$, *then not only does* $H_2(N, \mathbb{Z}) = \{0\}$ *but* $\pi_2(N) = \{0\}$.

This is due to Theorem 1 and 2, the classical result of closed geodesics and the theorem of Sacks–Uhlenbeck for the existence of stable minimal 2-spheres (cf. [S–U]).

Let $\xi \in \wedge_p T_x(N)$ be a unit simple p-vector. For any vector field V on N with the flow φ_t, we define a quadratic form Q_ξ by $Q_\xi(V) = (\mathrm{d}^2/\mathrm{d}t^2)|\varphi_{t*}\xi||_{t=0}$. For any $\mathscr{S} \in \mathscr{R}_p(N, A)$, we define a quadratic form $Q_{\mathscr{S}}$ by $Q_{\mathscr{S}}(V) = (\mathrm{d}^2/\mathrm{d}t^2)\mathbb{M}(\varphi_{t*}\mathscr{S})|_{t=0}$. Then we have $Q_{\mathscr{S}}(V) = \int Q_{\mathscr{S}_x}(V)\, \mathrm{d}\|\mathscr{S}\|$ (cf. [Fe]).

Now assume that N is isometrically immersed in a Euclidean space $\mathbb{E}^N$ and does not lie in any proper hyperplane of $\mathbb{E}^N$. We put $\mathscr{V} = \{\mathrm{grad}\, f_v \in \Gamma(T(N));\ v \in \mathbb{E}^N\}$, where $f_v(x) = \langle x, v\rangle$ for $x \in N$. There is a natural isomorphism $\mathscr{V} \cong \mathbb{E}^N$. Then the trace of Q_ξ on $\mathscr{V}$ with respect to the inner product induced from $\mathbb{E}^N$ is given as follows;

PROPOSITION 7.3 (cf. [O1]):

$$\mathrm{Tr}\, Q_\xi = \sum_{i=1}^{p} \sum_{\alpha=1}^{q} (2\|B(e_i, n_\alpha)\|^2 - \langle B(e_i, e_i), B(n_\alpha, n_\alpha)\rangle),$$

where B *denotes the second fundamental form of* N *and* $\{e_1, \ldots, e_p, n_1, \ldots, n_q\}$ *is an orthonormal basis of* $T_x(N)$ *with* $\xi = e_1 \wedge \cdots \wedge e_p$.

Denote by R the curvature tensor of N and define the sectional curvature of N as $K(X \wedge Y) = \langle R(X, Y)Y, X\rangle/\|X \wedge Y\|^2$. The Ricci tensor of N is defined by $\mathrm{Ric}\,(X, Y) = \Sigma_{i=1}^n \langle R(X, e_i)e_i, Y\rangle$, where $\{e_i\}$ is an orthonormal basis of $T_x(N)$. The mean curvature vector η of N is defined by $\eta = (1/n)\Sigma_{i=1}^n B(e_i, e_i)$. The equation of Gauss is given as follows;

$$\langle R(X, Y)Z, W\rangle = \langle B(X, W), B(Y, Z)\rangle - \langle B(X, Z), B(Y, W)\rangle$$

for $X, Y, Z, W \in T_x(N)$.

Put $e_{p+\alpha} = n_\alpha$ for $\alpha = 1, \ldots, q$. We use the following convention on the range of indices; $A, B, \ldots = 1, \ldots, n,\ i, j, \ldots = 1, \ldots, p,\ \alpha, \beta, \ldots =$

$p + 1, \ldots, n$. Then we compute

$$\begin{aligned}\operatorname{Tr} Q_\xi &= 2\sum_{i,A} \langle B(e_i, e_A), B(e_i, e_A)\rangle - 2\sum_{i,j} \langle B(e_i, e_j), B(e_i, e_j)\rangle \\ &\quad - \sum_{i,A} \langle B(e_i, e_i), B(e_A, e_A)\rangle + \sum_{i,j} \langle B(e_i, e_i), B(e_j, e_j)\rangle \\ &= 2\sum_{i,A} (\langle B(e_i, e_A), B(e_A, e_i)\rangle - \langle B(e_i, e_i), B(e_A, e_A)\rangle) \\ &\quad - 2\sum_{i,j} (\langle B(e_i, e_j), B(e_i, e_j)\rangle - \langle B(e_i, e_i), B(e_j, e_j)\rangle) \\ &\quad + \sum_{i,A} \langle B(e_i, e_i), B(e_A, e_A)\rangle - \sum_{i,j} \langle B(e_i, e_i), B(e_j, e_j)\rangle.\end{aligned}$$

By the equation of Gauss we get

$$\begin{aligned}\operatorname{Tr} Q_\xi &= -2\sum_{i} \operatorname{Ric}(e_i, e_i) + 2\sum_{i,j} K(e_i \wedge e_j) \\ &\quad + n\sum_{i} \langle B(e_i, e_i), \eta\rangle - \sum_{i,j} \langle B(e_i, e_i), B(e_j, e_j)\rangle \\ &= -2\sum_{i} \operatorname{Ric}(e_i, e_i) + \sum_{i,j} K(e_i \wedge e_j) \\ &\quad + \sum_{i} \langle B(e_i, e_i), \eta\rangle - \sum_{i,j} \|B(e_i, e_j)\|^2. \qquad (7.1)\end{aligned}$$

If N is a submanifold of a hypersphere $S^{N-1}(r)$ with the radius r, (7, 1) becomes

$$\begin{aligned}\operatorname{Tr} Q_\xi &= -2\sum_{i} \operatorname{Ric}(e_i, e_i) + \sum_{i,j} K(e_i \wedge e_j) \\ &\quad + n\sum_{i} \langle B(e_i, e_i), \eta\rangle - \sum_{i,j} \|B^s(e_i, e_j)\|^2 - (p/r^2),\end{aligned}$$

where B^s denotes the second fundamental form of N in $S^{N-1}(r)$. Further suppose that N is a minimal submanifold of $S^{N-1}(r)$. Then we have

$$\begin{aligned}\operatorname{Tr} Q_\xi &= -2\sum_{i} \operatorname{Ric}(e_i, e_i) + \sum_{i\neq j} K(e_i \wedge e_j) + p(m-1)/r^2 \\ &\quad - \sum_{i} \|B^s(e_i, e_i)\|^2 - \sum_{i\neq k} \|B^s(e_i, e_j)\|^2.\end{aligned}$$

Hence we get

$$\operatorname{Tr} Q_\xi \leqq -2pc + p(p-1)\kappa + p(n-1)/r^2 - p\beta, \tag{7.2}$$

where c = the minimum of the Ricci curvature of N, κ = the maximum of the sectional curvature of N, $\beta = \operatorname{Min}\{\|B^s(X, X)\|^2; X \text{ is a unit vector on } N\}$. Similarly, we have

$$\operatorname{Tr} Q_\xi \leqq -2qc + q(q-1)\kappa + q(n-1)/r^2 - q\beta. \tag{7.3}$$

Thus from (7.2) and (7.3) we obtain the following.

THEOREM 7.4: *Let N be an n-dimensional compact minimal submanifold of a sphere $S^{N-1}(r)$ with the radius r. If an integer k with $1 \leqq k \leqq n$ satisfies $k < \{\kappa + 2c - (n-1)/r^2 + \beta\}/\kappa$, then there exists no rectifiable stable p-current on N for $1 \leqq p \leqq k$ or $1 \leqq n - p \leqq k$. In particular $H_p(N, A) = H_{n-p}(N, A) = \{0\}$ for $1 \leqq p \leqq k$ and any finitely generated abelian group A.*

Let N be an n-dimensional compact minimal submanifold of a sphere with the radius r.

COROLLARY 7.5: *If $\operatorname{Ric} > (n-1)/2r^2$, then $H_1(N, A) = H_{n-1}(N, A) = \{0\}$ for any finitely generated abelian group A and N is simply connected.*

Proof. Since $c > (n-1)/2r^2$, $\kappa > 0$ and $\beta \geqq 0$, we get

$$\{\kappa + 2c - (n-1)/r^2 + \beta\}/\kappa \geqq \{\kappa + 2c - (n-1)/r^2\}/\kappa > 1.$$

Q.E.D.

We denote by μ the maximum of the curvature operator acting on 2-forms of N.

COROLLARY 7.6: *If $n/r^2 - 2c + 2\mu < 0$, then $H_i(N, A) = H_{n-i}(N, A) = \{0\}$ for $i = 1, 2, 3$ and finitely generated abelian group A, and $\pi_1(N) = \pi_2(N) = \pi_3(N) = \{0\}$.*

Proof. Since $\mu \geqq \kappa > 0$, we have

$$\{\kappa + 2c - (n-1)/r^2 + \beta\}/\kappa > (\kappa + 2\mu + 1/r^2 + \beta)/\kappa$$

$$\geqq (3\kappa + 1/r^2 + \beta)/\kappa > 3.$$

Q.E.D.

REMARK: (1) In [O2] we showed that if Ric $> n/r^2$, then N is harmonically unstable, in particular $\pi_1(N) = \pi_2(N) = \{0\}$. Corollary 7.5 is sharp in the following sense; if N is an n-dimensional real projective space imbedded in a unit sphere by its first standard minimal immersion, then $\pi_1(N) = \mathbb{Z}_2$, $H_1(N, \mathbb{Z}_2) = \mathbb{Z}_2$ and Ric $= \{(n-1)/2\}\{n/1(n+1)\}$.

(2) In [K-O-T] we showed that if $n/r^2 - 2c + 2\mu < 0$, then N is Yang-Mills unstable and $\pi_1(N) = \pi_2(N) = \{0\}$.

THEOREM 7.7: *Let N be an n-dimensional simply connected compact irreducible symmetric space belonging to the following list;* (1) S^n, (2) $P_2(\mathbb{C}\mathrm{ay})$, (3) $G_{p,q}(\mathbb{H})$, (4) $Sp(l)$, (5) $SU(3)/SO(3)$, (6) $SU(3)$, (7) $SU(6)/Sp(3)$, (8) E_6/F_4, (9) *Hermitian symmetric spaces. Denote by m the dimension of the Helgason sphere of N. If $\mathscr{S} \in \mathscr{R}_p(N, A)$ and $1 \leqq p \leqq m-1$ or $n-m+1 \leqq p \leqq m-1$, then $\mathscr{S}$ is unstable.*

Proof. Suppose that N is associated with a symmetric pair (U, L) and g is the invariant Riemannian metric on N induced by the Killing–Cartan form of U. By the theorem of Takahashi [Ta], N is isometrically realized as a minimal submanifold in a sphere $S^{m(1)}(\sqrt{n/\lambda_1}) \subset \mathbb{E}^{m(1)+1}$ by its first standard minimal immersion. Here λ_1 and $m(1)+1$ denote the first eigenvalue and its multiplicity for the Laplace–Beltrami operator of (N, g) acting on functions. The complete lists of n, κ and λ_1 for each case are given in Table I and [K-O-T]. It is known that $c = 1/2$ for every case. By direct computations, the value β for each case except Hermitian symmetric spaces is given as follows: (1) $\beta = 0$, (2) $\beta = 1/72$, (3) $\beta = (p-q)/4pq(p+q)$, (4) $\beta = 0$, (5) $\beta = 1/9$, (6) $\beta = 1/18$, (7) $\beta = 1/36$, (8) $\beta = 1/72$. Using these datas and Theorem 7.4, the straightforward computations imply the above conclusion.

Suppose that N is a Hermitian symmetric space. Denote by J the complex structure on N. Let $\mathscr{H}$ be the Lie algebra of all holomorphic vector fields on N and $\mathscr{K}$ the Lie subalgebra of $\mathscr{H}$ consisting of all Killing vector fields on N. By the theorem of Matsushima, we have a direct sum $\mathscr{H} = \mathscr{K} + J\mathscr{K}$. Set $\mathscr{V} = J\mathscr{K}$. We equipe an inner product on $\mathscr{V}$ induced from the invariant inner product of the Lie algebra $\mathscr{K}$. Lawson and Simons [L-S, p. 447, (5.4)] computed the trace of the quadratic form Q_ξ on $\mathscr{V}$ associated with any $\xi \in \wedge_p T_x(N)$ as follows;

$$\operatorname{tr} Q_\xi = \sum_{i=1}^{p} \sum_{\alpha=1}^{q} (K(e_i \wedge Jn_\alpha) - K(e_i \wedge n_\alpha)). \tag{7.4}$$

(7.4) becomes

$$\operatorname{tr} Q_\xi = \sum_{i=1}^{p} \left(\operatorname{Ric}(Je_i, Je_i) - \sum_{j=1}^{p} K(Je_i \wedge e_j) - \operatorname{Ric}(e_i, e_i) + \sum_{j=1}^{p} K(e_i \wedge e_j) \right)$$

$$= \sum_{i,j=1}^{p} (K(e_i \wedge e_j) - K(Je_i \wedge e_j)).$$

If $p = 1$, then we have

$$\operatorname{tr} Q_\xi = -K(Je_1 \wedge e_1) = -H(e_1) < 0.$$

If $p = 2$, then we have

$$\operatorname{tr} Q_\xi = 2K(e_1 \wedge e_2) - 2K(Je_1 \wedge e_2) - K(Je_1 \wedge e_1) + K(Je_2 \wedge e_2)$$

$$= -H(e_1 - Je_2, e_1 + Je_1) \leqslant 0.$$

Here $H(X)$ and $H(X, Y)$ denote the holomorphic sectional curvature and the holomorphic bisectional curvature of N, respectively.

Q.E.D.

REMARK. The above result is proved partially in [O1]. We conjecture that the same conclusion holds for every simply connected compact irreducible symmetric space.

Acknowledgement

The author would like to thank Dr H. Tasaki for stimulating conversation and valuable suggestion.

References

[C-L-N] B-Y. Chen, P-F. Leung and T. Nagano, Totally geodesic submanifolds of symmetric spaces III, preprint.

[Fe] H. Federer, Geometric measure theory, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York (1969).

[F-F] H. Federer and W.H. Fleming, Normal and integral currents, *Ann. of Math.* 72 (1960) 458–520.

[Fl] W.H. Fleming, Flat chains over a finite coefficient group, *Trans. Amer. Math. Soc.* 121 (1966) 160–186.

[H1] S. Helgason, Totally geodesic spheres in compact symmetric spaces, *Math. Ann.* 165 (1966) 309–317.

[H2] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, San Francisco, London (1978).

[H-W] R. Howard and S.W. Wei, On the existence and non-existence of stable submanifolds and currents in positively curved manifolds and the topology of submanifolds in Euclidean spaces, preprint.

[K] Y. Kimura, The nullity of compact Kaehler manifolds in complex projective spaces, *J. Math. Soc. Japan* 29 (1977) 561–580.

[K-N] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I, II, Wiley-Interscience, New York, (1963, 1969).

[K-O-T] S. Kobayashi, Y. Ohnita and M. Takeuchi, On instability of Yang-Mills connections, *Math. Z.* 193 (1986) 165–189.

[L-S] H.B. Lawson, Jr. and J. Simons, On stable currents and their application to global problems in real and complex geometry, *Ann. of Math.* 98 (1973) 427–450.

[M-P] W.G. Mckay and J. Patera, Tables of dimensions, indices, and branching rules for representations of simple Lie algebras, *Lecture Notes in Pure and Applied Mathematics 69*, Marcel Dekker, Inc., New York and Baesel (1981).

[N] H. Naitoh, Totally real parallel submanifolds in $P^n(\mathbb{C})$, Tokyo *J. Math.* 29 (1981) 291–306.

[O1] Y. Ohnita, Stable minimal submanifolds in compact rank one symmetric spaces, *Tohoku Math. J.* 38 (1986) 199–217.

[O2] Y. Ohnita, Stability of harmonic maps and standard minimal immersions, *Tohoku Math. J.* 36 (1986) 259–267.

[O-T] Y. Ohnita and H. Tasaki, Uniqueness of certain 3-dimensional homologically volume minimizing submanifolds in compact simple Lie groups, *Tsukuba J. Math.* 10 (1986) 11–16.

[S-U] J. Sacks and K. Uhlenbech, The existence of minimal immersions of 2-spheres, *Ann. of Math.* 113 (1981) 1–24.

[S] J. Simons, Minimal varieties in riemannian manifolds, *Ann. of Math.* 88 (1968) 62–105.

[Ta] T. Takahashi, Minimal immersions of Riemannian manifolds, *J. Math. Soc. Japan* 18 (1966), 380–385.

[Te1] M. Takeuchi, On the fundamental group and the group of isometries of a symmetric space, *J. Fac. Sci. Univ. Tokyo* 10 (1964) 88–123.

[Te2] M. Takeuchi, Modern theory of spherical functions (in Japanese), Iwanami, Tokyo, 1975.

[Te3] M. Takeuchi, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, *Tohoku Math. J.* 36 (1984) 293–314.

[Ts] H. Tasaki, Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces, *Tsukuba J. Math.* 9 (1985) 117–131.

[Wa] N.R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, Inc., New York (1973).

[We] S.W. Wei, Classification of stable currents in the product of spheres, preprint.