COMPOSITIO MATHEMATICA

MASATO KURIHARA

On two types of complete discrete valuation fields

Compositio Mathematica, tome 63, nº 2 (1987), p. 237-257
<http://www.numdam.org/item?id=CM_1987__63_2_237_0>

Compositio Mathematica 63: 237–257 (1987)

On two types of complete discrete valuation fields

MASATO KURIHARA

Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo, 113, Japan

Received 13 May 1986; accepted in revised form 9 March 1987

Introduction

In this paper, we divide complete discrete valuation fields of mixed characteristics $(0, p)$ whose residue fields are not necessarily perfect, into two types (type-I and type-II; for the definition, see §1 (1.3)), and show that there is much difference between the structures of Milnor K-groups and between the abelian extensions for these two types.

Let K be a complete discrete valuation field and A be the valuation ring of K with the maximal ideal m_A and the residue field F. We do not have a satisfactory ramification theory of K unless F is perfect. One way to study the ramification (especially wild ramification) in the imperfect residue field case, is to use local class field theory of Kato and Parsin and to study Milnor K-groups $K_q^M(K)$ for $q \geqslant 1$ (cf. [10] or §3; for the definition of Milnor K-groups, cf. Conventions). For $i \geqslant 1$, we denote by U_q^i (resp. V_q^i) the subgroup of the q-th Milnor K-group $K_q^M(K)$ generated by the elements of the form $\{1 + a, b_1, \ldots, b_{q-1}\}$ such that $a \in m_A{}^i$ and $b_j \in A^*$ (resp. $b_j \in K^*$) for $1 \leqslant j \leqslant q - 1$. Then, we have $V_q^{i+1} \subset U_q^i \subset V_q^i$ for $i \geqslant 1$ (cf. §2). When F is perfect, the group we consider is $K_1^M(K) = K^*$, in which case, $U_1^i (= V_1^i)$ corresponds to the i-th upper ramification group of the maximal abelian extension of K (cf. [17], [18]) and it is clear that $U_1^i/U_1^{i+1} \simeq F$. By analogy with the perfect residue case, it seems fundamental to know the structures of V_q^i/V_q^{i+1} for $i \geqslant 1$. If K is of equal characteristic, the structures of V_q^i/V_q^{i+1} are determined and described in terms of the residue field F (cf. [1]). But in the case of mixed characteristics, the appearance of them is quite different, it depends on whether K is of type-I or type-II for large i.

In the following, we always assume that K is of mixed characteristics with residue field F. In §2, we will have,

THEOREM 1. *Suppose that K is of type-I (resp. type-II and F has a finite p-base of order $q - 1$). Let U_q^i and V_q^i be the subgroup of $K_q^M(K)$ as above. Then, we* *have* $U_q^i = V_q^i$ *(resp.* $U_q^i = V_q^{i+1}$*) for sufficiently large i. In particular, in resp. case, for sufficiently large i such that* $(i, p) = 1$, *we have* $V_q^i/V_q^{i+1} = 0$.

We remark that if K is of equal characteristic $p > 0$ and F has a finite p-base of order $q - 1 > 0$, we have $V_q^i \neq U_q^i$ if $p \mid i$, and $U_q^i \neq V_q^{i+1}$ if i is prime to p, hence $V_q^i/V_q^{i+1} \neq 0$ for all i (cf. [1]).

By using the above theorem, we will see phenomena about abelian extensions of K that we do not encounter in the perfect residue field case. For example (cf. Th. (3.1)),

COROLLARY 2. *Let K be of type-II. Then, there is no totally ramified cyclic extension of degree* p^n *if n is sufficiently large.*

Recall that this never happens if the residue field F is perfect. In fact, in that case, there is a totally ramified $\mathbb{Z}_p$-extension of K.

The above Theorem 1 has the following (semi-)global application. Let X be a proper smooth variety over a local field k which has good reduction. (Here, a local field means a finite extension of the p-adic number field $\mathbb{Q}_p$.) In §4, we define a kind of idele class group $SK_1^{\text{top}}(X)$ which has an explicit presentation by Milnor K-groups. This group approximates the abelianized fundamental group $\pi_1^{ab}(X)$ by class field theory of X. We have a commutative diagram

$$\begin{array}{ccc} SK_1^{\text{top}}(X) & \xrightarrow{N} & k^* \\ \downarrow & & \downarrow \\ \pi_1^{ab}(X) & \longrightarrow & \mathrm{Gal}(k^{ab}/k), \end{array}$$

where the map labeled N is the norm map (which will be defined in §4) and the vertical arrows are the reciprocity maps which have dense images. The next theorem is considered as the local field version of the finiteness of $CH_0(Y)^{\deg 0}$ (the degree zero part of the group of zero cycles modulo rational equivalence) for a proper smooth variety Y over a finite field.

THEOREM 3 (cf. Th. (4.2.2)). *The norm map*: $SK_1^{\text{top}}(X) \to k^*$ *has finite kernel. Further, the natural map*: $\pi_1^{ab}(X) \to \mathrm{Gal}(k^{ab}/k)$ *has finite kernel.*

The second statement can also be proved by using the Albanese variety (cf. [2] Prop. (2.4)), but here, we give a purely class field theoretic proof of this result. In the case of $\dim X = 1$, Coombes [5] studied the etale coverings of

X by using a different definition of $SK_1^{\mathrm{top}}(X)$. It is due to him to analyze the idele class groups by using the "π-adic" filtration.

Conventions

For a ring R, R^* denotes the group of all invertible elements in R. For $q \geqslant 0$, the q-th Milnor K-group $K_q^M(R)$ is defined to be $(R^* \otimes \underset{q\text{-times}}{\cdots} \otimes R^*)/J$ where J is the subgroup generated by the elements of the form $a_1 \otimes \ldots \otimes a_q$ such that $a_i + a_j = 1$ for some $i \neq j$. The class of $a_1 \otimes \ldots \otimes a_q$ is denoted by $\{a_1, \ldots, a_q\}$.

For a complete discrete valuation field K of mixed characteristics $(0, p)$, v_K denotes the discrete valuation and e_K denotes the absolute ramification index ($= v_K(p)$). A finite extension L/K is called *fiercely ramified* if $[L:K] = [M:F]_{\mathrm{insep}}$ where M and F are the residue fields of L and K, respectively.

For a scheme T, T_i (resp. T^i) denotes the set of all points of dimension i (resp. codimension i) in T.

§1. Definition of type-I and type-II

Let K be a complete discrete valuation field of mixed characteristics $(0, p)$ and A, m_A and F be the valuation ring, the maximal ideal and the residue field, respectively. We denote by Ω_A^1 the absolute differential module ($= \Omega_{A/\mathbb{Z}}^1$) and $\hat{\Omega}_A^1$ is defined to be the completion of Ω_A^1 with respect to the m_A-adic topology. We shall describe the structure of this A-module in order to define the notion of type-I and type-II.

An A-module M is called pro-free if it is the m_A-adic completion of a free A-module.

LEMMA (1.1). *Let I be a p-base of F. For any lifting $\tilde{I}$ of I and any prime element π of A, $\hat{\Omega}_A^1$ is topologically generated by the elements $\mathrm{d}\pi$ and $\mathrm{d}t$ $(t \in \tilde{I})$. More precisely, $\hat{\Omega}_A^1$ is a quotient of a pro-free A-module with a base $(\mathrm{d}\pi, \mathrm{d}t\ (t \in \tilde{I}))$ by an A-submodule generated by one element of the form $a \cdot \mathrm{d}\pi + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t$ where $a, b_t \in A$ and $a \neq 0$. Put $m = \inf\{v_K(a)\} \cup \{v_K(b_t) \mid t \in \tilde{I}\}$. Then, as an A-module, the torsion part $(\hat{\Omega}_A^1)_{\mathrm{tors}}$ of $\hat{\Omega}_A^1$ is of finite length m, and generated by the element $(\pi^{-m}a)\mathrm{d}\pi + \Sigma_{t \in \tilde{I}}(\pi^{-m}b_t)\mathrm{d}t$.*

Proof. By [4] IX §2 Th. 1, there exists a Cohen subring A_0 of A containing $\tilde{I}$ (this means that A_0 is a complete discrete valuation subring containing $\tilde{I}$

such that p is a prime element of A_0 and $A_0/pA_0 \xrightarrow{\sim} A/m_A = F$). Since A_0 is formally smooth over $\mathbb{Z}$ ([7] Chap. 0 Th. (19.8.2)), $\Omega^1_{A_0} \otimes \mathbb{Z}/p^n$ is a free A_0/p^n-module (loc. cit. Cor. (20.4.1)), and we can take $\mathrm{d}t$ ($t \in \tilde{I}$) as its base, namely,

$$\Omega^1_{A_0} \otimes \mathbb{Z}/p^n \simeq \oplus_{t \in \tilde{I}} (A_0/p^n) \cdot \mathrm{d}t.$$

Let $f(T)$ be a monic minimal polynomial over A_0 of some prime element π of A. Note that $A \simeq A_0[T]/(f(T))$. The exact sequence

$$(f(T))/(f(T))^2 \otimes \mathbb{Z}/p^n \xrightarrow{d} \Omega^1_{A_0[T]}/(f(t)) \otimes \mathbb{Z}/p^n \to \Omega^1_A \otimes \mathbb{Z}/p^n \to 0$$

satisfies the Mittag-Leffler condition, so the inverse limit of this sequence is still exact. Thus, $\hat{\Omega}^1_A$ is topologically generated by the elements $\mathrm{d}\pi$ and $\mathrm{d}t$ ($t \in \tilde{I}$) having the one relation which comes from $\mathrm{d}(f(t)) = 0$. We write this relation of the form $a \cdot \mathrm{d}\pi + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t = 0$ in $\hat{\Omega}^1_A$. Note that $a = f'(\pi)$, hence $a \neq 0$. The assertions of Lemma (1.1) follows from these facts.

COROLLARY (1.2). *The following conditions are equivalent.*
(i) *The natural homomorphism* $(\hat{\Omega}^1_A)_{\mathrm{tors}} \to \Omega^1_F$ *is the* 0-*map.*
(ii) *For a lifting* $\tilde{I}$ *of a p-base of F and a prime element* π *of A, we have a relation in* $\hat{\Omega}^1_A$ *of the form* $a \cdot \mathrm{d}\pi + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t = 0$ *such that* $v_K(a) < v_K(b_t)$ *for all* $t \in \tilde{I}$.

DEFINITION (1.3). We call K of type-I if one of the equivalent conditions in Corollary (1.2) is satisfied. Otherwise, we call K of type-II.

EXAMPLE (1.3.1). If F is perfect, we have $\Omega^1_F = 0$, therefore, K is of type-I by (1.2) (i).

EXAMPLE (1.3.2). Suppose that we have a relation in $\hat{\Omega}^1_A$ $a \cdot \mathrm{d}\pi + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t = 0$ ($a \neq 0$) for some prime element π and for some lifting $\tilde{I}$ of a p-base. We put $m = \inf_{t \in \tilde{I}}(v_K(b_t) - v_K(a))$. Then, $L = K(\sqrt[p]{\pi})$ is of type-I if and only if $m \geqslant e_K + 1$. For example, let $K = \mathrm{Frac}(\widehat{\mathbb{Z}_p[T]_{(p)}})$ be the fraction field of the completion of the local ring of $\mathbb{Z}_p[T]$ at the prime ideal (p) where T is a variable. Then $K(\sqrt[p]{pT})$ is of type-II and $K(\sqrt[p]{p(1 + pT)})$ is of type-I.

PROPOSITION (1.4). *Let K/k be an extension of complete discrete valuation fields of mixed characteristics whose valuation rings we denote by A and R, respectively. If A is formally smooth over R, K and k are of the same type.*

Proof. Since A is formally smooth over R, $\hat{\Omega}^1_{A/R}$ is a pro-free A-module ([7] Cor. (20.4.11)) and the following sequence is exact and splits (loc. cit. Prop. (20.7.18)).

$$0 \to (\hat{\Omega}^1_R \otimes A)\hat{} \to \hat{\Omega}^1_A \to \hat{\Omega}^1_{A/R} \to 0$$

Thus, we have $(\hat{\Omega}^1_A)_{\mathrm{tors}} = (\hat{\Omega}^1_R \otimes A)\hat{}_{\mathrm{tors}}$. This shows that a generator of $(\hat{\Omega}^1_R)_{\mathrm{tors}}$ is still a generator of $(\hat{\Omega}^1_A)_{\mathrm{tors}}$. On the other hand, let F (resp. F_0) be the residue field of A (resp. R). Since the extension F/F_0 is separable, $\Omega^1_{F_0} \to \Omega^1_F$ is injective. So, we obtain the proposition from Corollary (1.2) (i).

Corollary (1.5). *Suppose that there exists a discrete valuation subfield k of K such that a prime element of the valuation ring R of k is still a prime element of A and the residue field of k is perfect. Then, K is of type-I.*

Proof. This follows from Proposition (1.4) and Example (1.3.1).

Corollary (1.6). *If L/K is tamely ramified, K and L are of the same type.*

Proof. By Proposition (1.4), we may assume that the residue field F of K is separably closed. In this case, L is defined by the equation $X^i = \pi_K$ where i is an integer prime to p and π_K is a prime element of A. Then, a solution π_L of this equation is a prime element of the valuation ring B of L. If in $\hat{\Omega}^1_A$ we have a relation $a \cdot \mathrm{d}\pi_K + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t = 0$, then in $\hat{\Omega}^1_B$ we have $ai\pi_L^{i-1} \cdot \mathrm{d}\pi_L + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t = 0$. This corollary follows from the fact that $0 \leqslant v_L(i\pi_L^{i-1}) < i$, and that $v_L(a)$ and $v_L(b_t)$ $(t \in \tilde{I})$ are divisible by i.

Proposition (1.7). *Let L/K be a totally ramified (resp. fiercely ramified cf. Conventions) extension. If K is of type-II (resp. of type-I), L is of type-II (resp. of type-I).*

Proof. Let B be the valuation ring of L. This proposition is immediate from the following claim.

(1.7.1) In each case, the natural homomorphism: $(\hat{\Omega}^1_A)_{\mathrm{tors}} \otimes B \to (\hat{\Omega}^1_B)_{\mathrm{tors}}$ is surjective.

For the proof of (1.7.1), we only consider the case when L/K is totally ramified. The other case can be proved by the same method. Let $a \cdot \mathrm{d}\pi + \Sigma_{t \in \tilde{I}} b_t \cdot \mathrm{d}t$ be a generator of $(\hat{\Omega}^1_A)_{\mathrm{tors}}$. Since K is of type-II, at least one of b_t

is a unit. Note that since L/K is totally ramified, $\hat{\Omega}^1_B$ is topologically generated by $d\pi_L$ and dt ($t \in \tilde{I}$) where π_L is a prime element of B (cf. Lemma (1.1)). Put $d\pi = \alpha \cdot d\pi_L + \Sigma_{t\in\tilde{I}}\beta_t \cdot dt$ in $\hat{\Omega}^1_B$. The elements α and β_t of B are divisible by π_L. In fact, $v_L(\alpha) = \mathrm{length}_B(\Omega^1_{B/A})$ and by considering the above relation in Ω^1_F, we have $\beta_t \equiv 0 \pmod{\pi_L}$. Now, the element $a \cdot d\pi + \Sigma_{t\in\tilde{I}}b_t \cdot dt = a\alpha \cdot d\pi_L + \Sigma_{t\in\tilde{I}}(a\beta_t + b_t) \cdot dt$ belongs to $(\hat{\Omega}^1_B)_{\mathrm{tors}}$ and generates this group because $a\beta_t + b_t$ is a unit when b_t is a unit. Thus, we obtain (1.7.1).

Q.E.D.

§2. Milnor K-groups

In this section, we study the structures of Milnor K-groups of complete discrete valuation fields of mixed characteristics.

Let K, A, F, m_A be as in §1. First, we define two filtrations on the q-th Milnor K-group $K^M_q(K)$. For $i \geqslant 1$, $U^iK^M_q(K)$ (resp. $V^iK^M_q(K)$) is defined to be the subgroup generated by the elements of the form $\{a_1, \ldots, a_q\}$ such that $a_1 - 1 \in m_A{}^i$, $a_2, \ldots, a_q \in A^*$ (resp. $a_1 - 1 \in m_A{}^i$, $a_2, \ldots, a_q \in K^*$). For an element a ($\neq 1$) of A and a prime element π, we can verify the formula

$$\{1 - a\pi, \pi\} = \{(1 - a\pi)(1 - a)^{-1}, -(1 - \pi)(1 - a)^{-1}\}.$$

It follows that

$$V^1 = U^1 \supset V^2 \supset U^2 \supset V^3 \supset \ldots.$$

On the subquotients of these groups, for a fixed prime element π, there are surjective homomorphisms (cf. [10] §1.3 Lemma 6).

$$\Omega^{q-2}_F \to V^iK^M_q(K)/U^iK^M_q(K) \tag{2.1.1}$$

$$\Omega^{q-1}_F \to U^iK^M_q(K)/V^{i+1}K^M_q(K) \tag{2.1.2}$$

Here, (2.1.1) (resp. (2.1.2)) is defined by

$$a \cdot \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-2}}{b_{q-2}} \mapsto \text{the class of } \{1 + \tilde{a}\pi^i, \tilde{b}_1, \ldots, \tilde{b}_{q-2}, \pi\}$$

$$\left(\text{resp. } a \cdot \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}} \mapsto \text{the class of } \{1 + \tilde{a}\pi^i, \tilde{b}_1, \ldots, \tilde{b}_{q-1}\}\right)$$

where $\tilde{}$ means a lifting to A. In [1] and [3], kernels of these homomorphisms are determined in equal characteristic case and for $i < e_K p/(p-1)$ in mixed characteristic case. In this paper, we cannot determine their kernels for $i > e_K p/(p-1)$, but will see that the appearance of the subquotients for sufficiently large i is much different from the above known cases.

THEOREM (2.2). *We put* $\mathrm{length}_A((\hat{\Omega}_A^1)_{\mathrm{tors}}) = m$.
(i) *For* $q < 0$, *if* K *is of type-I*, $V^i K_q^M(K) = U^i K_q^M(K)$ *for* $i > \inf(2(m+1), m+1+e_K p/(p-1))$.
(ii) *Suppose that* F *has a finite* p*-base of order* $q-1$. *If* K *is of type-II we have* $U^i K_q^M(K) = V^{i+1} K_q^M(K)$ *for* $i > \inf(2m, m+e_K p/(p-1))$.

REMARK (2.3). In the case $q = 1$ (namely, the case of $K_1^M(K) = K^*$), we have always $U^i = V^i$ and $U^i/U^{i+1} = F$ for $i \geqslant 1$. Compared with this, the group $K_q^M(K)$ for $q \geqslant 2$ is much more complicated and interesting. For example, assume that K is of type-II and F has a finite p-base of order $q-1$. Then, for $i \gg 0$ and $(i, p) = 1$, we have $gr_V^i = V^i/V^{i+1} = 0$! In fact, if i is prime to p, V^i/U^i is generated by the classes of the elements of the form $\{1 + u_1\pi^i, u_2, \ldots, u_{q-1}, \pi\}$ with $u_1, \ldots, u_{q-1} \in A^*$, which is equal to $-i^{-1}\{1 + u_1\pi^i, u_2, \ldots, u_{q-1}, -u_1\}$. This shows that $V^i/U^i = 0$ (i^{-1} makes sense because V^i/U^i is a p-torsion group.). On the other hand, we have $U^i = V^{i+1}$ by the theorem. Hence, $V^i/V^{i+1} = 0$.

For the proof of the theorem, we need the following two lemmas. We denote by $\hat{\Omega}_A^{q-1}$ the completion of Ω_A^{q-1}($=(q-1)$-th exterior power of Ω_A^1 over A).

LEMMA (2.4). *For a prime element* π *of* A *and* $j \geqslant 1$, *there is a surjective homomorphism* $\varrho_j \colon \hat{\Omega}_A^{q-1} \to U^j K_q^M(K)/V^{2j} K_q^M(K)$ *which sends* $a \cdot \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_{q-1}$ *to* $\{1 + ab_1 \cdot \cdots \cdot b_{q-1}\pi^j, b_1, \ldots, b_{q-1}\}$ *for* $a \in A$ *and* $b_1, \ldots, b_{q-1} \in A\backslash\{0\}$.

LEMMA (2.5). *For an element* c *of* A *such that* $v_K(c) > e_K/(p-1)$, *there exists a homomorphism*

$$\exp_c \colon \hat{\Omega}_A^{q-1} \to U^1 K_q^M(K)\hat{}$$

such that $\exp_c(a \cdot \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_{q-1}) = \{\exp(pcab_1 \cdot \cdots \cdot b_{q-1}), b_1, \ldots, b_{q-1}\}$ for $a \in A$ *and* $b_1, \ldots, b_{q-1} \in A\backslash\{0\}$. *Here*, $U^1 K_q^M(K)\hat{}$ *means the completion with respect to the topology defined by the filtration* $U^i K_q^M(K)$, exp *is the exponential homomorphism* $T \mapsto \Sigma_{n=0}^{\infty} T^n/n!$. (*Recall that for* $j > e_K/(p-1)$, exp: $m_A{}^j \to A^*$ *is well-defined.*)

Proof of lemmas. We use an exact sequence: $0 \to N \to A^{\otimes q} \xrightarrow{\psi} \Omega_A^{q-1} \to 0$ where $\psi(a \otimes b_1 \otimes \cdots \otimes b_{q-1}) = a db_1 \wedge \cdots \wedge db_{q-1}$ and N is the subgroup of $A^{\otimes q}$ generated by the elements of the form

(i) $a \otimes b_1 b_2 \otimes c_1 \otimes \cdots \otimes c_{q-2} - ab_1 \otimes b_2 \otimes c_1 \otimes \cdots \otimes c_{q-2} - ab_2 \otimes b_1 \otimes c_1 \otimes \cdots \otimes c_{q-2}$ and

(ii) $a \otimes b_1 \otimes \cdots \otimes b_{q-1}$ where $b_i = b_j$ for some $i \neq j$.

First, we shall prove Lemma (2.4). This presentation of Ω_A^{q-1} implies the existence of a homomorphism $\Omega_A^{q-1} \to U^j K_q^M(K)/V^{2j} K_q^M(K)$ as in (2.4). In fact, we have $\{1 + ab\pi^j, b\} = -\{1 + ab\pi^j, -a\pi^j\}$, therefore, the application $A^{\otimes q} \to U^j K_q^M(K)/V^{2j} K_q^M(K)$ which sends $a \otimes b_1 \otimes \cdots \otimes b_{q-1}$ to $\{1 + ab_1 \cdot \cdots \cdot b_{q-1}\pi^j, b_1, \ldots, b_{q-1}\}$, is a homomorphism. It is trivial that its kernel contains N. Its surjectivity is clear by the definition of the homomorphism. Finally, since the target group is annihilated by some power of p, we may replace Ω_A^{q-1} by $\hat{\Omega}_A^{q-1}$. This completes the proof of Lemma (2.4).

By the same method as above, for the proof of Lemma (2.5), it suffices to show that the application $A \to U^1 K_2^M(K)\hat{}$ which is defined by $x \mapsto \{\exp(pcx), x\}$ is a homomorphism, namely, additive.

We use the formula $\exp(T) = \Pi_{m \geq 1}(1 - T^m)^{-\mu(m)/m}$ (cf. [6] Chap. III §1) where μ is the Möbius function. Let P be the set of all natural numbers prime to p. We put $E(T) = \Pi_{m \in P}(1 - T^m)^{-\mu(m)/m} \in \mathbb{Z}_{(p)}[[T]]$. Then, by the formula above, we have

$$\exp(pcx) = \exp(cx)^p = E(cx)^p \cdot E((cx)^p)^{-1}.$$

Therefore,

$$\{\exp(pcx), x\} = p\{E(cx), x\} - \{E((cx)^p), x\}$$

$$= -\sum_{m \in P} (p\mu(m)/m)\{1 - (cx)^m, x\} + \sum_{m \in P} (\mu(m)/m)\{1 - (cx)^{pm}, x\}$$

Here, the multiplication by m^{-1} makes sense since $U^1 K_q^M(K)\hat{}$ is naturally considered as a $\mathbb{Z}_p$-module. We may assume K contains a primitive p-th root of unity ζ. Indeed, this follows from the fact that there is a norm homomorphism N: $K_q^M(K(\zeta)) \to K_q^M(K)$ such that $N \circ i = [K(\zeta): K]$ where $i: K_q^M(K) \to K_q^M(K(\zeta))$ is the natural map. We shall calculate the second term. From $1 - (cx)^{pm} = \Pi_{i=0}^{p-1}(1 - (cx\zeta^i)^m)$, we obtain

$$\{1 - (cx)^{pm}, x\} = \sum_{i=0}^{p-1} (1/m)\{1 - (cx\zeta^i)^m, x^m\} = -\sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, c\zeta^i\}$$

$$= -\{1 - (cx)^{pm}, c\} - \sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, \zeta^i\}.$$

It follows that

$$\begin{aligned}\{\exp(pcx), x\} &= \sum_{m\in P} (p\mu(m)/m)\{1 - (cx)^m, c\} \\ &\quad - \sum_{m\in P} (\mu(m)/m)\{1 - (cx)^{pm}, c\} \\ &\quad - \sum_{m\in P} (\mu(m)/m)\sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, \zeta^i\} \\ &= -p\{\exp(cx), c\} + \sum_{i=0}^{p-1}\{E(cx\zeta^i), \zeta^i\}\end{aligned}$$

Thus, we have only to show the following lemma.

LEMMA (2.5.1). *The application $A \to A^*/(A^*)^p$ which sends x to the class of $E(cx)$ is a homomorphism.*

Proof. In $\mathbb{Q}[[T]]$, we have $\exp(T) \equiv E(T) \pmod{\deg p}$, hence, the total degree of $E(T_1 + T_2) - E(T_1)E(T_2)$ in $\mathbb{Q}[[T_1, T_2]]$, a posteriori, in $\mathbb{Z}_{(p)}[[T_1, T_2]]$ is not less than p. This shows that $E(c(x + y)) \equiv E(cx)E(cy) \pmod{c^p}$. Since $1 + m_A{}^j \subset (A^*)^p$ for $j > e_K p/(p - 1)$, this completes the proof of Lemma (2.5.1). Q.E.D.

We now proceed to the proof of Theorem (2.2). Let K be of type-I. We shall show that $V^iK_q^M(K) = U^iK_q^M(K)$ for $i > 2(m + 1)$. By Lemma (1.1), we have in $\hat{\Omega}_A^1$ $a \cdot d\pi + \Sigma_{t\in\tilde{I}} b_t \cdot dt = 0$ such that $m = v_K(a) < v_K(b_t)$. By the homomorphism ϱ_j where $j = i - m - 1$, this relation induces in $K_q^M(K)/V^{2j}K_q^M(K)$ $\{1 + ac_1 \cdot \cdots \cdot c_{q-2}\pi^{i-m}, c_1, \ldots, c_{q-2}, \pi\} = -\Sigma_{t\in\tilde{I}}\{1 + tb_tc_1 \cdot \cdots \cdot c_{q-2}\pi^{i-m-1}, c_1, \ldots, c_{q-2}, t\}$ with $c_1, \ldots, c_{q-2} \in A\backslash\{0\}$. This shows that $V^iK_q^M(K) \subset U^iK_q^M(K) + V^{2j}K_q^M(K)$. But $V^{2j}K_q^M(K)$ is contained in $U^iK_q^M(K)$ because $2j = 2(i - m - 1) > i$. Thus, we have $V^iK_q^M(K) = U^iK_q^M(K)$. For $i > m + 1 + e_Kp/(p - 1)$, using the homomorphism $\exp_c$ (cf. Lemma (2.5)) where $v_K(c) = i - e_K - m - 1$ instead of ϱ_j, we have $V^iK_q^M(K) = U^iK_q^M(K)$. Thus (i) has been proved.

Next, we proceed to the proof of (ii). Let K be of type-II and $(t_i)_{1\leqslant i\leqslant q-1}$ be a lifting of a p-base of F and $i > 2m$. By the same method as above, the relation on π and t_i in $\hat{\Omega}_A^1$ implies by ϱ_{i-m} in Lemma (2.4) that the elements of the form $\{1 + c\pi^i, t_1, \ldots, t_{q-1}\}$ belong to $V^{i+1}K_q^M(K) + V^{2(i-m)}K_q^M(K) = V^{i+1}K_q^M(K)$. On the other hand, by (2.1.2), $U^iK_q^M(K)/V^{i+1}K_q^M(K)$ is generated by the elements of this form. Hence, we have $U^iK_q^M(K) = V^{i+1}K_q^M(K)$.

Using $\exp_c$ in Lemma (2.5), we have $U^i K_q^M(K) = V^{i+1} K_q^M(K)$ for $i > m + e_K p/(p-1)$. Q.E.D.

§3. Cyclic extensions of complete discrete valuation fields

We use the same notation as in §1 and §2. In this section, we prove the following two theorems by using Theorem (2.2) in §2. We put $\delta_1 = \inf(2(m+1), m+1+e_K p/(p-1))$ and $\delta_2 = \inf(2m, m+e_K p/(p-1))$ where $m = \mathrm{length}_A(\hat{\Omega}_A^1)_{\mathrm{tors}}$. Recall that for $i > \delta_1$, $V^i K_q^M(K) = U^i K_q^M(K)$ (resp. for $i > \delta_2$, $U^i K_q^M(K) = V^{i+1} K_q^M(K)$) when K is of type-I (resp. type-II) and F has a finite p-base of order $q-1$.

THEOREM (3.1). *Let L/K be a cyclic extension of degree p^n. If K is of type-II (resp. type-I) and L/K is totally (resp. fiercely) ramified, we have $n \leqslant \delta_2 (n \leqslant \delta_1)$.*

REMARK (3.1.1). The assertion for type-I in Theorem (3.1) is already known. In [16] and [8], for any cyclic extension L of K with residue field M, it was proved that $[M:F]_{\mathrm{insep}}$ is smaller than some constant dependent only on K. But the proofs of their theorems are completely different from ours, and our proof of type-I case and type-II case proceeds in parallel, so here, we treat type-I case together with type-II case to emphasize a symmetry between them.

REMARK (3.1.2). Let K be a complete discrete valuation field of equal characteristic. In this case, we fix a discrete valuation sub-field $k \subset K$ such that K/k is separable (not necessarily algebraic) and the residue field of k is perfect. Then, we can define the notion of type-I and type-II over k. Namely, K is called of type-I over k if the natural homomorphism: $(\hat{\Omega}_{A/R}^1)_{\mathrm{tors}} \to \Omega_F^1$ is the 0-map where A and R are the valuation rings of K and k, respectively. Otherwise, K is called of type-II. Theorem (3.1) still holds if we make an additional assumption. Let L/K be a totally (resp. fiercely) ramified cyclic extension and assume that K is of type-II (resp. type-I) and that there exists an element g of R such that $\mathrm{d}g$ generates $\hat{\Omega}_R^1$ as an $\hat{R}$-module and $g \in N_{L/K}(L^*)$. Then, the degree of L/K cannot be greater than some constant dependent only on K and k.

Next theorem is similar to (3.1), but says that it is difficult to extend a fiercely (resp. totally) ramified cyclic extension of a field of type-I (resp. type-II) to *any* cyclic extension of larger degree. For a cyclic extension L/K of degree p with Galois group G, we define the ramification number $s(L/K)$ to be

$\inf_{x\in L^*} v_L(s(x)x^{-1} - 1)$ where s is a generator of G. In the terminology of [18], if L/K is totally (resp. fiercely) ramified, $s(L/K) = i - 1$ (resp. i) where $i = \inf\{j \mid G_j = 0\}$.

THEOREM (3.2). *Let M/K be a cyclic extension of degree p such that $s(M/K) > e_M/p(p-1)$. If K is of type-I (resp. type-II) and M/K is fiercely ramified (resp. totally ramified), there is no cyclic extension L/K of degree p^n such that L/K contains M/K and $n > [\delta_1/e_K] + 1$ (resp. $[\delta_2/e_K] + 1$).*

REMARK (3.2.1). Let K be a complete discrete valuation field containing a primitive p-th root of unity and M/K be an extension defined by the equation $X^p = a$. Let $U^i(K^*/p)$ be the filtration on K^*/p induced from $U^iK_1^M(K)$. Then, $s(M/K) > e_M/p(p-1)$ if and only if $a \notin U^{e_K}(K^*/p)$.

REMARK (3.2.2). Let L/K be a cyclic extension of degree a power of p which does not contain unramified extensions. If $[L:K]$ is sufficiently large, there exist intermediate fields K' and K_1' such that $[K_1':K'] = p$ and $s(K_1'/K') > e_{K_1'}/p(p-1)$ (cf. [8] Lemma (4-1)). So, Theorem (3.2) also contains the fact that there is no fiercely (resp. totally) ramified cyclic extension of a field of type-I (resp. type-II) having a large degree.

For the proof of the theorems, we do not use local class field theory of Kato and Parsin directly, but the idea of the proof deeply depends on their theory. We will use the notation in [10] and [11], which we explain rapidly. For a field k of characteristic $\neq p$, we denote by $H^q_{p^n}(k)$ the Galois cohomology group $H^q(k, \mathbb{Z}/p^n(q-1))$ where $(q-1)$ means the Tate twist. If k is of characteristic $p > 0$, we define

$$H^q_{p^n}(k) \;=\; \operatorname{Coker}\,(W_n\Omega_k^{q-1} \xrightarrow{F-1} W_n\Omega_k^{q-1}/\mathrm{d}W_n\Omega_k^{q-2})$$

where $W_n\Omega_k^{\cdot}$ is the De Rham-Witt complex [9] and F is the operator of De Rham-Witt complex as usual (cf. [9] p. 569). Note that there is an isomorphism $H^1_{p^n}(k) = H^1(k, \mathbb{Z}/p^n)$ (cf. [18] p. 163).

When $\operatorname{char}(k) \neq p$ (resp. $\operatorname{char}(k) = p$), by composing the cohomological symbol: $K_q^M(k) \to H^q(k, \mathbb{Z}/p^n(q))$ (resp. the differential symbol: $K_q^M(k) \to W_n\Omega_k^q$) (cf. [11], [3]), the product structure of Galois cohomology (resp. De Rham-Witt complex) defines a pairing

$$H^s_{p^n}(k) \otimes K_q^M(k) \to H^{s+q}_{p^n}(k).$$

Now, we return to the situation of theorems. Since A is a direct limit of complete discrete valuation rings A_i whose residue fields are of finite transcendence degree over $\mathbb{F}_p$, a relation in $\hat{\Omega}^1_A$ comes from one of $\hat{\Omega}^1_{A_i}$ for some

i. So we may assume that A and A_i are of the same type and $\mathrm{length}_A(\hat{\Omega}^1_A)_{\mathrm{tors}} = \mathrm{length}_{A_i}(\hat{\Omega}^1_{A_i})_{\mathrm{tors}}$. Thus, we may assume F has a finite p-base. Put $[F:F^p] = p^{q-1} < \infty$. Consider the pairing

$$\psi: H^1_{p^n}(K) \otimes K^M_q(K) \to H^{q+1}_{p^n}(K). \tag{3.3.1}$$

This group $H^{q+1}_{p^n}(K)$ is rather "small". Let i be the composite of two canonical homomorphisms $W_n(F) \to H^1_{p^n}(F) \to H^1_{p^n}(K)$. We fix a prime element π and define a homomorphism i_q: $W_n\Omega^{q-1}_F \to H^{q+1}_{p^n}(K)$ such that $i_q(a \cdot \mathrm{d}\log b_1 \wedge \cdots \wedge \mathrm{d}\log b_{q-1}) = \psi(i(a) \otimes \{\tilde{b}_1, \ldots, \tilde{b}_{q-1}, \pi\})$ where $\tilde{b}_i$ is a lifting to A. This homomorphism is independent of the choice of π and induces an isomorphism $H^q_{p^n}(F) \xrightarrow{\sim} H^{q+1}_{p^n}(K)$ ([11] Th. 3). Therefore, we have:

LEMMA (3.3.2). *For any prime element π and any lifting of a p-base $(t_i)_{1 \leqslant i \leqslant q-1}$, there exists a surjective homomorphism $W_n(F) \to H^{q+1}_{p^n}(K)$ which sends a to $\psi(i(a) \otimes \{t_1, \ldots, t_{q-1}, \pi\})$.*

Concerning $H^{q+1}_{p^n}(K)$, we also use the fact that the exact sequence $0 \to \mathbb{Z}/p^{n-1}(q) \to \mathbb{Z}/p^n(q) \to \mathbb{Z}/p(q) \to 0$ induces an exact sequence ([11] Lemma 8)

$$0 \to H^{q+1}_{p^{n-1}}(K) \to H^{q+1}_{p^n}(K) \to H^{q+1}_p(K) \to 0. \tag{3.3.3}$$

Let $\chi \in H^1_{p^n}(K)$ be a character of order p^n. By the pairing (3.3.1), χ defines a homomorphism $\varphi_\chi: K^M_q(K) \to H^{q+1}_{p^n}(K)$.

For a finite extension L/K, there is a norm map $N_{L/K}: K^M_q(L) \to K^M_q(K)$. We will use the following properties:

(i) $N_{L/K}$ is compatible with the corestriction map in Galois cohomology by the cohomological symbol.

(ii) $N_{L/K}(U^1K^M_q(L)) \subset U^1K^M_q(K)$ and $N_{L/K}(V^{ei}K^M_q(L)) \subset V^iK^M_q(K)$ for $i > 0$ where e is the ramification index of L over K (for the proof, cf. [10] §1.2 Lemma 3, [12] Prop. 2).

By the property (i) above, we have $N_{L/K}K^M_q(L) \subset \mathrm{Ker}\,\varphi_\chi$ where L/K is the extension corresponding to χ.

We will need two more lemmas. A finite extension L/K is called purely wildly ramified if any subextension of L/K is wildly ramified.

LEMMA (3.3.4). *Let $\chi \in H^1_{p^n}(K)$ be a character such that the extension attached to χ is purely wildly ramified. Then, the homomorphism $\varphi_\chi: K^M_q(K) \to H^{q+1}_{p^n}(K)$ defined by χ as above is surjective. More precisely, $\varphi_\chi(U^1K^M_q(K)) = H^{q+1}_{p^n}(K)$.*

Proof of the lemma. Let M/K be the subextension of degree p. The following diagram is commutative.

$$\begin{array}{ccccccccc}
 & & K_q^M(M) & \xrightarrow{N_{M/K}} & K_q^M(K) & \longrightarrow & K_q^M(K)/N_{M/K}K_q^M(K) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_{\chi|M}} & & & & & & \\
 & & H_{p^{n-1}}^{q+1}(M) & & \downarrow{\scriptstyle \varphi_\chi} & & \downarrow{\scriptstyle \varphi_{p^{n-1}\chi}} & & \\
 & & \downarrow{\scriptstyle \mathrm{Cor}_{M/K}} & & & & & & \\
0 & \longrightarrow & H_{p^{n-1}}^{q+1}(K) & \longrightarrow & H_{p^n}^{q+1}(K) & \longrightarrow & H_p^{q+1}(K) & \longrightarrow & 0
\end{array}$$

Here, the bottom row is an exact sequence (3.3.3) and the corestriction map $\mathrm{Cor}_{M/K}\colon H_{p^{n-1}}^{q+1}(M) \to H_{p^{n-1}}^{q+1}(K)$ is surjective because the cohomological dimension of $K \leqslant q+1$ ([11] Th. 2). Considering $N_{M/K}(U^1K_q^M(M)) \subset U^1K_q^M(K)$, this lemma is reduced to the case $n = 1$. In this case, the lemma follows from Lemma (3.3.2) and [10] §3 Lemma 15. Q.E.D.

LEMMA (3.3.5). *Let χ and φ_χ be as above. If the extension attached to χ is totally (resp. fiercely) ramified, the images of $V^iK_q^M(K)$ and $U^iK_q^M(K)$ (resp. $U^iK_q^M(K)$ and $V^{i+1}K_q^M(K)$) by the homomorphism φ_χ coincide for $i \geqslant 1$.*

Proof of the lemma. Let L/K be the extension corresponding to χ. It suffices to show the following claim.

$$V^iK_q^M(K) \subset U^iK_q^M(K) + N_{L/K}K_q^M(L)$$

$$(\text{resp. } U^iK_q^M(K) \subset V^{i+1}K_q^M(K) + N_{L/K}K_q^M(L)). \tag{3.3.6}$$

If χ is totally ramified, the above is clear because we can take a prime element π which is a norm from L, and $V^iK_q^M(K)/U^iK_q^M(K)$ is generated by the elements of the form $\{1 + a\pi^i, c_1, \ldots, c_{q-2}, \pi\}$ with $a, c_1, \ldots, c_{q-2} \in A^*$ (cf. (2.1.1)). In the fiercely ramified case, we prove (3.3.6) by induction on the degree of L/K. Let M/K be the subextension of degree p. Then, we can take a lifting of a p-base $(t_i)_{1\leqslant i\leqslant q-1}$ such that $t_1 = N_{M/K}(t')$ where $t' \in M^*$. As above, $U^iK_q^M(K)/V^{i+1}K_q^M(K)$ is generated by the elements of the form $\{1 + a\pi^i, t_1, \ldots, t_{q-1}\}$ with $a \in A$ (cf. (2.1.2)). Since $\{1 + a\pi^i, t', t_2, \ldots, t_{q-1}\}$ is in $V^{i+1}K_q^M(M) + N_{L/M}K_q^M(L)$ by induction on $[L:K]$, the claim follows from the fact that $N_{M/K}(V^{i+1}K_q^M(M)) \subset V^{i+1}K_q^M(K)$.

For the proof of the theorems, we may replace F by a separable extension which preserves a p-base. So, we may assume that $H_p^q(F) = H_p^{q+1}(K) \neq 0$

(For example, replace F by $F(\bigcup_{n\geqslant 1}\{T^{p^{-n}}\})$ where T is a variable.). The exact sequence (3.3.3) shows that $H_{p^n}^{q+1}(K)$ has an element of order p^n.

Proof of Theorem (3.1). Let L/K be as in Theorem (3.1) and χ be the character corresponding to L/K. Since the argument proceeds in parallel, we only consider the case when L/K is totally ramified and K is of type-II. By Lemma (3.3.4), we obtain a surjective homomorphism φ_χ: $U^1K_q^M(K) \to H_{p^n}^{q+1}(K)$.
On the other hand, by Theorem (2.2) and Lemma (3.3.5), we know that the images of $U^iK_q^M(K)$ for $i > \delta_2$ by φ_χ all coincide. Since $U^iK_q^M(K) \subset p^nK_q^M(K)$ for sufficiently large i, this implies that φ_χ induces a surjective homomorphism $U^1K_q^M(K)/U^{\delta_2+1}K_q^M(K) \to H_{p^n}^{q+1}(K)$. The fact that $U^iK_q^M(K)/U^{i+1}K_q^M(K)$ is annihilated by p for all $i \geqslant 1$ and the target group has an element of order p^n, implies the inequality $\delta_2 \geqslant n$. Q.E.D.

Proof of Theorem (3.2). As above, we may assume that $[F:F^p] = p^{q-1} < \infty$ and $H_p^{q+1}(K) \neq 0$. Let ψ be a character corresponding to M/K and consider the homomorphism φ_ψ: $K_q^M(K)/N_{M/K}K_q^M(M) \to H_p^{q+1}(K)$. We put $t = f_{M/K}\cdot s(M/K)$ where $f_{M/K}$ is the degree of the residue extension of M/K. If M/K is fiercely ramified (resp. totally ramified), we have $\varphi_\psi(V^tK_q^M(K)) = H_p^{q+1}(K)$ and $\varphi_\psi(U^tK_q^M(K)) = 0$ (resp. $\varphi_\psi(U^tK_q^M(K)) = H_p^{q+1}(K)$ and $\varphi_\psi(V^{t+1}K_q^M(K)) = 0$) (cf. [10] §3.3 Proofs of Prop. 2 and Th. 1).
Let $\chi \in H_{p^n}^{q+1}(K)$ be a character such that the cyclic extension L attached to χ, contains M. The commutative diagram

$$\begin{array}{ccc} K_q^M(K)/N_{L/K}K_q^M(L) & \longrightarrow & K_q^M(K)/N_{M/K}K_q^M(M) \\ \varphi_\chi \Big\downarrow & & \Big\downarrow \varphi_\psi \\ H_{p^n}^{q+1}(K) & \xrightarrow{p^{n-1}} & H_p^{q+1}(K) \subset H_{p^n}^{q+1}(K) \end{array}$$

implies that $p^{n-1}\varphi_\chi(V^t) \neq 0$ and $p^{n-1}\varphi_\chi(U^t) = 0$ (resp. $p^{n-1}\varphi_\chi(U^t) \neq 0$ and $p^{n-1}\varphi_\chi(V^{t+1}) = 0$). Since the assumption on $s(M/K)$ implies $t > e_K/(p-1)$, we have $p^{n-1}U^t = U^{t+e_K(n-1)}$ and $p^{n-1}V^t = V^{t+e_K(n-1)}$.
Let K be of type-I (resp. type-II). By Theorem (2.2), we have $V^i = U^i$ (resp. $U^i = V^{i+1}$) for $i > \delta_1$ (resp. δ_2). Therefore, $t + e_K(n-1)$ cannot be greater than δ_1 (resp. δ_2). Hence, $e_K(n-1) < \delta_1$ (resp. δ_2), which implies $n \leqslant [\delta_1/e_K] + 1$ (resp. $[\delta_2/e_K] + 1$).

§4. A global application

(4.1) In this section, we prove Theorem 3 in Introduction as a (semi-)global application of Theorem (2.2) (i) which asserts $U^i = V^i$ for sufficiently large i in the case of type-I. We will use the class field theory of varieties over a local field. We cannot find an adequate reference for it, but it is not our aim to explain the construction of class field theory, so we omit the details of the proof concerning the construction, for example, of the reciprocity laws and the residue theorems etc. The proofs are essentially the same as in the case of the class field theory of schemes of finite type over $\mathbb{Z}$ in [14].

Let k be a complete discrete valuation field of mixed characteristics $(0, p)$. We denote by R the valuation ring of k and assume that its residue field F is perfect. Let $\mathfrak{X}$ be a proper smooth scheme over R which is geometrically connected and purely of relative dimension d, and let $X = \mathfrak{X} \otimes_R k$ and $Y = \mathfrak{X} \otimes_R F$. In the following, π always denotes a prime element of R.

Let $K_q^M(\mathcal{O}_\mathfrak{X})$ be the sheaf of Milnor K-groups for $q \geqslant 1$. (For a scheme T, the sheaf of Milnor K-groups $K_q^M(\mathcal{O}_T)$ is defined to be the sheaf associated to the presheaf $U \mapsto K_q^M(\Gamma(U, \mathcal{O}_T))$.) For $i \geqslant 0$, a sheaf $K_q^M(\mathcal{O}_\mathfrak{X}, \pi^i)$ on $\mathfrak{X}$ is defined to be $\operatorname{Ker}(K_q^M(\mathcal{O}_\mathfrak{X}) \to K_q^M(\mathcal{O}_\mathfrak{X}/\pi^i))$. The objects we consider in this section are the following groups. For $q \geqslant 1$, we define $I_q^{\mathrm{top}}(\mathfrak{X}, X)$ or simply $I_q^{\mathrm{top}}(X)$ by

$$I_q^{\mathrm{top}}(X) = \varprojlim_i H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_\mathfrak{X}, \pi^i)) \tag{4.1.1}$$

where the cohomology group at the right hand side is the Zariski cohomology with support on Y. There exists a natural filtration on $I_q^{\mathrm{top}}(X)$ induced by the definition. Namely, we define $U^i I_q^{\mathrm{top}}$ for $i \geqslant 0$ as follows.

$$U^i I_q^{\mathrm{top}} = \operatorname{Ker}(I_q^{\mathrm{top}}(X) \to H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_\mathfrak{X}, \pi^i))). \tag{4.1.2}$$

This group has an explicit presentation by Milnor K-groups, and can be considered as an idele class group, which we explain briefly.

Let $\mathcal{F}$ be a sheaf on $\mathfrak{X}_{\mathrm{Zar}}$. Then, the spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in Y^{p-1}} H_x^{p+q}(\mathfrak{X}, \mathcal{F}) \Rightarrow H_Y^{p+q}(\mathfrak{X}, \mathcal{F})$$

yields the following exact sequence

$$\bigoplus_{y \in Y_1} H_y^d(\mathfrak{X}, \mathcal{F}) \to \bigoplus_{x \in Y_0} H_x^{d+1}(\mathfrak{X}, \mathcal{F}) \to H_Y^{d+1}(\mathfrak{X}, \mathcal{F}) \to 0 \tag{4.1.3}$$

because $H_x^{d+1}(\mathfrak{X}, \mathcal{F}) = 0$ for $x \in Y^p$ with $p < d$. For a scheme T of dimension r, we define a P-chain to be a sequence $P = (p_0, \ldots, p_r)$ of points of T such that $p_i \in T_i$ and $\overline{\{p_0\}} \subset \cdots \subset \overline{\{p_r\}}$. Applying the localizing sequences repeatedly, we have a surjective homomorphism $\oplus_P (\mathscr{F})_{p_{d+1}}/(\mathscr{F})_{p_d} \to H_x^{d+1}(\mathfrak{X}, \mathscr{F})$ where P runs over the P-chains of $\mathfrak{X}$ of the form $P = (x, p_1, \ldots, p_{d+1})$. Let K be the function field of $\mathfrak{X}$ and for a P-chain $P = (p_0, \ldots, p_{d+1})$ of $\mathfrak{X}$, we denote by K_P the field K which is regarded as a discrete valuation field by the discrete valuation defined by p_d. Let η be the generic point of Y. By the fact described above, we have a surjective homomorphism

$$\bigoplus_{P'} K_q^M(K_{P'})/U^0 K_q^M(K_{P'}) \oplus \bigoplus_{P} K_q^M(K_P) \to H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi^i)). \tag{4.1.4}$$

Here, P' (resp. P) runs over the P-chains such that $p'_d \neq \eta$ (resp. $p_d = \eta$), and for a discrete valuation field L with valuation ring A, $U^i K_q^M(L)$ for $i \geqslant 1$ is defined as in §2 and $U^0 K_q^M(L)$ is the image of $K_q^M(A) \to K_q^M(L)$. P-chains with $p_d = \eta$ are regarded as P-chains of Y, so we sometimes identify them. The kernel of (4.1.4) can be described using the term Q-chain as in [14]. (In [14], the henselian topology was used but the argument proceeds similarly in the case of Zariski topology.). We can define another filtration $V^i I_q^{\mathrm{top}}$ on $I_q^{\mathrm{top}}(X)$ which is induced by (4.1.4) from $\oplus_P V^i K_q^M(K_P)$ where P runs over all P-chains of Y.

For a P-chain P of Y, the fixed prime element π and $i \geqslant 1$, there are two surjective homomorphisms (cf. (2.1.1) and (2.1.2))

$$\Omega_{F_P}^{q-1} \to U^i K_q^M(K_P)/V^{i+1} K_q^M(K_P) \tag{4.1.5}$$

$$\Omega_{F_P}^{q-2} \to V^i K_q^M(K_P)/U^i K_q^M(K_P) \tag{4.1.6}$$

where F_P is the residue field of K_P (hence, the function field of Y). The above homomorphisms induce two global surjective homomorphisms.

$$H^d(Y, \Omega_Y^{q-1}) \to U^i I_q^{\mathrm{top}}/V^{i+1} I_q^{\mathrm{top}} \tag{4.1.7}$$

$$H^d(Y, \Omega_Y^{q-2}) \to V^i I_q^{\mathrm{top}}/U^i I_q^{\mathrm{top}} \tag{4.1.8}$$

This can be verified by using the explicit presentation of both groups as in [14] Lemma (1.6.3). (Cf. a similar homomorphism is constructed in [5] Th. (2.9).)

PROPOSITION (4.1.9). *For i sufficiently large (for example, $i > 2(m+1)$ where $m = \mathrm{length}_R(\hat{\Omega}_R^1)$), (4.1.8) is the 0-map. Further, if the absolute ramification index e_k is smaller than $p - 1$, (4.1.8) is the 0-map for any $i \geqslant 1$.*

Proof. By definition of (4.1.8), it suffices to show that (4.1.6) is the 0-map for $i > 2(m + 1)$. Since $\mathfrak{X}$ is smooth over R, the claim follows from Corollary (1.5) and Theorem (2.2) (cf. the proof of Th. (1.4)). For the second assertion, if $e_k < p - 1$, we have $p \cdot U^i = U^{i+e_k}$ and $p \cdot V^i = V^{i+e_k}$ for $i \geqslant 1$. Since we can take r and j such that $i = re_k + j$, $r \geqslant 0$, and $0 < j < p$, the claim reduces to the case $0 < i < p$. In this case, it is trivial that $V^i = U^i$ (cf. Remark (2.3)). Q.E.D.

Next, we consider $gr^0 I_q^{\mathrm{top}} = I_q^{\mathrm{top}}(X)/U^1 I_q^{\mathrm{top}}(X)$. By (4.1.2), we have $gr^0 I_q^{\mathrm{top}} = H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi))$. From the exact sequence:

$$0 \to K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi) \to K_q^M(\mathcal{O}_{\mathfrak{X}}) \to K_q^M(\mathcal{O}_Y) \to 0,$$

taking cohomology groups, we obtain an exact sequence

$$H^d(Y, K_q^M(\mathcal{O}_Y)) \to gr^0 I_q^{\mathrm{top}} \to H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) \to 0. \tag{4.1.10}$$

We shall compute the group $H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}))$.

We have the following isomorphisms ([13] Th. 2)

$$H_x^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) = K_{q-d-1}^M(\kappa(x)) \quad \text{for} \quad x \in Y_0$$

$$H_y^{d}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) = K_{q-d}^M(\kappa(y)) \quad \text{for} \quad y \in Y_1$$

where $\kappa(x)$ is the residue field at x and we define $K_m^M(\kappa(x)) = 0$ if $m < 0$. By (4.1.3), we have an exact sequence

$$\bigoplus_{y \in Y_1} K_{q-d}^M(\kappa(y)) \to \bigoplus_{x \in Y_0} K_{q-d-1}^M(\kappa(x)) \to H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) \to 0. \tag{4.1.11}$$

In particular, we obtain

$$H_Y^{d+1}(\mathfrak{X}, K_{d+1}^M(\mathcal{O}_{\mathfrak{X}})) = CH_0(Y) \tag{4.1.12}$$

(the group of zero cycles modulo rational equivalence).

(4.2) We define $SK_1^{\mathrm{top}}(X) = I_{d+1}^{\mathrm{top}}(X)$. This group is more useful than the group $SK_1^M(X)$ which is defined to be $H^d(X, K_{d+1}^M(\mathcal{O}_X))$, when we consider the arithmetic of X. Let k be a local field (which means a finite extension of $\mathbb{Q}_p$). By using this group $SK_1^{\mathrm{top}}(X)$, we can construct the class field theory of X. In fact, we can construct the reciprocity map: $SK_1^{\mathrm{top}}(X) \to \pi_1^{ab}(X)$ by using the "explicit" presentation of $SK_1^{\mathrm{top}}(X)$ (cf. (4.1.4)) and patching the reciprocity maps of higher dimensional local fields (cf. [10]) as in the case of arithmetical schemes (cf. [14]). This reciprocity map has dense image. Indeed, the quotient of $\pi_1^{ab}(X)$ by the closure of the image of this homomorphism corresponds to the abelian covering in which any closed point of $\mathfrak{X}$ splits completely. Such a covering is necessarily trivial because $\pi_1^{ab}(\mathfrak{X}) = \pi_1^{ab}(Y)$ and we already know that over a smooth scheme over a finite field, such a covering is trivial (cf. [15], [14]).

Again by using the "explicit" presentation of $SK_1^{\mathrm{top}}(X)$, we can construct the norm map N: $SK_1^{\mathrm{top}}(X) \to k^*$, which is obtained by patching the residue homomorphisms (cf. [12]). This time, the residue theorem affirms the well-definedness of N. Then, we have the following commutative diagram

$$\begin{array}{ccc} SK_1^{\mathrm{top}}(X) & \xrightarrow{N} & k^* \\ \downarrow & & \downarrow \\ \pi_1^{ab}(X) & \longrightarrow & \mathrm{Gal}(k^{ab}/k) \end{array} \tag{4.2.1}$$

where the vertical arrows are the reciprocity maps.

THEOREM (4.2.2). *Let k be a finite extension of the p-adic number field $\mathbb{Q}_p$, and $\mathfrak{X}$, X, Y be as above. Then, the kernel of the norm map N: $SK_1^{\mathrm{top}}(K) \to k^*$ is finite.*

Proof. The filtration $U^iK_1^M(k)$ on k^* is denoted by U_k^i as usual. Since the residue preserves the filtration (cf. [12]), N also. To be precise, $N(V^iSK_1^{\mathrm{top}}) \subset U_k^i$ for $i \geqslant 1$. Hence, we can consider $gr^i(N)$: $V^iSK_1^{\mathrm{top}}/V^{i+1}SK_1^{\mathrm{top}} \to U_k^i/U_k^{i+1}$.

(4.2.3) For $gr^0SK_1^{\mathrm{top}} = SK_1^{\mathrm{top}}(X)/U^1SK_1^{\mathrm{top}}$, we have the following diagram which is commutative by definition of the homomorphisms.

$$\begin{array}{ccccccccc} & & H^d(Y, K_{d+1}^M(\mathcal{O}_Y)) & \longrightarrow & gr^0SK_1^{\mathrm{top}} & \longrightarrow & CH_0(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F^* \longrightarrow gr^0(k^*) & = & k^*/U_k^1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

In this diagram, the top row is exact by (4.1.10) and (4.1.12), and the bottom row is a well-known exact sequence. The left vertical arrow is bijective by the generalized Moore's exact sequence (cf. [14], in which the henselian topology is used but we can replace it by the Zariski topology in the proof of bijectivity). The right vertical arrow is the degree map of Chow group, and the class field theory of Y tells us that its kernel is finite (cf. [14] Th. (6.1)). Thus, $gr^0(N)$ has finite kernel.

(4.2.4) Next, we consider $gr^i SK_1^{\mathrm{top}} = V^i SK_1^{\mathrm{top}} / V^{i+1} SK_1^{\mathrm{top}}$ for $i \geqslant 1$. We already know its finiteness by (4.1.7) and (4.1.8) because $H^q(Y, \Omega_Y^r)$ is finite. Moreover, we have the following commutative diagram.

$$\begin{tikzcd} H^d(Y, \Omega_Y^d) \arrow[r, "\varrho"] \arrow[dr, "\text{trace}"', "\simeq"] & gr^i SK_1^{\mathrm{top}} \arrow[d, "gr^i(N)"] \\ & gr^i(k^*) = F \end{tikzcd}$$

In this diagram, ϱ is the map in (4.1.7) and $H^d(Y, \Omega_Y^d) \to F$ is the trace map of Serre duality, which is an isomorphism. On the other hand, by Proposition (4.1.9), if i is sufficiently large, ϱ is surjective. Hence, ϱ and $gr^i(N)$ are bijective for $i \gg 0$.

Since the filtration $V^i SK_1^{\mathrm{top}}$ is separated, the theorem follows from (4.2.3) and (4.2.4). Q.E.D.

COROLLARY (4.2.5). *If $e_k < p - 1$, we have*

$$\operatorname{Ker}(SK_1^{\mathrm{top}}(X) \to k^*) = \operatorname{Ker}(CH_0(Y) \to \mathbb{Z}).$$

Proof. This follows from Proposition (4.1.9) and the proof of Theorem (4.2.2).

The fact that the reciprocity maps in (4.2.1) have dense images implies the following corollaries.

COROLLARY (4.2.6). *The kernel of the natural homomorphism: $\pi_1^{ab}(X) \to \operatorname{Gal}(k^{ab}/k)$ is finite.*

COROLLARY (4.2.7). *Every abelian etale covering of X comes from an abelian extension of k and an abelian etale covering of Y if $e_k < p - 1$.*

REMARK (4.2.8). This result (4.2.7) was essentially proved by Coombes [5] in the case of $\dim X = 1$ (cf. [5] the proof of (3.6), and (2.11), (2.12)). Our method of proof of (4.2.7) is essentially the same as his.

(4.3) Finally, we give a remark concerning the case $q = d$. We define $CH_0^{\mathrm{top}}(X) = I_d^{\mathrm{top}}(X)$. On the other hand, let Alb_X be the Albanese variety of

X. By analogy with $N: SK_1^{\text{top}}(X) \to k^*$, I expect that there exists a canonical homomorphism having finite kernel

$$CH_0^{\text{top}}(X)^{\deg 0} \to \mathrm{Alb}_X(k)$$

which induces on $CH_0(X)^{\deg 0}$ the classical Albanese mapping.

Acknowledgements

I wish to express my hearty thanks to Professor K. Kato for his valuable advice and encouragement. I owe much in §4 to him. I also wish to thank O. Hyodo and T. Saito for their helpful discussions and encouragement.

References

1. S. Bloch: Algebraic K-theory and crystalline cohomology. *Publ. Math. I.H.E.S.* 47 (1977) 187–268.
2. S. Bloch: Algebraic K-theory and classfield theory for arithmetic surfaces. *Ann. Math.* 114 (1981) 229–265.
3. S. Bloch and K. Kato: p-adic etale cohomology. *Publ. Math. I.H.E.S.* 63 (1986) 107–152.
4. N. Bourbaki: *Alèbre commutative*, Chap. 8 et 9. Masson, Paris (1983).
5. K.R. Coombes: Local class field theory for curves. *Contemporary Math.* 55, Part I (1986) 117–134.
6. M. Demazure: Lectures on p-divisible groups. *Lecture Notes in Math.*, 302. Springer, Berlin (1972).
7. A. Grothendieck: Eléments de Géométrie algébraique, IV (Première Partie). *Publ. Math. I.H.E.S.* 20 (1964).
8. O. Hyodo: Wild ramification in the imperfect residue field case, preprint.
9. L. Illusie: Complexe de De Rham-Witt et cohomologie crystalline. *Ann. Sc. Ec. Norm. Sup.* 4ème série, 12 (1979) 501–661.
10. K. Kato: A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo* 27 (1980) 603–683.
11. K. Kato: Galois cohomology of complete discrete valuation fields. *Lecture Notes in Math.*, 967. Springer (1982).
12. K. Kato: Residue homomorphisms in Milnor K-theory. *Adv. St. Pure Math.* 2 (1983) 153–172.
13. K. Kato: Milnor K-theory and the Chow group of zero cycles. *Contemporary Math.* 55, Part I (1986) 241–253.
14. K. Kato and S. Saito: Global class field theory of arithmetic schemes. *Contemporary Math.* 55, Part I (1986) 255–331.
15. S. Lang: Unramified class field theory over function fields in several variables. *Ann. Math.* 64 (1956) 285–325.

16. H. Miki: *On cyclic extensions of p-power degree over complete p-adic fields* (in Japanese), Master's thesis. Univ. Tokyo (1973).
17. J.P. Serre: Sur les corps locaux à corps résiduel algébriquement clos. *Bull. Soc. Math. France* 89 (1961) 105–154.
18. J.P. Serre: *Corps locaux* (3e édition). Hermann, Paris (1968).