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## 1. Introduction

It is the purpose of this paper to prove a generalization to $\sigma$-compact manifolds of a well known result due to M. Brown (see [4]), which asserts the existence of a special kind of continuous, "non-pathological" surjections from the unit $n$-dimensional cube onto a given compact connected manifold $M^n$.

In the more general setting when $M^n$ is $\sigma$-compact, the space $\mathscr{E}(M)$ of ends of $M^n$ plays an important role: Since $\mathscr{E}(M)$ is a totally disconnected, compact, metrizable space, a set $E$ contained in the boundary of the unit cube $I^n$ can be constructed in such a way that $E$ is homeomorphic to $\mathscr{E}(M)$. Now $I^n \backslash E$ and $M^n$ are two manifolds with the same set of ends. Broadly speaking, our result states that $M^n$ is the identification space obtained from $I^n \backslash E$ by identifying points within the boundary of $I^n \backslash E$ alone.

The set $\mathscr{E}(M)$ is empty exactly when $M$ is compact. In this case, the arguments are reduced to those given by M. Brown for compact manifolds. Some applications are mentioned afterwards.


## §1. The set of ends

The concept of the set of ends of a space is due to Freudenthal. Here we recall some basic notions.

Let $X$ be a locally compact, Hausdorff space. Denote by $\mathscr{K}(X)$ the set of all compact subsets of $X$ partially ordered by inclusion. If $K \in \mathscr{K}(X)$, denote by $\mathscr{C}(X \backslash K)$ the set of connected components of $X \backslash K$ considered as a discrete topological space.

If $K, L \in \mathscr{K}(X)$ with $K \subset L$, then there is a well defined continuous function

$$\varrho_K^L : \mathscr{C}(X \backslash L) \to \mathscr{C}(X \backslash K)$$

such that for each $V \in \mathscr{C}(X \backslash L)$, $\varrho_K^L(V)$ is the unique component of $X \backslash K$ containing $V$. In this manner, the collection

$$\{\mathscr{C}(X \backslash K), \varrho_K^L \,|\, K, L \in \mathscr{K}(X) \quad \text{and} \quad K \subset L\}$$

constitutes an inverse system of topological spaces indexed over the directed set $\mathscr{K}(X)$.

An *end* of $X$ is, by definition, a point in the inverse limit space of this system. In other words, an end of $X$ is a function $e$ which assigns to each compact set $K$ of $X$ a non-empty connected component $e(K)$ of $X \backslash K$, in such a way that $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. Let $\mathscr{E}(X)$ be the set of all ends. There is a topology on $X \cup \mathscr{E}(X)$ having as a basis of neighbourhoods of $e_0 \in \mathscr{E}(X)$ the $\mathscr{N}_K(e_0) = e_0(K) \cup \{\text{ends } e \,|\, e(K) = e_0(K)\}$, $K \in \mathscr{K}(X)$. With this topology $X \cup \mathscr{E}(X)$ is a Hausdorff space containing $\mathscr{E}(X)$, with its inverse limit topology, as a closed (nowhere dense) subspace.

If $f : X \to Y$ is a continuous *proper* function (i.e., $F \subset Y$ compact implies $f^{-1}(F)$ compact), then $f$ is extended uniquely and continuously to a function

$$f \cup f_\varepsilon : X \cup \mathscr{E}(X) \to Y \cup \mathscr{E}(Y)$$

such that for $e \in \mathscr{E}(X)$ and $F \subset Y$ compact $f_\varepsilon(e)F$ is the (unique) component of $Y \backslash F$ containing $f(e(f^{-1}(F)))$.

Let $X$ be a space, and let $K \in \mathscr{K}(X)$. A connected component $V$ of $X \backslash K$ is said to be *bounded* if its closure is compact, and otherwise we say that $V$ is *unbounded*. Define

$$\hat{K} = X \backslash \cup \{V \in \mathscr{C}(X \backslash K) \,|\, V \text{ is unbounded}\}.$$

The proof of the following lemma may be found in Berlanga and Epstein [2].

1.1 LEMMA. *Let $X$ be a connected, locally connected, locally compact, Hausdorff space and let $K \in \mathscr{K}(X)$. Then $X \backslash K$ has only finitely many unbounded components and $\hat{K}$ is compact.*

1.2 REMARK. It follows that $\mathscr{E}(X)$ is compact since $\hat{\mathscr{K}}(X) = \{\hat{K} \,|\, K \in \mathscr{K}(X)\}$ is cofinal in $\mathscr{K}(X)$ and each $\mathscr{C}(X \backslash \hat{K})$ is finite. It is also known that $X \cup \mathscr{E}(X)$ is compact and that $\mathscr{E}(X)$ is totally disconnected. Also if $X$ is metric $X \cup \mathscr{E}(X)$ is metrizable.

## §2. Definitions

Let $X$ be a subset of a topological space $Y$. We define $\mathring{X}$ and Cl $X$ to be, respectively, the topological interior and the topological closure of $X$ in $Y$. Call $X$ a (closed) *n-cell* if $X$ is homeomorphic to the unit *n*-cube $I^n = [0, 1]^n$. For a subset $X$ of a manifold $M$ we define Int $X$ to be $(M \backslash \partial M) \cap \mathring{X}$, where $\partial M$ denotes the boundary of $M$.

## §3. The main theorem

*Let $M^n$ be a connected, second countable manifold of dimension n. Then there exists a compact set $E \subset \partial I^n$ and a continuous proper surjection $\psi : I^n \backslash E \to M$ such that*

(1) $\psi|_{\text{Int } I^n} : \text{Int } I^n \to \psi \, (\text{Int } I^n)$ *is a homeomorphism;*

(2) $\psi \, (\text{Int } I^n) \cap \psi(\partial I^n \backslash E) = \varnothing;$

(3) $\psi$ *extends naturally to* $\tilde{\psi} : I^n \to M \cup \mathscr{E}(M)$ *in such a way that* $\tilde{\psi}|_E$ *is a homeomorphism from E onto* $\mathscr{E}(M)$.

*Furthermore, if $n \geqslant 2$ then E can be chosen to be contained in* $[1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$.

## §4. Definitions, lemmas and proof of the main theorem

4.1. DEFINITIONS. An $(n - 1)$-dimensional submanifold $B$ of an *n*-manifold $M$ is *bicollared in M* if there is a homeomorphism $P$ of $B \times \langle -1, 1 \rangle$ onto a neighbourhood of $B$ in $M$ such that $P(b, 0) = b$, for all $b \in B$. If $B$ is closed in $M$ we require also that $P$ can be extended to a closed embedding of $B \times [1, 1]$ into $M$.

If $B$ is the boundary of an *n*-dimensional submanifold $C$ of $M$, then $B \times \langle -1, 0]$ and $B \times [0, 1 \rangle$ denote the *inner* and *outer* collars of $B$. In general, we will not distinguish $(b, t) \in B \times \langle -1, 1 \rangle$ from $P((b, t))$.

Define $\mathscr{H}(M)$ to be the group of homeomorphisms of $M$ onto itself. If $h : M \to M$ is a homeomorphism, then supp $h$ denotes the *support* of $h$, that is, the closure of the set of points of $M$ which are actually moved by $h$.

The following result is a straightforward generalization of Lemma 2 in M. Brown [4] (or Lemma 6 in Berlanga and Epstein [2]).

**4.2. LEMMA.** *Let $M^n$ be a manifold with $n \geqslant 3$ and let $d$ be a metric on $M$. Let $C^n$ be a closed $n$-dimensional manifold with bicollared boundary $\partial C$ in $M$.*

*Let $\varepsilon > 0$ be given and suppose $\Lambda = \{D_j\}_{j \in J}$ is a locally finite family of sets in $M$ such that each $D_j$ is a closed $n$-cell of diameter less than $\varepsilon/2$ whose interior intersects $C$. Let $X = \{x_i\}_{i \in L}$ be a locally finite set of points in $\bigcup_j$ Int $D_j$.*

*Suppose that $0 < \gamma < 1$. Then there is an $\varepsilon$-homeomorphism $f$ in $\mathscr{H}(M)$ such that $f(C) \supset f(\mathring{C}) \supset C \cup X$ and supp $f \subset (\bigcup_j \text{Int } D_j \backslash C) \cup \partial C \times \langle -\gamma, \gamma \rangle$. In particular, $f$ fixes pointwise the inner $n$-manifold bounded by $\partial C \times \{-\gamma\}$.*

**4.3 LEMMA.** *Let $M$ be a connected, second countable, $n$-dimensional manifold with $n \geqslant 3$ and let $X \subset \text{Int } M$ be a locally finite set of points. Then there exists a compact set $E \subset [1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\} \subset \partial I^n$, and a proper embedding $\psi_* : I^n \backslash E \to M$ with bicollared boundary such that $\psi_* (\text{Int } I^n) \supset X$ and $\psi_*$ extends naturally to $\tilde{\psi}_* : I^n \to M \cup \mathscr{E}(M)$ in such a way that $\tilde{\psi}_*|_E$ is a homeomorphism from $E$ onto $\mathscr{E}(M)$.*

*Proof.* Define a *clean (closed) $n$-cube* in $I^n$ to be a cube $C$ of the form $[0, \beta]^n + v$, for some $\beta > 0$ and $v \in \mathbb{R}^n$, such that $C \subset I^n$ and $C \cap \partial I^n = ([0, \beta]^{n-1} \times \{\beta\}) + v$.

Observe that if $C_1, \ldots, C_k$ is a disjoint collection of clean cubes then $\text{Cl}\,(I^n \backslash \bigcup_i C_i)$ is homeomorphic to $I^n$. We divide the proof in three steps.

*Step 1.* Let $\{K_i\}_{i \in \mathbb{N}}$ be any collection of $K_i \in \mathscr{K}(M)$ such that $M = \bigcup_i K_i$ and $K_i \subset \mathring{K}_{i+1}$; further properties of the $K_i$ will be specified in Step 2. It is not difficult now to define a sequence $\{L_i\}_{i \in \mathbb{N}}$ of $n$-cells in $I^n$ with $L_i \subset \mathring{L}_{i+1}$ and such that

(a) The complement of $\mathring{L}_i$ is the finite disjoint union of clean cubes of diameter less or equal $1/2^i$, and such that, for each $A \in \mathscr{C}(I^n \backslash L_i)$, we have, $A \cap [1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\} \neq \varnothing$. Hence, $E = \bigcap_i I^n \backslash L_i$ is contained in $[1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$;

(b) For each $i \in \mathbb{N}$ there exists a bijection $\lambda_i : \mathscr{C}(I^n \backslash L_i) \to \mathscr{C}(M \backslash \hat{K}_i)$ such that the diagrams

$$
\begin{array}{ccc}
\mathscr{C}(I^n \backslash L_j) & \xrightarrow{\lambda_j} & \mathscr{C}(M \backslash \hat{K}_j) \\
\varrho \downarrow & & \varrho \downarrow \\
\mathscr{C}(I^n \backslash L_i) & \xrightarrow{\lambda_i} & \mathscr{C}(M \backslash \hat{K}_i)
\end{array}
$$

commute $(i < j)$.

The reader can readily verify the following assertion:

*Assertion.* $E = \bigcap_i I^n \backslash L_i$, $\mathscr{E}(I^n \backslash E)$ and $\mathscr{E}(M)$ are homeomorphic. Furthermore, the identity map $I^n \backslash E \to I^n \backslash E$ extends naturally to a homeomorphism of $I^n = (I^n \backslash E) \cup E$ onto $(I^n \backslash E) \cup \mathscr{E}(I^n \backslash E)$.

We now proceed into Step 2 of this lemma.

*Step 2.* Let the $K_i$ be constructed as to satisfy also the following properties:
– $\mathring{K}_i$ is connected;
– $M \backslash \hat{K}_i$ has exactly the same number of components as $\mathring{K}_{i+1} \backslash \hat{K}_i$.
If $\psi_0 : I^n \to \mathring{K}_0$ is an embedding with bicollared boundary, then there exists a homeomorphism $h_1$ of $M$ with compact support such that
(1) supp $h_1 \cap \psi_0(L_0) = \varnothing$;
(2) supp $h_1 \subset \mathring{K}_1$;
(3) If $A \in \mathscr{C}(I^n \backslash L_1)$ then
    (a) $h_1(\psi_0(A)) \subset \lambda_0(\varrho_0^1(A))$;
    (b) $h_1(\psi_0(A))$ and $\lambda_1(A)$ are not separated in $M$ by $h_1(\psi_0(I^n \backslash A)) \cup \hat{K}_0$, (that is, $h_1(\psi_0(A))$ and $\lambda_1(A)$ lie in the same connected component of $M \backslash (h_1(\psi_0(I^n \backslash A)) \cup \hat{K}_0)$.

*Proof.* Let $A_1, A_2, \ldots, A_k$ be the components of $I^n \backslash L_1$. It is not difficult to construct a family of disjoint arcs, say $\{\gamma_i : [0, 2] \to M \mid 1 \leqslant i \leqslant k\}$ and a family $\{U_i \mid 1 \leqslant i \leqslant k\}$ of disjoint connected open sets in $\mathring{K}_1$ such that, for each $i$,

$U_i \cap \psi_0(I^n) \subset \psi_0(A_i)$;

$\gamma_i([0, 1]) \subset U_i$;

$\gamma_i([1, 2]) \subset \mathring{K}_2 \backslash \hat{K}_0$;

$\gamma_i(0) \in \psi_0(A_i)$;

$\gamma_i(1) \in \lambda_0(\varrho_0^1(A_i))$;

$\gamma_i(2) \in \lambda_1(A_i)$.

This can be done because $M \backslash \psi_0(I^n)$ is connected and an $n$-dimensional manifold cannot be disconnected by a set of dimension $n - 2$ (see Hurewicz and Wallman [5]).

Since the group of compactly supported homeomorphisms of a connected manifold acts transitively on interior points, we can find, for each $i$, a homeomorphism $h_{1,i}$ compactly supported on $U_i$ which sends $\gamma_i(0)$ to $\gamma_i(1)$.

For each $i = 1, 2, \ldots, k$, let $\tau_i \in [1, 2]$ be the last parameter such that its image under $\gamma_i$ lies in $h_{1,i}(\psi_0(I^n))$.

Consequently, there is a unique $x_i$ in $\partial I^n \cap A_i$ with $\gamma_i(\tau_i) = h_{1,i}(\psi_0(x_i))$. Now choose a clean closed cube $B_i$ such that $x_i \in B_i \subset A_i$ and $h_{1,i}(\psi_1(B_i)) \subset \lambda_0(\varrho_0^1(A_i))$.

With a homeomorphism of $M$ sending $\psi_0(I^n)$ onto itself and supported in a small neighbourhood of $\psi_0(\mathrm{Cl}\ A_i)$ we can shrink $\psi_0(\mathrm{Cl}\ A_i)$ onto $\psi_0(B_i)$ before applying $h_{1,i}$. Therefore, without loss of generality we can assume that $A_i = B_i$ and that supp $h_{1,i} \cap$ supp $h_{1,j} = \varnothing$ for $i \neq j$. Hence, $h_{1,i}(\psi_0(A_i)) \subset \lambda_0(\varrho_0^1(A_i))$ and $h_{1,i}(\psi_0(A_i))$, $\lambda_1(A_i)$ are not separated in $M$ by $h_{1,i}(\psi_0(I^n\backslash A_i)) \cup \hat{K}_0$.

Finally, the homeomorphism $h_1 = h_{1,1} \circ h_{1,2} \circ \ldots \circ h_{1,k}$ has the required properties.
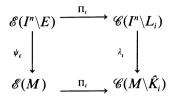
*Step 3.* By induction, we can construct a sequence $\{h_i\}_{i \in \mathbb{N}}$ of homeomorphisms with compact support such that, for each $i$,

(1) supp $h_{i+1} \cap (h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ \psi_0(L_i)) = \varnothing$;
(2) supp $h_{i+1} \subset \mathring{\hat{K}}_{i+1}$;
(3) supp $h_{i+1} \cap \hat{K}_{i-1} = \varnothing$;
(4) If $A \in \mathscr{C}(I^n\backslash L_{i+1})$ then
   (a) $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(A) \subset \lambda_i(\varrho_i^{i+1}(A))$;
   (b) $h_{i+1} \circ \ldots \circ h_1 \circ \psi_0(A)$ and $\lambda_{i+1}(A)$ are not separated in $M$ by $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(I^n\backslash A) \cup \hat{K}_i$.

Define $\psi_i = h_i \circ \ldots \circ h_1 \circ \psi_0$, $i \in \mathbb{N}$. Therefore, the following properties hold:

(5) $\psi_i|_{L_i} = \psi_{i+k}|_{L_i}$ for all $i, k \in \mathbb{N}$;
(6) $\psi_{i+k}(A) \subset \lambda_i(\varrho_i^{i+1}(A))$ for all $i \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$, and all $A \in \mathscr{C}(I^n\backslash L_{i+1})$.
   It follows that $\lim_{i \to \infty} \psi_i = \psi_*$ exists in $\bigcup_i L_i$ and is such that
(7) $\psi_*|_{L_i} = \psi_i|_{L_i}$ for all $i \in \mathbb{N}$;
(8) $\psi_*(A) \subset \lambda_i(\varrho_i^{i+1}(A))$ for all $i \in \mathbb{N}$ and all $A \in \mathscr{C}((I^n\backslash E)\backslash L_{i+1})$;
(9) $\psi_*^{-1}(\hat{K}_i) \subset L_{i+1}$.

Property (7) says that $\psi_*$ is continuous and injective. Property (9) (which follows from (8)) tells us that $\psi_* : I^n\backslash E \to M$ is proper, and therefore induces a map $\psi_* \cup \psi_\varepsilon : (I^n\backslash E) \cup \mathscr{E}(I^n\backslash E) = I^n \to M \cup \mathscr{E}(M)$ such that if $e$ is an end of $I^n\backslash E$, $\psi_\varepsilon(e)\hat{K}_i$ is the component of $M\backslash\hat{K}_i$ containing $\psi_*(e(\psi_*^{-1}(\hat{K}_i)))$, hence, by (9), it is equal to the component of $M\backslash\hat{K}_i$ containing $\psi_*(e(L_{i+1}))$, but, by (8), this is just $\lambda_i(\varrho_i^{i+1}(e(L_{i+1}))) = \lambda_i(e(L_i))$. That is, we have proved that the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{E}(I^n\backslash E) & \xrightarrow{\;\;\Pi_i\;\;} & \mathscr{C}(I^n\backslash L_i) \\
\Big\downarrow{\scriptstyle \psi_\varepsilon} & & \Big\downarrow{\scriptstyle \lambda_i} \\
\mathscr{E}(M) & \xrightarrow{\;\;\Pi_i\;\;} & \mathscr{C}(M\backslash \hat{K}_i)
\end{array}$$

Since each $\lambda_i$ is bijective, $\psi_\varepsilon$ must be a homeomorphism. Therefore, we have constructed a proper embedding $\psi_*: I^n\backslash E \to M$ inducing a homeomorphism on ends.

In order to complete the proof of Lemma 4.3 we need to produce a bicollar of $\psi_*(\partial I^n\backslash E)$ and we need to "expand" the image $C$ of $I^n\backslash E$ in $M$ as to contain $X$ in its interior.

Let $E'$ be the projection of $E$ into $I^{n-1}$, so $E = E' \times \{1\}$. It is not difficult to see that the spaces $W = [-1, 2]^n\backslash(E' \times [1/2, 2])$ and $T = W \cap I^n$ are homeomorphic to $I^n\backslash E$ and that the inclusion map $T \to W$ is a proper map inducing a homeomorphism on ends.

Therefore without loss of generality, we can assume that the domain of the map $\psi_*$ is $W$. But now $\psi_*|_T$ has the same properties of $\psi_*$ with the advantage that $\partial T$ has a natural bicollar contained in $M$.

It now only remains to "expand" the image of $\psi_*$. To this purpose we can construct a locally finite family $\Lambda_0 = \{D_j\}_{j\in J}$ of closed $n$-cells such that $X$ is contained in $\bigcup_j \text{Int } D_j$ and $\text{Int } D_j \cap C \neq \varnothing$ for all $j \in J$.

Therefore, by an application of Lemma 4.2, say with $\gamma = 1/2$ and $\varepsilon = \infty$, we get the desired expansion.

*4.4. Proof of the main theorem.* When the dimension of the manifold $M$ is less or equal two, the theorem follows from the classification of second countable manifolds of dimensions one and two (see Ahlfors and Sario [1]).

Assume now that the dimension of $M$ is greater or equal to three. Let $d$ be a complete metric on $M$. Let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of locally finite covers of $M$ such that each element of $\Lambda_i$ is a closed $n$-cell of diameter less than $1/2^{i+1}$ and $\text{Int } M = \cup\{\text{Int } D \mid D \in \Lambda_i\}$. For each $i$, let $X_i$ be a locally finite set of points such that $X_i \subset \text{Int } M$ and $\text{Int } D \cap X_i \neq \varnothing$ if $D \in \Lambda_i$.

Let $C_1$ be the image under $\psi_*$ where $\psi_*$ is the embedding given by the above lemma, and assume that $X_1 \subset \text{Int } C_1$. Applying Lemma 4.2 with $X = X_2$, $\Lambda = \Lambda_1$ and $\gamma$ small, we get a $1/2$-homeomorphism $f_1$ of $M$ onto itself such that

$$M \supset C_2 = f_1(C_1) \supset f_1(\mathring{C}_1) \supset C_1 \cup X_2 \quad \text{and} \quad f_1|_{(1-\gamma)C_1} = \text{Id},$$

where

$$(1 - \gamma)C_1 = C_1 \backslash \partial C_1 \times \langle -\gamma, 0].$$

Repeated applications of 4.2 give a sequence $f_1, f_2, \ldots$ of homeomorphisms of $M$ such that for each $m \in \mathbb{N}\backslash\{0\}$,

- $f_m$ is a $(1/2)^m$-homeomorphism;
- $M \supset f_m \circ \ldots \circ f_1(C_1) \supset f_m \circ \ldots \circ f_1(\mathring{C}_1)$
$$\supset C_1 \cup \bigcup_i \{X_i \,|\, 1 \leqslant i \leqslant m + 1\};$$
- $f_{m+1}$ restricted to $f_m \circ \ldots \circ f_1((1 - \gamma/2^m)C_1)$ is the identity.

  Clearly $f_m \circ \ldots \circ f_1$ converges to a map $\psi$ such that
- $\psi(C_1) = \lim_{m \to \infty} f_m \circ \ldots \circ f_1(C_1) = M$;
- $\psi$ is a homeomorphism on $\mathring{C}_1$;
- $\psi^{-1}(\psi(\partial C_1)) = M \backslash \mathring{C}_1$;

So that when $\psi$ is restricted to $C_1$ we get the required map.

4.5 REMARK. Let $\psi : I^n \backslash E \to M$ be a mapping given by the main theorem above. Then, measures (having the boundary of the unit $n$-cube as a null set) and homeomorphisms of the unit $n$-cube fixing $\partial I^n$ point-wise can be thrown, respectively, into measures and homeomorphisms of $M$ via $\psi$. This provides us with a tool for the topological and algebraic study of various groups of (measure preserving) homeomorphisms of $M$ (see [3]).

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