COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 63, nº 2 (1987), p. 145-158
<http://www.numdam.org/item?id=CM_1987__63_2_145_0>

Compositio Mathematica 63: 145–158 (1987)

Gauss sums and algebraic cycles

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Received 4 September 1986; accepted 21 November 1986

Introduction

Let L be a function field in one variable over a finite field k and X be a complete, smooth, geometrically irreducible curve over L of positive genus. Let $A = J(X)$ denote the Jacobi variety of X, $L_S(A, s)$ be an L-series of A, r be the rank of the Mordell-Weil group $A(L)$ and $Ш = Ш(A, L)$ be the Tate–Shafarevich group. The Birch–Swinnerton–Dyer conjecture, as formulated by Tate [3, 18] asserts that $Ш$ is a finite group and

$$\lim_{s\to 1} \frac{L_S(A, s)}{(s-1)^r} = \frac{[Ш]\,|\det \langle \alpha_i, \alpha_j \rangle|}{[A(L)_{\text{tor}}]^2}, \tag{1}$$

where [ ] denotes the cardinality of a finite group, $\{\alpha_1, \ldots \alpha_r\}$ is a basis for $A(L)$ modulo torsion, $\langle , \rangle$: $A(L)/A(L)_{\text{tor}} \times A(L)/A(L)_{\text{tor}} \to \mathbb{R}$ denotes the (non-degenerate) Néron–Tate height pairing for the self-dual abelian variety A, and $|\det \langle \alpha_i, \alpha_j \rangle|$, $1 \leqslant i, j \leqslant r$ denotes the absolute value of the determinant of this pairing.

The purpose of this paper is to give a refinement of (1) under hypotheses which are known to imply its validity. For each prime l unequal to the characteristic of k we define an invariant $\Delta(l)$ in terms of Gauss sums arising from the l-adic étale cohomology of a surface associated to X. The same construction applied to the crystalline cohomology of the surface yields an invariant $\Delta(p)$. Roughly speaking, the Δ's provide a factorization of $|\det \langle \alpha_i, \alpha_j \rangle|^{1/2}$ as a product of local terms. This leads to a factorization of the right side of (1).

Let $\mathscr{A}$ denote the Néron model of A over L, for each reducible fiber $\mathscr{A}_c$ of $\mathscr{A}$ let m_c denote the number of components of $\mathscr{A}_c$, and $m = \text{l.c.m.}\,\{m_c\}$. Let $Ш(l)$ and $A(L)_{\text{tor}}(l)$ denote, respectively, the l-primary components of

Ш and of $A(L)_{\text{tor}}$, then

$$\mathbf{e}(-r/8)(2m)^r \lim_{s\to 1} \sqrt{\frac{L_S(A, s)}{(s-1)^r}} = \prod_l \frac{[Ш(l)]^{1/2}\Delta(l)}{[A(L)_{\text{tor}}(l)]} \tag{2}$$

where $\mathbf{e}(x) = e^{2\pi i x}$, the square root is the positive one, and the product extends over all primes l, including l equal to the characteristic of k.

Let C be a complete, smooth, geometrically irreducible curve over k with function field L and $\mathscr{X}$ be the minimal model of X over C. $\mathscr{X}$ is a smooth projective surface over k. The local structure of $\mathscr{X}$ over C and the Arakelov–Hriljac construction of the local Néron pairing on $\mathscr{X}$ are briefly recalled in Section 1. The relation between the intersection theory on $\mathscr{X}$ and the height pairing on $A(L)$ are studied in Section 2. A particular basis, adapted to the arithmetic applications in Sections 4 to 6, for $NS(\mathscr{X}) \otimes \mathbb{Q}$ is constructed. A glance at the intersection matrix with respect to this basis shows that the positive definiteness of the height pairing on $A(L)/A(L)_{\text{tor}}$ is an immediate consequence of the Hodge index theorem on $\mathscr{X}$.

Basic facts concerning the Fourier transform of a character of second degree on a locally compact abelian group and the Weil reciprocity law for rational quadratic forms are recalled in Section 3. In Section 4 this theory is applied to characters of second degree which arise from the cup products in the l-adic étale cohomologies and in the crystalline cohomology of $\mathscr{X}$. Here the assumption that k is finite enters for the first time. We assume also that the characteristic of k is odd and that the cycle map $NS(\mathscr{X}) \otimes \mathbb{Z}_l \to H^2_{\text{flat}}(\bar{\mathscr{X}}, T_l\mu)^G$ is bijective for some l (including l equal to the characteristic of k). For each prime l unequal to the characteristic of k we define Gauss sums arising from the images of various subgroups of $NS(\mathscr{X}) \otimes \mathbb{Q}$ in the étale cohomology group $H^2_l(\bar{\mathscr{X}}, \mathbb{Q}_l)(1)^G$. The invariant $\Delta(l)$ which appears in (2) is a quotient of these Gauss sums. A similar argument is then applied to the images of the same subgroups of $NS(\mathscr{X}) \otimes \mathbb{Q}$ in the crystalline group $H^2(\mathscr{X}/W_k) \otimes K(1)^F$. This leads to the invariant $\Delta(p)$. Formula (2) then follows from the Hodge index theorem and the reciprocity law; the proof is in Section 5. Finally, in Section 6 it is shown that $\mathbf{e}(-r/8)(2m)^r|\det\langle\alpha_i, \alpha_j\rangle|^{1/2}$ can be expressed as a quotient of Gauss sums defined in terms of an adelic cohomology of $\mathscr{X}$.

For some elliptic rational and elliptic $K3$ surfaces the $\Delta(l)$'s and $\Delta(p)$ can be evaluated explicitly. These examples will be discussed in another paper.

§1

DEFINITION. Let $\mathcal{o}$ be a discrete valuation ring. A *curve over* $\mathcal{o}$ is a pair $(\mathcal{Y}, f)$ where $\mathcal{Y}$ is a connected scheme and $f: \mathcal{Y} \to \operatorname{Spec} \mathcal{o}$ is a morphism proper, flat and of finite type such that the fibres of f are algebraic curves. $(\mathcal{Y}, f)$ is said to be a *regular curve* if all the local rings of $\mathcal{Y}$ are regular.

Let $\mathcal{Y}_\eta$ and $\mathcal{Y}_s$ denote, respectively, the generic and the closed fibres of $\mathcal{Y}$. $\mathcal{Y}_s$ may be singular, reducible and non-reduced even if $\mathcal{Y}$ is smooth and geometrically irreducible. Shafarevich ([15], Lecture 6) has developed an intersection theory on $\mathcal{Y}$ which we use later in this section.

Let L denote the quotient field of $\mathcal{o}$. Assume that

i) the integral closure of $\mathcal{o}$ in any finite algebraic extension of L is a finite $\mathcal{o}$-module,

ii) the residue field of $\mathcal{o}$ is perfect,

and let X be a curve over L, complete, smooth and geometrically irreducible. Then there exists a regular curve $\mathcal{X}'$ over $\mathcal{o}$ such that $\mathcal{X}'_\eta$ is L-isomorphic to X, ([1], §1 Resolution Theorem). In case the genus of X is positive there exists a minimal model for $\mathcal{X}'$, unique up to isomorphism over Spec $\mathcal{o}$, such that $\mathcal{X}_\eta$ is L-isomorphic to X. The existence of $\mathcal{X}$ was first established for curves of genus one containing a rational point ([13], Chapitre III, Théorème 1). For arbitrary curves of positive genus the result is proven in [10], Theorem 4.4; [16], pp. 131, 155.

For a divisor D on X let $\boldsymbol{D}$ denote the closure of D in $\mathcal{X}$ and $\operatorname{Div}_0(X)_L$ denote the group of divisors of degree zero on X rational over L. Let Div $\mathcal{X}$ and $\operatorname{Div}_s \mathcal{X}$ denote, respectively, the group of divisors on $\mathcal{X}$ and the group of divisors on $\mathcal{X}$ with support contained in $\mathcal{X}_s$.

PROPOSITION 1. *If X has an L-rational point P, let* $\operatorname{Div}_0(X)_L(P)$ *denote the subgroup of* $\operatorname{Div}_0(X)_L$ *consisting of divisors which do not contain P as a component and* $\operatorname{Div}_s(\mathcal{X})(P) \otimes \mathbb{Q}$ *the subspace of* $\operatorname{Div}_s(\mathcal{X}) \otimes \mathbb{Q}$ *generated by components of $\mathcal{X}_s$ which do not intersect $\boldsymbol{P}$. Then there exists a unique homomorphism*

$$\Phi: \operatorname{Div}_0(X)_L(P) \to \operatorname{Div}_s(\mathcal{X})(P) \otimes \mathbb{Q}$$

such that for all $D \in \operatorname{Div}_0(X)_L(P)$, $\boldsymbol{D} + \Phi(D)$ *has intersection product zero with each component of $\mathcal{X}_s$.*

PROOF. As in the proof of Theorem 1.3 of [6] observe that the intersection product is non-degenerate on $\operatorname{Div}_s(\mathcal{X})(P) \otimes \mathbb{Q}$. For $D \in \operatorname{Div}_0(X)_L(P)$ let $\Phi(D)$ be the unique element of $\operatorname{Div}_s(\mathcal{X})(P) \otimes \mathbb{Q}$ such that $\Phi(D) \cdot F = -\boldsymbol{D} \cdot F$ for all $F \in \operatorname{Div}_s(\mathcal{X})$.

DEFINITION. Using the hypothesis and notations of the proposition let

$$\delta\colon \mathrm{Div}_0(X)_L(P) \to \mathrm{Div}(\mathscr{X}) \otimes \mathbb{Q}$$

be the homomorphism $\delta(D) = \boldsymbol{D} + \Phi(D) - (\boldsymbol{D} \cdot \boldsymbol{P})\mathscr{X}_s$.

PROPOSITION 2. *For each pair of elements* D, E $\in \mathrm{Div}_0(X)_L(P)$ *with disjoint supports*

$$\delta(D) \cdot \delta(E) = \langle D, E \rangle,$$

where $\cdot$ *denotes the Shafarevich intersection product [16, p. 85] and* $\langle , \rangle$ *denotes the Néron pairing [8, Chapter 11, Theorems 3.6 and 3.7].*

PROOF. Let $\mathbb{Q}\mathscr{X}_s$ denote the subspace of $\mathrm{Div}_s(\mathscr{X}) \otimes \mathbb{Q}$ generated by $\mathscr{X}_s$. There is a canonical isomorphism $\alpha\colon \mathrm{Div}_s(\mathscr{X}) \otimes \mathbb{Q}/\mathbb{Q}\mathscr{X}_s \to \mathrm{Div}_s(\mathscr{X})(P) \otimes \mathbb{Q}$. Let $\Phi_s\colon \mathrm{Div}_0(X)_L \to \mathrm{Div}_s(\mathscr{X}) \otimes \mathbb{Q}/\mathbb{Q}\mathscr{X}_s$ be the homomorphism defined by Hriljac in [6], Theorem 1.3, then $\alpha \circ \Phi_s|_{\mathrm{Div}_0(X)_L(P)} = \Phi$. The proposition now follows from Theorem 1.6 of [6].

§2

(2.1) Throughout the rest of the paper L will denote a field of algebraic functions in one variable over a perfect field k, hence the valuation rings of L satisfy the conditions i) and ii) of Section 1. Let X be a curve over L, complete, smooth and geometrically irreducible, and C be a curve over k, also complete, smooth and geometrically irreducible, with function field L. Let $\mathscr{X}$ denote the minimal model of X over C, $p{:}\mathscr{X} \to C$ be the k-rational projection, and X_L be the generic fiber of p. $\mathscr{X}$ is a smooth projective surface over k. We assume that there is a k-rational section $\sigma\colon C \to \mathscr{X}$; let $P = \sigma(C) \cap X_L$. Then $\sigma_*(C)$ is a k-rational divisor on $\mathscr{X}$ of degree one and P may be regarded as a L-rational point of X.

DEFINITION. Let

$$\delta\colon \mathrm{Div}_0(X)_L(P) \to \mathrm{Div}\,(\mathscr{X}) \otimes \mathbb{Q}$$

be the homomorphism $\delta(D) = \boldsymbol{D} + \Sigma_{v\in\Omega}\Phi_v(D) - (\boldsymbol{D} \cdot \sigma)F$, where $\boldsymbol{D}$ denotes the scheme theoretic closure of D in $\mathscr{X}$ and Ω denotes the set of discrete valuation rings in L. For $v \in \Omega$, Φ_v denotes the map Φ defined in Proposition 1 of Section 1 and F denotes a complete fiber of p.

For a pair of elements $D, E \in \mathrm{Div}_0(X)_L(P)$ with disjoint supports $\Sigma_{v \in \Omega}\, \delta_v(D) \cdot \delta_v(E) \deg v$ (where $\cdot$ denotes the Shafarevich intersection product used in Section 1) is equal to the usual intersection product $\delta(D) \cdot \delta(E)$ on $\mathscr{X}$, which is, of course, defined for any pair of elements in $\mathrm{Div}(\mathscr{X}) \otimes \mathbb{Q}$.

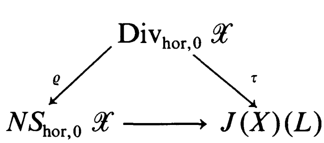
THEOREM. *Let θ be a theta divisor on $J(X)$, the Jacobian of X, and N: $J(X) \times J(X) \to \mathbb{Q}$ be the Néron-Tate height pairing on $J(X)$ with respect to $\theta + \theta^-$, then for $D, E \in \mathrm{Div}_0(X)_L$*

$$-\delta(D) \cdot \delta(E) = N(\mathrm{cl}\, D, \mathrm{cl}\, E),$$

where cl *D denotes the class of D.*

PROOF. The theorem follows from [6], Proposition 2 of Section 1 and Theorem 3.1. (Hriljac's proof, following [8], Chapter 5, Theorem 5.2 and 5.3.2, applies in the function field as well as in the number field case.)

(2.2) In order to write the intersection matrix of $\mathscr{X}$ in a form appropriate for the arithmetic applications in Sections 4 to 6 we now construct a basis for $NS(\mathscr{X}) \otimes \mathbb{Q}$ ($NS\mathscr{X}$ denotes the Néron-Severi group of $\mathscr{X}$). Let $\bar{\mathscr{X}} = \mathscr{X} \times \bar{k}$ and $\mathrm{Div}_{\mathrm{hor}}\, \bar{\mathscr{X}}$ be the subgroup of $\mathrm{Div}\, \bar{\mathscr{X}}$ generated by irreducible curves W on $\bar{\mathscr{X}}$ such that $\bar{p}$: $W \to \bar{V}$ is surjective, and $\mathrm{Div}_{\mathrm{hor},0}\, \mathscr{X}$ be the subgroup of $\mathrm{Div}_{\mathrm{hor}}\, \bar{\mathscr{X}}$ generated by k-rational divisors which intersect each complete fiber of p with total intersection multiplicity zero. For $D \in \mathrm{Div}_0\, \mathscr{X}$ let $[D_L]$ be the linear equivalence class as a divisor on X_L of $D_L = D \cap X_L$, then the map t: $\mathrm{Div}_{\mathrm{hor},0}\, \mathscr{X} \to J(X)(L)$ defined by $\tau(D) = [D_L]$ is surjective and the kernel of consists of the divisors in $\mathrm{Div}_{\mathrm{hor},0}\, \mathscr{X}$ which are linearly equivalent to zero on $\bar{\mathscr{X}}$. (See the proof of [3], Lemma 4.2, for these facts.) On the other hand, let ϱ: $\mathrm{Div}_{\mathrm{hor},0}\, \mathscr{X} \to NS\mathscr{X}$ be the canonical map of a divisor to its algebraic equivalence class on $\bar{\mathscr{X}}$ and denote by $NS_{\mathrm{hor},0}\, \mathscr{X}$ the image of ϱ. There is then a unique surjective map from $NS_{\mathrm{hor},0}\, \mathscr{X}$ to $J(X)(L)$ which makes the diagram



commute.

Let B denote the L/k-trace of $J(X)$. By the Mordell–Weil–Lang–Néron theorem ([9], Theorem 1), $J(X)(L)/B(k)$ is a finitely generated group. We assume from now on that $B(k)$ is a finite group. (In Sections 4 to 6 k will be a finite field so $B(k)$ will necessarily be finite.)

Let $D_i, i = 1, \ldots r$ be representatives in $\mathrm{Div}_{\mathrm{hor},0}\, \mathscr{X}$ of a basis for $J(X)(L)$ modulo torsion. Let F be a non-singular fiber of p, S_j, $j = 1, \ldots t$, be the singular fibers, $X_1^{S_j}$ denote the component of S_j which intersects the section σ and $X_2^{S_j}, \ldots X_{m_j}^{S_j}$ be the other components of S_j. Then by [3], Proposition 4.6, the images in $NS\mathscr{X}$ of $\sigma, \sigma - (\sigma \cdot \sigma)F, \{X_m^{S_j}\}, j = 1, \ldots t,$ $m > 1$, and $\{D_i\}$, $i = 1, \ldots r$ generate a free subgroup of $NS\mathscr{X}$ of finite index. Thus the images of $\sigma, \sigma - (\sigma \cdot \sigma)F$, $\{X_m^{S_j}\}$ and $\{\delta(D_i)\}$ form a basis for $NS(\mathscr{X}) \otimes \mathbb{Q}$. The intersection matrix M with respect to this basis has the form

$$\begin{bmatrix} \begin{array}{|cc|} \hline \sigma \cdot \sigma & 0 \\ 0 & -\sigma \cdot \sigma \\ \hline \end{array} & & & & \\ & \boxed{I_1} & & & \\ & & \ddots & & \\ & & & \boxed{I_t} & \\ & & & & \boxed{\delta(D_i) \cdot \delta(D_j)} \end{bmatrix}$$

where I_j denotes the intersection matrix of the $X_m^{S_j}$, $m = 1, \ldots m_j$. Since $\sigma - (\sigma \cdot \sigma)F$ has positive self-intersection, all blocks except the first one, are negative definite by the Hodge index theorem. In particular, the last block is negative definite. Combining this observation with the theorem gives a geometric proof of the positive definiteness of the Néron–Tate height on $J(X)(L)/J(X)(L)_{\mathrm{tor}}$. (Recall we have assumed that $B(k)$ is finite.)

§3

(3.1) Let G be a locally compact abelian group and T be the group of complex numbers of norm one. A continuous function $f\colon G \to T$ is called a character of second degree if the map $(x, y) \to f(x + y)/f(x)f(y)$ is bimultiplicative. Let G^* denote the topological dual of G. For a character of second degree f let $\varrho\colon G \to G^*$ be the morphism defined by $\varrho(y)(x) = f(x + y)/f(x)f(y)$; in case ϱ is an isomorphism, f is said to be non-degenerate ([20], no. 1). We shall deal only with non-degenerate f. When such an f is regarded as a tempered distribution its Fourier transform is given by

$$f^*(x^*) = \gamma(f)|\varrho|^{-1/2}f(\varrho^{-1}(x^*))^{-1},$$

where $|\varrho|$ denotes the Haar module of ϱ and $\gamma(f) \in T$ ([20], no. 14, Théorème 2). For a closed subgroup H of G let $H^\perp$ denote the subgroup of G^* consisting of characters which are trivial on H and set $H^\varrho = \varrho^{-1}(H^\perp)$. In case $f(x) = 1$ for all $x \in H$ it is easily seen that $H \subseteq H^\varrho$, that the restriction of f to H^ϱ is periodic with respect to H, and that f induces a character of second degree on H^ϱ/H (for details see [15], IV, 1). In case H^ϱ/H is finite, $\gamma(f)$ may be computed from a generalized Gauss sum ([15], IX, 2):

$$\gamma(f) = [H^\varrho : H]^{-1/2} \sum f(x), \quad x \in H^\varrho/H. \tag{3}$$

(3.2) Let K denote $\mathbb{R}$ or $\mathbb{Q}_l$, V be a finite dimensional K-vector space, q: $V \to K$ be a quadratic form and χ: $K \to T$ be a non-trivial character, then $\chi \circ q$ is a character of second degree which is non-degenerate if and only if q is. In this case $\gamma(\chi \circ q)$ is an eighth root of unity ([20], nos. 26, 28). For fixed χ, $\gamma(\chi \circ q)$ depends only on the class of q in the Witt group of K, and, in fact, the map $q \to \gamma(\chi \circ q)$ induces a character on this group ([20], no. 25). In case $K = \mathbb{R}$, let $\chi_\infty(x) = \mathbf{e}(-x) = e^{-2\pi i x}$, then $\gamma(\chi_\infty \circ q) = \mathbf{e}(-s/8)$ where s denotes the signature of q ([20], no. 26). Recall that a lattice in V is a $\mathbb{Z}$-module in V generated by a basis of V over $\mathbb{R}$. In case $K = \mathbb{Q}_l$, let χ_l be the Tate character ([17], 2.2): let $\mathbb{Q}^{(l)} = \{x | x = a/l^n, a, n \in \mathbb{Z}, n \geqslant 0\}$, then $\mathbb{Q}^{(l)} \cap \mathbb{Z}_l = \mathbb{Z}$, $\mathbb{Q}^{(l)} + \mathbb{Z}_l = \mathbb{Q}_l$, and $\mathbb{Q}_l/\mathbb{Z}_l \simeq \mathbb{Q}^{(l)}/\mathbb{Z}$, then χ_l is uniquely determined by the conditions $\chi_l(x) = \mathbf{e}(x)$ for $x \in \mathbb{Q}^{(l)}$ and $\chi_l|_{\mathbb{Z}_l} = 1$. By a lattice in V we shall mean a $\mathbb{Z}_l$-module in V generated by a basis for V over $\mathbb{Q}_l$, or equivalently, a compact, open $\mathbb{Z}_l$-module in V.

PROPOSITION. *Let K be $\mathbb{R}$ or $\mathbb{Q}_l$, V be a finite dimensional K-vector space, and q be a non-degenerate quadratic form on V. Let $\chi = \chi_\infty$ in case $K = \mathbb{R}$, $\chi = \chi_l$ in case $K = \mathbb{Q}_l$, $f(x) = \chi(q(x)/2)$, and ϱ: $V \to V^*$ be the morphism associated to f. Let H be a lattice in V such that $f(x) = 1$ for all $x \in H$ and D be the determinant of the bilinear form $B(y, z) = q(y + z) - q(y) - q(z)$ with respect to a basis for H. Then $|D| = [H^\varrho : H]$ in case $K = \mathbb{R}$ and $|D|_l = |[H^\varrho : H]|_l$ in case $K = \mathbb{Q}_l$, where $|\ |_l$ denotes the normalized absolute value.*

PROOF. Since Ker $\chi_l = \mathbb{Z}_l$, B: $H \times H \to 2\mathbb{Z}_l$ in case $K = \mathbb{Q}_l$; similarly B: $H \times H \to 2\mathbb{Z}$ in case $K = \mathbb{R}$. Let $H' = \{x \in V | B(H, x) \subseteq \mathbb{Z}\}$ in case

$K = \mathbb{R}$ and $H' = \{x \in V | B(H, x) \subseteq \mathbb{Z}_l\}$ in case $K = \mathbb{Q}_l$, then $H \subseteq H'$. Choose a basis $h_1, \ldots h_n$ for H and let $h'_1, \ldots h'_n$ be the dual basis for H'. Let $B(h_i, h_j) = m_{ij}$, M be the matrix (m_{ij}) and $N = (n_{ij}) = M^{-1}$, then $h'_i = \Sigma n_{ij} h_j$. In case $K = \mathbb{R}$, $[H' : H] = |\det N|^{-1} = |D|$ and in case $K = \mathbb{Q}_l$, $[H' : H]|_l = |\det N|_l^{-1} = |D|_l$ by [21], Chapter I, §2, Theorem 3, Corollary 3. Since $\varrho\,(z)(y) = \chi(B(z, y))$ for $\chi = \chi_\infty, \chi_l$ we have $H' = H^\varrho$.

(3.3) Let W be a finite dimensional $\mathbb{Q}$-vector space and q: $W \to \mathbb{Q}$ be a non-degenerate quadratic form on W. For a place l of $\mathbb{Q}$, finite or infinite, let $W_l = W \otimes \mathbb{Q}_l$ and denote by q_l: $W_l \to \mathbb{Q}_l$ the quadratic form induced by q on W_l. Let χ be the unique character of the adele ring $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ which is trivial on the principal adeles and such that $\chi|_\mathbb{R} = \chi_\infty$ and $\chi|_{\mathbb{Q}_l} = \chi_l$ ([21], Chapter IV, §2, proof of Theorem 3). Set $\gamma_l(q) = \gamma(\chi_l \circ q_l)$. The Weil reciprocity law states that $\Pi_l \gamma_l(q) = 1$, where the product is extended over all l ([20], no. 30, Proposition 5; [7], §4, Satz 4.1).

§4

(4.1) From this point on we assume that k is finite and of characteristic p, $p \neq 2$; let $G = \mathrm{Gal}\,(\bar{k}/k)$ and $\bar{\mathscr{X}} = \mathscr{X} \times \bar{k}$. For a prime l (l possibly equal to p) and m a positive integer let μ_{l^m} be the sheaf of l^m-th roots of unity and $H^n(\bar{\mathscr{X}}, T_l\mu) = \lim_{\overleftarrow{m}} H^n_{\text{flat}}(\bar{\mathscr{X}}, \mu_{l^m})$. In case $l \neq p$, $H^n(\bar{\mathscr{X}}, T_l\mu)$ may be interpreted as an étale cohomology group ([4], Théorème 11.7).

We now assume, cf. [18] p. 98

– (T) For some l the cycle map c_l: $NS(\mathscr{X}) \otimes \mathbb{Z}_l \to H^2(\bar{\mathscr{X}}, T_l\mu)^G$ is bijective.

By [11], Theorem 4.1, if this is the case for one l, it is the case for all l, or equivalently, the cycle map in crystalline cohomology c_p: $NS(\mathscr{X}) \otimes_\mathbb{Z} \mathbb{Q}_p \to H^2(\mathscr{X}/W) \otimes K(1)^F$ is an isomorphism ([11], Remark 5.4; here W denotes the Witt vectors of k, K the fraction field of W, and F the (p)-linear injective map on $H^2(\mathscr{X}/W) \otimes K(1)$ induced by the Frobenius endomorphism of $\bar{\mathscr{X}}$).

For a prime l, $l \neq p$, let $H^n_l(\bar{\mathscr{X}}) = H^n_l(\bar{\mathscr{X}}, \mathbb{Q}_l)$ be the l-adic étale cohomology of $\bar{\mathscr{X}}$ and $P_2(\mathscr{X}, t)$ be the characteristic polynomial of the endomorphism of $H^2_l(\bar{\mathscr{X}})$ induced by the Frobenius endomorphism of $\bar{\mathscr{X}}$. Let Br $\mathscr{X}$ denote the Brauer group of $\mathscr{X}$, $\varrho(\mathscr{X})$ be the rank of $NS\mathscr{X}$, $|\det I|$ be the absolute value of the determinant of the intersection matrix for a basis of $NS\mathscr{X}$ modulo torsion, $\boldsymbol{Pic}^0_\mathscr{X}$ be the connected component of the identity of the Picard scheme of $\mathscr{X}$, and $\alpha(\mathscr{X}) = \chi(\mathscr{X}, \mathscr{O}_\mathscr{X}) - 1 + \dim_k \boldsymbol{Pic}^0_\mathscr{X}$. Let q be the cardinality of k. Artin and Tate ([19], (C)) have conjectured:

– (AT) Br $\mathscr{X}$ is finite and

$$\lim_{s\to 1} \frac{P_2(\mathscr{X}, q^{-s})}{(1 - q^{1-s})^{\varrho(\mathscr{X})}} = \frac{[\mathrm{Br}\ \mathscr{X}]\ |\det I|}{q^{\alpha(x)}[NS\ (\mathscr{X})_{\mathrm{tor}}]^2}.$$

In [11], Theorems 4.1 and 6.1, it is shown that (T) implies (AT). In Section 2 we assumed that p: $\mathscr{X} \to C$ has a k-rational section so p is cohomologically flat in dimension zero. Hence by [3], Theorem 6.1 (AT) implies that Ш $(J(X), L)$ is finite and formula (1) holds.

Let p_J: $\mathscr{J} \to C$ denote the Néron model of $J(X)$ and for $c \in C(k)$, let m_c denote the number of components in the fiber $p_J^{-1}(c)$ and $m =$ l.c.m.$\{m_c\}$. As in Section 2.1 let N: $J(X) \times J(X) \to \mathbb{Q}$ denote the Néron-Tate height pairing on $J(X)$ relative to $\theta + \theta^-$; by [14], Chapitre III, §3, Proposition 2 (iii), and Chapitre III, §4, Théorème 1, N takes values in $(1/m)$ $\mathbb{Z}$. Let Γ' denote the subgroup of $NS\ (\mathscr{X}) \otimes \mathbb{Q}$ generated by the images of the divisors σ, $\sigma - (\sigma \cdot \sigma)F$, $\{X_m^{S_i}\}$ and $\{\delta(D_i)\}$ defined in Section 2.2 and set $\Gamma = 2m\Gamma'$. By the theorem of Section 2.1 the intersection matrix with respect to Γ has entries in $2\mathbb{Z}$.

(4.2) In this paragraph we apply the theory reviewed in Section 3 to the images of various subgroups of Γ in the twisted l-adic étale cohomology of $\bar{\mathscr{X}}$. The analogous construction in crystalline cohomology will be discussed in the next paragraph. Recall that $H_l^2(\bar{\mathscr{X}})(1)^G$ is canonically G-isomorphic to $H^2(\bar{\mathscr{X}}, T_l\mu)^G \otimes \mathbb{Q}_l$, that the intersection product on $NS\ \mathscr{X}$ is compatible via the cycle map with the cup product $H_l^2(\bar{\mathscr{X}})(1)^G \times H_l^2(\bar{\mathscr{X}})(1)^G \to H^4(\bar{\mathscr{X}})(2)$ and that this last group is canonically isomorphic via the trace map to $\mathbb{Q}_l$ ([12], Chapter VI, §9, 11; [18], §2). The intersection pairing on $NS\ \mathscr{X}$ is non-degenerate, so from (T) it follows that the pairing $H_l^2(\bar{\mathscr{X}})(1)^G \times H_l^2(\bar{\mathscr{X}})(1)^G \to \mathbb{Q}_l$ is also.

Definition. For a prime l, $l \neq p$, and $\alpha \in H_l^2(\bar{\mathscr{X}})(1)^G$ let $f_l(\alpha) = \chi_l(\alpha \cup \alpha)/2)$ and $\varrho_l = H_l^2(\bar{\mathscr{X}})(1)^G \to H_l^2(\bar{\mathscr{X}})(1)^{G*}$ be the morphism associated to f_l. Let $H_l = c_l(\Gamma \otimes \mathbb{Z}_l) \subseteq H_l^2(\bar{\mathscr{X}})(1)^G$, $H_l^{\perp}$ be the lattice in $H^2(\bar{\mathscr{X}})(1)^{G*}$ consisting of characters which are trivial on H_l, and $H_l^{\varrho_l} = \varrho_l^{-1}(H_l^{\perp})$.

Proposition 1. *Let* $g_l = \Sigma f_l(\alpha)$, $\alpha \in H_l^{\varrho_l}/H_l$, *then* $G_l = \gamma(f_l) \times |2m|_l^{-\varrho(\mathscr{X})}|\det M|_l^{-1/2}$, *where* $\gamma(f_l)$ *is an eighth root of unity and* M *is the intersection matrix constructed in Section 2.2.*

Proof. The proposition follows from (3) and the proposition of section 3.2.

REMARK. $\gamma(f_l)$ depends only on the cup product in $H_l^2(\bar{\mathscr{X}})(1)^G$ and not on the lattice Γ in $NS(\mathscr{X}) \otimes \mathbb{Q}$.

The next definition and proposition treat the first block of the intersection matrix M of Section 2.2.

DEFINITION. Let Γ_0 be the subgroup of Γ generated by the images in $NS(\mathscr{X}) \otimes \mathbb{Q}$ of the divisors $2m\sigma$ and $2m(\sigma \cdot \sigma)F$, $H_{l,0} = c_l(\Gamma_0 \otimes \mathbb{Z}_l) \subseteq H_l^2(\bar{\mathscr{X}})(1)^G$, $H_{l,0}^{\perp}$ be the lattice in $c_l(\Gamma_0 \otimes \mathbb{Q}_l)^*$ consisting of characters which are trivial on $H_{l,0}$, $\varrho_{l,0}$: $c_l(\Gamma_0 \otimes \mathbb{Q}_l) \to c_l(\Gamma_0 \otimes \mathbb{Q}_l)^*$ be the morphism associated to $f_l|_{c_l(\Gamma_0 \otimes \mathbb{Q}_l)}$, and $H_{l,0}^{\varrho_{l,0}} = \varrho_{l,0}^{-1}(H_{l,0}^{\perp})$.

PROPOSITION 2. *Let* $G_{l,0} = \Sigma f_l(\alpha)$ *for* $\alpha \in H_{l,0}^{\varrho_{l,0}}/H_{l,0}$, *then* $G_{l,0} = |2m|_l^{-2}|\sigma \cdot \sigma|_l^{-1}$, *where* $\sigma \cdot \sigma$ *is the self-intersection of the section* σ: $C \to \mathscr{X}$.

REMARK. Here the "γ-factor" is one because the block is a hyperbolic plane.

The blocks $I_1, \ldots I_t$ of M are treated in the same way as the first one and lead to Gauss sums $G_{l,j}, j = 1, \ldots t$ with $G_{l,j} = \gamma(f_{l,j})|2m|_l^{1-m_j}|\det I_j|_l^{-1/2}$.

(4.3) In order to carry over the preceeding construction from étale to crystalline cohomology let $H^{2r}(\mathscr{X}/W)_K(r) = H^{2r}(\mathscr{X}/W) \otimes K(r)$ and recall that the intersection product on $NS\ \mathscr{X}$ is compatible via the cycle map with the cup product $H^2(\mathscr{X}/W)_K(1) \times H^2(\mathscr{X}/W)_K(1) \to H^4(\mathscr{X}/W)_K(2)$ ([5], II, 4). The trace map gives a canonical isomorphism of this last group to K and one sees as in paragraph 2 that the pairing $H^2(\mathscr{X}/W)_K(1)^F \otimes H^2(\mathscr{X}/W)_K(1)^F \to K$ is non-degenerate.

DEFINITION. Let $n = [k: \mathbb{F}_p]$, $t = (1/n)Tr_{k/\mathbb{Q}_p}$, and $*$: $H^2(\mathscr{X}/W)_K(1)^F \otimes H^2(\mathscr{X}/W)_K(1)^F \to \mathbb{Q}_p$ denote the composite

$$H^2(\mathscr{X}/W)_K(1)^F \times H^2(\mathscr{X}/W)_K(1)^F \to K \xrightarrow{t} \mathbb{Q}_p.$$

For $\alpha \in H^2(\mathscr{X}/W)_K(1)^F$ let $f_p(\alpha) = \chi_p((\alpha * \alpha)/2)$ and ϱ_p: $H^2(\mathscr{X}/W)_K(1)^F \to H^2(\mathscr{X}/W)_K(1)^{F*}$ be the morphism associated to f_p. Let $H_p = c_p(\Gamma \otimes \mathbb{Z}_p)$, $H_p^{\perp}$ be the lattice in $H^2(\mathscr{X}/W)(1)^{F*}$ consisting of characters which are trivial on H_p, and $H_p^{\varrho_p} = \varrho_p^{-1}(H_p^{\perp})$.

The arguments in paragraph 2 now carry over to crystalline theory. Let $G_p = \Sigma f_p(\alpha)$, $\alpha \in H_p^{\varrho_p}/H_p$, then $G_p = \gamma(f_p)|2m|_p^{-\varrho(\mathscr{X})}|\det M|_p^{-1/2}$ where $\gamma(f_p)$ is an eighth root of unity. Define Gauss sums $G_{p,j}, j = 0, \ldots t$ as in paragraph 2, then $G_{p,0} = |2m|_p^{-2}|\sigma \cdot \sigma|_p^{-1}$ and $G_{p,j} = \gamma(f_{p,j})|2m|_p^{1-m_j}|\det I_j|_p^{-1/2}$.

§5

The determinant of the pairing $\Gamma \times \Gamma \to \mathbb{Q}$ induced by the intersection product on $NS\mathscr{X}$ is a unit in $\mathbb{Z}_l$ for almost all l. Denote this set of primes by Ω. By (T) the determinants of the corresponding pairings $H_l \times H_l \to \mathbb{Q}_l$ are units in $\mathbb{Z}_l$ for $l \in \Omega$, $l \neq p$, and hence by the proposition in Section 3.2 $H_l^{\varrho_l} = H_l$ and $G_l = 1$ for these l's.

DEFINITION. Let Δ denote the last block of the intersection matrix M of Section 2.2. For a prime, l, including $l = p$, let $\Delta(l) = G_l \cdot G_{l,0}^{-1} \cdot \Pi_j G_{l,j}^{-1}$, and $\gamma(\Delta, l) = \gamma(f_l) \, \Pi_j \, \gamma(f_{l,j})^{-1}$ denote the argument of $\Delta(l)$. Let

$$R(l) = \frac{[Ш(l)]^{1/2} \Delta(l)}{[J(X)(L)_{\mathrm{tor}}(l)]}.$$

REMARK. By the results of Sections 4.2 and 4.3 $\Delta(l)$ has modulus $|2m|_l^{-r} |\det \Delta|_l^{-1/2}$, where r is the rank of $J(X)(L)$. Since we have assumed that $p: \mathscr{X} \to C$, has a section $Ш = \mathrm{Br}\mathscr{X}$ ([19], Theorem 3.1), so for $l \neq 2$ $[Ш(l)]$ is a square ([11], Remark 2.5).

PROPOSITION. $\Pi_l \, \gamma(\Delta, l) = \mathrm{e}(-r/8)$.

Proof. The Weil reciprocity law gives $\Pi_l \, \gamma(f_l) = \mathbf{e}(s(M)/8)$, where $s(M)$ denotes the signature of M, and $s(M) = 2 - \varrho(\mathscr{X})$ by the Hodge index theorem. Each of the blocks I_j, $j = 1, \ldots t$ of M is negative definite, so the reciprocity law gives $\Pi_l \, \gamma(f_{l,j})^{-1} = \mathbf{e}(m_j - 1/8)$. As $\varrho(\mathscr{X}) = 2 + \Sigma\, (m_j - 1) + r$, the proposition follows.

THEOREM. *Let k be a finite field of characteristic p, $p \neq 2$, C be a complete, smooth, geometrically irreducible curve over k with function field $k(C) = L$. Let X be a complete, smooth, geometrically irreducible curve over L, A be the Jacobian variety of X, and $\mathscr{X}$ be the minimal model of X over C. If the projection $\mathscr{X} \to C$ has a k-rational section and (T) holds, then*

$$\mathbf{e}(-r/8)(2m)^r \lim_{s \to 1} \sqrt{\frac{L_S(A, s)}{(s-1)^r}} = \prod_l R(l),$$

where the square root is the positive one and the product extends over all primes l including $l = p$.

Proof. The results reviewed in Section 4.1 show that (T) implies (1) under the hypotheses. The theorem now follows from the remark, the product formula, and the proposition.

§6

The purpose of this section is to show that $\mathbf{e}(-r/8)(2m)^r|\det \Delta|_{\mathbb{R}}^{1/2}$ can be written as a quotient of Gauss sums defined in terms of an adelic cohomology similar to the one introduced in [2], §1.

Definition. Let $\mathbb{A}_p$ denote the restricted direct product of $\{\mathbb{Q}_l | l \neq p\}$ with respect to $\{\mathbb{Z}_l\}$, $H^2(\bar{\mathscr{X}}, \mathbb{A}_p)(1)^G$ be the restricted direct product of $\{H_l^2(\bar{\mathscr{X}})(1)^G | l \neq p\}$ with respect to $\{c_l(NS(\mathscr{X}) \otimes \mathbb{Z}_l)\}$, and

$$H^2_{\mathbb{A}_p}(\mathscr{X})(1) \;=\; H^2(\mathscr{X}/W)_K(1)^F \times H^2(\bar{\mathscr{X}}, \mathbb{A}_p)(1)^G.$$

The $*$ pairing of Section 4.3 and the cup products on the étale cohomologies yield a non-degenerate pairing $\otimes$: $H^2_{\mathbb{A}_p}(\mathscr{X})(1) \times H^2_{\mathbb{A}_p}(\mathscr{X})(1) \to \mathbb{Q}_p \times \mathbb{A}_p \subseteq \mathbb{A}_{\mathbb{Q}}$ compatible with the intersection product on $NS\mathscr{X}$.

Definition. For $\alpha \in H^2_{\mathbb{A}_p}(\mathscr{X})(1)$ let $f(\alpha) = \chi((\alpha \otimes \alpha)/2)$ where χ is the character of Section 3.3 restricted to $\mathbb{Q}_p \times \mathbb{A}_p$ and ϱ: $H^2_{\mathbb{A}_p}(\mathscr{X})(1) \to H^2_{\mathbb{A}_p}(\mathscr{X})(1)^*$ be the morphism associated to f. Let $H = H_p \times \Pi_{l \neq p} H_l$, $H^{\perp}$ be the subgroup of $H^2_{\mathbb{A}_p}(\mathscr{X})(1)^*$ consisting of characters which are trivial on H, and $H^{\varrho} = \varrho^{-1}(H^{\perp})$.

Proposition. *Let* $G_{\mathbb{A}} = \Sigma f(\alpha)$, $\alpha \in H^{\varrho}/H$, *then* $G_{\mathbb{A}} = \mathbf{e}(2 - \varrho(\mathscr{X})/8)$ $(2m)^{\varrho(\mathscr{X})}|\det M|_{\mathbb{R}}^{1/2}$.

Proof. Recall that $H^2_{\mathbb{A}_p}(\mathscr{X})(1)^*$ is canonically isomorphic to the product of $H^2(\mathscr{X}/W)_K(1)^{F*}$ times the restricted direct product of $\{H_l^2(\mathscr{X})(1)^{G*} | l \neq p\}$ with respect to $\{(c_l(NS(\mathscr{X}) \otimes \mathbb{Z}_l))^{\perp}\}$ ([17], Theorem 3.2.1). Hence $H^{\varrho}/H = H_p^{\varrho_p}/H_p \times \Pi_{l \notin \Omega'} H_l^{\varrho_l}/H_l$, where $\Omega' = \Omega \cup \{p\}$ and Ω is the set of primes defined at the beginning of Section 5. By (3), Proposition 1 of Section 4.1, the analogous result in Section 4.2, and the product formula H^{ϱ}/H has cardinality $(2m)^{\varrho(\mathscr{X})}|\det M|_{\mathbb{R}}$. Now it suffices to show that $G_{\mathbb{A}}$ has argument $\mathbf{e}(2 - \varrho(\mathscr{X})/8)$.

Let $H^2_{\mathbb{A}_p}(\mathscr{X})(1)' = H^2(\mathscr{X}/W)_K(1)^F \times \Pi_{l\notin\Omega'} H^2_l(\bar{\mathscr{X}})(1)^G$ and $\otimes' H^2_{\mathbb{A}_p}(\mathscr{X})(1)' \times H^2_{\mathbb{A}_p}(\mathscr{X})(1)' \to \mathbb{Q}_p \times \Pi_{l\notin\Omega'}\mathbb{Q}_l$ be the pairing induced by $\otimes$. For $\alpha' \in H^2_{\mathbb{A}_p}(\mathscr{X})(1)'$ let $f'(\alpha') = (\chi_p \times \Pi_{l\notin\Omega'}\chi_l)((\alpha' \otimes \alpha')/2)$, where the χ's are the Tate characters, and ϱ' is the morphism associated to f'. Set $H' = H_p \times \Pi_{l\notin\Omega'} H_l$ and let $G'_{\mathbb{A}} = \Sigma f'(\alpha')$, $\alpha' \in \varrho'^{-1}(H'^{\perp})/H'$. Clearly, $G'_{\mathbb{A}} = G_{\mathbb{A}}$, in particular, their arguments are equal. By [15], IX, 1(iii), $G'_{\mathbb{A}}$ has argument $\gamma(f_p)\Pi_{l\notin\Omega'}\gamma(f_l)$ and by the proof of the proposition in Section 5 this equals $\mathbf{e}(2 - \varrho(\mathscr{X})/8)$.

The sum $G_{\mathbb{A}}$ was defined using the subgroup Γ of $NS(\mathscr{X}) \otimes \mathbb{Q}$, which enters in the definition of H. As in Section 4 use $\Gamma_0 \subseteq \Gamma$ to define a sum $G_{\mathbb{A},0}$, then the proof of the proposition shows $G_{\mathbb{A},0} = (2m)^2|\sigma \cdot \sigma|_{\mathbb{R}}$. Treating the other blocks $I_j, j = 1, \ldots t$ of M in the same manner leads to sums $G_{\mathbb{A},j}$ with $G_{\mathbb{A},j} = \mathbf{e}(m_j - 1/8)(2m)^{m_j - 1}|\det \mathrm{I}_j|^{1/2}_{\mathbb{R}}$. The final result is:

Theorem. $\mathscr{G}_{\mathbb{A}} \cdot G^{-1}_{\mathbb{A},0} \cdot \Pi_j G^{-1}_{\mathbb{A},j} = \mathbf{e}(-r/8)(2m)^r|\det \Delta|^{1/2}_{\mathbb{R}}$.

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