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1. Introduction

This paper is based on a detailed study of the proof [11] by Toledo and Tong of the Hirzebruch-Riemann-Roch (HRR) formula. The main idea of their proof involves a geometrical construction, for any holomorphic vector bundle E over a complex manifold X, of a certain Čech cocycle on an open cover of X. When X is compact this cocycle integrates to give the Euler characteristic

$$\chi(X, E) = \sum_i (-1)^i \dim H^i (X, E).$$

The most difficult part of the proof is the identification of the cocycle as a local expression for the characteristic class $\mathrm{Todd}(X)\mathrm{ch}(E)$ in Hodge cohomology, as given by Atiyah's theory [1], [2].

While this is similar in spirit to the heat equation proof of Patodi [10] for Kaehler manifolds the actual method is quite different. Toledo and Tong define their cocycle for an open cover of X corresponding to a complex-analytic atlas and holomorphic local trivializations of E. Its value on a particular simplex of the cover is an expression in the corresponding local coordinates of X, the transition matrices of E and their mutual partial derivatives. There is no assumption of further structure, such as a metric, on X or E.

The cocycle itself is obtained by an iterative process from the given coordinate systems and involves the construction of an appropriate 'twisting cochain'. While not hard to describe, this appears to lead to extremely complicated formulae. This is dealt with in [11] by establishing certain qualitative features of the resulting expressions. It is then shown, again by analogy with the original heat equation method, that any Čech cocycle defined by a local formula with these features must represent a characteristic class. Standard examples force this class to be $\mathrm{Todd}(X)\mathrm{ch}(E)$.

More recently the heat equation method for the Dirac operator has been refined by Getzler [5], Berline and Vergne [3] and Bismuth [4]. These authors show *directly* that the integrand is given by the appropriate geometric formula.

Here an analogous result is obtained for the Toledo–Tong construction. For an open cover of X with corresponding complex-analytic charts, let A be the Atiyah cocycle of the holomorphic tangent bundle. Up to second-order terms, which can be neglected from the point of view of this method, the twisting cochain itself is shown to be given by an action of the cocycle

$$\frac{A}{1 - e^{-A}}$$

on the local Koszul complexes which form the basis of the construction. The precise statement is given in Theorem 3.11 below. This formula in fact follows directly from the construction of the twisting cochain given in [11], and the inductive proof depends on little more than an interesting interaction between the symmetry of the various partial derivatives which occur and the skew-symmetry of the Koszul complexes. Of course this formula also gives the characteristic power series for the Todd genus so it is not surprising that the HRR formula follows directly from this result: a straightforward generating function argument shows that the final cocyle indeed corresponds to the product of the Todd genus and Chern character. No abstract characterization of the cocycle is needed, so this gives a particular elementary proof of the HRR formula. The same method also works as an alternative to the invariant theory arguments used in the proof of HRR for coherent sheaves [8], or Grothendieck-Riemann-Roch for projections [9]. The proof [13] of the holomorphic Lefschetz Formula can similarly be reduced to an explicit calculation.

Notation

For a complex manifold X the usual notations $\mathcal{O}_X$ and Ω^r_X are used for the structure sheaf and the sheaf of holomorphic r-forms respectively, except that Ω^r_X is regarded as a complex, zero except in degree $-r$. In general no distinction is made between a locally-free sheaf and the corresponding vector bundle. An intersection $U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$ of sets belonging to an open cover will also be denoted by $U_{\alpha_0 \ldots \alpha_p}$. Cohomology degree always corresponds to the *total* degree of the associated cochains so that, for example, the global sections of Ω^r_X give $H^{-r}(X, \Omega^r_X)$ and $H^k(X, \mathcal{O}_X)$ is dual to $H^{-k}(X, \Omega^n_X)$

under Serre duality. These seem to be the natural conventions for residue and duality constructions, and are compatible with the gradings on the various 'twisted' complexes which appear here. Also, the natural convention for products (2.4) agrees in cohomology with the usual wedge product of global forms under the Dolbeault isomorphism.

2. Review of the Toledo–Tong proof

Let X be a complex manifold, $\mathscr{F}$ a coherent sheaf on X and $\mathscr{V} = \{V_\alpha\}$ a Stein open cover of X. Suppose that a locally-free resolution $F_\alpha^{\cdot}$ of $\mathscr{F}$ is given over each V_α with differential a_α^0 of degree $+1$. We briefly recall the notion of a twisting cochain in this situation. For a more detailed account see Section 1 of [7]. Let

$$C^{\cdot}(\mathscr{V}, \operatorname{Hom}^{\cdot}(F, F)) = \sum_{p,q} C^p(\mathscr{V}, \operatorname{Hom}^q(F, F)) \tag{2.1}$$

be the space of cochains u which on the simplex $\alpha_0 \ldots \alpha_p$ of the cover define holomorphic vector bundle maps of each $F_{\alpha_p}^r$ into $F_{\alpha_0}^{r+q}$ over $V_{\alpha_0 \ldots \alpha_p}$. For example, the differentials a_α^0 of the local resolutions correspond to a cochain a^0 of bidegree (0, 1).

Since the $F_\alpha^{\cdot}$ all resolve the same sheaf, a familiar result from homological algebra gives the existence of a (1, 0)-cochain a^1 which on each $V_{\alpha\beta}$ is a chain map from $F_\beta^{\cdot}$ to $F_\alpha^{\cdot}$ inducing the identity on the cohomology $\mathscr{F}$. The most natural attempt to define a Čech differential D on the space of cochains by the formula

$$(Du)_{\alpha_0 \ldots \alpha_{p+1}}$$

$$= a^1_{\alpha_0\alpha_1} u_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{k=1}^{p} (-1)^k u_{\alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{p+1}} + (-1)^{p+1} u_{\alpha_0 \ldots \alpha_p} a^1_{\alpha_p \alpha_{p+1}}$$

fails since $D^2 \neq 0$ in general. One of the main innovations of [11] deals with this problem by the introduction of a 'twisting cochain'. This is a cochain a in (2.1) of total degree $+1$, with components a^k of bidegree $(k, 1 - k)$ for $k \geqq 0$ and which satisfies the 'twisting cochain equation':

$$\delta a + a \cdot a = 0. \tag{2.2}$$

Here the operator δ and cup-product are defined for cochains u, v in (2.1) by the formulae

$$(\delta u)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{k=1}^{p} (-1)^k u_{\alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{p+1}} \tag{2.3}$$

and

$$(u \cdot v)_{\alpha_0 \ldots \alpha_{p+r}} = (-1)^{qr} u_{\alpha_0 \ldots \alpha_p} v_{\alpha_p \ldots \alpha_{p+1}} \tag{2.4}$$

if u, v have bidegree (p, q), (r, s) respectively. It is required that the a^0 are the given differentials and $a^1_{\alpha\alpha}$ is the identity map over each V_α.

The existence of a^k for $k > 0$ can be shown inductively by solving the sequence of equations

$$a^0 \cdot a^k + a^k \cdot a^0 = -\delta a^{k-1} - \sum_{m=1}^{k-1} a^m \cdot a^{k-m} \tag{2.5}$$

for the a^k. Over a simplex $\alpha_0 \ldots \alpha_k$ the left side is just the differential of $a^k_{\alpha_0 \ldots \alpha_k}$ in the complex $\mathrm{Hom}^\cdot (F_{\alpha_k}, F_{a_0})$. The right side can be checked inductively to be a cocycle and the existence of a^k then follows by proving the appropriate acyclicity property of the complex of homomorphisms. For example, the components of a^1 are chain maps as above and the $a^2_{\alpha\beta\gamma}$ are chain homotopies between $a^1_{\alpha\gamma}$ and $a^1_{\alpha\beta} a^1_{\beta\gamma}$. The existence of further terms is therefore a natural extension of a familiar result about projective resolutions. However, the formula (2.2) also corresponds to the fact that the operator D_a defined on (2.1) by

$$D_a u = \delta u + a \cdot u + (-1)^{k+1} u \cdot a,$$

for u of total degree k, satisfies $D_a^2 = 0$. More generally, if $G^\cdot_\alpha$ are locally-free resolutions of a second coherent sheaf $\mathscr{G}$ over the sets of the same cover with associated twisting cochain b then we can consider the complex $C^\cdot(\mathscr{V}, \mathrm{Hom}^\cdot (F, G))$ defined by analogy with (2.1) and with differential

$$D_{a,b} = \delta u + b \cdot u + (-1)^{k+1} u \cdot a.$$

This differential has components of bidegree $(k, 1 - k)$ for $k \geqq 0$, so preserves the filtration by the Čech degree. The resulting spectral sequence is easily identified with the spectral sequence relating the local and global Ext functors, so the total cohomology of the complex is $\mathrm{Ext}^k (X, \mathscr{F}, \mathscr{G})$. This applies in particular when $\mathscr{F}$ or $\mathscr{G}$ is already locally-free, when the obvious choice of twisting cochain will always be assumed.

In [11] this is applied with X replaced by $X \times X$ and $\mathscr{F}$ the sheaf $\Delta_* \mathcal{O}_X$, where Δ is the inclusion of the diagonal. Suppose X has complex dimension n and let $\mathscr{U} = \{U_\alpha\}$ be a Stein open cover of X for which there are complex-analytic coordinates z_α^i on each U_α. Take $\mathscr{V}$ as the open cover of $X \times X$ consisting of the cover $\mathscr{V}'$ of a neighbourhood of the diagonal by the sets $U_\alpha \times U_\alpha$ together with a Stein open cover $\mathscr{V}''$ of the complement of the diagonal. In this case there is a natural choice for the local resolutions of $\Delta_* \mathcal{O}_X$ which has the following geometrical description. Let p_1, p_2 be the projections onto the factors of $X \times X$. On $U_\alpha \times U_\alpha$ write the local coordinates $p_1^* z_\alpha^i$ and $p_2^* z_\alpha^i$ as z_α^i and ζ_α^i respectively. Let K_α^{-r} be the restriction of $p_1^* \Omega_X^r$ to $U_\alpha \times U_\alpha$ and define the vector field Z_α on the same set by

$$Z_\alpha = \sum_{i=1}^{n} (z_\alpha^i - \zeta_\alpha^i)\partial/\partial z_\alpha^i. \tag{2.6}$$

The differential a_α^0 on $K_\alpha^{\cdot}$ is taken to be the interior product with Z_α. This is just the resolution of $\Delta_* \mathcal{O}_X$ by the Koszul complex associated to the basis dz_α^i of $p_1^* \Omega_X^1$ and the functions $z_\alpha^i - \zeta_\alpha^i$ generating the ideal sheaf of the diagonal. On the sets of $\mathscr{V}''$ the zero complex gives a suitable resolution.

There are many ways to extend these local differentials to a twisting cochain for $\Delta_* \mathcal{O}_X$. Toledo and Tong use a particular construction which will be analysed in detail in the next section. The cohomology sheaves of the local complexes $\mathrm{Hom}^{\cdot}(K_\alpha, p_1^* \Omega_X^n)$ over each $U_\alpha \times U_\alpha$ vanish in all degrees $k \neq 0$, while in degree 0 the cohomology is isomorphic to $\Delta_* \mathcal{O}_X$ via the canonical maps

$$\begin{aligned} \mathrm{Hom}\,(K^{-n}, p_1^*\Omega_X^n) &\to \Delta_* \Delta^* \,\mathrm{Hom}\,(K^{-n}, p_1^*\Omega_X^n) \\ &\to \Delta_*\, \mathrm{Hom}\,(\Omega_X^n, \Omega_X^n) \\ &\to \Delta_* \mathcal{O}_X. \end{aligned}$$

The spectral sequence for the Čech filtration on the corresponding global complex $C^{\cdot}(\mathscr{V}, \mathrm{Hom}^{\cdot}(K, p_1^*\Omega_X^n))$ collapses to give isomorphisms

$$\mathrm{Ext}^k\,(X \times X, \Delta_* \mathcal{O}_X, p_1^* \Omega_X^n) \to H^k(X, \mathcal{O}_X). \tag{2.7}$$

By Serre-Grothendieck duality this is adjoint to an isomorphism

$$H_c^{-k}(X \times X, \Delta_* \Omega_X^n) \leftarrow H_c^{-k}(X, \Omega_X^n)$$

where the subscript indicates cohomology with compact support. The main point to be checked is that this coincides with the natural identification of these two spaces. This a standard result of Grothendieck duality and is essentially equivalent to the parametrix argument of [11].

Functoriality of the duality pairing also shows that the map from $H^k(X, \mathcal{O}_X)$ into $H^k(X \times X, p_1^*\Omega_X^n)$ obtained by composing the inverse of (2.7) with the map induced by the homomorphism $\mathcal{O}_{X \times X} \to \Delta_* \mathcal{O}_X$ is adjoint to the restriction map

$$H_c^{-k}(X \times X, p_2^*\Omega_X^n) \to H_c^{-k}(X, \Omega_X^n).$$

Inversion of (2.7) at the cochain level therefore gives a geometric interpretation of the Gysin map for the inclusion of the diagonal.

For a locally-free sheaf E over X a similar argument using the complex

$$C^{\cdot}(\mathscr{V}, \operatorname{Hom}^{\cdot}(K \otimes p_1^*E, p_1^*\Omega_X^n \otimes p_2^*E))$$

gives isomorphisms

$$\operatorname{Ext}^k(X \times X, \Delta_* E, p_1^*\Omega_X^n \otimes p_2^*E) \to H^k(X, \operatorname{Hom}(E, E))$$

and a Gysin map into $H^k(X \times X, p_1^* \operatorname{Hom}(E, \Omega_X^n) \otimes p_2^*E)$. For X compact the standard argument of Lefschetz using the Künneth formula and duality gives the result that the image λ of the identity section of $H^0(X, \operatorname{Hom}(E, E))$ under the Gysin map has the property that

$$\chi(X, E) = \int_X \operatorname{Trace} \Delta^*\lambda.$$

Integration on $H^0(X, \Omega_X^n)$ can be interpreted as the composition of the Dolbeault map with the usual integration of $2n$-forms. The HRR formula then follows once Trace $\Delta^*\lambda$ can be identified with the component of $\operatorname{Todd}(X)\operatorname{ch}(E)$ in $H^0(X, \Omega_X^n)$, where the characteristic classes are given by Atiyah's definition [1], [2]. This result appears below as a direct consequence of elementary geometric properties of a particular twisting cochain.

As noted in [12] the map (2.7) is the case $m = 0$ of a family of "residue maps" res_m, defined for $0 \leqq m \leqq n$ by first restricting $C^r(\mathscr{V}, \operatorname{Hom}^{-m}(K, p_1^*\Omega_X^n))$ to the diagonal and then using the identifications

$$\Delta^* \operatorname{Hom}^{-m}(K, p_1^*\Omega_X^n) = \operatorname{Hom}(\Omega_X^{n-m}, \Omega_X^n) = \Omega_X^m$$

to map into $C^r(\mathscr{U}, \Omega_X^m)$. One property of the twisting cochain used in [11] is that $\Delta^* a^k = 0$ unless $k = 1$, while $\Delta^* a^1_{\alpha\beta}$ induces the identity map on each Ω_X^r. Therefore each res_m is the chain map, and in particular give a map

$$\text{Ext}^0(X \times X, \Delta_* \mathcal{O}_X, p_1^* \Omega_X^n) \to \sum_{k=0}^{n} H^0(X, \Omega_X^k).$$

In degree $k = n$ this is just the composition of the map induced by the quotient $\mathcal{O}_{X \times X} \to \Delta^* \mathcal{O}_X$ with restriction to the diagonal.

Similar remarks apply in the case of a locally-free sheaf E. If $\eta(E)$ is the Atiyah class in $H^0(X, \text{Hom}\,(E, E) \otimes \Omega_X^1)$, let $\text{Ch}(E)$ be the class

$$\sum_{k \geqq 0} \eta(E)^k/k! \quad \text{in} \quad \sum_{k \geqq 0} H^0(X, \text{Hom}\,(E, E) \otimes \Omega_X^k).$$

In [12] the stronger result

$$\sum_{k=0}^{n} \text{res}_k(\lambda) \;=\; \text{Todd}(X)\text{Ch}(E)$$

is proved, where $\text{Todd}(X)$ is the total Todd class. This formula is also a consequence of the computations which follow.

3. Properties of the twisting cochain

The construction of a twisting cochain for $\Delta_* \mathcal{O}_X$ given below is similar to that of [11], except that here the inductive argument is also used to obtain explicit formulae for the terms of the cochain. Because the whole construction is local and all the terms involved are eventually restricted to the diagonal, only the values of the cochain on the simplices of the cover $\mathscr{V}'$ are of interest. On other simplices the values can be taken arbitrarily to satisfy (2.5).

Over $U_\alpha \times U_\alpha$ the equation (2.5) is solved using an explicit chain homotopy for each complex $K_\alpha^\bullet$. In the following description the subscript α stays fixed, so will be omitted. As before z^i, ζ^i denote the same coordinates on the first and second factors of the product $U \times U$.

Assume that U is convex with respect to these coordinates and let ϱ_t be the contraction of $U \times U$ onto the diagonal given by

$$\varrho_t(z, \zeta) \;=\; (\zeta + t(z - \zeta), \zeta)$$

for $0 \leqq t \leqq 1$. Here $z + t(z - \zeta)$ is the point with coordinates $z^i + t(z^i - \zeta^i)$. For $t > 0$ the velocity field of ϱ_t is Z/t where Z is the vector field associated to the z^i as in (2.6). For a differential form ω on $U \times U$ and $0 < t \leqq 1$ define

$$p_t\omega = (\varrho_t^* \mathrm{d}\omega)/t.$$

This has a continuous extension to the whole of [0, 1] and the standard homotopy formula (see [6], for example) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_t^*\omega = i_Z p_t\omega + p_t i_Z\omega.$$

The linear operator P on forms, defined by

$$P\omega = \int_0^1 (\varrho_t^* \mathrm{d}\omega/t)\,\mathrm{d}t$$

therefore satisfies

$$\omega - \varrho_0^*\omega = i_Z P\omega + P i_Z\omega.$$

Of course $\varrho_0^*\omega$ is zero unless ω is a 0-form, in which case it is the constant function with value $\omega(\zeta)$.

For a multi-index $I = i_1 \ldots i_r$ with $1 \leqq i_1 < \cdots < i_r \leqq n$ let $\mathrm{d}z^I$ be the r-form $\mathrm{d}z^{i_1} \wedge \ldots \wedge \mathrm{d}z^{i_r}$. Multi-indices appearing in summations will be summed over strictly increasing multi-indices only, so that any r-form has a unique expression

$$\omega = \sum_I w_I \mathrm{d}z^I.$$

The degree r of the multi-index I will be denoted by $|I|$ and the notation I_k for $i_1 \ldots \hat{i}_k \ldots i_r$ will also be used. For I empty we take $\mathrm{d}z^I = 1$ and $|I| = 0$. In terms of the components ω_I of ω the operator P is given by

$$(P\omega)_I(z, \zeta)$$

$$= \sum_{l=1}^{r} (-1)^{l+1} \int_0^1 t^{|I|-1} (\partial\omega_{I_l}/\partial z^{i_l})(\zeta + t(z - \zeta), \zeta)\,\mathrm{d}t. \tag{3.1}$$

REMARK 3.2

a) Since $\mathrm{d} \circ P = 0$ it follows that $P^2 = 0$ and also that the skew-symmetrization of

$$\frac{\partial}{\partial z^{i_0}} (P\omega)_{i_1 \ldots i_r}$$

over $i_0, i_1, \ldots, i_r$ is identically zero.

b) All differentiation and integration takes place with the second coordinate fixed, so P commutes with restriction to a fibre $U \times \{\zeta\}$.

c) P is also invariant with respect to translation of the z, and hence also the ζ, coordinates by a constant vector.

Geometrically, b) and c) mean that in constructing the twisting cochain it will be enough to fix a point p in $U_{\alpha_0 \ldots \alpha_p}$, assume all the $z^i_{\alpha_j}(p)$ are zero and find a twisting cochain for the resolutions of the sheaf $\mathcal{O}_p$ of p given by restricting the Koszul resolutions of $\Delta_* \mathcal{O}_X$ to $\zeta = 0$. Similar remarks will be seen to apply in the construction of the associated cocycle, and restriction to the diagonal corresponds to taking the value of this cocycle at $z = 0$.

Let P_α be the chain contraction of $K^{\cdot}_\alpha$ given by (3.1) with $z^i = z^i_\alpha$. If $v_{\alpha\beta}$ is a cocycle in $\mathrm{Hom}^{\cdot}(K_\beta, K_\alpha)$ of degree $k < 0$ then P_α can be used to solve

$$a^0_\alpha u_{\alpha\beta} + (-1)^k u_{\alpha\beta} a^0_\beta = v_{\alpha\beta} \tag{3.3}$$

for $u_{\alpha\beta}$. Assume $u_{\alpha\beta}$ has been define on K^r_β for $r > s$ so that (3.3) holds. Then on K^s_β we can take

$$u_{\alpha\beta} = P_\alpha(v_{\alpha\beta} + (-1)^{k+1} u_{\alpha\beta} a_\beta) \tag{3.4}$$

where P_α acts on $\mathrm{Hom}^{\cdot}(K_\beta, K_\alpha)$ via

$$(P_\alpha v_{\alpha\beta})(\mathrm{d}z^I_\beta) = P_\alpha(v_{\alpha\beta}(\mathrm{d}z^I_\beta)).$$

Since $K^r_\beta = 0$ for $r > 0$ there is no problem in starting the induction. The same argument works for $k = 0$ also, provided there exists $u_{\alpha\beta}$ satisfying $a^0_\alpha u_{\alpha\beta} = v_{\alpha\beta}$ on K^0_β with which to start the process.

This can be used, at least in principle, to find an explicit solution to the equations (2.5) in the present case. Order the components of the cochain lexicographically by pairs of integers, so that (r, s) corresponds to the piece of a^r acting on Koszul degree $-s$. Since a^0 is given the induction can be started with $a^1_{\alpha\beta}$ the identity map on each $K^0_\beta = \mathcal{O}_{X\times X} = K^0_\alpha$. Then, according to (3.4) we can take

$$a^k_{\alpha_0\ldots\alpha_k} = (-1)^{k+1}P_{\alpha_0}\left[\delta a^{k-1} + \sum_{l=1}^{k} a^l \cdot a^{k-l}\right]_{\alpha_0\ldots\alpha_k}. \tag{3.5}$$

Each term on the right precedes the term on the left for this ordering, so successive applications generate a twisting cochain. It is clear however that, except in very low dimensions, this will lead to extremely complicated formulae. For example, in dimension n the resulting expressions for the twisting cochain will involve up to $n(n+1)/2$ iterations of the formula (3.5). The calculations which follow depends on the crucial fact that the first-order derivatives of each term at the origin depend only on the first order derivatives of the preceding terms. Each P_α involves partial differentiation so this is contrary to what is suggested by (3.5), but has the effect of reducing the calculations to a manageable size. The following three properties of the resulting cochain are used repeatedly in subsequent calculations (and were also observed in [12]).

PROPOSITION 3.6

i) *$a^k_{\alpha_0\ldots\alpha_k}$ is zero on $K^0_{\alpha_k}$ for all $k \neq 1$.*

ii) *At the origin $a^1_{\alpha\beta}$ induces the identity on the fibre of $K^{-r}_\beta = \Omega^r_X = K^{-r}_\alpha$, for all r.*

iii) *At the origin $a^k_{\alpha_0\ldots\alpha_k}$ vanishes on the fibre of $K^r_{\alpha_k}$ for all $k \neq 1$ and all r.*

Proof of 3.6 i). By definition the map $a^1_{\alpha\gamma} - a^1_{\alpha\beta}a^1_{\beta\gamma}$ is zero on K^0_γ so from (3.5) the same is true of $a^1_{\alpha\beta\gamma}$. For $k > 2$ it follows inductively that *all* the terms on the right of (3.5) vanish indentically on $K^0_{\alpha_k}$.

Both ii) and iii) follow from the next lemma, which gives the effect of a chain homotopy on products of terms of the twisting cochain. For multi-indices I, J let

$$(a^k_{\alpha_0\ldots\alpha_k})^I_J = ((a^k_{\alpha_0\ldots\alpha_k})(dz^I_{\alpha_k}))_J$$

and let $\phi_{\alpha\beta}$ be the Jacobian of the coordinate transformations from z_α to z_β, so that

$$(\phi_{\alpha\beta})^i_j = \partial z^i_\beta/\partial z^j_\alpha.$$

For a section of $\mathrm{Hom}^{\cdot}(K_\beta, K_\alpha)$ let ∂_m denote the partial derivative $\partial/\partial z_\alpha^m$, applied component-wise for the trivilization given by the sections dz_α^I and dz_β^J so that, for example,

$$(\partial_m a^p_{\alpha_0\ldots\alpha_p})^I_K = \partial/\partial z^m_{\alpha_0}(a^p_{\alpha_0\ldots\alpha_p})^I_K.$$

Because each $a^p_{\alpha_0\ldots\alpha_p}$ is in the imge of P_{α_0}, Remark 3.2 a) gives

$$\sum_{m,K} (\partial_m a^p_{\alpha_0\ldots\alpha_p})^I_K \, dz^m_{\alpha_0} \wedge dz^K_{\alpha_0} = 0 \tag{3.7}$$

for $p > 0$, while from the definition of a^0,

$$\sum_{m,K} (\partial_m a^0_\alpha)^I_K \, dz^m_\alpha \wedge dz^K_\alpha = |I| \, dz^I_\alpha. \tag{3.8}$$

The next lemma is the basis of the inductive argument.

LEMMA 3.9. *For* $p > 0$

$$(P_{\alpha_0}(a^p_{\alpha_0\ldots\alpha_p} a^q_{\alpha_p\ldots\alpha_{p+q}})^I_J(Z_{\alpha_0}) dz^J_{\alpha_0}$$

$$= \int_0^1 t^{|J|-1} \sum_{m,r,K,L} [(a^p_{\alpha_0\ldots\alpha_p})^K_L (\phi_{\alpha_0\alpha_p})^m_r (\partial_m a^q_{\alpha_p\ldots\alpha_{p+q}})^I_K](tz_{\alpha_0}) dt \, dz^r_{\alpha_0} \wedge dz^L_{\alpha_0}$$

Proof. The formula (3.1) for P_{α_0} gives directly

$$\sum_{l,J,K} \int_0^1 (-1)^{l+1} t^{|J|-1} [(\partial_{j_l} a^p_{\alpha_0\ldots\alpha_p})^K_{J_l} (a^q_{\alpha_p\ldots\alpha_{p+q}})^I_K$$

$$+ \sum_m (a^p_{\alpha_0\ldots\alpha_p})^K_{J_l} (\phi_{\alpha_0\alpha_p})^m_{j_l} (\partial_m a^q_{\alpha_p\ldots\alpha_{p+q}})^I_K](tz_{\alpha_0}) dt \, dz^J_{\alpha_0}.$$

The first set of terms vanishes by (3.7) above while the second term gives the required expression.

COROLLARY 3.10. *For* $p > 0$,

$$(P_{\alpha_0}(a^p_{\alpha_0\ldots\alpha_p} a^q_{\alpha_p\ldots\alpha_{p+q}})^I_J(0) \, dz^J_{\alpha_0}$$

$$= |J|^{-1} \sum_{m,K,L} [(a^p_{\alpha_0,\ldots\alpha_p})^K_L (\partial_m a^q_{\alpha_p\ldots\alpha_{p+q}})^I_K](0) \, dz^m_{\alpha_p} \wedge dz^L_{\alpha_0}$$

This is immediate from the above.

Proof of 3.6 ii). This is equivalent to

$$\sum_J (a^1_{\alpha\beta})^I_J(0)\ \mathrm{d}z^J_\alpha \;=\; \mathrm{d}z^I_\beta.$$

Case $|I| = 0$ is given. Induction on $|I|$, formula (3.8) and Corollary 3.10 give

$$\sum_J (P_\alpha(a^1_{\alpha\beta}a^0_\beta))^I_J(0)\ \mathrm{d}z^J_\alpha \;=\; \mathrm{d}z^I_\beta$$

which implies the required result.

Proof of 3.6 iii). The proof of iii) also follows inductively, using ii). Each P_α commutes with δ and for $l > 0$ each $a^l_{\alpha_0\ldots\alpha_l}$ is in the image of P_{α_0}. Therefore it remains to check that each term of the form

$$(P_{\alpha_0}(a^p_{\alpha_0\ldots\alpha_p}a^q_{\alpha_p\ldots\alpha_{p+q}}))^I_J(0)$$

vanishes whenever $p + q > 1$, assuming that the components of a^p and a^q appearing in this expression already satisfy ii) or iii) as appropriate. If $p > 1$ this is immediate from Corollary 3.10. For $p = 1$ the expression specializes to

$$|J|^{-1} \sum_{m,K} (\partial_m a^q_{\alpha_1\ldots\alpha_{q+1}})^I_K(0)\ \mathrm{d}z^m_{\alpha_1} \wedge \mathrm{d}z^K_{\alpha_1}$$

but this is zero by (3.7).

The next theorem is the main formula relating the Atiyah class of the tangent bundle to the twisting cochain. The proposition above gives complete information on the 0-jet of the twisting cochain at the chosen point (so on the diagonal also). This theorem gives the 1-jet and in fact contains all the information needed for the computation of the Todd genus. The calculations are again carried out in a neighbourhood of a fixed point of X where all coordinate systems are assumed to be zero.

First define Φ^p in $C^p(\mathcal{U}, \mathrm{Hom}(\Delta^*K^{-1}, \Delta^*K^{-1} \otimes \Omega^p_X))$ by

$$(\Phi^p_{\alpha_0\ldots\alpha_p})^i_j \;=\; \sum_{k_1\ldots k_{p-1}} d(\phi_{\alpha_0\alpha_1})^{k_1}_j \wedge \mathrm{d}(\phi_{\alpha_1\alpha_2})^{k_2}_{k_1} \wedge \ldots \wedge \mathrm{d}(\phi_{\alpha_{p-1}\alpha_p})^i_{k_{p-1}},$$

for $p > 0$, and let $(\Phi^0_\alpha)^i_j$ be the identity map δ^i_j. Each Φ^p is a cocycle which, up to sign, represents the pth power of the adjoint of the Atiyah class of the tangent bundle. In particular the traces of these cocyles generate the characteristic ring of TX in Hodge cohomology and it is through these expressions that the Todd genus will make its appearance. The next result is the key step.

THEOREM 3.11. *For all k, $r \geqq 0$ there exist constants $C_{k,r}$ such that, for $|I| = r$,*

$$\Delta^*(\partial_s a^k_{\alpha_0 \ldots \alpha_k})(dz^I_{\alpha_k}) = C_{k,r} \sum_l (-1)^{l+1} (\Phi^k_{\alpha_0 \ldots \alpha_k})^{i_l}_s \wedge dz^{I_l}_{\alpha_k}.$$

Moreover $C_{k,r}$ is zero for $r = 0$ and independent of r for $r > 0$. If $C_{k,r} = (-1)^{k(k-1)/2} A_k$ for $r > 0$ then, as formal power series,

$$\sum_{k \geqq 0} A_k x^k = \frac{x}{1 - e^{-x}}. \tag{3.12}$$

Note that the choice of sign in the definition of A_k is quite natural, since if $d\phi$ is the Čech cocycle with $((d\phi)_{\alpha\beta})^i_j = d(\phi_{\alpha\beta})^i_j$ then, according to the sign convention for products (2.4), $\Phi^k = (-1)^{k(k-1)/2} d\phi \cdot \cdots \cdot d\phi$ (k factors).

Proof of 3.11. By translation invariance it is only necessary to prove equality at the common origin of the coordinate systems. For $r = 0$ the left side of the formula is always zero by Proposition 3.6 i), so we can take $C_{k,0} = 0$ for all k. A direct calculation also shows that with $C_{0,r} = 1$ the formula is valid for $k = 0$ and all $r > 0$.

The remainder of the proof is quite similar to that of Proposition 3.6. For $p > 0$ and $q \geqq 0$ it turns out that the 1-jet of

$$P_{\alpha_0}(a^p_{\alpha_0 \ldots \alpha_p} a^q_{\alpha_p \ldots \alpha_{p+q}}) \tag{3.13}$$

at the origin of coordinates depends only on the 1-jets of a^p and a^q. Because the terms $P_{\alpha_0}(\delta a^{k-1})_{\alpha_0 \ldots \alpha_k}$ are always zero, induction on the lexicographic ordering again gives the result and also establishes a recurrence relation between the coefficients involved. This leads to the generating function (3.12).

So assume that the components of a^p and a^q appearing in (3.13) satisfy a formula of the required type. It remains to show that the 1-jet of (3.13) is given by a similar formula.

First assume $p > 1$, so that $a^p_{\alpha_0 \dots \alpha_p}(0) = 0$ and Lemma 3.9 gives

$$(|J| + 1) \sum_J \partial/\partial z^s_{\alpha_0}(P_{\alpha_0}(a^p_{\alpha_0 \dots \alpha_p} a^q_{\alpha_p \dots \alpha_{p+q}}))^I_J(0)\, dz^J_{\alpha_0}$$

$$= \sum_{m,K,L} [(\partial_s a^p_{\alpha_0 \dots \alpha_p})^K_L (\partial_m a^q_{\alpha_p \dots \alpha_{p+q}})^I_K](0)\, dz^m_{\alpha_p} \wedge dz^L_{\alpha_0}$$

$$= (-1)^p C_{p,q+r-1} \sum_{l,m,K} (-1)^{l+1} [(\Phi^p_{\alpha_0 \dots \alpha_p})^{k_l}_s (\partial_m a^q_{\alpha_p \dots \alpha_{p+q}})^I_K](0)\, dz^m_{\alpha_p} \wedge dz^{K_l}_{\alpha_p}.$$

Setting $M = mK$ gives

$$(-1)^{p+1} C_{p,q+r-1} \sum_M \sum_{l>1} |M| (-1)^{l+1} [(\Phi^p_{\alpha_0 \dots \alpha_p})^{m_l}_s (\partial_{m_1} a^q_{\alpha_p \dots \alpha_{p+q}})^I_{M_1}](0)\, dz^{M_l}_{\alpha_p}.$$

If summation extends to $l = 1$ then the expression becomes skew symmetric in the upper M indices. If $q > 0$ it vanishes by (3.7), and for $q = 0$ formula (3.8) gives

$$(-1)^{p+1} C_{p,r-1} \sum_{l,M} (-1)^{l+1} |I| (\Phi^p_{\alpha_0 \dots \alpha_p})^{m_l}_s \wedge dz^{M_l}_{\alpha_p} \tag{3.14}$$

since $|M| = |I|$ in this case.

So for $q > 0$ the original summation over $l > 1$ gives

$$(-1)^p C_{p,q+r-1} \sum_{m,K} (\Phi^p_{\alpha_0 \dots \alpha_p})^m_s (\partial_m a^q_{\alpha_p \dots \alpha_{p+q}})^I_K\, dz^K_{\alpha_p}$$

$$= (-1)^p C_{p,q+r-1} C_{q,r} \sum_l (-1)^{l+1} (\Phi^{p+q}_{\alpha_0 \dots \alpha_{p+q}})^{i_l}_s \wedge dz^{I_l}_{\alpha_{p+q}}.$$

For $q = 0$ the result is the same except for the additional term (3.14).

Case $p = 1$ is quite similar, except that the non-vanishing of $a^1_{\alpha\beta}$ leads to two extra terms:

$$\sum_{m,l,J,K} (-1)^{l+1} [(a^1_{\alpha_0\alpha_1})^K_{J_l} (\partial_s \phi_{\alpha_0\alpha_1})^m_{j_l} (\partial_m a^q_{\alpha_1 \dots \alpha_{q+1}})^I_K](0) dz^J_{\alpha_0}$$

and

$$\sum_{l,m,u,J,K} (-1)^{l+1} [(a^1_{\alpha_0\alpha_1})^k_{J_l} (\phi_{\alpha_0\alpha_1})^m_{j_l} (\phi_{\alpha_0\alpha_1})^u_s (\partial_u \partial_m a^q_{\alpha_1 \dots \alpha_{q+1}})^I_K](0)\, dz^J_{\alpha_0}.$$

Of these, the second term is equal to

$$\sum_{j,m,u,K} [(\phi_{\alpha_0\alpha_1})^m_j(\phi_{\alpha_0\alpha_1})^u_s(\partial_u\partial_m a^q_{\alpha_1\ldots\alpha_{q+1}})^I_K](0)\mathrm{d}z^j_{\alpha_0} \wedge \mathrm{d}z^K_{\alpha_1}$$

$$= \sum_{j,u,K} [(\phi_{\alpha_0\alpha_1})^u_s(\partial_u\partial_m a^q_{\alpha_1\ldots\alpha_{q+1}})^I_K](0)\mathrm{d}z^m_{\alpha_1} \wedge \mathrm{d}z^K_{\alpha_1}$$

which vanishes for all q by (3.7) or (3.8). The first term is

$$\sum_{j,m,K} [(\partial_s\phi_{\alpha_0\alpha_1})^m_j(\partial_m a^q_{\alpha_1\ldots\alpha_{q+1}})^I_K](0)\mathrm{d}z^j_{\alpha_0} \wedge \mathrm{d}z^K_{\alpha_1}$$

$$= \sum_{m,K} [\mathrm{d}(\phi_{a_0\alpha_1})^m_s(\partial_m a^q_{\alpha_1\ldots\alpha_{q+1}})^I_K](0)\mathrm{d}z^K_{\alpha_1}$$

$$= C_{q,r}\sum_l (-1)^{l+1}(\Phi^{q+1}_{\alpha_0\ldots\alpha_{q+1}})^{i_l}_s \wedge \mathrm{d}z^{I_l}_{\alpha_{q+1}}.$$

These calculations are summarized by the formula

$$\sum_J (|J| + 1)\, \partial/\partial z^s_{\alpha_0}(P_{\alpha_0}(a^p_{\alpha_0\ldots\alpha_p}a^q_{\alpha_p\ldots\alpha_{p+q}}))^I_J(0)\mathrm{d}z^J_{\alpha_0}$$

$$= c(p, q, r)\sum_l (-1)^{l+1}(\Phi^{p+q}_{\alpha_0\ldots\alpha_{p+q}})^{i_l}_s(0) \wedge \mathrm{d}z^{I_l}_{\alpha_{p+q}}$$

where

$$c(p, q, r) = \begin{cases} (-1)^p C_{p,q+r-1}C_{q,r} & (p > 1, q > 0) \\ (-1)^p C_{p,r-1}(C_{0,r} - r) & (p > 1, q = 0) \\ C_{q,r}(1 - C_{1,q+r-1}) & (p = 1, q > 0) \\ (r - 1)C_{1,r-1} + C_{0,r} & (p = 1, q = 0) \end{cases}$$

For $p > 1$ the components of a^p are defined by

$$a^p_{\alpha_0\ldots\alpha_p} = \sum_{k=1}^{p} (-1)^{pk+1}P_{\alpha_0}(a^k_{\alpha_0\ldots\alpha_k}a^{p-k}_{\alpha_k\ldots\alpha_p})$$

which proves the first part of the theorem with

$$(p + r)C_{p,r} = \sum_{k=1}^{p} (-1)^{pk+1}c(k, p - k, r).$$

Next we check that for $r \geqslant 1$ the coefficients $C_{p,r}$ are independent of r. The formula above and the recursive definition $a^1_{\alpha\beta} = P_\alpha(a^1_{\alpha\beta} a^0_\beta)$ give

$$(r + 1)C_{1,r} = (r - 1)C_{1,r-1} + C_{0,r}$$

Since $C_{1,0} = 0$ and $C_{0,r} = 1$ it follows inductively that $C_{1,r} = 1/2$ for all $r > 0$. Assuming inductively that for $q < p$ the coefficients $C_{q,r}$ are independent of $r > 0$, this gives

$$(p + r)C_{p,r} = \sum_{k=1}^{p-1} (-1)^{pk+k+1} C_k C_{p-k} + (r - 1)C_{p,r-1} + (-1)^{p+1} C_{p-1}.$$

For $r = 1$ this reduces to

$$(p + 1)C_{p,1} = \sum_{k=1}^{p-1} (-1)^{pk+k+1} C_k C_{p-k} + (-1)^{p+1} C_{p-1}$$

and by induction this then holds for all r. Setting $C_p = (-1)^{p(p-1)/2} A_p$ in the recurrence relation gives $A_0 = 1$, $A_1 = 1/2$ and for $p > 1$:

$$(p + 1)A_p + \sum_{k=1}^{p-1} A_k A_{p-k} = A_{p-1}.$$

In terms of the formal power series $A = \Sigma_{k \geqq 0} A_k x^k$ this is equivalent to the differential equation

$$\mathrm{d}(xA)/\mathrm{d}x + A^2 = xA + 2A$$

for which (3.12) is the unique power series solution satisfying $A_0 = 1$.

4. Construction of the cocycle

The final part of the argument involves the construction of a 0-cocycle τ in the complex

$$C^\cdot(\mathscr{V}, \mathrm{Hom}^\cdot(K \otimes p_1^* E, p_1^* \Omega_X^n \otimes p_2^* E)). \tag{4.1}$$

For clarity we first consider the case where E is trivial. If τ^k is the component of τ of bidegree $(k, -k)$ then for the argument of Section 2 we requite τ to have the property that, on the diagonal, τ^0_α induces the identity map between $\Delta^* K^{-n}$ and Ω^n_X. A cocycle is determined inductively by solving

$$\tau^k_{\alpha_0 \ldots \alpha_k} a^0_{\alpha_k} = \left[\delta \tau^{k-1} - \sum_{l=0}^{k-1} \tau^l \cdot a^{k-l} \right]_{\alpha_0 \ldots \alpha_k},$$

where δ now denotes the alternating sum over the faces $\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_k$ for $0 \leqslant i < k$. As in the construction of the twisting cochain itself this can be done using explicit homotopy contractions of the local complexes $\mathrm{Hom}^\cdot\,(K^\cdot_\alpha, p_1^* \Omega^n_X)$. The appropriate operator Q_α is adjoint to the contraction P_α used in that case and over a simplex $\alpha_0 \ldots \alpha_k$ is given by the formula

$$(Q_{\alpha_k} f_{\alpha_0 \ldots \alpha_k})^I(z, \zeta)$$

$$= \sum_{m=1}^{n} \int_0^1 t^{n-|I|-1} (\partial f^{mI}_{\alpha_0 \ldots \alpha_k} / \partial z^m_{\alpha_k})(\zeta_{\alpha_k} + t(z_{\alpha_k} - \zeta_{\alpha_k}), \zeta_{\alpha_k}) \mathrm{d}t. \tag{4.2}$$

As before we can restrict to a fibre ζ_{α_k} = constant and assume that all coordinates are zero where this fibre meets the diagonal. The partial differentiation is carried out with respect to the trivializations of $K^{-r}_{\alpha_k}$ given by the sections $\mathrm{d}z^{i_1}_{\alpha_k} \wedge \ldots \wedge \mathrm{d}z^{i_r}_{\alpha_k}$ and of Ω^n_X given by $\mathrm{d}z^1_{\alpha_0} \wedge \ldots \wedge \mathrm{d}z^n_{\alpha_0}$. The operator Q_{α_k} satisfies

$$[(Q_{\alpha_k} f_{\alpha_0 \ldots \alpha_k}) a^0_{\alpha_k} + Q_{\alpha_k}(f_{\alpha_0 \ldots \alpha_k} a^0_{\alpha_k})]^I(z)$$

$$= \begin{cases} f^I(z) - f^I(0) & \text{if } |I| = n \\ f^I(z) & \text{otherwise.} \end{cases}$$

Because $(\tau^0_\alpha a^1_{\alpha\beta} - \tau^0_\beta)(0) = 0$ a cocycle of the required form can be determined inductively by taking

$$\tau^k_{\alpha_0 \ldots \alpha_k} = Q_{\alpha_k} \left[\delta \tau^{k-1} - \sum_{l=0}^{k-1} \tau^l \cdot a^{k-l} \right]_{\alpha_0 \ldots \alpha_k}.$$

It is easily checked that $Q_\alpha \tau^0_\alpha = 0$ and that $Q^2_\alpha = 0$. Since Q_{α_k} commutes with δ the formula above simplifies to

$$\tau^k_{\alpha_0 \ldots \alpha_k} = \sum_{l=0}^{k-1} (-1)^{kl+l+1} Q_{\alpha_k} (\tau^l_{\alpha_0 \ldots \alpha_l} a^{k-l}_{\alpha_l \ldots \alpha_k}). \tag{4.3}$$

PROPOSITION 4.4. *For $l < k$,*

$$(n - |I|)Q_{\alpha_k}(\tau^l_{\alpha_0\ldots\alpha_l}a^{k-l}_{\alpha_l\ldots\alpha_k})(\mathrm{d}z^I_{\alpha_k})(0)$$

$$= \tau^l_{\alpha_0\ldots\alpha_l}(\mu^{k-l}_{\alpha_l\ldots\alpha_k} \wedge \mathrm{d}z^I_{\alpha_k})(0)$$

where μ^r in $C^r(\mathscr{U}, \Omega^r_X)$ is given by

$$\mu^r_{\alpha_0\ldots\alpha_r} = C_r \text{ trace } \phi_{\alpha_r\alpha_0}\Phi^r_{\alpha_0\ldots\alpha_r}$$

and C_r and Φ^r are defined in the previous section.

Proof. For $k - l > 1$ the cochain a^{k-l} vanishes at the origin so (4.2) and the chain rule transform the expression on the left into

$$\sum_{m,p=1}^{n} \sum_J \tau^l_{\alpha_0\ldots\alpha_l}[(\phi_{\alpha_k\alpha_l})^p_m(\partial_p a^{k-l}_{\alpha_l\ldots\alpha_k})^{mI}_J \mathrm{d}z^J_{\alpha_l}](0).$$

Now from Theorem 3.11,

$$\sum_J (\partial_p a^{k-l}_{\alpha_l\ldots\alpha_k})^{mI}_J \mathrm{d}z^J_{\alpha_l}$$

$$= C_{k-l}\left[(\Phi^{k-l}_{\alpha_l\ldots\alpha_k})^m_p \wedge \mathrm{d}z^I_{\alpha_k} + \sum_l (-1)^l(\Phi^{k-l}_{\alpha_l\ldots\alpha_k})^{i_l}_p \wedge \mathrm{d}z^m_{\alpha_k} \wedge \mathrm{d}z^{I_l}_{\alpha_k}\right].$$

The first term on the right gives the required formula while the second leads to a sum of terms each containing an expression

$$\sum_{p=1}^{n} (\Phi^{k-l}_{\alpha_l\ldots\alpha_k})^{i_l}_p \wedge \mathrm{d}z^p_{\alpha_l}$$

which is zero by the symmetry of the partial derivatives appearing in Φ^{k-l}.

In case $k - l = 1$ there is an additional term

$$\sum_{p=1}^{n} \sum_J \partial/\partial z^p_{\alpha_{k-1}}(\tau^{k-1}_{\alpha_0\ldots\alpha_{k-1}})^{pJ}(a^1_{\alpha_{k-1}\alpha_k})^I_J(0)$$

which is always zero because $\tau^{k-1}_{\alpha_0\ldots\alpha_{k-1}}$ is in the image of $Q_{\alpha_{k-1}}$, and $Q^2_{\alpha_{k-1}} = 0$.

It now follows inductively that, under the identification at the origin of $K^{k-n}_{\alpha_k}$ with Ω^{n-k}_X, the sections $\tau^k_{\alpha_0\ldots\alpha_k}$ of Hom $(K^{k-n}_{\alpha_k}, \Omega^n_X)$ operate as wedge product by certain k-forms $T^k_{\alpha_0\ldots\alpha_k}$.

THEOREM 4.5 (HRR). *The T^k are cocycles representing characteristic classes of the tangent bundles TX in $H^0(X, \Omega_X^k)$. If $x_1, \ldots, x_n$ are the formal Chern roots of TX then*

$$\sum_{k \geqq 0} T^k(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i/(1 - e^{-x_i}). \tag{4.6}$$

Proof. The choice of τ^0 gives $T_\alpha^0 = 1$, and further T^k are determined recursively by (4.3) and Proposition 4.4, which give

$$kT^k_{\alpha_0 \ldots \alpha_k} = \sum_{l=0}^{k-1} (-1)^{kl-l+1} T^l_{\alpha_0 \ldots \alpha_l} \wedge \mu^{k-l}_{\alpha_l \ldots \alpha_k}. \tag{4.7}$$

Wedge product gives an isomorphism of Ω_X^k with Hom $(\Omega_X^{n-k}, \Omega_X^n)$ so the $T^k_{\alpha_0 \ldots \alpha_k}$ form a Čech cocycle in $C^k(\mathscr{U}, \Omega_X^k)$ and according to our sign conventions (2.4) the formula (4.7) can be written as

$$kT^k = -\sum_{l=0}^{k-1} T^l \cdot \mu^{k-l}. \tag{4.8}$$

Clearly $T = \Sigma_{k \geqq 0} T^k$ is a polynomial in the μ^r and therefore a characteristic class of TX. We show that it coincides with the Todd genus (4.6). We first express μ in terms of the formal Chern roots x_i of TX. According to the argument given in [2], for example, the Atiyah cocycle η in $C^1(\mathscr{U}$, Hom $(TX, TX \otimes \Omega_X^1))$ is given by

$$\eta_{\alpha\beta}(\partial/\partial z^i_\beta) = \sum_{j=1}^{n} d(\phi_{\beta\alpha})^j_i \partial/\partial z^j_\alpha.$$

The change in order of α, β introduces a minus sign so according to our sign conventions

$$\text{trace } \Phi^k = (-1)^{k(k-1)/2} \sum_{i=1}^{n} (-x_i)^k$$

so that

$$\mu = \sum_{k \geqq 0} \mu^k = \sum_{i=1}^{n} - x_i/(1 - e^{x_i}).$$

But in terms of the Chern roots (4.8) takes the form

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x^i} (\log T) = \sum_{i=1}^{n} 1 + x^i/(1 - e^{x_i}).$$

This has solutions of the form $T = \Pi_{i=1}^{n} F(x_i)$ where $F(x)$ satisfies

$$x \frac{dF}{dx} = (1 + x - e^x)F/(1 - e^x)$$

and this has solution $F(x) = x/(1 - e^{-x})$ satisfying $F(0) = 1$.

Finally we consider the case where E is non-trivial. Assume that each $E_\alpha = E|U_\alpha$ has a fixed holomorphic trivialization with transition matrices $b_{\alpha\beta}\colon E_\beta \to E_\alpha$ over $U_{\alpha\beta}$. On $U_\alpha \times U_\alpha$ the identity matrix gives an isomorphism between $p_1^* E_\alpha$ and $p_2^* E_\alpha$ which restricts to the diagonal as the identity map on E_α. The tensor product of this map and the previous τ_α^0 is a 0-cochain in the complex (4.1). Taking this as the new τ^0, the construction of a total cocycle τ proceeds exactly as before, except that each $a^k_{\alpha_0 \ldots \alpha_k}$ must be replaced by $a^k_{\alpha_0 \ldots \alpha_k} \otimes b_{\alpha_0 \alpha_k}$. In this construction the partial derivatives are again taken component-wise for the fixed trivializations of the E_α. All the a^k for $k > 1$ vanish on restriction to the diagonal, so

$$Q_{\alpha_k}(\tau^l_{\alpha_0 \ldots \alpha_l}(a^{k-l}_{\alpha_l \ldots \alpha_k} \otimes b_{\alpha_l \alpha_k}))(0)$$

$$= (Q_{\alpha_k}(\tau^l_{\alpha_0 \ldots \alpha_l} a^{k-l}_{\alpha_l \ldots \alpha_k}) \otimes b_{\alpha_l \alpha_k})(0)$$

for $k - l > 1$. For $k - l = 1$ there is an extra term on the right which comes from differentiating the components of b. On $dz^I_{\alpha_k}$ this takes the value

$$\sum_{m=1}^{n} \sum_{J} \tau^{k-1}_{\alpha_0 \ldots \alpha_{k-1}} [(a^1_{\alpha_{k-1}\alpha_k})^{mI}_J \otimes \partial b_{\alpha_{k-1}\alpha_k}/\partial z^m_{\alpha_k} \, dz^J_{\alpha_{k-1}}]$$

$$= \tau^{k-1}_{\alpha_0 \ldots \alpha_{k-1}} (db_{\alpha_{k-1}\alpha_k} \wedge dz^I_{\alpha_k}).$$

By analogy with the previous case, the $\Delta^* \tau^k$ therefore operate as wedge product by cocycles S^k in $C^k(\mathscr{U}, \operatorname{Hom}(E, E \otimes \Omega^k_X))$. This space of cochains is an algebra over $C^\cdot(\mathscr{U}, \Omega^\cdot_X)$ in the obvious way.

THEOREM 4.9. *The S^k are polynomials in the cocycles μ^r and the Atiyah cocycle of E corresponding to the given trivializations. In terms of the Chern roots x_i*

of TX and the Atiyah class y of E the S^k are given by

$$\sum_{k \geqq 0} S^k(x_1, \ldots, x_n, y) = T(x_1, \ldots, x_n)e^y.$$

Proof. As for the tangent bundle, the Atiyah class of E is represented by the 1-cocycle $-db$ and the recursive definition (4.3) gives

$$kS^k = -\sum_{l=0}^{k-1} S^l \cdot \mu^{k-l} - S^{k-1} \cdot db.$$

This shows that the S^k are of the required form. In terms of the Chern roots this relation is equivalent to

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} (\log S) + y \frac{\partial}{\partial y} (\log S)$$

$$= \sum_{i=1}^{n} (1 + x^i/(1 - e^{x_i})) + y.$$

Solving with $S(0) = 1$ gives the required result.

It follows that in cohomology

$$\sum_{k=0}^{n} \mathrm{res}_k(\tau) = \mathrm{Todd}(X)\mathrm{Ch}(E)$$

where $\mathrm{Ch}(E)$ is represented by $\Sigma_{k \geqslant 0} (-db)^k/k!$. This formula was also proved in [12] by a different argument. After taking traces this gives the HRR formula for the bundle E, as explained in Section 2.

If E is replaced by a coherent sheaf $\mathscr{F}$ the local trivializations and transition matrices can be replaced by local resolutions and a twisting cochain for $\mathscr{F}$. The tensor product by endomorphisms of E becomes the twisted product operation of [7]. All the preceding calculations go through as before.

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