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Convergence of Riemannian manifolds

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1. Introduction

In this paper we consider sequences of compact n-dimensional Riemannian manifolds. We are going to study the convergence of such sequences and the properties of possible limit spaces. In particular, we shall investigate under which assumptions the limit space again carries the structure of a manifold of the same dimension.

The concept of convergence of Riemannian manifolds was introduced by M. Gromov (see [7]). We shall recall the precise definitions in Chapter 3. But let us first consider a few examples.

1.1 Example: Take a sequence of smoothed truncated cones in $\mathbb{R}^3$, converging to a cone. We observe that a singularity arises in the limit, due to the fact that the curvature does not stay bounded. Let us therefore restrict our attention to sequences with bounded sectional curvatures $|K_M| \leqslant \Lambda^2$. We also assume bounded diameters $\operatorname{diam}(M) \leqslant d$, otherwise the compactness is lost in the limit.

1.2 Example: Take a sequence of two-dimensional flat tori becoming thinner and thinner, eventually collapsing to S^1. Here the assumptions on the curvature ($=0$) and the diameter are satisfied, but the limit space is of lower dimension. This phenomenon of collapsing is not a subject of this paper, see e.g. [7], or [11]. It can be ruled out by a lower bound on the volumes of the manifolds.

1.3 Definition: Let $\mathfrak{M}(n, d, \Lambda, V)$ denote the class of compact n-dimensional Riemannian manifolds M with diameter $\operatorname{diam}(M) \leqslant d$, volume $\operatorname{vol}(M) \geqslant V$, and sectional curvature $|K_M| \leqslant \Lambda^2$.

1.4 Remark: The hypothesis $\operatorname{vol}(M) \geqslant V$ can be replaced by $i(M) \geqslant i_0$, where $i(M)$ denotes the injectivity radius of M. (See e.g. [14]).

1.5. THEOREM: *Cheeger's Finiteness Theorem* ([5], [12]): *$\mathfrak{M}$ contains only a finite number of diffeomorphism classes. This number can be estimated explicitly in terms of n, d, Λ, V.*

We now turn to the results on compactness. In [7], thm. 8.28, M. Gromov stated the following theorem. (For the definition of the Lipschitz topology, see Chapter 3 of these notes).

1.6. THEOREM: *Gromov's Compactness Theorem: With respect to the Lipschitz topology, $\mathfrak{M}(n, d, \Lambda, V)$ is relatively compact in a larger class of n-dimensional manifolds, namely $C^{1,1}$-manifolds with C^0-metrics.*

In Gromov's proof of this theorem, he uses as a main tool a theorem about the equivalence of Hausdorff and Lipschitz convergence ([7], thm. 8.25), but a number of details remained open. Recently, A. Katsuda [10] succeeded to work out Gromov's proof of thm. 8.25 in full detail.

Katsuda's paper is rather long. In chapter 3 of this paper, we shall give a shorter proof of this theorem (Thm. 3.9). It is very similar to the author's proof of Cheeger's Finiteness Theorem in [12]. The largest part of the work has already been done in [12].

In Gromov's version of the Compactness Theorem 1.6, the regularity properties are not optimal. However, as we shall see in Chapter 2, the regularity of the limit metric plays an important role for the applications of the theorem. In Chapter 4, we are going to prove the following improved version of Thm. 1.6.

1.7. THEOREM: *Let $0 < \alpha < 1$. Then any sequence in $\mathfrak{M}$ contains a subsequence converging w.r.t. the Lipschitz topology to an n-dimensional differentiable manifold M with metric g of Hölder class $C^{1+\alpha}$.*

This result is optimal in terms of Hőlder conditions, i.e. the theorem does not hold for $C^{1,1}$ instead of $C^{1+\alpha}$. We shall prove this in Chapter 5. At least it is quite obvious that we cannot expect more than $C^{1,1}$ in general, as the following example illustrates.

1.8. Example: Put two spherical caps onto a cylinder in $\mathbb{R}^3$. The resulting compact surface, with the induced metric, can obviously be obtained as a limit of a sequence of smooth surfaces with bounded curvatures, diameters, and volumes. Nevertheless, the metric is not C^2, otherwise the curvature would have to be continuous.

Remark

After the preparation of this paper, the author learned that R.E. Greene and H. Wu obtained the same result as Theorem 1.7 independently. (R.E. Greene and H. Wu, Lipschitz convergence of Riemannian manifolds, preprint).

We wish to remark that a preliminary version of this paper was completed in February 1985. This preliminary paper did in particular not contain chapter 5, which consists of more recent results.

2. Applications

2.1. "Pinching just below 1/4" (Berger [2])

For any *even* number n, there exists $\epsilon(n) > 0$ such that any compact simply connected n-dimensional Riemannian manifold M with $1/4 - \epsilon(n) \leqslant K_M \leqslant 1$ is homeomorphic to the standard sphere S^n, or diffeomorphic to $P^n\mathbb{C}$, $P^n\mathbb{H}$, or to the Cayley plane $P^2\mathbb{C}a$.

In Berger's proof, he assumes to have a sequence of metrics with pinching constants converging to 1/4. The compactness theorem is used to yield a 1/4-pinched limit metric. Then a well-known rigidity theorem can be applied. The main difficulty is to show the smoothness of the limit metric, which is only C^0 a priori according to Gromov. So we see that regularity properties of the limit metrics are crucial for the applications of the compactness theorem. Let us mention two further applications:

2.2. "Diameter-pinching" (Brittain [3])

There exists an $\epsilon > 0$ depending only on n, $\max|K_M|$, and a positive number V_0, such that if $\mathrm{Ric}(M) \geqslant n-1$, $\mathrm{vol}(M) \geqslant V_0$, and $\mathrm{diam}(M) \geqslant \pi - \epsilon$, then M is diffeomorphic to S^n.

2.3. "Volume-pinching" (Katsuda [11])

Let M be an n-dimensional compact Riemannian manifold with
$\mathrm{Ric}(M) \geqslant n-1$ and $|K_M| \leqslant \Lambda^2$. There exists $\epsilon > 0$ depending on n and Λ such that $\mathrm{vol}(M) \geqslant \mathrm{Vol}(S^n) - \epsilon$ implies that M is diffeomorphic to S^n.

3. Hausdorff and Lipschitz topology

Let us recall the definitions of Lipschitz distance and Hausdorff distance of metric spaces (see [7]).

3.1. Definition: Let X, Y be metric spaces, $f: X \to Y$ a Lipschitz map. Then $\mathrm{dil}\, f := \sup \dfrac{d(f(x), f(x'))}{d(x, x')}$ is called *dilatation* of f.

$d_L(X, Y) := \inf\{ |\ln \operatorname{dil} f| + |\ln \operatorname{dil} f^{-1}| \mid f$ bi-Lipschitz hom.$\}$ is the *Lipschitz distance* between X and Y, if bi-Lipschitz homeomorphisms exist. Otherwise $d_L(X, Y) = \infty$.

3.2. PROPOSITION: *For compact metric spaces, d_L is a distance.*

There is the following important result of *A. Shikata* [15]:

3.3. THEOREM: *There exists an $\epsilon > 0$ depending only on n such that any two compact n-dimensional Riemannian manifolds M, $\overline{M}$ with $d_L(M, \overline{M}) < \epsilon$ are diffeomorphic.The explicit estimate for ϵ has been improved by H. Karcher (see* [9]).

We now turn to the Hausdorff distance.

3.4. Definition: (a) Let Z be a metric space, $A, B \subseteq Z$. Let $U_\epsilon(A) := \{z \in Z \mid d(z, A) < \epsilon\}$. $d_H^Z(A, B) := \inf\{\epsilon > 0 \mid U_\epsilon(A) \supseteq B$ and $U_\epsilon(B) \supseteq A\}$ is the classical *Hausdorff distance* for subspaces of a single metric space.
(b) Let X, Y now be arbitrary metric spaces.
$d_H(X, Y) := \inf\{d_H^Z(f(X), g(Y))\}$, where the inf is taken over all metric spaces Z and all isometries $f: X \to Z$, $g: Y \to Z$.

3.5. PROPOSITION: *For compact metric spaces, d_H is a distance.*

The reason for introducing both distances is that it is often easier first to prove d_H-convergence – while d_L-convergence is stronger:

3.6 PROPOSITION: *Any sequence of compact d_L-convergent metric spaces is also d_H-convergent.*

We will soon prove the important fact that in our class $\mathfrak{M}(n, d, \Lambda, V)$ both topologies coincide – a theorem due to *M. Gromov* ([7], thm. 8.25), which is crucial for his proof of the compactness theorem.

But let us first introduce a notion that makes the Hausdorff distance much handier.

3.7. Definition: An *ϵ-net* N in a metric space X is a subset $N \subseteq X$ such that $\bigcup_{x \in N} U_\epsilon(x) = X$. In our context, *$\epsilon$-nets* will always be finite sets of points in a compact Riemannian manifold.

Roughly speaking, d_H-convergence of compact metric spaces reduces to d_L-convergence of ϵ-nets. More precisely:

3.8. PROPOSITION: *If X and X^k, $k = 1, 2, \ldots$ are compact metric spaces and if for any $\epsilon > 0$ there exists an ϵ-net N of X which is the d_L-limit of a sequence of ϵ-nets N^k of X^k, then X^k is d_H-convergent to X.*

All *proofs* of the preceding propositions can be found in [7].

3.9. THEOREM: *In $\mathfrak{M}(n, d, \Lambda, V)$ the Hausdorff and Lipschitz topologies coincide. More precisely: Given $\rho > 0$, there exists a $\zeta(n, d, \Lambda, V, \rho)$ such that $d_H(M, \bar{M}) < \zeta$ implies $d_L(M, \bar{M}) < \rho$ for all $M, \bar{M} \in \mathfrak{M}(n, d, \Lambda, V)$.*

3.10. Remark: In principle, all estimates are explicit, in particular $\zeta(n, d, \Lambda, V, \rho)$. Nevertheless we omit explicit calculations for the sake of brevity. We mainly have to verify that the hypothesis of the crucial Lemma 2 of [12] is satisfied in this slightly different situation. Let us again state that lemma.

Assume two manifolds M and $\bar{M}$ in $\mathfrak{M}$ being covered by N convex balls of radius R, such that the $R/2$-balls still cover and the $R/4$-balls are disjoint. Let z_i and $\bar{z}_i$ be the centers, $u_i: \mathbb{R}^n \to T_{z_i}M$ and $\bar{u}_i: \mathbb{R}^n \to T_{\bar{z}_i}\bar{M}$ linear isometries, and $\phi_i := \exp_{z_i} \circ u_i: B_R(0) \subseteq \mathbb{R}^n \to B_R(z_i) \subseteq M$, equally $\bar{\phi}_i$, normal coordinates. P_{ij} denotes parallel translation along the shortest geodesic joining z_i and z_j.

3.11. LEMMA: *There exist R, ϵ_0, ϵ_1 in terms of n, d, Λ, V such that the conditions*
(i) $d(\phi_j^{-1}\phi_i, \bar{\phi}_j^{-1}\bar{\phi}_i) \leqslant \epsilon_0$ and
(ii) $\| u_j^{-1}P_{ij}u_i - \bar{u}_j^{-1}\bar{P}_{ij}\bar{u}_i \| \leqslant \epsilon_1$ for all i, j imply that M and $\bar{M}$ are diffeomorphic.

The diffeomorphism F is constructed by averaging the locally defined diffeomorphisms $F_i = \bar{\phi}_i\phi_i^{-1}$. From the calculations in [12] it follows that $\| dF \|$ is arbitrarily close to 1, provided R and ϵ_0 are small enough. Hence we have:

3.12. LEMMA: *There exist R, ϵ_0, ϵ_1 in terms of n, d, Λ, V, ρ such that conditions (i) and (ii) imply $d_L(M, M) \leqslant \rho$.*

Proof of Theorem 3.9: Given n, d, Λ, V and ρ, we are going to show the existence of an η such that $d_H(M, \bar{M}) \leqslant \eta$ implies $d_L(M, \bar{M}) \leqslant \rho$. Take R, ϵ_0, ϵ_1 as small as dictated by lemmas 3.11 and 3.12. Then choose a net of centers $\{z_i\} \subset M$ such that the $R/8$-balls are disjoint and the $R/4$-balls cover M. The problem is now to find a net of centers $\{\bar{z}_i\} \subset \bar{M}$ and proper identifications of the tangent spaces $T_{z_i}M$ and $T_{\bar{z}_i}\bar{M}$, i.e. suitable mappings u_i, $\bar{u}_i$ such as to satisfy conditions (*i*) and (*ii*). For that purpose we introduce an *auxiliary net* $\{p_k\} \subset M$, containing $\{z_i\}$ and being, say, δ-dense, δ much smaller than R. If $d_H(M, \bar{M})$ is small enough, there exists a, say, 2δ-net $\{\bar{p}_k\} \subset \bar{M}$ having arbitrary small Lipschitz distance from $\{p_k\}$ – the latter holds in particular for the subnet $\{z_i\}$ and the corresponding subnet $\{\bar{z}_i\} \subset \{\bar{p}_k\}$. The balls $B(\bar{z}_i, R/16)$ are certainly disjoint and the balls $B(\bar{z}_i, R/2)$

cover $\overline{M}$. Finally take $\{B(z_i, R)\}$ and $\{B(\bar{z}_i, R)\}$ as coverings of M and $\overline{M}$. On both manifolds, the $R/2$-balls still cover, as required for lemma 3.11. The fact that only the $R/16$-balls are disjoint instead of the $R/4$-balls, leads to a worse estimate for the number of balls (See [12]). As a consequence, the above mentioned estimate for $\| dF \|$ requires even smaller R and ϵ_0. ([12], p. 80). Assume R and ϵ_0 to be chosen small enough from the start.

We are now looking for suitable linear isometries u_i, $\bar{u}_i$. For the identifications of tangent spaces $I_i = \bar{u}_i u_i^{-1}$ we demand that $F_i = \bar{\phi}_i \phi_i^{-1} = \exp_{\bar{z}_i} I_i \exp_{z_i}^{-1}$ sends $\{p_k\} \cap B(z_i, R)$ almost onto the corresponding net in $\overline{M}$. We are going to show that such I_i exist and that the so defined local mappings F_i satisfy the hypothesis of lemma 3.11. Roughly speaking, if the mappings are good on a very dense net, they are good enough everywhere.

Now fix an i. Consider the nets $\{x_k := \exp_{z_i}^{-1} p_k\} \subset B(0, R) \subset \mathbb{R}^n$ and $\{\bar{x}_k := \exp_{\bar{z}_i}^{-1} \bar{p}_k\}$. We are looking for a linear isometry I_i that sends $\{x_k\}$ almost onto $\{\bar{x}_k\}$. That is a purely euclidean problem, which was solved by T. Yamaguchi ([17]). His proof was rather technical, because he computed all constants explicitly. We will give a shorter version of the proof at the end of this section.

3.13. LEMMA: *Given $\nu > 0$, there exists a $\delta > 0$ and a $\mu > 0$ such that for any two δ-nets $\{x_k\}$ in $B(0, R) \subset \mathbb{R}^n$, and $\{\bar{x}_k\} \subset \mathbb{R}^n$ with the same number of points, satisfying $x_0 = \bar{x}_0 = 0$ and*

$$1 - \mu \leqslant \frac{|\bar{x}_k - \bar{x}_l|}{|x_k - x_l|} \leqslant 1 + \mu \quad \textit{for all } k, l,$$

there exists a linear isometry $I \in \mathrm{O}(n)$ such that

$$|I(x_k) - \bar{x}_k| \leqslant \nu \quad \textit{for all } i.$$

Applying Lemma 3.13 to the nets $\{x_k\}$ and $\{\bar{x}_k\}$ in the fixed euclidean spaces $T_{z_i}M$ and $T_{\bar{z}_i}\overline{M}$, we can choose u_i arbitrarily and obtain $\bar{u}_i := I_i u_i$.
The rest of the proof of Theorem 3.9 now follows from the next lemma.

3.14. LEMMA: *There exist constants in terms of n, d, Λ, V s.t.*
(*i*) $d(\phi_j^{-1}\phi_i, \bar{\phi}_j^{-1}\bar{\phi}_i) \leqslant \mathrm{const}_0(\delta + \nu)$,
(*ii*) $\| u_j^{-1} P_{ij} u_i - \bar{u}_j^{-1} \bar{P}_{ij} \bar{u}_i | \leqslant \mathrm{const}_1 R + \mathrm{const}_2 \dfrac{\delta + \nu}{R}$.

Therefore, if R and then δ are chosen small enough from the start and if $d_H(M, \overline{M})$ is so small as to yield a sufficiently small ν by Lemma 3.13, Lemmas 3.11 and 3.12 can be applied to complete the proof of the theorem. □

Proof of Lemma 3.14: For technical reasons to be understood in (*ii*), we prove (*i*) for $B(0, 2R)$ instead of $B(0, R)$ and assume that the choice of I_i has been made with regard to these larger balls. By Lemma 3.13 and Rauch's comparison theorem, which yields a bound on $\| d \exp_{\bar{z}_i} \|$, we have $d(F_i(p_k), \bar{p}_k) = d(\exp_{\bar{z}_i} I_i x_k, \exp_{\bar{z}_i} \bar{x}_k) \leqslant \text{const} \cdot \nu$, where $\{p_k\}$, as above, denotes the auxiliary net.

By the triangle inequality,

$$d\left(\bar{\phi}_j \phi_j^{-1}(p_k), \bar{\phi}_i \phi_i^{-1}(p_k)\right) = d\left(F_j(p_k), F_i(p_k)\right) \leqslant 2 \cdot \text{const} \cdot \nu.$$

Again by Rauch's comparison theorem,

$$\begin{aligned} &d\left(\phi_j^{-1}\phi_i\left(u_i^{-1}x_k\right), \bar{\phi}_j^{-1}\bar{\phi}_i\left(u_i^{-1}x_k\right)\right) \\ &\quad = d\left(\phi_j^{-1}(p_k), \bar{\phi}_j^{-1}\bar{\phi}_i\left(\phi_i^{-1}(p_k)\right)\right) \\ &\quad \leqslant \operatorname{dil}\left(\bar{\phi}_j^{-1}\right) d\left(\bar{\phi}_j \phi_j^{-1}(p_k), \bar{\phi}_i \phi_i^{-1}(p_k)\right) \leqslant \text{const}' \cdot \nu. \end{aligned}$$

Thus we have shown that $| \phi_j^{-1}\phi_i - \bar{\phi}_j^{-1}\bar{\phi}_i |$ is small on the $\operatorname{dil}(\phi_i^{-1})$ δ-dense net $\{u_i^{-1}x_k\}$. As we have a bound on $\operatorname{dil}(\phi_j^{-1}\phi_i - \bar{\phi}_j^{-1}\bar{\phi}_i)$, we obtain for all $x \in B(0, 2R)$:

$$\begin{aligned} &\left|\left(\phi_j^{-1}\phi_i - \bar{\phi}_j^{-1}\bar{\phi}_i\right)(x)\right| \\ &\quad \leqslant \text{const}'\nu + \operatorname{dil}\left(\phi_j^{-1}\phi_i - \bar{\phi}_j^{-1}\bar{\phi}_i\right) \cdot \operatorname{dil}\left(\phi_i^{-1}\right) \cdot \delta \leqslant \text{const}_0(\delta + \nu). \end{aligned}$$

(*ii*) is proved by comparing parallel translation on the manifold and in the tangent space, see [3], p. 101, where the following formula is derived from Jacobi field estimates:
$d(\exp_{z_i}(w + \exp_{z_i}^{-1} z_j), \exp_{z_j} P_{ij} w) \leqslant \text{const}'' \cdot R \mid w \mid$, where say, $\mid w \mid \leqslant R$. Let $w = u_i v$. Then $\exp_{z_i}(w + \exp_{z_i}^{-1} z_j) = \phi_i(v + \phi_i^{-1}(z_j))$ and $\exp_{z_j} P_{ij} w = \phi_i(u_j^{-1} P_{ij} u_i v)$, hence

$$\begin{aligned} &| u_j^{-1} P_{ij} u_i v - \bar{u}_j^{-1} \bar{P}_{ij} \bar{u}_i v | \\ &\quad \leqslant \operatorname{dil}\left(\phi_j^{-1}\right) \cdot \text{const}'' R \mid v \mid + \operatorname{dil}\left(\bar{\phi}_j^{-1}\right) \cdot \text{const}'' R \mid v \mid \\ &\qquad + \mid \phi_j^{-1}\phi_i\left(v + \phi_i^{-1}(z_j)\right) - \bar{\phi}_j^{-1}\bar{\phi}_i\left(v + \bar{\phi}_i^{-1}(\bar{z}_j)\right) \mid. \end{aligned}$$

As $\{z_j\} \subset \{p_k\}$ and $\{\bar{z}_j\} \subset \{\bar{p}_k\}$, we have

$$\left|\left(v + \phi_i^{-1}(z_j)\right) - \left(v + \bar{\phi}_i^{-1}(z_j)\right)\right| \leqslant \nu,$$

and therefore

$$\begin{aligned} &\left|\phi_j^{-1}\phi_i\left(v + \phi_i^{-1}(z_j)\right) - \bar{\phi}_j^{-1}\bar{\phi}_i\left(v + \bar{\phi}_i^{-1}(\bar{z}_j)\right)\right| \\ &\quad \leqslant \text{const}_0(\delta + \nu) + \operatorname{dil}\left(\bar{\phi}_j^{-1}\bar{\phi}_i\right)\nu, \end{aligned}$$

where we have used (i) for $x = v + \phi_i^{-1}(z_j) \in B(0, 2R)$. Finally,

$$|u_j^{-1}P_{ij}u_iv - \bar{u}_j\bar{P}_{ij}\bar{u}_iv| \leqslant \left(dil\ \phi_j^{-1} + dil\ \bar{\phi}_j^{-1}\right) const''R^2 + const_0(\delta + \nu) + dil\left(\phi_j^{-1}\phi_i\right)\nu$$

for $|v| \leqslant R$, hence

$$\| u_j^{-1}P_{ij}u_i - \bar{u}_j^{-1}\bar{P}_{ij}\bar{u}_i \| \leqslant \mathrm{const}_1 R + \mathrm{const}_2(\delta + \nu)/R. \qquad \square$$

Proof of Lemma 3.13; Assume without loss of generality, $R > 1$. Take an arbitrary orthonormal frame $\{e_s\}$ of $\mathbb{R}^n$. There exists a subset $\{a_1, \ldots, a_n\}$ of the net $\{x_k\}$ with $|a_s - e_s| \leqslant \delta$. Let $\{\bar{a}_1, \ldots, \bar{a}_n\}$ denote the corresponding points in $\{\bar{x}_k\}$. $\{a_s\}$ and $\{\bar{a}_s\}$ have small Lipschitz distance, therefore $\{\bar{a}_s\}$ is still linearly independent, even "almost orthonormal", because the triangles $(0, a_s, a_t)$ have almost the same edgelengths as $(0, \bar{a}_s, \bar{a}_t)$.

The Schmidt orthonormalization process therefore yields an orthonormal frame $\{\bar{e}_s\}$ close to $\{\bar{a}_s\}$, say $|\bar{a}_s - \bar{e}_s| \leqslant \bar{\delta}$. We define I by $I(e_s) := \bar{e}_s$. Thus I is a linear isometry with
$|I(a_s) - \bar{a}_s| \leqslant \delta + \bar{\delta}$, $s = 0, 1, \ldots, n$.

Now let x_k be an arbitrary net point. The triangle inequality yields $\| I(x_k) - \bar{a}_s| - |x_k - a_s \| \leqslant \delta + \bar{\delta}$. Moreover, the small Lipschitz distance of the given nets implies the existence of a small δ' such that

$\| \bar{x}_k - \bar{a}_s| - |x_k - a_s \| \leqslant \delta'$, hence

$$\| I(x_k) - \bar{a}_s| - |\bar{x}_k - \bar{a}_s \| \leqslant \delta + \bar{\delta} + \delta'.$$

Consider the mapping $\varphi : \bar{B}(0, R) \to \mathbb{R}^{n+1}$,

$$\varphi(x) := (|x|, |x - \bar{a}_1|, \ldots, |x - \bar{a}_n|).$$

Note that $a_0 = \bar{a}_0 = 0$. φ is injective and continuous, $\bar{B}(0, R)$ is compact, therefore $\varphi^{-1} : \varphi(\bar{B}(0, R)) \to \bar{B}(0, R)$ is continuous. Hence each $I(x_k)$ is arbitrarily close to $\bar{x}_k$, provided δ and μ are small enough. That completes the proof of the lemma. $\square$

4. The compactness theorem

4.1. Let us again state theorem 1.7:

THEOREM: *Let* $0 < \alpha < 1$. *Then any sequence in* $\mathfrak{M}$ *contains a subsequence which is* d_L*-convergent to an n-dimensional differentiable manifold M with metric g of class* $C^{1+\alpha}$.

We obtain optimal regularity properties by using "optimal coordinates", namely *harmonic coordinates*.

In [8], *Jost* and *Karcher* proved the existence of harmonic coordinates $H : B(p, R) \to T_pM$ on *a-priori sized* balls around any point $p \in M$, where the radius R depends only on geometrical quantities of M (see Thm. 4.3). These coordinates are called harmonic because the component functions are harmonic. Just as the well-known exponential coordinates, they are canonically defined, i.e. without any arbitrary choices. Harmonic coordinates are optimal in the following sense:

4.2. THEOREM. (*J. Kazdan, D. De Turck* [6]): *If a Riemannian metric is of class $C^{k+\alpha}$ in some coordinates, so it is in harmonic coordinates.*

Crucial for our purpose are the following properties of harmonic coordinates (see [8], [16]).

4.3. THEOREM: (*J. Jost, H. Karcher* [8]): *Let M be a compact Riemannian manifold, $0 < \alpha < 1$. About any point $p \in M$ there exists a ball $B(p, R)$ of fixed radius R, on which harmonic coordinates exist and have the following properties:*

- (*i*) *There exist a uniform $C^{2+\alpha}$-Hölder bound for the transition functions,*
- (*ii*) *a uniform $C^{1+\alpha}$-bound for the metric, and*
- (*iii*) *a uniform C^{α}-bound for the Christoffel symbols, where the radius and the Hölder bounds depend on the dimensional, the injectivity radius, and curvature bounds of M.*

Proof of Theorem 1.7: Assume R' to be small that harmonic coordinates exist on balls of radius, say, $5R'$ and satisfy (*i*)–(*iii*) of 4.3. Let H_i^k be these coordinates on balls $B(z_i^k, R') \subset M^k$, and set $\phi_i^k := (H_i^k)^{-1} u_i^k$, where u_i^k are again linear isometries $\mathbb{R}^n \to T_{z_i^k} M^k$. Now $(\phi_i^k)^{-1}(B(z_i^k, R'))$ is no longer a ball, but contains a ball $B(0, R) \subset \mathbb{R}^n$, R close to R'. $\phi_i^k(B(0, R))$ is again not a ball, but contains a ball with radius R'', R'' close to R. After passing to a subsequence, we can assume each manifold to be covered by balls with radius R'', such that the $R''/2$-balls still cover and the $R''/4$-balls are disjoint. As in [12] we can achieve that the coverings all have the same nerve. The Hölder bounds of 4.3 are now universal for the whole sequence. The transition functions can be considered as mappings $\phi_{ij}^k : B(0, R) \to B(0, cR) \subset \mathbb{R}^n$, c a constant. But different from [12], we do not only have a uniform C^1-bound for the ϕ_{ij}^k, but a uniform $C^{2+\alpha}$-bound, because harmonic coordinates are that much better than exponential coordinates. By Ascoli's theorem there exists a subsequence such that for all pairs (i, j) for which transition functions exist, they converge in the C^2-topology to limit functions $\phi_{ij}^\infty : B(0, R) \to B(0, cR)$ of class $C^{2+\alpha}$. Once more taking a subsequence and once more applying Ascoli's theorem, the metrics also converge – considered as functions on $B(0, R)$- to limit "metrics" g_i^∞ of class $C^{1+\alpha}$ on each coordinate ball, i.e. on each copy of $B(0, R)$. The distinct copies of $B(0, R)$ are now glued together via the transition functions ϕ_{ij}^∞. Consider the restrictions $\tilde{\phi}_{ij} := \phi_{ij}^\infty |_{\phi_{ji}^\infty(B(0, R)) \cap B(0, R)}$ and define $x \sim y : \Leftrightarrow \exists \tilde{\phi}_{ij}$ s.t. $\tilde{\phi}_{ij}(x) = y$. Set $M^\infty := \bigcup_{i=1}^{N} B(0, R)/_\sim$. If ϕ_i^∞ denotes the canonical projection, restricted to the i-th copy of $B(0, R)$, M^∞ becomes a $C^{2+\alpha}$-manifold with the ϕ_i^∞ as coordinates. $\phi_j^\infty(\phi_i^\infty)^{-1} = \tilde{\phi}_{ij}$ are the transition functions. As they converge in C^1 (even C^2), the limit "metrics" g_i^∞ have the right transformation behavior, i.e. they define a (0, 2)-tensor field g^∞ on M^∞. Clearly g^∞ is symmetric. As the metrics g_i^k are close to the euclidean metric on $B(0, R)$, this is also true for g_i^∞. In particular, the g_i^∞ are positive definite. So g^∞ is again a metric, but only of class $C^{1+\alpha}$. We are now going to show that (M^k, g^k) is d_H-convergent to (M^∞, g^∞). Let $\{p_l^\infty\}$ be a δ-net in M^∞. We can find a δ'-net $\{p_l^k\}$ in M^k, δ' arbitrarily close to δ, such that $|d(p_l^\infty, p_m^\infty) - d(p_l^k, p_m^k)|$ is arbitrarily small if k is large enough. Just take any point $(\phi_i^\infty)^{-1}p_l$ representing $p_l \in M^\infty$ and let $p_l^k = \phi_i^k(\phi_i^\infty)^{-1}p_l$. $\{p_l^\infty\}$ is a finite net, and therefore $d(p_l^\infty, p_m^\infty)$ is bounded away from zero. Thus $\dfrac{d(p_l^k, p_m^k)}{d(p_l^\infty, p_m^\infty)}$ goes to 1, i.e. $\{p_l^k\}$ is d_L-convergent to $\{p_l^\infty\}$. Hence M^k is d_h-convergent to M^∞. In particular, $\{M^k\}$ is a d_H-Cauchy sequence. By theorem 3.9, it is also a d_L-Cauchy sequence, hence d_L-convergent to some metric space, which can be nothing but the d_H-limit space M^∞, because for compact metric spaces X, Y, $d_H(X, Y) = 0$ implies X and Y are isometric (Prop. 3.5).

Crucial is the non smoothness of the limit metric, whereas the non-smoothness of M^∞ itself is not essential, because by a classical result of *Whitney's*, the maximal C^2-atlas defined by our C^2-atlas on M^∞ contains a smooth sub-atlas, which defined a C^∞-structure on M^∞. By M we denote M^∞ endowed with this C^∞-structure.

We are now going to present a reformulation of the compactness theorem (suggested by *D. Brittain*), which avoids the notions of Hausdorff and Lipschitz topology, and state some more facts that are important for the applications of the theorem.

4.4. Theorem: *Let $\{(M^k, g^k)\}$ be a sequence of manifolds in $\mathfrak{M}(n, d, \Lambda, V)$, $0 < \alpha < 1$. There exists a subsequence $\{(M^l, g^l)\}$ with the following properties*:

(*i*) *Each M^l is diffeomorphic to a single fixed manifold M.*

(*ii*) *There exist diffeomorphism $F^l: M \to M^l$ such that $\{(F^l)^* g^l\}$ converges in C^1 to a $C^{1+\alpha}$-metric g on M.*

(*iii*) *diam(M^l) converges to* diam(M).

(*iv*) *For the injectivity radii we have* $\limsup i(M^l) \leqslant i(M)$.

(*v*) *If* $\exp^l$ *denotes the exponential map of M^l,* $\exp$ *that of (M, g), and* $\tilde{\exp}^l = (F^l)^* \exp^l$, *then* $\tilde{\exp}_p^l$ *converges to* $\exp_p$ *uniformly on compact subsets of T_pM, and* $\exp_p$ *is Lipschitz.*

Proof: (*i*) We define the diffeomorphisms $F^l: M^\infty \to M^1$ as in [12] and in the Proof of Theorem 3.9, namely $F^l(p)$ is defined as the center of mass of the points $F_i^l(p) := \phi_i^l(\phi_i^\infty)^{-1}(p)$, weighted appropriately. (*i*) of Lemma 3.10 is obviously satisfied, because the transition functions ϕ_{ij}^l converge in C^0 to ϕ_{ij}^∞. Condition (*ii*) of 3.10 had been used to estimate $\|dF_i^l - dF_j^l\|$, but now, with harmonic coordinates, such an estimate follows directly from the C^1-convergence of ϕ^l_{ij}, for $(\phi^l_j)^{-1}\phi^l_i \to (\phi^\infty_j)^{-1}\phi^\infty_i$ in C^1 implies $\phi^l_i(\phi^\infty_i)^{-1}$ is C^1-close to $\phi^l_j(\phi^\infty_j)^{-1}$ if l is large enough. Thus the construction of [12] yields a diffeomorphism $F^l: M^\infty \to M^l$.

(*ii*) is proved in local coordinates. Set $\tilde{g}^l:(F^l)^*g^l$. In coordinates, $\tilde{g}^l_i := (\phi^\infty_i)^*\tilde{g}^l$, while $g^l_i = (\phi^l_i)^*g^l$. Thus $\tilde{g}^l_i = [(\phi^l_i)^{-1}F^l\phi^\infty_i]^*g^l_i$ is the transformation formula for the metrics in coordinates. Therefore, if $(\phi^l_i)^{-1}F^l\phi^\infty_i$ is C^2-close to the identity, then $\tilde{g}^l_i$ is C^1-close to g^l_i. But $(\phi^l_i)^{-1}F^1\phi^\infty_i = (\phi^\infty_i)^{-1}(F^l_i)^{-1}F^l\phi^\infty_i$, therefore it suffices to show that F^l is C^2-close to F^l_i. This follows from the fact that the transition functions are even C^2-convergent, whence F^l_i is C^2-close to any F^l_j. Unfortunately this does not yet imply that F^l_i is also C^2-close to the average F^l, because $F^l(p)$ is defined as the unique zero y of $\sum_{j=1}^{N} \exp_y^{-1}F^l_j(p)\psi^l_j(p)$. Therefore, in the definition of the center of mass too, the exponential mapping needs to be replaced by the canonical harmonic coordinate mapping

$$H_y: B(y, R) \subset M^l \to T_yM^l.$$

With this improved definition, all former arguments remain true, but now the center becomes C^2-close to $F^l_i(p)$.

So far we have shown that $\tilde{g}^l_i$ is C^1-close to g^l_i if l is large enough. As the metrics g^l_i converge in C^1 to g^∞_i, this is also true for the $\tilde{g}^l_i$. Hence $\tilde{g}^l$ is C^1-convergent to g^∞.

(*iii*) and (*iv*) are proved in [13] for C^1-convergent metrics. (*v*) The uniform convergence follows again from Ascoli's theorem, for Rauch's comparison theorem yields a universal bound on $\|d\,\widetilde{\exp}^l_p\|$ on each ball $\bar{B}^l(0, R)$ in T_pM^∞, w.r.t. the metric $\tilde{g}^l$. As $\tilde{g}^l \to g^\infty$, we may take fixed balls $\bar{B}(0, r)$ w.r.t. the metric g^∞. After choosing a subsequence, $\widetilde{\exp}^l_p$ converges uniformly on $\bar{B}(0, r)$, but for different r's we obtain different Rauch-bounds and thus also different subsequences. Taking for r all integers successively and choosing each sequence as a subsequence of the preceding one, we obtain a sequence of sequences, from which we finally extract the diagonal sequence. Assume $\{M^l\}$ to this sequence. Any compact $K \subset T_pM^\infty$ is contained in $\bar{B}(0, n)$ for all n greater than some n_K, therefore $\widetilde{\exp}^l_p$ converges uniformly on K to a mapping which must be $\exp^\infty_p$. The universal Lipschitz bound on $\bar{B}(0, n_K)$ passes to the limit.

Sakai [13] proves the convergence of geodesics with methods from ordinary differential equations, under the assumption of C^1-convergence of the metrics. To prove convergence of $d\,\widetilde{\exp}^l_p$, he needs C^2-convergence of the metrics, which is not satisfied in our situation.

Finally, we note again that the non-smoothness of M^∞ is not essential. Let M be M^∞ with the above mentioned C^∞-structure and pull back all the metrics to M.

5. Sharpness of the $C^{1+\alpha}$ result

In this chapter we are going to show that the $C^{1+\alpha}$ result on the limit metrics is sharp in terms of Hőlder conditions. More precisely, we will present a very simple counterexample of a limit metric that is $C^{1+\alpha}$ for any $\alpha < 1$ but not $C^{1,1}$. Nevertheless we are able to derive a stronger result, in terms of Sobolev spaces.

5.1. Example: Consider the following function of two real variables, where $x = (x_1, x_2)$.

$$E(x) := \left(x_1^2 + x_2^2\right)^{(x_1^2 - x_2^2)}.$$

E is obviously positive, $\lim_{x \to 0} E(x) = 1$, $\lim_{x \to 0} \operatorname{grad}_x E = 0$, and apart from the origin, E is C^∞.

In polar coordinates, the expression for E reads

$$E(r, \varphi) = r^{2r^2 \cos 2\varphi}.$$

At $x = 0$, E is $C^{1+\alpha}$ for any $\alpha < 1$ but not $C^{1,1}$:

$$\frac{|\operatorname{grad}_x E - \operatorname{grad}_0 E|^2}{|x|^{2\alpha}} = \frac{4E^2(x)}{r^{2\alpha}}\left(4r^2(\cos^2 2\varphi + \ln r)\ln r + \cos^2 2\varphi\right).$$

This expression tends to zero for any $\alpha < 1$, but to infinity for $\alpha = 1$.

The function E is now used to define metrics. By $g_{11} = g_{22} = E$, $g_{12} = 0$, a metric on $\mathbb{R}^2$ is defined, expressed in isothermal coordinates. It is important that the coordinates x_1, x_2 are harmonic with respect to g, for by theorem 4.2 this rules out the possibility of making the metric smoother by reparametrization.

It remains to show that the curvature of g stays bounded in the neighborhood of zero, although it is not defined at the origin itself. In isothermal coordinates, the Gaussian curvature reads

$$K = -\frac{1}{2E}\Delta \ln E.$$

In our case, $\Delta \ln E = 8 \cos 2\varphi$, so K is indeed bounded.

In order to obtain a sequence of metrics g^k converging to g, we apply mollifiers J_{ϵ_k} to g, with $\epsilon_k \to 0$. Since this smoothing process, namely convolution with narrow kernels, preserves Hölder conditions, we obtain a sequence $g^k = J_{\epsilon_k} g$ of C^∞-metrics with uniform Hölder bounds, converging to g in the $C^{1+\alpha}$ topology. Clearly the bound on K carries over.

So far we have shown that the $C^{1+\alpha}$-estimates for the metric in terms of curvature bounds which Jost and Karcher obtained in [8] cannot be sharpened to $\alpha = 1$. The desired example for the compactness theorem is now easily established. We consider any compact, two-dimensional Riemannian manifold, and in some arbitrary neighborhood we replace the given metric by g^k, putting the two metrics together by a partition of 1 in the usual way. Let us now derive our final result concerning the regularity of limit metrics. It is an immediate consequence of the following theorem.

5.2. THEOREM: (*I.G. Nikolaev* [19]). *Let M be a space of bounded curvature, Ω a domain in which harmonic coordinates are defined. Then in Ω the components g_{ij} of the metric are functions of class $H_p^2(\Omega)$ for all $p \geqslant 1$.*

Bounded curvature $K_1 \leqslant K \leqslant K_2$ is here to be understood in a generalized sense, namely in terms of the angular excess of triangles, compared with triangles of the same edgelenghs in the space forms of curvature K_1 and K_2. On compact differentiable manifolds, C^1-limits of smooth Riemannian metrics with bounded curvature $|K| \leqslant \Lambda^2$ satisfy this condition.

Moreover, if all metrics in a C^1-convergent sequence are expressed in harmonic coordinates, then this must also hold for the limit metric, because the condition for harmonic coordinates reads

$$g^{ij}\Gamma_{ij}^l = 0, \quad l = 1, \dots, n.$$

In our proof of the compactness theorem, the limit metric was in fact obtained as a C^1-limit of a sequence of smooth metrics with bounded curvature, expressed in harmonic coordinates. Hence its components are H_p^2-functions by Nikolaev's theorem. Thus we have shown:

5.3. THEOREM: *In the compactness theorem* 1.6 *or* 4.4, *the components of the limit metric, expressed in harmonic coordinates, are contained in the Sobolev spaces H_p^2 for any $p \geqslant 1$. In particular, all notions or curvature are almost everywhere defined.*

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