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Characterization of Jacobian varieties in arbitrary characteristic

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0. Introduction

In this work we fix an algebraically closed field k of *arbitrary characteristic* and assume all varieties defined over k.

The work is devoted to proving two characterization theorems for Jacobians of algebraic curves over the field k.

The characterization theorems are as follows:

Let (X, D) be a principally polarized abelian variety of dimension g. The dualizing sheaf ω_D of D defines a *canonical rational map*:

$$f: D \dashrightarrow \mathbb{P}^*_{g-1} = \text{Proj } S(H^0(D, \omega_D)) = \text{Proj } S(\Lambda^{g-1} H^0(X, \Omega_X)).$$

Let $\mathbb{P}_{g-1} = \text{Proj } SH^0(X, \Omega_X)$ be the dual projective space and, for every $x \in \mathbb{P}_{g-1}$, let H_x the hyperplane of $\mathbb{P}^*_{g-1}$ defined by incidence.

We assume that D is symmetric and $\dim D_{\text{sing}} \leqslant g-3$.

THEOREM 3.1. *(X, D) is the Jacobian variety of a non-hyperelliptic curve if and only if there exists an irreducible curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $2g-2$ such that:*

For every point $p \in C$, $f^{-1}(H_p)$ breaks into two different conjugated components.

This theorem, together with Andreotti's results [Andreotti, 1958], implies that the curve C must be the dual variety, R^*, of the ramification locus of the canonical map f.

THEOREM 3.4. (*hyperelliptic case*). *The p.p. abelian variety (X, D) is the Jacobian of an hyperelliptic curve if and only if there exists a curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $g-1$ (not contained in hyperplanes) such that: For every geometric point $p \in C$, $f^{-1}(H_p)$ breaks in two irreducible conjugated components Y_p', Y_p'' and there exists a finite number of points $p_0 \in C$, verifying $Y'_{p_0} = Y''_{p_0}$.*

If (X, D) satisfies these conditions, there exists a covering $\overline{C} \xrightarrow{\pi} C$ of degree 2 and (X, D) is the Jacobian variety of $\overline{C}$.

Finally, we prove a second theorem which is more adapted than Theorem 3.1. to the study of moduli problems:

The natural homomorphism $H^0(D, \omega_D) \otimes_k H^0(D, \omega_D) \to H^0(D, \mathcal{O}_D(2D))$ induces a morphism $\mathbb{P}_{g-1} \times \mathbb{P}_{g-1} \to \mathbb{P}^{N-1} = \operatorname{Proj} S(H^0(D, \mathcal{O}_D(2D))^*)$ and hence a morphism $C \times C \to \mathbb{P}^{N-1}$ for every curve $C \hookrightarrow \mathbb{P}_{g-1}$ whose scheme-theoretic image is a surface S. *Assume that* $\dim X > 3$, then one has:

THEOREM 3.5.
1. *(X, D) is the Jacobian variety of a non-hyperelliptic curve if and only if there exists an irreducible and projectively normal curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $2g-2$ such that the surface S is contained in the scheme-theoretic image $\bar{X}$ of the rational map $X \dashrightarrow \mathbb{P}^{N-1} (x \mapsto D(x) \cap D + D(-x) \cap D)$. Notice that $\bar{X}$ is a 'blowing-up' of the Kummer variety $X/\{\pm 1\}$.*
2. *The p.p. abelian variety (X, D) is the Jacobian of an hyperelliptic curve if and only if there exists an irreducible curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $g-1$ (not contained in hyperplanes) such that $S \subset \bar{X}$.*

We shall now comment on the meaning of these results and on the information that may be provided by study of the structure of the Jacobians.

1. The three theorems announced are valid for fields of arbitrary characteristic, such that they may be useful for dealing with the problem of Schottky without constraints on characteristic.

2. Despite the undoubtable relationship between Theorem 3.1. and Mumford's conjecture ([Mumford, 1976], proved by Welters over $\mathbb{C}$, [Welters, 1983]), two essential differences should be noted:

Firstly, the existence is not assumed of a curve C in the abelian variety X fulfilling certain relationships, but rather that curve C is embedded in the projective space $\mathbb{P}_{g-1}$ and, furthermore, it is determined uniquely by the polarization D ($C = R^*$, the dual variety of the ramification locus R). The immersion of C in X is performed in the proof of the theorem. The importance of this theorem is that it seems to be possible to state the condition of the existence of a curve C from Theorem 3.1. by algebraic relationships between the theta functions, a possibility which is not altogether clear if the curve is already assumed to be immersed in X (since among other things, such an immersion is not unique).

Secondly, in Mumford's conjecture [Mumford, 1976; Welters, 1983], the condition $D(u) \cap D \subset D(x) \cup D(y)$ is imposed, evidently stronger than the condition of reducibility on the divisors of the canonical series of D imposed in Theorem 3.1.

Such reasons lead one to believe that dealing with the problem of characterization in terms of the divisors of the canonical series of D is more natural. Such a notion receives further support in view of the fact that the ideas on which Theorem 3.1. is based have for the first time permitted the statement and proof of a characterization theorem for hyperelliptic Jacobians, Theorem 3.4., valid in arbitrary characteristic.

3. The results mentioned above permit us to state and prove Theorem 3.5. in which the condition of reducibity of Theorem 3.1. is substituted by inclusion of S^2C (embedded in $\mathbb{P}^{N-1}$ by the Segre morphism) in the variety $\overline{X} \subset \mathbb{P}^{N-1}$ obtained by projecting from the 'origin' the Kummer variety $X/\{\pm 1\}$. We believe that this statement may shed some light on certain moduli problems (such as the Schottky problem) since the condition $S^2C \subset \overline{X}$ can be readily expressed in equations, once the ideals of S^2C and $\overline{X}$ in $\mathbb{P}^{N-1}$ have been determined (which in turn is standard).

Apart from these results, in §2 we prove that Andreotti's result on the ramification locus of the Gauss morphism of W_{g-1} permits one to reconstruct the surface $C - C$ of the Jacobian of a non-hyperelliptic curve *exclusively* in terms of its polarization. This problem has been dealt with, over $\mathbb{C}$, by [Gunning, 1982, 1984].

1. Dualizing sheaf and Picard scheme of a principal polarization

Let (X, D) be an abelian variety principally polarized (p.p) ($D^g = g!$, $g = \dim X$) over an algebraically closed field of arbitrary characteristic k. We shall assume in the rest of this paper that D is a symmetric polarization ($(-1)^*D = D$) and $\dim D_{\text{sing}} \leqslant g-3$ (that is, D is regular in codimension one). Observe that these conditions imply that D is irreducible and normal (since D is a Cartier divisor it will be a Gorenstein scheme). We also assume that $g \geqslant 3$.

Since D is a Cartier divisor on an abelian variety, the dualizing sheaf of D is $\omega_D \simeq \mathcal{O}_D(D)$ and one has a natural isomorphism $H^0(D, \omega_D) \simeq H^1(X, \mathcal{O}_X)$. The natural morphism of sheaves $H^0(D, \omega_D) \otimes_k \mathcal{O}_D \to \omega_D$ induces the *canonical rational map:*

$$f\colon D \dashrightarrow \operatorname{Proj} S\big(H^1(X, \mathcal{O}_X)\big)$$

Let $\varphi'\colon T_D \to \varphi^* T_X = \mathcal{O}_D \otimes_k H^0(X, \Omega_X)^*$ be the tangent morphism induced by the immersion $D \hookrightarrow X$; taking dual and exterior algebras one has a epimorphism of sheaves:

$$\varphi'_{g-1}\colon \Lambda^{g-1}(\varphi^* T_X)^* \simeq \Lambda^{g-1}\big(H^0(X, \Omega_X)\big) \otimes_k \mathcal{O}_D \to \Lambda^{g-1} T_D^* \to 0$$

which induces a rational map:

$$\varphi_{g-1}\colon D \dashrightarrow \operatorname{Proj} S\big(\Lambda^{g-1} H^0(X, \Omega_X)\big)$$

this is the '*rational Gauss map*'.

1.1.

The natural isomorphism Proj $S(H^1(X, \mathcal{O}_X)) \xrightarrow{\sim}$ Proj $S(\Lambda^{g-1}H^0(X, \Omega_X))$ *and the isomorphism* $\omega_D \simeq \Lambda^{g-1}T_D$, [Hernández Ruipérez, 1981; Hartshorne, 1970], *allow us to identify the maps* $f = \varphi_{g-1}$.

Standard arguments prove that the canonical rational map is generically finite [Ran, 1982].

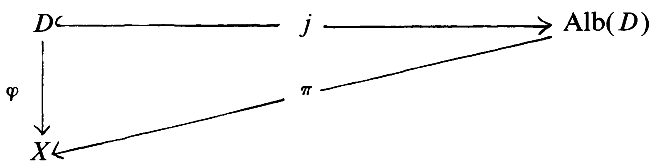
Let $\hat{X} = \mathrm{Pic}^0(X)$ be the dual abelian variety of X. Since D is a principal polarization, one has an isomorphism $\mu_D: X \to \hat{X}$ given by $\mu_D(x) =$ class of $D(x) - D$ ($D(x)$ being the image of D by the translation $\alpha \mapsto \alpha + x$).

The immersion $\varphi: D \hookrightarrow X$ induces a morphism between the Picard schemes $\varphi^*: \hat{X} \to \mathrm{Pic}^0(D)$.

THEOREM 1.2. $\varphi^* \cdot \mu_D: X \to \mathrm{Pic}^0(D)$ *is an isomorphism of abelian varieties.*

Proof. Let us prove first that dim $\mathrm{Pic}^0(D) = g$. The tangent space to $\mathrm{Pic}^0(D)$ at the origin is $H^1(D, \mathcal{O}_D)$. The exact sequence $0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ induces an isomorphism $\varphi': H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(D, \mathcal{O}_D)$ which is precisely the morphism induced by φ^* between the tangent spaces. This shows, that dim $\mathrm{Pic}^0(D) \leqslant g$. However, since the divisor D generates the abelian variety X, the Albanese variety of D is of dimension $\geqslant g$ [Grothendieck, 1962; Serre, 1958–1959] and by the general properties of the Albanese variety, $\mathrm{Pic}^0(D)$ is an abelian variety of dimension g.

The immersion $\varphi: D \hookrightarrow X$ induces a commutative diagram



where π is a separable isogeny. In fact, π is an isomorphism: if it were not, $\pi^{-1}(D)$ would be an ample disconnected divisor of Alb(D) (the ampleness is by [Hartshorne, 1970] and $\pi^{-1}(D)$ is disconnected since $\pi^{-1}D \to D$ is an étale covering with a section); however, every effective ample divisor in a smooth variety of dimension $\geqslant 2$ is connected [Mumford, 1966b].

Taking into account that Alb(D) $= \mathrm{Pic}^0(\mathrm{Pic}^0(D))$ one finishes the proof.

To avoid cumbersome notation, we shall alway denote

$$\mathbb{P}^*_{g-1} = \mathrm{Proj}\ S\big(H^0(D, \omega_D)\big), \quad \mathbb{P}_{g-1} = \mathrm{Proj}\ S\big(H^0(X, \Omega_X)\big) \quad \text{and}$$

$$\Gamma_{g-1} \hookrightarrow \mathbb{P}^*_{g-1} \times \mathbb{P}_{g-1}$$

the correspondence of incidence. Note that $(-1)_X$ acts on $H^0(X, \Omega_X)$ as the multiplication by (-1), hence by the identification $f=\varphi_{g-1}$, we will have that $(-1)^*$ leaves the canonical series of D invariant: that is, if K_D is a canonical divisor of D, then $(-1)^*(K_D)=K_D$.

1.2. The Jacobian case

Let C be a smooth curve of genus g over the field k, and W_{g-1} the theta divisor of its Jacobian $J_{g-1}(C)$.

In this case we can say more about the Gauss and the canonical maps:

Let D_{g-1} be the universal divisor on $C\times S^{g-1}C$ ($(g-1)^{\text{th}}$-symmetric product) and $\pi: C\times S^{g-1}C\to S^{g-1}C$ the natural projection. From the exact sequence $0\to\mathcal{O}\to\mathcal{O}(D_{g-1})\to\mathcal{O}_{D_{g-1}}(D_{g-1})\to 0$ over $\dot{C}\times S^{g-1}C$ one obtains, by taking π_* and restricting to U_{g-1} (the open subset of $S^{g-1}C$ where the Abel morphism $\varphi_{g-1}: S^{g-1}C\to W_{g-1}$ is an isomorphism which, by Kempf's Riemann singularity theorem [Kempf, 1973], is the open subset of smooth points of W_{g-1}) an exact sequence of *locally free* sheaves on U_{g-1}:

$$0\to\pi_*\mathcal{O}_{D_{g-1}}(D_{g-1})\,|_{U_{g-1}}\overset{\varphi'}{\to} R^1\pi_*\mathcal{O}\,|_{U_{g-1}}\to R^1\pi_*\mathcal{O}(D_{g-1})\,|_{U_{g-1}}\to 0 \qquad (1)$$

The morphism of schemes $\varphi_{g-1}: U_{g-1}\to \operatorname{Proj} S(H^1(C,\mathcal{O}_C))=\mathbb{P}^*_{g-1}$ is exactly the Gauss morphism over the open subset U_{g-1}.

From well-known results on duality and on the infinitesimal structure of Hilbert schemes [Grothendieck, 1962; Hernández Ruipérez, 1981; Mattuck, 1965; Mumford, 1966a] one obtains canonical isomorphisms $\omega_{g-1}\simeq \Lambda^{g-1}[\pi_*\mathcal{O}_{D_{g-1}}(D_{g-1})]^*$ and $H^0(S^{g-1}C,\omega_{g-1})\overset{\sim}{\to}\Lambda^{g-1}H^0(C,\Omega_C)$, ($\omega_{g-1}$ being the dualizing sheaf of $S^{g-1}C$).

These facts imply that the natural epimorphism $H^0(S^{g-1}C,\omega_{g-1})\otimes_k \mathcal{O}_{g-1_{C|U_{g-1}}}\to\omega_{g-1|_{U_{g-1}}}\to 0$ is identified with the epimorphism obtained from (1) by taking duals and exterior algebras. Hence we arrive at a more explicit construction of the canonical morphism $f: U_{g-1}\to\mathbb{P}^*_{g-1}=\operatorname{Proj} S(\Lambda^{g-1}H^0(C,\Omega_C))$ over the open subset U_{g-1} in terms of the dualizing sheaf, Ω_C, of C.

2. The surface $C-C$ of a non-hyperelliptic Jacobian

This section is devoted to proving that non-hyperelliptic Jacobians satisfy the conditions of the characterization theorems of Section 3. We shall also find a geometric procedure to construct the surface $C-C$ of the jacobian in the non-hyperelliptic case uniquely in terms of the theta divisor W_{g-1}.

Let us consider the automorphism $\vartheta: C\times C\times J_{g-1}(C)\to C\times C\times J_{g-1}(C)$ defined by $\vartheta(p,q,\alpha_{g-1})=(p,q,(p-q)+\alpha_{g-1})$ and write $\overline{D}'=\vartheta(\overline{D})$, where $\overline{D}=C\times C\times W_{g-1}$. Let $\overline{D}_{g-1}$ be the Cartier divisor on $\overline{D}$ given by $\overline{D}_{g-1}=\overline{D}\cap D'$, and Z_{g-1} the divisor on $C\times W_{g-1}$, the image of the universal divisor D_{g-1} on $C\times S^{g-1}C$. Denote by Z^*_{g-1} the image of Z_{g-1} by the automorphism $1\times\sigma: C\times W_{g-1}\to C\times W_{g-1}$ (σ being the involution of J_{g-1} given by $\alpha\mapsto k-\alpha$, k being the canonical series of C), and by $\pi_i: C\times C\times W_{g-1}\to C\times W_{g-1} (i=1, 2)$, the natural projections. Using the See-saw lemma ([Mumford, 1974], pg. 54), the intersection formulas for W_{g-1} [Weil, 1957; Matsusaka, 1958; Mumford, 1976] can be rewritten in the following form:

LEMMA 2.1. (*Weil, Matsusaka*) *With the above notations one has*:
1. $\overline{D}_{g-1}=\pi_1^*Z_{g-1}+\pi_2^*Z^*_{g-1}$.
2. *If* $\delta: C\hookrightarrow C\times C$ *is the diagonal immersion, then*:

$$(\delta\times 1)^{-1}D_{g-1|_{p\times W_{g-1}}}=\overline{D}_{g-1|_{p\times p\times W_{g-1}}}=W_{g-2}(p)+W^*_{g-2}(p)$$

In particular, for every point $p\in C$, $W_{g-2}(p)+W^*_{g-2}(p)$ *is a canonical divisor of* W_{g-1}; *precisely, it is the self-intersection of* W_{g-1} *'along the tangent direction to* C *at the point* p' (*as is shown in Theorem* 2.2.2.).

We assume C to be non-hyperelliptic. Observe that $\mathbb{P}^*_{g-1}=\operatorname{Proj} S(H^0(C, \Omega_C)^*)$ is precisely the subscheme of $S^{2g-2}C$ representing the canonical series of C ([Grothendieck, 1962], 232). Thus, the canonical map f can be considered as a map $f: U_{g-1}\to S^{2g-2}C$. Let $D_{2g-2}\subset C\times S^{2g-2}C$ be the universal divisor, then $(1\times f)^{-1}D_{2g-2}$ is a divisor on $C\times U_{g-1}$ flat over U_{g-1} and of relative degree $2g-2$, and can be extended to a unique divisor on $C\times W_{g-1}$, also denoted by $(1\times f)^{-1}D_{2g-2}$.

THEOREM 2.2. *Let* C *be non-hyperelliptic, with the above notations one has*:
1. $(1\times f)^{-1}D_{2g-2}=(\delta\times 1)^{-1}\overline{D}_{g-1}$ *as divisors on* $C\times W_{g-1}$.
2. $f^{-1}(H_p)=W_{g-2}(p)+W^*_{g-2}(p)$,

for every point $p\in C$ (*considering* C *embedded in* $\mathbb{P}_{g-1}$ *by its dualizing sheaf*), *where* H_p *stands for the hyperplane of* $\mathbb{P}^*_{g-1}$ *defined by* p *through incidence.*

Proof
1. Both divisors are flat over U_{g-1} of relative degree $2g-2$. Then, proving that for every point $\alpha\in U_{g-1}$, one has $(1\times f)^{-1}D_{2g-2|_{C\times\{\alpha\}}}=(\delta\times 1)^{-1}\overline{D}_{g-1|_{C\times\{\alpha\}}}$, suffices.
   But $(1\times f)^{-1}D_{2g-2|_{C\times\{\alpha\}}}=f(\alpha)$ (1.2), and $(\delta\times 1)^{-1}\overline{D}_{g-1|_{C\times\{\alpha\}}}$ is the unique canonical divisor K_α on C containing α. Since $K_\alpha=\varphi_{g-1}(\alpha)$, by the description in 1.3., one concludes by the identification $f=\varphi_{g-1}$.
2. By the above construction, $(1\times f)^{-1}D_{2g-2|_{\{p\}\times W_{g-1}}}=f^{-1}(H_p)$ and the result follows from Lemma 2.1.

As we shall see in §3, Theorem 2.2. implies that the non-hyperelliptic Jacobians satisfy the conditions of Theorem 3.1.

2.3. Reconstruction of $C - C$ for a non-hyperelliptic Jacobian from the theta divisor

Let C be non-hyperelliptic. [Andreotti, 1958] proved that the curve C embedded in $\mathbb{P}_{g-1}$ is the dual variety of the ramification locus, R, of the canonical map $f: W_{g-1} \dashrightarrow \mathbb{P}^*_{g-1}$ (this result is valid in arbitrary characteristic) thus the theta divisor determines the embedding $C \hookrightarrow \mathbb{P}^*_{g-1}$. Let us denote by $\Gamma_C \hookrightarrow \mathbb{P}^*_{g-1} \times C$ the graph of incidence. By Theorem 2.2., $Y = (1 \times f)^{-1}\Gamma_C \to C \times W_{g-1}$ has two components $Y = Y' + Y''$ such that $\sigma(Y') = Y''$.

$Z = \pi_1^{-1}(Y') + \pi_2^{-1}(Y'')$ is a Cartier divisor over W_{g-1} parametrized by $C \times C (\pi_i : C \times C \times W_{g-1} \to C \times W_{g-1}$ being the projections), and it thus induces a morphism of schemes:

$$\psi : C \times C \to \mathrm{Pic}^P(W_{g-1})$$

$\mathrm{Pic}^P(W_{g-1})$ being the connected component of the Picard scheme containing the dualizing sheaf ω_{g-1}.

The composition of ψ and the isomorphism $\mathrm{Pic}^P(W_{g-1}) \xrightarrow{\sim} J_0(C)$ (Theorem 1.2) is a morphism $\mu: C \times C \to J_0(C)$ which coincides with the morphism $(p, q) \to p - q$. This concludes the construction of $C - C$ from W_{g-1}.

Notice that the Torelli theorem as proved by [Andreotti, 1958] is weaker than the one proved by [Matsusaka, 1958]. Andreotti proved that *two non-hyperelliptic curves C, C' whose polarized Jacobian varieties are isomorphic, are isomorphic.*

Matsusaka's theorem ensures that the immersion of C and C' into their respective jacobians are essentially the same (up to translations and the involution σ).

From the construction made in 2.3. if follows immediately the following Corollary, which allows us to recover Matsusaka's statement:

COROLLARY 2.4. *Let $\tau: J_0(C) \to J_0(C)$ be an automorphism of the Jacobian of a non-hyperellliptic curve C such that $\tau(W_{g-1}) = W_{g-1}$ and $\tau(0) = 0$. Then, $\tau(C - C) = C - C$. In particular, either τ or $(-1) \cdot \tau$ is induced by an automorphism of C and the translation by a point $p \in C$.*

3. Characterization of Jacobian varieties

Through this section (X, D) there will be a p.p. abelian variety such that D is a *symmetric polarization* and $\dim D_{\mathrm{sing}} \leqslant g - 3$.

With the notations of Section 1, given a point $\alpha \in \mathbb{P}_{g-1}$, we shall denote by $H_\alpha = \Gamma_{g-1}(\alpha)$ the hyperplane which it defines in $\mathbb{P}^*_{g-1}$ by incidence. If C is a curve of $\mathbb{P}_{g-1}$, we shall denote: $\Gamma_C = \Gamma_{g-1|\mathbb{P}^*_{g-1} \times C}$.

We shall designate by R the scheme-theoretic closure of the locus of ramification of the canonical rational map f.

THEOREM 3.1. *The p.p. abelian variety (X, D) is the Jacobian of a nonhyperelliptic curve if and only if there exists an irreducible curve C of degree $2g-2$ of $\mathbb{P}_{g-1}$ such that*:

$Y = f^{-1}(\Gamma_C)$ is a Cartier divisor on $C \times D$ with two different components $Y = Y' + Y''$, such that $(-1)_X(Y') = Y''$.

That is, for every point $p \in C$ (geometric or not) we have that $f^{-1}(H_p) = Y_p' + Y_p''$ is a divisor on D with two different components such that $(-1)_X(Y_p') = Y_p''$. If (X, D) satisfies these conditions, then C is a smooth curve, $C \hookrightarrow \mathbb{P}_{g-1}$ its canonical immersion, (X, D) its polarized Jacobian and C is the dual variety of the ramification locus R.

Proof of Theorem 3.1. The necessity is the Theorem 2.2. Let us assume that (X, D) satisfies the hypothesis of the theorem.

Let $C_1 = C_2 = C$ and let us consider the natural projections $\pi_i : C_1 \times C_2 \times D \to C_i \times D$ $(i = 1, 2)$. Note that Y' and Y'' are Weil divisors which in general would not be Cartier divisors.

$Z = \pi_1^{-1}(Y') + \pi_2^{-1}(Y'')$ is a Weil divisor on D parametrized by $C \times C$. The restriction of Z to $\Delta_C \times D$ coincides with Y which is Cartier divisor. Hence, the set of points $(p, q) \in C \times C$ such that $Z|_{p \times q \times D}$ is a Cartier divisor is a non-empty open subset $U \subset C \times C$, which must contain Δ_C, (E.G.A. IV_3, 13.3.1).

Thus, we will have a natural morphism of schemes:

$$\psi : U \to \operatorname{Pic}(D)$$

defined by the Cartier divisor $Z_U \hookrightarrow U \times D$.

If $\overline{C}$ is the desingularization of C, ψ will induce a morphism which we shall continue to designate by $\psi : \overline{C} \times \overline{C} \to \mathbb{P}\mathrm{ic}(D)$. By construction $\psi(\Delta_{\overline{C}}) =$ [class of ω_D]. Composing ψ with the isomorphism of Theorem 1.2., one obtains a morphism of schemes:

$$\mu : \overline{C} \times \overline{C} \to X$$

We should now note that the scheme-theoretic image of μ is a surface. In fact, we have that for every point $p \in \overline{C}$, $\mu : \overline{C} \times \{p\} \to X$ is injective over the open subset $V \subset \overline{C}$ of the smooth points of C. Note that $Z|_{p \times q \times D} = Y_p' + Y_q''$. If $Y_p' + Y_q'' \sim Y_p' + Y_p''$ (linear equivalence on D), as the involution $(-1)_X$ leaves the divisors of the canonical serie of D invariant, one has:

$$\begin{array}{ccc} (-1)_X(Y_p') + (-1)_X(Y_q'') & = & Y_p' + Y_q'' \\ \| & & \\ Y_p'' + Y_q' & & \end{array}$$

hence $Y_p' = Y_q'$ and $Y_p'' = Y_q''$; that is: $f^{-1}(H_p) = f^{-1}(H_q)$ and then one would have $p = q$ (by definition of f).

Now we assume that the image of the diagonal $\mu(\Delta_{\bar{C}})$ is the origin of X and that $0 \in D$. Having fixed a point $p_0 \in V$, we shall have two curves C_0 and C_0' in X: the images of $\bar{C} \times p_0$ and of $p_0 \times \bar{C}$ by the morphism μ. Both are birationally isomorphic to C and, moreover, $(-1)_X(C_0) = C_0'$ (since $\mu(p_0, p_0) = 0$). If we set $Y_0 = Y_0' + Y_0'' = f^{-1}(H_{p_0})$, we will have that:

$$Y_p' = T_p(Y_0'), \quad Y_p'' = T_{-p}(Y_0'') \ (p \in C_0)$$

(T_p and T_{-p} being the translations). These equalities follow by applying the Rigidity lemma ([Mumford, 1974], pag. 43) to the morphism $\mu : \bar{C} \times \bar{C} \to X$, which shows that $\mu(p, q) = \mu(p, p_0) + \mu(p_0, q)$, (see Remark 3.2. concerning these equalities).

Hence $C_0 \hookrightarrow D$ and we have finite morphisms $C_0 \times Y_0' \xrightarrow{s} D$ $((p, \alpha) \mapsto p + \alpha)$ and $C_0 \times Y_0'' \xrightarrow{s'} D$ $((p, \alpha) \mapsto -p + \alpha)$ whose degrees are $s^{-1}(0) = (C_0 \cap Y_0')_D$ (intersection over D). Given a general $\alpha \in D$, $(C_0 \cap Y_0)_D = s^{-1}(\alpha) \cup s'^{-1}(\alpha) = \{ p \in C_0 \text{ such that } Y_p \ni \alpha \} = \{ p \in C \text{ such that } p \in f(\alpha)^* \} = 2g - 2$ (where $f(\alpha)^*$ is the hyperplane of $\mathbb{P}_{g-1}$ defined by $f(\alpha)$. We thus have that $\frac{D^2}{2} * C_0 \equiv (g-1)D$ (where $*$ is the Pontriagin product and $\equiv$ the algebraic equivalence), and by the 'duality' between the Pontriagin product and the intersection product [Kleiman, 1968; Beauville, 1983], we will have that $D^{g-1} \equiv (g-1)! C_0$. By the criterion of [Matsusaka, 1959] one finishes.

Remark 3.2. In the case $k = \mathbb{C}$, once the equalities $Y_p' = T_p(Y_0')$, $Y_p'' = T_{-p}(Y_0'')$ have been proven, one could conclude the proof by applying Welters' criterion [Welters, 1983]. However, even in this case, our proof is a direct consequence of Matsusaka's criterion avoiding Gunning's criterion [Gunning, 1982] which is essentially transcendent. This discussion is senseless in positive characteristic.

Remark 3.3. There is an important difference between Theorem 3.1. and Mumford's conjecture proved by [Welters, 1983]. In Mumford's conjecture, the condition $D(u) \cap D \subset D(x) \cup D(y)$ is imposed and it is obviously stronger than the reducibility of the canonical divisors imposed in Theorem 3.1.

THEOREM 3.4. (hyperelliptic case) *The p.p. abelian variety (X, D) is the Jacobian of an hyperelliptic curve if and only if there exists a curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $g-1$ (not contained in hyperplanes) such that for each geometric point $p \in C$, $f^{-1}(H_p)$ is a divisor on D with two irreducible components Y_p', Y_p'' such that $(-1)_X(Y_p') = Y_p''$ and there exists a finite number of points $p_0 \in C$ satisfying $Y_{p_0}' = Y_{p_0}''$.*

If (X, D) satisfies these conditions, then C is a rational normal curve, there exists a branch covering $\bar{C} \to C$ of degree 2 whose ramification locus is the set of the points p_0, and (X, D) is the Jacobian of $\bar{C}$.

Proof. Let $\text{Hilb}^P(D)$ and $\text{Hilb}^Q(D)$ be the components of the Hilbert scheme with Hilbert polynomials P, Q (P being the Hilbert polynomial of Y_p' and Q the Hilbert polynomial of Y_p with respect to the ample sheaf $\mathcal{O}_D(4D)$) and $Z \hookrightarrow D \times \text{Hil}^P(D)$ the universal closed subscheme defined by an ideal $0 \to \not{p}_Z \to \mathcal{O}_{D \times \text{Hilb}^P(D)}$. The involution $(-1)_X$ acts on $D \times \text{Hilb}^P(D)$; one then has the ideal $0 \to \not{p}_Z \cap (-1)_X \not{p}_Z \to \mathcal{O}$ which defines a closed subscheme $\overline{Z} \hookrightarrow D \times \text{Hilb}^P(D)$ and, hence a rational map $\text{Hilb}^P(D) \overset{\pi}{\dashrightarrow} \text{Hilb}^Q(D)$.

Let $Y = f^{-1}(\Gamma_C) \hookrightarrow C \times D$, as in Theorem 3.1.; this divisor defines an embedding $C \hookrightarrow \text{Hilb}^Q(D)$. (Notice that C is contained in the image of π.) We define the curve $\overline{C}$ to be $\overline{\pi^{-1}(C)}$; obviously one has a finite morphism $\pi: \overline{C} \to C$ of degree 2 and then $\overline{C}$ is either hyperelliptic, rational, or has two irreducible components isomorphic to C.

By construction, the projection $Y \to C$ factors through $\overline{C}$, $Y \overset{p}{\to} \overline{C} \overset{\pi}{\to} C$, and Y breaks into two components over $\overline{C}$: that is, $Y = Y_1 + Y_2$ is a subscheme of $\overline{C} \times D$ and $(-1)_X(Y_1) = Y_2$.

Now proceeding as in the proof of Theorem 3.1. we obtain a morphism of schemes $\mu: \overline{C} \times \overline{C} \to X$ whose image is a surface. By the hypothesis C is rational; this excludes that $\overline{C}$ be rational or can have two components (because in these cases the image of μ would be a single point). Then $\overline{C}$ is hyperelliptic and the rest of the proof runs as in Theorem 3.1.

We have a natural morphism of schemes:

$$X \overset{\pi}{\to} \mathbb{P}^N = \text{Proj } S\big(H^0(X, \mathcal{O}_X(2D))^*\big) \quad (N = 2^g - 1).$$

If $x \in X$ is a geometric point, one has:

$\pi(x) =$ point of $\mathbb{P}^N$ defined by the divisor $D(x) + D(-x)$ (see [Mumford, 1970] for more details and for the scheme-theoretic definition of π).

By construction, the scheme-theoretic image of the morphism π is the Kummer variety $X' = X/\{\pm 1\}$.

Since D is invariant by $(-1)_X$, the restriction of π to D will also be the quotient map by $\{\pm 1\}$:

$$D \overset{\pi}{\to} D' = D/\{\pm 1\} \subset \mathbb{P}^N.$$

Moreover, by restricting to D the invertible sheaf $\mathcal{O}_X(2D)$ we obtain $\mathcal{O}_D(2D) = \omega_D^2$ and by repeating the scheme theoretic construction of π as in [Mumford, 1970] we obtain rational maps:

$$\overline{\pi}: X \dashrightarrow \mathbb{P}^{N-1} = \text{Proj } S\big(H^0(D, \mathcal{O}_D(2D)) *\big)$$

$$\overline{\pi}: D \dashrightarrow \mathbb{P}^{N-1} = \text{Proj } S\big(H^0(D, \mathcal{O}_D(2D)) *\big)$$

defined over the open subset $X - \{0\}$ (0 being the origin of X).

Over the geometric points of X $\bar{\pi}$ is given by:

$$\bar{\pi}(x) = (D(x) + D(-x)) \cap D, \quad x \neq 0$$

We shall denote by $\bar{X}$ the scheme-theoretic image of X by the morphism $\bar{\pi}$.

The exact sequence of cohomology

$$0 \to H^0(X, \mathcal{O}_X(D)) \xrightarrow{\mu_0} H^0(X, \mathcal{O}_X(2D)) \xrightarrow{\mu} H^0(D, \mathcal{O}_D(2D)) \to 0$$

induces a rational map:

$$\mu: \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$$

which may be identified with the 'projection of vertex p_0', $p_0 \in \mathbb{P}^N$ being the point defined by the morphism μ_0; that is, $p_0 = \pi(0)$. Observe that μ induces a birational map $\mu: X' \dashrightarrow \bar{X}$. Furthermore, the total transform of p_0 by $\mu: X' \dashrightarrow \bar{X}$ is the subvariety $\bar{X}_0$ defined by:

$$\bar{X}_0 = \mathbb{P}_{g-1} = \operatorname{Proj} SH^0(D, \mathcal{O}_D(D))^* \hookrightarrow \mathbb{P}^{N-1}$$

$$D_v \mapsto 2D_v.$$

That is, $\bar{X}_0$ is the projectivized Zariski tangent space to X at 0.

Let us consider the divisor $Y \hookrightarrow C \times D$ constructed in 3.1. One defines the Cartier divisor $\bar{Y} \hookrightarrow C \times C \times D$ as $\bar{Y} = \pi_1^{-1}(Y) + \pi_2^{-1}(Y)$, where $\pi_i: C \times C \times D \to C \times D$ $(i = 1, 2)$ are the two natural projections.

Since by construction $\bar{Y}$ is a divisor of the linear series ω_D^2 'parametrized' by $C \times C$, it will define a morphism:

$$\psi: C \times C \to \mathbb{P}^{N-1}$$

That is, ψ is the morphism induced by the homomorphism of vector spaces $H^0(D, \omega_0) \otimes_k H^0(D, \omega_D) \to S^2H^0(D, \omega_D) \hookrightarrow H^0(D, \mathcal{O}_D(2D))$. We shall designate by S the scheme-theoretic image of ψ (obviously, S is a surface isomorphic to S^2C).

THEOREM 3.5. *Let (X, D) be a p.p. abelian variety of dimension $\geqslant 3$. One has*:

1. *(X, D) is the Jacobian of a non-hyperelliptic curve if and only if there exists an irreducible curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $2g-2$ (non degenerate and projectively normal) such that the surface S of $\mathbb{P}^{N-1}$ is contained in $\bar{X}$.*
2. *(X, D) is the Jacobian of an hyperelliptic curve if and only if there exists an irreducible curve $C \hookrightarrow \mathbb{P}_{g-1}$ of degree $g-1$ (non degenerate) such that the surface S is contained in $\bar{X}$.*

Proof. If (X, D) is the Jacobian of a smooth algebraic curve, it satisfies the conditions of the theorem by Section 2.

Reciprocally, let us assume that (X, D) fulfills the conditions of the statement. The theorem will follow by proving that the hypothesis of Theorem 3.1. is also fulfilled.

If $S \subset \overline{X}$, given two geometric points $p, q \in C$, there exists a point $x \in X$ such that:

$$Y_p + Y_q = D(x) \cap D + D(-x) \cap D \qquad (*)$$

However, if Y_p and Y_q were irreducible, we would have $Y_p = D(x) \cap D$, $Y_q = D(-x) \cap D$, but this is only possible if $2x = 0$ since Y_p and Y_q are invariant by the involution $(-1)_X$.

Hence we will have decomposition $Y_p = Y_p' + Y_p''$, $Y_q = Y_q' + Y_q''$ such that $(-1)_X(Y_p') = Y_p''$ for every geometric point $p \in C$, and the proof of 2. concludes by 3.1.

The proof of 1. will follow by proving that the decomposition $Y_p = Y_p' + Y_p''$ (such that $(-1)_X Y_p' = Y_p''$) also holds for a generic point p of C.

If this decomposition were not fulfilled for a generic point p, the curve $\overline{C} = \pi^{-1}(C_0) \subset X$ (C_0 being the curve $\psi(C \times p_0)$ for a geometric point $p_0 \in C$) would be an irreducible covering, $\overline{C} \xrightarrow{\pi} C_0$, of degree 2. Now, as at the end of proof of Theorem 3.1, one would have $2D^{(g-1)} \equiv (g-1)!\overline{C}$ and, with the notations of [Hoyt, 1963], $\alpha(\overline{C}, D) = 2\delta_X$. This would imply the existence of an isogeny from the Jacobian of $\overline{C}$ onto $\overline{X}$, which is absurd the geometric genus of C being $\geqslant g-1$.

Remark 3.6. Theorems 3.1., 3.4., 3.5. can be slightly modified to characterize the p.p. abelian varieties which are products of jacobians. This is obtained, roughly speaking, by removing the irreducibility conditions and imposing to the components smiliar conditions to those of the present statement, and aplying Hoyt's generalization of Matsusaka's criterion [Hoyt, 1963].

Remark 3.7. If the base field is $\mathbb{C}$, Theorems 3.1., 3.4., 3.5. are still true if we remove the conditions on the degree of the curve C. This can be proved with the same arguments by applying Gunning's results [Gunning, 1982].

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