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Fields of definition of algebraic varieties in characteristic zero

ALEXANDRU BUIUM
Department of Mathematics, INCREST, Bdul Păcii 220, 79622 Bucharest, Rumania

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1. Introduction

Let K be an algebraically closed field of characteristic zero and V a K-variety (by this we mean an irreducible reduced quasiprojective K-scheme). A subfield K_1 of K will be called a field of definition for V if there exists a K_1-variety V_1 such that V is K-isomorphic to $V_1 \otimes_{K_1} K$. The aim of this paper is to show how one can compute fields of definition for V with the help of derivations on the function field $K(V)$ of V (here a derivation δ on a field L means a $\mathbb{Q}$-linear map $\delta : L \to L$ such that $\delta(\lambda_1\lambda_2) = \lambda_1\delta\lambda_2 + \lambda_2\delta\lambda_1$ for all $\lambda_1, \lambda_2 \in L$).

For any set Δ of derivations on $K(V)$ define

$$K^\Delta = \{\lambda \in K;\ \delta\lambda = 0 \text{ for all } \delta \in \Delta\}.$$

Clearly K^Δ is an algebraically closed subfield of K. A special role will be played by the set $\Delta(V)$ of all derivations δ on $K(V)$ which are integral on V in the sense that $\delta(\mathcal{O}_{V,p}) \subset \mathcal{O}_{V,p}$ for all $p \in V$ (here $\mathcal{O}_{V,p}$ denotes the local ring of V at p). Indeed our main result is the following:

THEOREM 1: *Suppose V is smooth and projective. Then $K^{\Delta(V)}$ is a field of definition for V and any other algebraically closed field of definition for V must contain $K^{\Delta(V)}$.*

The following alternative description of $\Delta(V)$ will be useful: $\Delta(V) = H^0(V, T_{V/\mathbb{Q}})$ (where for any scheme W over a field L we denote by $T_{W/L}$ the sheaf $\mathrm{Hom}_{\mathcal{O}_W}(\Omega_{W/\mathrm{Spec}(L)}, \mathcal{O}_W)$ of L-derivations from $\mathcal{O}_W$ into $\mathcal{O}_W$; if $W = \mathrm{Spec}(A)$ we shall write $T_{A/L} = H^0(W, T_{W/L})$).

Theorem 1 will be proved in Section 3.

In Section 4 we shall discuss the possibility of extending Theorem 1 to singular and to open varieties. We would like to note that in the case of open varieties the right substitute for $\Delta(V)$ will be the set $\Delta(V, \log)$ of all 'logarithmic' (instead of 'integral') derivations (see Section 4 for precise definitions and results).

In Section 5 we shall discuss the problem of finding the smallest algebraically closed 'field of definition' for a complete local ring (again we send to Section 5 for definitions and results).

The main motivation for our work concerns algebraic differential equations without movable singularities (cf. [Matsuda, 1980; Buium, 1984]). More precisely Theorem 1 may be taken as a starting point for a generalisation of the 'one variable theory' from [Matsuda, 1980] to the case of several variables (see [Buium, 1984] for the case of two variables). We shall achieve this program in a separate paper (Buium, in prep.).

Our proof of Theorem 1 is not purely algebro-geometric it will involve a 'reduction to the complex field $\mathbb{C}$'. Then the main step towards Theorem 1 will be the following result which has an interest in itself and which will be proved in Section 2:

THEOREM 2: *Let $f: X \to S$ be a smooth projective morphism with connected fibres, between smooth $\mathbb{C}$-varieties. Then there is a diagram with cartesian squares (Fig. 1),*

$$\begin{array}{ccccc} X & \leftarrow & X' & \rightarrow & X'' \\ f\downarrow & & \downarrow & & \downarrow f'' \\ S & \overset{\alpha}{\leftarrow} & S' & \overset{\beta}{\rightarrow} & S'' \end{array}$$

Fig. 1.

such that β is a surjective map of $\mathbb{C}$-varieties, S'' is smooth, α is an étale covering of a Zariski open set of S, f'' is a smooth projective morphism and for any $t \in S''$ the Kodaira-Spencer map

$$\rho_t : T_t S'' \to H^1\left(X_t'', T_{X_t''/\mathbb{C}}\right)$$

is injective (where $T_t S''$ = tangent space of S'' at t, $X_t'' = (f'')^{-1}(t)$, $T_{X_t''/\mathbb{C}}$ = tangent bundle of X_t'').

We would like to note that Theorem 2 was proved in ([Viehweg, 1983], p. 574) under a very restrictive assumption on the local Torelli map of f at the generic point of S.

2. Proof of Theorem 2

In this section we prove Theorem 2. Points of $\mathbb{C}$-varieties will always mean closed points. Choose an invertible sheaf $\mathscr{L}$ on X which is ample relative to f, put $\mathscr{L}_t = \mathscr{L}|_{X_t}$ $(X_t = f^{-1}(t))$ and let $\lambda_t \in \mathrm{Pic}(X_t)/\mathrm{Pic}^{\tau}(X_t)$ be the class of $\mathscr{L}_t$ modulo numerical equivalence.

Claim 1

The set

$$R=\{(t,\ s)\in S\times S;\ (X_t,\ \lambda_t)\simeq(X_s,\ \lambda_s)\}$$

is constructible in $S\times S$ (note that if no X_t was ruled then R would be Zariski closed in $S\times S$; this follows from [Matsusaka, 1968]).

An argument for this goes as follows. Let $p_i: Z=S\times S\to S$, $i=1, 2$, be the canonical projections and let $Y_i\to Z$ be obtained from $X\to S$ by base change with p_i. Let U be the Z-scheme representing the functor $Z'\to \mathrm{Isom}_{Z'}(Y_1\times_Z Z',\ Y_2\times_Z Z')$ [Grothendieck, 1957–1962]; recall that U is a countable disjoint union of Z-schemes U_n of finite type. Let $\mathscr{L}_i$ be the pull-back of $\mathscr{L}$ on $V_i=Y_i\times_Z U$ and let $F: V_1\to V_2$ be the universal isomorphism. Clearly the sets

$$U_n'=\left\{u\in U_n;\ \left((F^*\mathscr{L}_2)\otimes\mathscr{L}_1^{-1}\right)_u\equiv 0\right\}$$

are open in U_n (here '$\equiv$' denotes the numerical equivalence) and we have $R=\mathrm{Im}(U'\to Z)$ where U' is the union of all U_n' for $n\geqslant 1$. So, by Chevalley's constructibility theorem, we shall be done if we prove that U_n' are empty for all except a finite number of n's. Now for any $u\in U'$ let $z(u)=(t(u),\ s(u))$ denote the image of u under $U\to Z$ and let $\Gamma_u\subset X_{t(u)}\times X_{s(u)}$ be the graph of the corresponding isomorphism which we denote also by $u: X_{t(u)}\to X_{s(u)}$. Consider on $Y_1\times_Z Y_2$ the sheaf $q_1^*\mathscr{L}\otimes q_2^*\mathscr{L}(q_i: Y_i\to X$ being the canonical projections); this sheaf is ample relative to Z and denote by $\mathcal{O}_{\Gamma_u}(1)$ its restricton to Γ_u. Now if $1\times u: X_{t(u)}\to\Gamma_u\subset X_{t(u)}\times X_{s(u)}$ is the graph map then:

$$(1\times u)^*\left(\mathcal{O}_{\Gamma_u}(m)\right)=\mathscr{L}_{t(u)}^m\otimes u^*\left(\mathscr{L}_{s(u)}^m\right)\equiv\mathscr{L}_{t(u)}^{2m}.$$

Hence the Hilbert polynomial $m\to\chi(\Gamma_u,\ \mathcal{O}_{\Gamma_u}(m))$ equals to a polynomial $m\to\chi(X_{t(u)},\ \mathscr{L}_{t(u)}^{2m})$ which does not depend on u. This implies that U_n' is empty for sufficiently big n.

Claim 2

Replacing S by a Zariski open subset of it we may suppose there exists a morphism $\psi: S\to M$ into a $\mathbb{C}$-variety M such that for any $s\in S$ we have

$$\psi^{-1}\psi(s)=\{t\in S;\ (s,\ t)\in R\}.$$

This can be done by standard manipulation of Chow varieties (see Rosenlicht, 1956] p. 406 for similar arguments). The idea is to embed S as a locally closed subset of a projective space $\mathbb{P}$ and to take the Zariski closure $\bar{R}$ of R in $\mathbb{P} \times S$; by Claim 1, for each irreducible component $\bar{R}_j$ of $\bar{R}$ the projection $\bar{R}_j \to S$ will give a family of cycles of codimension m_j and degree d_j in $\mathbb{P}$ (m_j, d_j being some integers) and hence a rational map from S to the corresponding Chow variety $C(m_j, d_j)$. Using constructibility of R one can make an elementary analysis showing that, after shrinking S in the Zariski topology, the resulting morphism

$$\psi : S \to \prod_j C(m_j, d_j) \to \prod_q C\left(q, \sum_{m_j = q} d_j\right)$$

has the property required in Claim 2.

Claim 3

Replacing S by an étale open set of it one can find a morphism $\eta : S \to N$ onto a variety N such that η has a section and such that for any $t \in S$ the set

$$S_t = \{ s \in S;\ X_s \simeq X_t \}$$

is a union of at most countably many fibres of η.

Indeed, since the set of classes of numerically equivalent divisors on a fixed variety is countable, S_t is a union of at most countably many fibres of the map ψ from Claim 2. Now we are done by replacing M by an étale open set N of $\psi(S)$ and replacing S by $S \times_M N$.

Claim 4

We may suppose in Claim 3 that in addition there exists a smooth projective morphism $g : Y \to N$ such that X is S-isomorphic to $Y \times \cdots {}_N S$; in particular we shall have that for any $u \in N$ the set

$$N_u = \{ v \in N;\ Y_v \simeq Y_u \}$$

is at most countable.

The argument in this step is similar to the one in ([Viehweg, 1983], p. 576). Take $\gamma : N \to T \subset S$ a section of $S \to N$, put $X_T = X \times_S T$, $X_N = X_T \times_T N$, $X' = X_N \times_N S$. Then for any $t \in S$, the fibres of $X \to S$ and $X' \to S$ above t are isomorphic; this means that the S-scheme $U = U_1 \cup U_2 \cup \ldots$ representing

$$S' \to \mathrm{Isom}_{S'}(X \times_S S',\ X' \times_S S')$$

maps onto S. By Baire's theorem there is at least a finite type piece U_n of U

dominating S. Now we are done by replacing S by some covering of a locally closed irreducible subscheme of U_n which is étale over S, and by putting $Y = X_N$.

Claim 5

For any t in a Zariski open set of N (notations being as in Claim 4) the Kodaira-Spencer map ρ_t associated to $g: Y \to N$ at t is injective (this will of course close the proof of Theorem 2!).

Indeed if the morphism $\rho: T_{N/\mathbb{C}} \to R^1 g_*(T_{Y/N})$ is injective at the generic point of N we are done. If not, we may choose, after shrinking N in the Zariski topology, a line bundle L contained in Ker(ρ). By Frobenius there is a germ of analytic curve C whose analytic tangent bundle T_C equals to the restriction of L to C. By ([Kodaira and Spencer, 1958], 6.2) the family $Y \times_N C \to C$ must be analytically locally trivial, contradicting Claim 4 which states that N_u is at most countable for $u \in N$.

3. Proof of Theorem 1

The fact that any algebraically closed field of definition K_1 for V contains $K^{\Delta(V)}$ is quite easy and general (it does not require smoothness or projectivity of V). Indeed it will be sufficient to prove that any K_1-derivation θ on K must vanish on $K^{\Delta(V)}$. But if $V \simeq V_1 \otimes K_1 K$ (V_1 being some K_1-variety) we see that θ extends to a derivation $\delta: K(V) \to K(V)$ defined by

$$\delta(\lambda \otimes y) = \lambda \otimes (\theta y) \quad \text{for all } \lambda \in K_1(V_1), \quad y \in K.$$

Now δ is integral on V, hence will vanish on $K^{\Delta(V)}$ and we are done. So in the remainder of this section we concentrate ourselves on proving that $K^{\Delta(V)}$ is a field of definition for V. This is of course equivalent to proving that K^{Δ} is a field of definition for V whenever Δ is a subset of $\Delta(V)$.

We assume first that K^{Δ} is uncountable. Consequently K^{Δ} will contain a subfield k which is isomorphic to $\mathbb{C}$. One can easily construct a smooth projective morphism of k-varieties $f: X \to S$ such that the function field $k(S)$ of S is contained in K and $V \simeq X \times_S \text{Spec}(K)$. Apply Theorem 2 to f and put $K' = k(S')$, $K'' = k(S'')$. Since K' is a finite extension of $k(S)$, there is an embedding $K' \to K$ extending the inclusion $k(S) \to K$. Put $V'' = X'' \times_{S''} \text{Spec}(K'')$. We have a field extension $K'' \to K' \to K$ and V is K-isomorphic to $V'' \otimes_{K''} K$ so we shall be done if we prove that K^{Δ} contains K''. Now there is standard exact sequence [Grothendieck, 1964] Ch. 0, 20.5.7:

$$0 \to \pi^* \Omega_{K/k} \to \Omega_{V/k} \to \Omega_{V/K} \to 0 \qquad (*)$$

where $\pi: V \to \mathrm{Spec}(K)$ is the canonical structure morphism. A similar sequence exists for $V'' \to \mathrm{Spec}(K'')$. These sequences plus the injectivity of the Kodaira-Spencer maps associated to f'' at the points of S'' yeld a diagram with exact rows and columns (Fig. 2).

$$\begin{array}{ccccc} & & T_{K/K''} & & \\ & & \downarrow \psi & & \\ H^0(V, T_{V/k}) & \overset{\varphi}{\to} & T_{K/k} & \longrightarrow & H^1(V, T_{V/K}) \\ & & \downarrow & & \| \\ & 0 \to & T_{K''/k} \otimes_{K''} K & \to & H^1(V, T_{V''/K''} \otimes_{K''} K) \end{array}$$

Fig. 2.

A diagram chase shows that φ and ψ have the same image in $T_{K/k}$. Since Δ is a subset of $H^0(V, T_{V/k})$ we get in particular that $K'' \subset K^\Delta$.

Theorem 1 is proved in the case K^Δ uncountable.

Suppose now K^Δ is countable. Then there is an embedding $K^\Delta \to \mathbb{C}$; the ring $K \otimes_{K^\Delta} \mathbb{C}$ will be a domain and denote by L its field of quotients.

Now it is easy to see (use the exact sequence $(*)$ with $k = \mathbb{Q}$) that for any $\delta \in \Delta$ we have $\delta(K) \subset K$ so one can define a derivation δ' on L by the formula

$$\delta'(\lambda \otimes y) = (\delta\lambda) \otimes y \quad \text{for all } \lambda \in K \quad \text{and } y \in \mathbb{C}.$$

Moreover one can define a derivation δ'' on $L(V \otimes_K L)$ by the formula

$$\delta''(u \otimes v) = (\delta u) \otimes v + u \otimes (\delta' v) \quad \text{for all } u \in K(V), \quad v \in L.$$

Clearly δ'' is integral on $V \otimes_K L$ and let Δ'' be the set of all such δ'' as δ runs through Δ. Now $L^{\Delta''}$ contains $1 \otimes \mathbb{C}$ hence it is uncountable so by the first part of our proof $L^{\Delta''}$ is a field of definition for $V \otimes_K L$. We have four fields (Fig. 3).

$$\begin{array}{ccc} K^\Delta & \hookrightarrow & K \\ \downarrow & & \downarrow \\ L^{\Delta''} & \hookrightarrow & L \end{array}$$

Fig. 3.

and note that K and $L^{\Delta''}$ are linearly disjoint over K^Δ (this may be proved exactly as in [Kolchin, 1973], p. 87 using the Wronskian argument). So we shall be done if we prove the following general fact:

LEMMA 1: *Let V be a smooth projective K-variety and let K_0, K_1 and K_2 be algebraically closed subfields of K such that (Fig. 4)*

$$\begin{array}{ccc} K_0 & \hookrightarrow & K_1 \\ \downarrow & & \downarrow \\ K_2 & \hookrightarrow & K \end{array}$$

Fig. 4.

and such that K_1 and K_2 are linearly disjoint over K_0.

Suppose K_1 and K_2 are fields of definition for V. Then K_0 is also a field of definition for V.

Proof. Choose an ample $\mathscr{L} \in \mathrm{Pic}(V)$. Suppose V is K-isomorphic to $V_i \otimes_{K_i} K$, $i = 1, 2$. Then there exists $\mathscr{L}_i \in \mathrm{Pic}(V_i)$ such that $\mathscr{L}_i \otimes_{K_i} K \equiv \mathscr{L}$; clearly $\mathscr{L}_i$ are still ample. One can find projective morphisms $f_i : X_i \to S_i$ of K_0-varieties such that $K_0(S_i) \subset K_i$, V_i, is K_i-isomorphic to $X_i \times_{S_i} \mathrm{Spec}(K_i)$ and such that $\mathscr{L}_i$ is the pull back of some $\mathscr{M}_i \in \mathrm{Pic}(X_i)$ with $\mathscr{M}_i$ ample relative to f_i. Put $T = S_1 \times S_2$, $Y_i = X_i \times_{S_i} T$. By linear disjointness of K_1 and K_2 over K_0 the morphism $K_1 \otimes_{K_0} K_2 \to K$ is injective, hence $\mathrm{Spec}(K) \to T$ is dominant. Since $Y_1 \times_T K$ is K-isomorphic to $Y_2 \times_T K$, it follows that $\mathrm{Spec}(K) \to T$ factors through some finite type component U_n of the object U representing the functor $T' \to \mathrm{Isom}_{T'}(Y_1 \times_T T', Y_2 \times_T T')$. But since the isomorphism $Y_1 \times_T K \simeq Y_2 \times_T K$ preserves the polarisations induced by $\mathscr{M}_1$ and $\mathscr{M}_2$ we conclude that the image of $\mathrm{Spec}(K) \to U_n$ is contained in $U_n' = U' \cap U_n$ where U' is the closed subset of U whose geometric points are precisely those points for which the corresponding isomorphism preserves polarisations (see the proof of Claim 1 in Section 2).

Now the image of $U_n' \to T$ contains an open subset T_0 of T in other words for any $(s_1, s_2) \in T_0$ the fibres of $Y_1 \to T$ and $Y_2 \to T$ above (s_1, s_2) are isomorphic as polarized varieties. But these fibres identify with $f_1^{-1}(s_1)$ and $f_2^{-1}(s_2)$ respectively with polarisations given by $\mathscr{M}_1$, $\mathscr{M}_2$. Now fix $(s_1^0, s_2^0) \in T_0$ and put $S_2' = \{s_2 \in S_2;\ (s_1^0, s_2) \in T_0\}$; then $X_2' := X_2 \times_{S_2} S_2' \to S_2'$ has all its closed fibres isomorphic as polarized varieties (with polarisation given by $\mathscr{M}_2$). Let X_2'' be $F \times S_2'$, $F = f_2^{-1}(s_2^0)$ and let H be the object representing the functor $B \to \mathrm{Isom}_B(X_2' \times_{S_2'} B, X_2'' \times_{S_2'} B)$. Then let H' be the closed subset of H whose geometric points correspond to those isomorphisms which preserve polarisations (we take on $X_2'' \to S_2'$ the polarisation induced from that of F). As noted in Claim 1, Section 2, H' is of finite type over S_2' (and not only locally of finite type). Since the map $H' \to S_2'$ is surjective, we can find a component of H' dominating S_2' and hence an étale map $\tilde{S}_2 \to S_2'$ such that $\tilde{X}_2 = X_2' \times_{S_2'} \tilde{S}_2 \to \tilde{S}_2$ is $\tilde{S}_2$-isomorphic to $\tilde{S}_2 \times_{K_0} F$. Since K is algebraically closed we may embed $K_0(\tilde{S}_2)$ in K and we get

$$V \simeq X_2 \times_{S_2} K = \tilde{X}_2 \times_{\tilde{S}_2} K = F \otimes_{K_0} K \quad (\text{over } K)$$

4. Singular varieties and open varieties

A general strategy of treating singular varieties and open varieties is to treat first pairs consisting of a smooth projective variety plus an effective divisor (sometimes supposed with normal crossings). As a general principle too, global objects have to be replaced by objects with a logarithmic behaviour along the divisor.

This is precisely what we shall do now; namely we shall give a variant of our theory from §§1–3 for pairs (V, D) where V is a smooth projective K-variety (K being as usual algebraically closed of characteristic zero) and D is an effective Cartier divisor on V. A subfield K_1 of K will be called a field of definition for (V, D) if there exists a K_1-variety V_1, a divisor D_1 on V_1 and a K-isomorphism $V \simeq V_1 \otimes_{K_1} K$ such that $q^* D_1 = D$ where $q: V \to V_1$ is the projection. Clearly if K_1 is a field of definition for (V, D) it is also a field of definition for the open variety $V \setminus D$. Now for (V, D) as above we say that a derivation δ on $K(V)$ is logarithmic on (V, D) if it is integral on V and if for any $p \in V$ and any local equation $f \in \mathcal{O}_{V,p}$ of D at p we have

$$f^{-1}\delta f \in \mathcal{O}_{V,p}$$

(this is the same as to say that δ takes the ideal sheaf $\mathcal{O}_V(-D)$ into itself!). Denote by $\Delta(V, D)$ the set of logarithmic derivations on (V, D); note that $\Delta(V, D) \subset \Delta(V)$ and that $\Delta(V, D_1) = \Delta(V, D_2)$ provided D_1 and D_2 have the same support; this follows from the fact that primes associated to differential ideals in a differential ring are differential ([Matsumura, 1982], p. 232).

Now denote by $T_{V/K}$ (log D) the subsheaf of the tangent sheaf $T_{V/K}$ of V consisting of those derivations which take $\mathcal{O}_X(-D)$ into itself (see also [Kawamata, 1978]).

The following Theorem reduces to Theorem 1 if $D = 0$.

THEOREM 3: *Let V be a smooth projective K-variety and D an effective divisor on V. Suppose the injective map*

$$H^0\big(V, T_{V/K}(\log D)\big) \to H^0\big(V, T_{V/K}\big)$$

it also surjective. Then $K^{\Delta(V,D)}$ *is the smallest algebraically closed field of definition for* (V, D).

Note that the surjectivity of the map above occurs in each of the following cases:

a) $D = 0$.
b) $H^0(V, T_{V/K}) = 0$.
c) $D = \Sigma D_i$, D_i are smooth subvarieties of V crossing normally and $H^0(D_i, N_{D_i}) = 0$ (where N_{D_i} is the normal sheaf of D_i). Indeed in this case the cokernel of the map from Theorem 3 injects into $\bigoplus_i H^0(D_i, N_{D_i})$ (cf. [Kawamata, 1978]).

Proof of Theorem 3. The only non-trivial fact to prove it that $K_0 = K^{\Delta(V,D)}$ is a field of definition for (V, D). Since $K^{\Delta(V)} \subset K^{\Delta(V,D)}$ we get by Theorem 1 that K_0 is a field of definition for V i.e. V is K-isomorphic to $V \otimes_{K_0} K$ for some V_0. For any $\delta \in \Delta(V, D)$ we have $\delta(K) \subset K$ so we may consider the derivation $\delta^* \in \Delta(V)$ defined by

$$\delta^*(\lambda \otimes y) = \lambda \otimes \delta y \quad \text{for all } \lambda \in K_0(V_0), \quad y \in K.$$

Then $\delta - \delta^* \in H^0(V, T_{V/k})$. By hypothesis $(\delta - \delta^*)(\mathcal{O}_V(-D)) \subset \mathcal{O}_V(-D)$. Since $\delta(\mathcal{O}_V(-D)) \subset \mathcal{O}_V(-D)$ we get $\delta^*(\mathcal{O}_V(-D)) \subset \mathcal{O}_V(-D)$. Now we may conclude by the following general:

LEMMA 2: *Let K be a field, Δ a set of derivations on K, $K_0 = \{\lambda \in K, \delta\lambda = 0$ for all $\delta \in \Delta\}$ and let A_0 be a K_0-algebra. Put $A = A_0 \otimes_{K_0} K$ and define for any $\delta \in \Delta$ a derivation $\delta^*: A \to A$ by the rule $\delta^*(\lambda \otimes y) = \lambda \otimes \delta y$ for all $\lambda \in A_0$, $y \in K$. Suppose I is an ideal in A such that $\delta^*(I) \subset I$ for all $\delta \in \Delta$. Then $I = I_0 \otimes_{K_0} K$ for some ideal I_0 in A_0.*

Proof. Put $I_0 = I \cap A_0$ and $J = I_0 \otimes_{K_0} K$. Suppose $I \backslash J \neq \emptyset$. Let $(e_k)_k$ be a basis of A_0 as a K_0-vector space and take an element $a = \Sigma e_k \otimes a_k \in I \backslash J$ $(a_k \in K)$ for which the number

$$\#\{k;\ a_k \neq 0\}$$

is minimal. We may of course assume there is an index k_0 such that $a_{k_0} = 1$. Now for all $\delta \in \Delta$,

$$\sum e_k \otimes \delta a_k = \delta^*\left(\sum e_k \otimes a_k\right) \in I$$

so by minimality of a we have that $\Sigma e_k \otimes \delta a_k \in J$. Since $a \notin J$ there is at least an index k_1 and there is a derivation $\delta \in \Delta$ such that $\delta a_{k_1} \neq 0$. By minimality of a we get that

$$a - a_{k_1}(\delta a_{k_1})^{-1}\left(\sum e_k \otimes \delta a_k\right) \in J$$

from which we get $a \in J$, contradiction. The lemma is proved.

Using Theorem 3 we shall prove the following:

THEOREM 4: *Let V be a normal projective K-variety of dimension two. Then $K^{\Delta(V)}$ is the smallest algebraically closed field of definition for V.*

Proof. Let $f: W \to V$ be Zariski's canonical resolution; so f is obtained as a composition $W = V_n \to V_{n-1} \to \cdots \to V_1 = V$ where V_i is obtained from V_{i-1} by first normalizing V_{i-1} and then blowing up the (reduced) ideal of the singular locus Σ_{i-1} of $(V_{i-1})^{nor}$. By a theorem of [Seidenberg, 1966] $\Delta(V_{i-1}) \subset \Delta((V_{i-1})^{nor})$. By another theorem of Seidenberg ([Matsumura, 1982], p. 233) for any $y \in \Sigma_{i-1}$ and for any $\delta \in \Delta((V_{i-1})^{nor})$ we have $\delta(m_y) \subset m_y$ (here m_y = maximal ideal of $\mathcal{O}_y$). An elementary local computation shows then that $\Delta(V_{i-1})^{nor}) \subset \Delta(V_i)$. So after all we deduce that $\Delta(V) \subset \Delta(W)$. Put $D = f^{-1}(\Sigma_1)$ set-theoretically; then D is the support of a reduced divisor which we still call D. Since $W \setminus D \simeq V \setminus \Sigma_1$ we immediately get that $\Delta(W) \subset \Delta(V)$ so we get $\Delta(V) = \Delta(W)$. We claim that $\Delta(W) = \Delta(W, D)$.

Indeed if $\delta \in \Delta(W)$ then $\delta \in \Delta(V)$ so by Seinberg's theorem $\delta(m_y) \subset m_y$ for all $y \in \Sigma_1$. Consequently $\delta(m_y \mathcal{O}_W) \subset m_y \mathcal{O}_W$. We conclude using the fact that the radical of a differential ideal in a differential ring is still a differential ideal ([Matsumura, 1982], p. 232). Now the equality $\Delta(W) = \Delta(W, D)$ implies in particular that the map $H^0(W, T_{W/K}(\log D)) \to H^0(W, T_{W/K})$ is an isomorphism. Applying Theorem 3 we get that $K_0 = K^{\Delta(V)}$ is a field of definition for (W, D) so there is a smooth projective K_0-variety W_0 such that $W \simeq W_0 \otimes_{K_0} K$ and there is a divisor D_0 on W_0 with $D = q^* D_0$, $(q : W \to W_0)$. Then we claim that there is a birational morphism $f_0 : W_0 \to V_0$ onto a normal surface V_0 which is an isomorphism above $V \setminus f_0(D_0)$ and such that $f_0(D_0)$ is a finite set.

Indeed there exist projective morphisms $f_S : X \to Y$, $g : X \to S$, $h : Y \to S$, $g = f_S h$ where g and h are projective, S is an affine algebraic K_0-scheme with $K_0(S) \subset K$ and $f_S \times_S \operatorname{Spec}(K) : X \times_S \operatorname{Spec}(K) \to Y \times_S \operatorname{Spec}(K)$ identifies with $f : W \to V$. Then the desired $f_0 : W_0 \to V_0$ may be obtained by taking the morphism $g^{-1}(s) \to (h^{-1}(s))^{nor}$ induced from f_S where $s \in S$ is a sufficiently general K_0-point of S. Now it is easy to see that V is K-isomorphic to $V_0 \otimes_{K_0} K$ and we are done.

The following seems quite plausible:

CONJECTURE 1: *If V is a normal projective K-variety then $K^{\Delta(V)}$ is the smallest algebraically closed field of definition for V.*

Now we close by discussing the case of open non-singular varieties. Let U be a non-singular K-variety. By a compactification of U we mean a triple (V, D, φ) with V non-singular and projective, D a divisor on V and φ a K-isomorphism $U \simeq V \setminus D$.

For any such compactification, $\Delta(V, D)$ identifies via φ with a set of derivations on $K(U)$. Define

$$\Delta(U, \log) = \bigcup \Delta(V, D)$$

the union being taken after all possible compactifacations (V, D, φ) of U. It is easy to see that $K^{\Delta(U, \log)}$ is contained in any algebraically closed field of definition for U. We hope the following to be true:

CONJECTURE 2: *If U is a non-singular K-variety, $K^{\Delta(U, \log)}$ is the smallest algebraically closed field of definition for U.*

We can prove Conjecture 2 in various special cases. For instance:

THEOREM 5: *Conjecture 2 holds in any of the following cases:*
1) *U is an affine curve.*
2) *U is an affine surface of general type.*

To prove Theorem 5 we need some preparation.

We say that $(V_1, D_1, \varphi_1) \leqslant (V_2, D_2, \varphi_2)$ for two compactifications of U if the rational map $\varphi_1\varphi_2^{-1}: V_2 \to V_1$ is everywhere defined. It is easy to see that in this situation $\Delta(V_2, D_2) \subset \Delta(V_1, D_1)$ as subsets in $\Delta(U)$. So if the set of compactifications of U has a smallest element (V_1, D_1, φ_1) we have

$$\Delta(U, \log) = \Delta(V_1, D_1).$$

Note that a smallest element as above does not necessarily exist (compare with [Kawamata, 1978]).

Now for a smooth projective K-variety V, let $\sigma: V \to V$ be a K-automorphism and let $\sigma^*: K(V) \to K(V)$ the corresponding K-automorphism of $K(V)$. Take D an efffective divisor on V. Furthermore consider a set Δ of derivations on $K(V)$. Denote by Δ^σ the set $\{(\sigma^*)^{-1}\delta\sigma^*;\ \delta \in \Delta\}$. Then it is easy to check that:
a) $K^\Delta = K^{\Delta^\sigma}$
b) $\Delta(V, D))^\sigma = \Delta(V, \sigma(D))$.
In particular $K^{\Delta(V,D)}$ is a field of definition for (V, D) if and only if $K^{\Delta(V,\sigma(D))}$ is a field of definition for $(V, \sigma(D))$.

Now let's start the proof of Theorem 5.

Proof. Suppose U is an affine curve.

In this case there is essentially a unique compactification (V, D, φ) with D reduced so $\Delta(U, \log) = \Delta(V, D)$.

Put g = genus of V. If $g \geqslant 2$, $H^0(V, T_{V/K}) = 0$ and we conclude by Theorem 3. Suppose $g = 1$.

Put $K_0 = K^{\Delta(V,D)}$; by Theorem 1, there is a K-isomorphism $V \simeq V_0 \otimes_{K_0} K$ with V_0 an elliptic curve over K_0. Let $p_0 \in V_0(K_0)$ be a K_0-point of V_0 and $p \in V(K)$ the unique K-point of V lying over p_0. By transitivity of $\mathrm{Aut}_K(V)$ on V and by the preparation above, we may suppose $p \in D$. For any $\delta \in \Delta(V, D)$ let $\delta^* \in \Delta(V)$ be the derivation defined as in the proof of Theorem 3 (so $\delta^*(\lambda \otimes y) = \lambda \otimes \delta y$ for $\lambda \in K_0(V_0)$, $y \in K$). Since $\delta - \delta^* \in H^0(V, T_{V/K}) = H^0(V_0, T_{V_0/K_0}) \otimes_{K_0} K$ we get $\delta - \delta^* = f\theta$ with $f \in K$, $\theta = a$ generator of $H^0(V_0, T_{V_0/K_0})$. Now if t is a parameter of the maximal ideal m_{p_0} of $\mathcal{O}_{V_0,p_0}$ then $\theta t \notin m_{p_0}$. On the other hand $\delta^*(m_p) \subset m_p$ because $m_p = m_{p_0} \otimes K$ hence $f\theta(m_p) \subset m_p$. In particular $\theta t \otimes f = f\theta(t \otimes 1) \in m_{p_0} \otimes K$ which implies $f = 0$, hence $\delta = \delta^*$. Now we may conclude by Lemma 2.

Suppose now $g = 0$. If $\#D \leqslant 3$, $\mathbb{Q}$ is a field of definition for $(\mathbb{P}^1_K, D)$ and we are done. Suppose $\#D \geqslant 4$ and take $p_1, p_2, p_3 \in D$. Since $\mathrm{Aut}_K(\mathbb{P}^1_K)$ is triply transitive we may assume that each p_i ($i = 1, 2, 3$) lies over a K_0-point p_i^0 of $\mathbb{P}^1_{K_0}$ ($K_0 = K^{\Delta(V,D)}$). For any $\delta \in \Delta(V, D)$ define δ^* as above; then we have $\delta - \delta^* = a_0\theta_0 + a_1\theta_1 + a_2\theta_2$ with $a_0, a_1, a_2 \in K$ and $\theta_0, \theta_1, \theta_2 \in H^0(\mathbb{P}^1_{K_0}, T_{\mathbb{P}^1_{K_0}/K_0})$,

$$\theta_0 t = 1$$

$$\theta_1 t = t$$

$$\theta_2 t = t^2$$

where $\mathbb{P}^1_{K_0} = \operatorname{Proj} K_0[t_0, t_1]$, $t = t_1/t_0$. Once again $(\delta - \delta^*)(m_{p_i}) \subset m_{p_i}$ and if $m_{p_i} = (t - \lambda_i)$ for $\lambda_i \in K_0$ we get

$$a_0 + a_1\lambda_i + a_2\lambda_i^2 = 0 \quad \text{for } i = 1, 2, 3.$$

This implies $a_0 = a_1 = a_2$ and we conclude again by Lemma 2.

We would like to note that in a similar vein but using some additional tricks one can treat complements of divisors in projective spaces and abelian varieties of dimension $\geqslant 2$ (cf. Buium, in prep.).

Let's consider the case when U is as in 2) and embed U in a smooth projective surface V. Contracting succesively the exceptional curves of the first kind in $V \setminus U$ we may suppose $V \setminus U$ does not contain such curves.

Since U is affine, $D = V \setminus U$ is a divisor and one can easily see that if $i: U \to V$ is the inclusion then (V, D, i) is the smallest compactification of U. By our preparation and since $H^0(V, T_{V/K}) = 0$ we may conclude by Theorem 3. Clearly, the same argument works for a large class of surfaces U, not necessarily of general type.

5. Complete local rings

In this section we discuss the local analog of our theory.

As in §1, let K be an algebraically closed field of characteristic zero. A K-singularity will mean any local neotherian complete K-algebra whose residue field is a trivial extension of K; so A is K-isomorphic to $K[[X_1, \ldots, X_n]]/J$ for some $n \geqslant 1$ and some ideal J. A subfield K_1 of K will be called a field of definition for if there exists a K-isomorphism as above with J generated by elements of $K_1[[X_1, \ldots, X_n]]$.

Now let $\Delta(A)$ be the set of all derivations $\delta: A \to A$ for which $\delta(K) \subset K$ and define

$$K^{\Delta(A)} = \{\lambda \in K;\ \delta\lambda = 0 \quad \text{for all } \delta \in \Delta(A)\}.$$

Clearly $K^{\Delta(A)}$ is an algebraically closed subfield of K. We hope the following to be true:

CONJECTURE 3: *If A is a normal isolated K-singularity, $K^{\Delta(A)}$ is the smallest algebraically closed field of definition for A.*

Now it is easy to see (using an argument analog to that given in the beginning of Section 3) that $K^{\Delta(A)}$ is always contained in any algebraically closed field of definition for A; so the hard part of Conjecture 3 says that $K^{\Delta(A)}$ is a field of definition for A. Note also that if Conjecture 3 holds for A and if k is an algebraically closed subfield of K and $\{t_\alpha\}_\alpha$ is a transcendence basis of K/k then k is a field of definition for A if and only if $\partial/\partial_\alpha^t : K \to K$ lift to derivations $\delta_\alpha : A \to A$.

We are able to prove Conjecture 3 in two special cases:

THEOREM 6: *Conjecture 3 holds in each of the following cases:*
1) *A is a homogeneous singularity.*
2) *A is a quasi-homogeneous surface singularity.*

Recall that a K-singularity is called homogeneous (quasihomogeneous respectively) if there is a K-isomorphism $A \simeq K[[X_1, \ldots, X_n]]/J$ with J generated by homogeneous polynomials (respectively by polynomials which are quasi-homogeneous with respect to some weights $w_1, \ldots, w_n$ associated to $X_1, \ldots, X_n$).

Theorem 6 will be proved by reduction to the global case.

Suppose first A is a quasi-homogeneous surface singularity, $A = K[[X_1, \ldots, X_n]]/(F_1, \ldots, F_m)$, F_j being quasihomogeneous with respect to the weights $w_1, \ldots, w_n$. Put $B = K[X_1, \ldots, X_n]/(F_1, \ldots, F_m) = \bigoplus_{k=0} B_k$ where B_k is the piece of degree k with respect to the weights. Now there are natural K-linear maps $\varphi_k : A \to B_k$ which take the class of a series $f \in K[[X_1, \ldots, X_n]]$ into the class of the polynomial f_k, where f_k is the sum of all monomials of f having degree k (with respect to $w_1, \ldots, w_n$). For any derivation $\delta \in \Delta(A)$ one can construct in a canonical way a derivation $\tilde{\delta} : B \to B$ with $\tilde{\delta}(B_k) \subset B_k$ and such that δ and $\tilde{\delta}$ coincide on K; indeed for any $b \in B$ write $b = \Sigma b_k$, $b_k \in B_k$ and put

$$\tilde{\delta}(b) = \sum \varphi_k(\delta b_k).$$

It is trivial to check that $\tilde{\delta}$ has the desired properties. Put $W = \operatorname{Proj}(B[T])$ where weight$(T) = 1$ and extend $\tilde{\delta}$ to a derivation still denoted by $\tilde{\delta}$ on $B[T]$ such that $\tilde{\delta}T = 0$. Now W is a projective surface and we consider its normalisation $V = W^{\text{nor}}$. Clearly $\tilde{\delta}$ induces a derivation (still denoted by $\tilde{\delta}$) which belongs to $\Delta(W)$. By Seidenberg's theorem [Seidenberg, 1966] this derivation induces a derivation $\tilde{\delta} \in \Delta(V)$. But now $K^{\Delta(V)} \subset K^{\Delta(A)}$ so by Theorem 4, $K_0 = K^{\Delta(A)}$ is a field of definition for V hence V is K-isomorphic to $V_0 \otimes_{K_0} K$ where V_0 is some projective normal K_0-surface. So there exists a K_0-point $p_0 \in V_0$ such that the only K-point of V lying above it is the isolated singular point p corresponding to the irrelevant ideal of B. Let U_0 be an open affine neighbourhood of p_0 in V_0, $U_0 = \mathrm{Spec}(K_0[X_1, \ldots, X_N]/(G_1, \ldots, G_M)$, $p_0 = (X_1 - \lambda_1, \ldots, X_N - \lambda_N)$, $\lambda_j \in K_0$. Then we have K-isomorphisms

$$A \simeq \hat{\mathcal{O}}_{V,p} \simeq \left((K[X_1, \ldots, X_N]/(G_1, \ldots, G_m))_{(X_1 - \lambda_1, \ldots, x_N \lambda_n)}\right)^{\wedge}$$

$$\simeq K[[X_1, \ldots, X_n]]/(\sigma G_1, \ldots, \sigma G_M)$$

where $\sigma : K[[X_1, \ldots, X_N]] \to K[[X_1, \ldots, X_N]]$ takes X_j into $X_j + \lambda_j$ and we are done because $\sigma G_j \in K_0[[X_1, \ldots, X_n]]$.

The proof of Theorem 6 in the homogeneous case is similar and we omit it; instead of using Theorem 4 one has to blow up the vertex of the projective cone W associated to the graded ring of A and to apply Theorem 1 to this blown up cone.

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